Appendix A

Generalized System Equations for a Linear Predictor

The block diagram of a linear predictor is shown in Figure A1. Here $\hat{x}(n)$ is the predicted value of $x(n)$ and $e(n)$ is the $D$ step ahead forward prediction error. The delay $D$ determines how many steps ahead the predictor is designed to predict $x(n)$.

![Figure A.1: Block Diagram of the General Prediction Filter.](image)

The predicted signal:

$$
\hat{x}(n) = -\sum_{k=1}^{M} a^M_k x(n-D-k+1)
$$

(A.1.1)

where $M$ is the order of the predictor and $a_1^M - a_2^M - a_3^M \ldots - a_M^M$ are the predictor coefficients. The $D$ step forward prediction error $e(n)$ is the difference between the desired signal $d(n)$ and the predicted signal, i.e.,

$$
e(n) = d(n) - \hat{x}(n)
$$

$$
= d(n) + \sum_{k=1}^{M} a^M_k x(n-D-k+1)
$$

(A.1.2)

The predictor coefficients are selected such that the mean square value of $e(n)$, i.e., $E\{e(n)^2\}$, is minimized. Therefore, for any set of predictor coefficients $a^M_k \ (1 \leq k \leq M)$,
\[
\frac{\delta}{\delta a_k^M} E\{e(n)^2\} = 0, \quad 1 \leq k \leq M \quad (A.1.3)
\]

Due to the linearity of the expectation and differentiation operators we can interchange these two operations and (A.1.3) can be written as:

\[
\frac{\delta}{\delta a_k^M} E\{e(n)^2\} = E\left\{\frac{\delta}{\delta a_k^M} e(n)^2\right\}
\]

\[
= E\left\{d(n) + \sum_{k=1}^{M} a_k^M x(n-D-k+1) a_k^M x^*(n-D-k+1) \right\} + \sum_{k=1}^{M} a_k^M x(n-D-k+1) a_k^M x^*(n-D-k+1) \right\} 
\]

Since the predictor is estimating the present value of the input sample, the desired signal \(d(n) = x(n)\). Therefore, from (A.1.3) and (A.1.4) we can write,

\[
2 \Re \left[ a_k^M \left\{ r_{xx}(D+l-1) + \sum_{k=1}^{M} a_k^M r_{xx}(l-k) \right\} \right] = 0 \quad , \quad l=1,2,...,L \quad (A.1.5)
\]

\[
\Rightarrow r_{xx}(D+l-1) + \sum_{k=1}^{M} a_k^M r_{xx}(k-l) = 0
\]

where \(r_{xx}(k)\) is the value of the auto-correlation sequence of \(x(n)\) for the \(k\)-th lag. The set of equations in (A.1.5) is known as the normal equations. Equation (A.1.5) indicates \(L\) distinct equations, each for a different value of \(l\). By solving these \(L\) equations, the predictor coefficients can be determined for the minimum mean square error criterion. Now by using the identity \(r_{xx}(-k) = r_{xx}^*(k)\) the \(L\) equations of (A.1.5) can be arranged in matrix format as shown below.

\[
\begin{bmatrix}
  r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(M-1) \\
  r_{xx}^*(1) & r_{xx}(0) & \cdots & r_{xx}(M-2) \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{xx}^*(L-1) & r_{xx}^*(L-2) & \cdots & r_{xx}^*(L-M) \\
\end{bmatrix}
\begin{bmatrix}
  a_1^M \\
  a_2^M \\
  \vdots \\
  a_M^M \\
\end{bmatrix}
= \begin{bmatrix}
  r_{xx}(D) \\
  r_{xx}(D+1) \\
  \vdots \\
  r_{xx}(D+L-1) \\
\end{bmatrix} \quad (A.1.6)
\]
Equation (A.1.6) can be expressed compactly as:

$$ \mathbf{Ra} = \mathbf{b} \quad (A.1.7) $$

where $\mathbf{R}$ is the $(L \times M)$ correlation matrix and $\mathbf{a}$ is the predictor coefficient vector to be determined. The vector $\mathbf{b}$ contains elements of the auto-correlation of the input sequence $x(n)$ and the particular lags of the auto-correlation values depend on how many samples ahead we want to predict. The matrix $\mathbf{R}$ has the special property that the $(i,j)$th element of the matrix

$$ r(i, j) = r_{xx}(i - j). $$

Again, since $r_{xx}(-k) = r_{xx}^*(-k)$, $r(i, j) = r^*(j, i)$. A matrix with these properties is called a Toeplitz matrix.
Appendix B

Let us consider an estimation process that is being used to estimate a desired signal $A$. To reduce the complexity of the problem, $A$ will be considered to be a stationary signal with a constant value. Let the estimated value of the desired signal be $V$. Now $V$ can be written as:

$$V = A + X$$  \hspace{1cm} (A2.1)

In (A2.1) $X$ is the estimation error, which is a random variable (RV). For convenience, let us consider $X$ to be a zero mean, white RV with variance $\sigma_x^2$. Let us consider a new random variable $U$ generated by transforming the RV $V$ in the following way:

$$U = \frac{1}{V} = \frac{1}{A + X}$$  \hspace{1cm} (A2.2)

If there were no estimation error in the system, $U$ would be a constant with a value of $1/A$. Therefore the desired value of $U$ is $1/A$. In the presence of the estimation error $X$, $U$ will be a random variable with a constant component. To analyze the effect of the estimation error $X$, $U$ will be divided into two parts: the first part is the desired value of $U$ and the second part is the error signal resulting from the estimation error, i.e.,

$$U = \frac{1}{A} + e$$  \hspace{1cm} (A2.3)

Now if we assume that the estimation error $X$ is less than the desired value $A$, (A2.2) can be expanded by using the Maclaurin series as follows:

$$U = \frac{1/A}{1 + X/A} = \frac{1}{A} \sum_{n=0}^{\infty} \left( -\frac{X}{A} \right)^n, \quad |X| < |A|$$  \hspace{1cm} (A2.4)

Comparing (A2.3) and (A2.4) the error term of (A2.3) can be written as:
To analyze the effect of the estimation error on $U$, the first and the second order moments of the error term $e$ (in (A2.5)) will be determined.

**First Order Moment**

To determine the moments of the error signal we will always start with (A2.4), the expression for $U$. The first order moment or the mean of the random variable $U$, can be expressed as

$$\mu_u = E\{U\} = E\left\{ \frac{1}{A} \sum_{n=1}^{\infty} \left( -\frac{X}{A} \right)^n \right\}$$

(A2.6)

Exploiting the linearity of the expectation and summation operations, (A2.6) can be written as:

$$\mu_u = E\{U\} = \frac{1}{A} \sum_{n=0}^{\infty} E\left\{ \left( -\frac{X}{A} \right)^n \right\}$$

(A2.7)

Since $X$ is assumed to be zero-mean and white, and $A$ is assumed to be constant, the expected value of the terms in (A2.7) with odd power $n$ will be zero. Again, the expected values of $(X/A)^{2m}$, for $m=2n$, are negligible compared to $E\{(X/A)^2\}$, when $\frac{X}{A} << 1$. Therefore, the mean value of the error term $e$ can be expressed compactly as follows:

$$\mu_u = E\{U\} = \frac{1}{A} + \frac{1}{A} E\left\{ \left( \frac{X}{A} \right)^2 \right\} = \frac{1}{A} + \frac{\sigma_x^2}{A^3}$$

(A2.8)

In (A2.8), the $1/A$ term represents the desired value of $U$ in the absence of the estimation error. Therefore, the mean value of the estimation error can be written as:
\[ \mu_e = E\{e\} = \frac{\sigma_x^2}{A^3} \]  

(A2.9)

From (A2.9) it can be said that, even though \( X \) is a zero mean random variable, the mean of \( e \) is not zero and the mean value depends on the variance of \( X \) and the magnitude of \( A \).

**Second Order Moment**

According to the assumption of a stationary desired amplitude \( A \), the second order central moment of \( U \) should be zero in the absence of estimation error. Therefore, any non-zero value of the second order central moment of \( U \) will be the result of the estimation error. To find the central moment we will first determine the second order moment of \( U \), which is given by:

\[
E\{U^2\} = \frac{1}{A^2} E\left\{ \sum_{n=0}^{\infty} \left( -\frac{X}{A} \right)^n \sum_{m=0}^{\infty} \left( -\frac{X}{A} \right)^m \right\}
\]

\[
= \frac{1}{A^2} E\left\{ \left[ 1 - \frac{X}{A} + \frac{X^3}{A^3} + \cdots \right] \times \left[ 1 - \frac{X}{A} - \frac{X^3}{A^3} + \cdots \right] \right\}
\]

\[
= \frac{1}{A^2} E\left\{ 1 - \frac{2X}{A} + \frac{3X^2}{A^2} - \frac{4X^3}{A^3} + \frac{5X^4}{A^4} - \cdots \right\}
\]

(A2.10)

By using the same assumptions that were used to simplify (A2.7), (A2.10) can be written as:

\[
E\{U^2\} = \frac{1}{A^2} + \frac{1}{A^2} \frac{3\sigma_x^2}{A^2} = \frac{1}{A^2} + \frac{3\sigma_x^2}{A^4}
\]

(A2.11)

To find the effect of the estimation error, the second order central moment of \( U \) will be calculated about the value of \( U \) in the errorless case. Therefore,

\[
\sigma_u^2 = E\{U^2\} - \frac{1}{A^2} = \frac{1}{A^2} + \frac{3\sigma_x^2}{A^4} - \frac{1}{A^2}
\]

\[
= \frac{3\sigma_x^2}{A^4}
\]

(A2.12)
From (A2.12) it can be concluded that the variance of $U$ is greater than the variance of the estimation error $\sigma^2_x$, if $A \leq \frac{4}{\sqrt{3}} \approx 1.3161$. Figure B. 1 shows the effect of $A$ and the variance of the estimation error $\sigma^2_{\hat{X}}$ on the variance of $U$, $\sigma^2_U$. In Figure B. 1, all the values of $\sigma^2_U$ above 40 were truncated to 40. $X$ was considered a zero-mean Gaussian random process while generating Figure B. 1. It is clear from Figure B. 1 that for the smaller values of $\sigma^2_x$, the values of $A$ below which $\sigma^2_{\hat{U}} > \sigma^2$, pretty much coincide with the bound derived in (A2.12). For higher values of $\sigma^2$, the higher order statistics of $X$ are not negligible and thus the assumptions made to derive (A2.12) are no longer valid. In this situation the value of $A$, below which $\sigma^2_{\hat{U}} > \sigma^2$, will not be 1.3161. Figure B. 1 shows that this value of $A$ starts to increase with an increase in $\sigma^2_x$.

![Figure B. 1: Variance of $U$ as a Function of $A$ and the Variance of $X$.](image-url)