Chapter 2 -- Derivation of Basic Equations

2.1 Overview

In structural analysis, plates and shells are defined as bodies which are bounded by two surfaces, where the distance between the bounding surfaces is small compared to their lateral dimensions. The set of all material points equidistant from the two bounding surfaces is termed the middle surface (MS). If the MS is curved, then the body is a shell; otherwise it is a plate. The thinness of the body justifies use of a specialized structural theory, in which the shell or plate is treated essentially like a surface surrounded by a small thickness envelope.

The theory of shells is a complicated subject, and there are numerous books devoted solely to its exposition, including the texts by Flügge (1973) and Novozhilov (1959) which cover the linear theory, and Librescu (1975) and Mushtari and Galimov (1961) which also cover the case of geometrically nonlinear response. In this chapter, we attempt to provide only those details required for the work at hand. The interested reader is directed to any of the above referenced texts for further study. We begin with an introduction to the theory of surfaces, then proceed to derive a shell theory by descent from the 3D theory of elasticity, in lines of curvature coordinates. The derived theory will be presented in both linear and nonlinear forms, for composite laminate shells.

2.2 Introductory theory of surfaces

Consider a surface in three-dimensional space. Positions of points within the surface may be described three-dimensionally, with reference to a global system, or two-dimensionally, with reference to a coordinate system which is local to the surface. Such a 2D system is known as a gaussian coordinate system, and must be defined so that there is a unique one-to-one correspondence between material points of the surface and gaussian coordinate pairs. Let us take a rectangular cartesian coordinate system with coordinates (x,y,z) as our fixed global system, and take coordinates (ξ1,ξ2) as our gaussian coordinates, as shown in Fig. 2.1.
Although there are any number of choices which could be made for the gaussian coordinate system \((\xi_1, \xi_2)\), in the development which follows we will use only coordinates wherein the \(\xi_1\)– and \(\xi_2\)–curves passing through any point are mutually orthogonal.

Let \(\vec{r}\) be a vector which describes the position of a point \(A\) within the surface. See Fig. 2.2. We may write \(\vec{r}\) as
\[ \dot{r} = x(\xi_1, \xi_2)\hat{i} + y(\xi_1, \xi_2)\hat{j} + z(\xi_1, \xi_2)\hat{k} \]  \hspace{1cm} (2.1)

where \( \hat{i}, \hat{j}, \hat{k} \) are unit vectors in the x-, y-, and z- directions, respectively. Denote by \( \dot{r}_1 \) the derivative vector \( \frac{\partial}{\partial \xi_1} \dot{r} \); similarly, \( \dot{r}_2 = \frac{\partial}{\partial \xi_2} \dot{r} \). Clearly, in view of the definition of a derivative, \( \dot{r}_1 \) is tangent to the \( \xi_1 \)-curve, and \( \dot{r}_2 \) is tangent to the \( \xi_2 \)-curve. Now, consider a second point \( B \) also within the surface, separated from \( A \) by a vector \( d\dot{r} \). The distance \( ds \) between \( A \) and \( B \) is given by

\[ ds^2 = d\dot{r} \cdot d\dot{r} \]  \hspace{1cm} (2.2)

In view of equation (2.1), we find

\[ d\dot{r} = \frac{\partial}{\partial \xi_1} \dot{r} d\xi_1 + \frac{\partial}{\partial \xi_2} \dot{r} d\xi_2 \]  \hspace{1cm} (2.3)

The equation (2.3) is used in equation (2.2) to get

\[ ds^2 = A_1^2 d\xi_1^2 + A_2^2 d\xi_2^2 \]  \hspace{1cm} (2.4)

where

\[ A_1^2 = \dot{r}_1 \cdot \dot{r}_1 = |\dot{r}_1|^2 \]  \hspace{1cm} (2.5)
\[ A_2^2 = \dot{r}_2 \cdot \dot{r}_2 = |\dot{r}_2|^2 \]

or

\[ A_1 = |\dot{r}_1| \]  \hspace{1cm} (2.6)
\[ A_2 = |\dot{r}_2| \]

and we have used the relation \( \dot{r}_1 \cdot \dot{r}_2 = 0 \), which arises from the orthogonality of the \( \xi_1 \)- and \( \xi_2 \)-curves. The terms \( A_1 \) and \( A_2 \) are known as the metrics of the surface. They relate infinitesi-
mal distances within the surface to infinitesimal changes in the values of the gaussian coordinates. For example, if $d\mathbf{\hat{r}}$ is directed along a $\xi_1$–curve, then $ds = A_1 d\xi_1$. In general terms, we may write

$$d\mathbf{\hat{r}} = A_1 d\xi_1 \mathbf{\hat{t}}_1 + A_2 d\xi_2 \mathbf{\hat{t}}_2$$

(2.7)

where $\mathbf{\hat{t}}_1, \mathbf{\hat{t}}_2$ are unit vectors in the directions of $\mathbf{\hat{r}}_{1,}, \mathbf{\hat{r}}_{2,}$, respectively.

Unit vectors tangent to and normal to the surface (i.e., the basis vectors of the surface) may be found by normalizing $\mathbf{\hat{r}}_{1,}$ and $\mathbf{\hat{r}}_{2,}$:

$$\mathbf{\hat{t}}_1 = \frac{\mathbf{\hat{r}}_{1,}}{A_1}$$

$$\mathbf{\hat{t}}_2 = \frac{\mathbf{\hat{r}}_{2,}}{A_2}$$

(2.8)

$$\mathbf{\hat{n}} = \pm \frac{(\mathbf{\hat{r}}_{1,} \times \mathbf{\hat{r}}_{2,})}{(A_1 A_2)}$$

where $\mathbf{\hat{t}}_1, \mathbf{\hat{t}}_2$ are tangent to the $\xi_1$- and $\xi_2$-curves, and $\mathbf{\hat{n}}$ is normal to the surface. The sign of $\mathbf{\hat{n}}$ may be chosen for convenience to place the normal as either “inwardly” or “outwardly” directed, as desired. The unit basis vectors of the surface change direction from point to point within the surface as a result of curvature.

![Diagram](image)

**Fig. 2.3 Derivatives of the Normal Vector**

We will require knowledge of the derivatives of the basis vectors with respect to the gaus-
sian coordinates. These derivatives appear in their simplest form when expressed in lines of curvature coordinates. The coordinates $\xi_1, \xi_2$ are lines of curvature coordinates if, in addition to orthogonality, they pass the following test: for every $\xi_i$-curve on the surface ($i = 1, 2$), a plane which contains that curve must also contain the normal to the surface at every point on that curve. These conditions ensure that adjacent normals to the surface, taken along a coordinate curve, intersect in the $\hat{t}_i \cdot \hat{n}$ plane. That is, $\hat{n}_1 \cdot \hat{t}_2 = 0$ and $\hat{n}_2 \cdot \hat{t}_1 = 0$. For lines of curvature coordinates, we may make the following argument (See Novozhilov (1959), Chapter 1) to find the derivatives of the normal vectors. Two neighboring points along a $\xi_1$-curve are separated by a small coordinate value $\Delta \xi_1$, and have normal vectors as shown in Fig. 2.3. The difference between the two normal vectors is denoted by $\Delta \hat{n}$, and the length of the arc which joins the two points is $A_1 \Delta \xi_1$. The radius of curvature of the $\xi_1$-curve is $R_1$. As $\Delta \xi_1$ becomes small, the arc which joins points $A$ and $B$ becomes a straight line, directed along $\hat{t}_1$, and the vector $\Delta \hat{n}$ also becomes parallel to $\hat{t}_1$. Then, using similar triangles, we see

$$\frac{\Delta \hat{n}}{\hat{n}} = \frac{A_1 \Delta \xi_1 \hat{t}_1}{R_1}$$

The vector $\hat{n}$ has unit magnitude. In the limit as $\Delta \xi_1 \rightarrow 0$, we get

$$\frac{\partial}{\partial \xi_1}(\hat{n}) = \frac{A_1}{R_1} \hat{t}_1$$

(2.9)

Likewise,

$$\frac{\partial}{\partial \xi_2}(\hat{n}) = \frac{A_2}{R_2} \hat{t}_2$$

(2.10)

where $R_2$ is the radius of curvature of a $\xi_2$-curve.
Fig. 2.4 Tangent and Normal Vectors of a Gaussian Coordinate Curve

Calculation of the derivatives of the tangent vectors is as follows. First, it will be seen that the derivative of the tangent vector is perpendicular to the tangent vector. Then, scalar products of basis vectors with the derivative will be taken to get the components of the derivative vectors.

Consider a small arc of a $\xi_1$-curve, with basis and normal vectors $\hat{t}_1, \hat{n}$ as shown in Fig. 2.4. Because the magnitude of $\hat{t}_1$ is unity, we may write

$$|\hat{t}_1|^2 = \hat{t}_1 \cdot \hat{t}_1 = 1$$

Taking the derivative of the above expression, we get

$$\frac{\partial |\hat{t}_1|^2}{\partial \xi_1} = 2 \hat{t}_1 \cdot \frac{\partial \hat{t}_1}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} (1) = 0,$$

and by similar reasoning,

$$\frac{\partial |\hat{t}_1|^2}{\partial \xi_2} = 2 \hat{t}_1 \cdot \frac{\partial \hat{t}_1}{\partial \xi_2} = \frac{\partial}{\partial \xi_2} (1) = 0$$

which proves that the derivatives of the tangent vectors are perpendicular to the tangent vectors.

Next, consider the mixed derivative of the position vector $\hat{r}$ with respect to the gaussian coordinates. If $\hat{r}$ is continuous and single-valued, then the order of differentiation is immaterial. Thus,

$$\frac{\partial^2 \hat{r}}{\partial \xi_1 \partial \xi_2} = \frac{\partial}{\partial \xi_1} \frac{\partial \hat{r}}{\partial \xi_2} = \frac{\partial}{\partial \xi_2} \frac{\partial \hat{r}}{\partial \xi_1} = \frac{\partial^2 \hat{r}}{\partial \xi_2 \partial \xi_1}$$

or, using equation (2.8),
\[
\frac{\partial}{\partial \xi_2}(A_1 \hat{t}_1) = \frac{\partial}{\partial \xi_1}(A_2 \hat{t}_2)
\]

whence

\[
\frac{\partial \hat{t}_2}{\partial \xi_1} = \frac{1}{A_2} \left[ \frac{\partial}{\partial \xi_2}(A_1 \hat{t}_1) - \frac{\partial A_2}{\partial \xi_2} \right]
\]

Now,

\[
\hat{t}_2 \cdot \frac{\partial \hat{t}_1}{\partial \xi_1} = \frac{\partial}{\partial \xi_1}(\hat{t}_2 \cdot \hat{t}_1) - \frac{\partial \hat{t}_2}{\partial \xi_1} \cdot \hat{t}_1 = \frac{\partial \hat{t}_2}{\partial \xi_1} \cdot \hat{t}_1
\]

with the last step arising from the orthogonality of the \(\xi_1\)- and \(\xi_2\)-curves. Thus,

\[
\hat{t}_2 \cdot \frac{\partial \hat{t}_1}{\partial \xi_1} = -\frac{1}{A_2} \left[ \hat{t}_1 \cdot \frac{\partial}{\partial \xi_2}(A_1 \hat{t}_1) - \frac{\partial A_2}{\partial \xi_2} (\hat{t}_2 \cdot \hat{t}_1) \right] = -\frac{A_1 \hat{t}_1}{A_2 \partial \xi_2} \cdot \hat{t}_1 - \frac{1}{A_2 \partial \xi_2} (\hat{t}_1 \cdot \hat{t}_1)
\]

Using the orthogonality of the tangent vector with its derivative, we finally get

\[
\hat{t}_2 \cdot \frac{\partial \hat{t}_1}{\partial \xi_1} = -\frac{1}{A_2 \partial \xi_2} \frac{\partial A_1}{\partial \xi_2}
\]

Similar manipulations may be performed to get the remaining derivatives and products. If this approach is taken, we get the Gauss-Weingarten relations:
In equation (2.11) and henceforth, the notation \((\cdot)_1\) will represent the derivative of \((\cdot)\) with respect to \(\xi_1\). Similarly, \((\cdot)_2\) represents the derivative of \((\cdot)\) with respect to \(\xi_2\). It may be noted from equation (2.11) that 
\[
\hat{n} \cdot \vec{r}_{12} = \hat{n} \cdot \vec{r}_{21} = 0.
\]
This is the mathematical condition which, along with their mutual orthogonality, define the coordinate curves as *lines of curvature*.

There are, finally, two other useful relations to be derived from consideration of the derivatives of the basis vectors. These two relations are known as the Gauss-Codazzi relations, and stem from the continuous, single-valued nature of the basis vectors. First, noting that 
\[
\hat{n}_{12} = \hat{n}_{21},
\]
and using equation (2.11), we may derive the relations of Codazzi:

\[
\frac{\partial}{\partial \xi_2} \left( \frac{A_1}{R_1} \right) = \frac{1}{R_2} \frac{\partial A_1}{\partial \xi_2},
\]

\[
\frac{\partial}{\partial \xi_1} \left( \frac{A_2}{R_2} \right) = \frac{1}{R_1} \frac{\partial A_2}{\partial \xi_1}.
\]

Then by asserting that \(\hat{t}_{1,12} = \hat{t}_{1,21}\) or \(\hat{t}_{2,12} = \hat{t}_{2,21}\) along with equation (2.11), we get

\[
\begin{align*}
\begin{pmatrix}
\hat{t}_{1,1} \\
\hat{t}_{1,2} \\
\hat{t}_{2,1} \\
\hat{t}_{2,2} \\
\hat{n}_{1} \\
\hat{n}_{2}
\end{pmatrix}
&= 
\begin{bmatrix}
0 & \frac{A_{1,2}}{A_2} & \frac{A_1}{R_1} \\
0 & \frac{A_{2,1}}{A_1} & 0 \\
\frac{A_{1,2}}{A_2} & 0 & \frac{A_2}{R_2} \\
\frac{A_{2,1}}{A_1} & 0 & \frac{A_2}{R_2} \\
\frac{A_1}{R_1} & 0 & 0 \\
0 & \frac{A_2}{R_2} & 0
\end{bmatrix}
\begin{pmatrix}
\hat{t}_1 \\
\hat{t}_2 \\
\hat{n}
\end{pmatrix},
\end{align*}
\]
which is known as the Gauss relation.

\[
\frac{\partial}{\partial \xi_1} \left( \frac{1}{A_1} \frac{\partial A_2}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left( \frac{1}{A_2} \frac{\partial A_1}{\partial \xi_2} \right) = \frac{A_1 A_2}{R_1 R_2},
\]

For a general surface of revolution with the \(z\)-axis as the axis of rotation, we may find expressions for the radii of curvature as follows. Choose for \((\xi_1, \xi_2)\) the coordinates \((\eta, \theta)\) as shown in Fig. 2.5, where \(\eta\) corresponds to the latitude of a point and \(\theta\) corresponds to the longitude. Given this choice of coordinates, we may write the vector \(\mathbf{r}'\) as

\[
\mathbf{r}'(\eta, \theta) = x(\eta, \theta)\hat{i} + y(\eta, \theta)\hat{j} + z(\eta, \theta)\hat{k}
\]
or

\[
\mathbf{r}'(\eta, \theta) = R(\eta) \cos \eta \cos \theta \hat{i} + R(\eta) \cos \eta \sin \theta \hat{j} + R(\eta) \sin \eta \hat{k}
\]

where, due to axisymmetry, \(|\mathbf{r}'| = R(\eta)\). Now, with equation (2.14), we may find the metrics \(A_1\) and \(A_2\) by equation (2.6); we find the unit basis vectors by equation (2.8), using the negative sign on \(\hat{n}\) in order to have a positive outward normal. By the Gauss-Weingarten relations, (2.11), we see that
which leads us (with some algebra) to

\[
\frac{1}{R_1} = -\hat{n} \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \eta}
\]

\[
\frac{1}{R_2} = -\hat{n} \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \theta}
\]

in which primes indicate differentiation with respect to \(\eta\). As previously stated, the terms \(R_1, R_2\) are the radii of curvature of the lines of curvature; the reciprocals of these radii are known as the principal curvatures of the surface.

Finally, consider that there exists a second surface, parallel to and separated from the first by a small distance \(\zeta\), measured normal to the first (reference) surface. If the vector \(\hat{r}\) denotes the location of a point on the reference surface, then a correspondent point on the parallel surface is located by

\[
\hat{r} = \hat{r} + \zeta \hat{n}
\]

See Fig. 2.6.
Tangent vectors to the parallel surface are found by
\[ \vec{R}_{,1} = \vec{r}_{,1} + \zeta \hat{n}_{,1}, \]
and so on. By using equations (2.8) and (2.11), we get
\[ \vec{R}_{,1} = H_1 \hat{t}_1 \]
\[ \vec{R}_{,2} = H_2 \hat{t}_2 \]
\[ R_{,\zeta} = H_3 \hat{n} \]  \hspace{1cm} (2.16)
where
\[ H_1 = A_1 \left( 1 + \frac{\zeta}{R_1} \right) \]
\[ H_2 = A_2 \left( 1 + \frac{\zeta}{R_2} \right) \] \hspace{1cm} (2.17)
\[ H_3 = H_{,\zeta} = 1 \]
H_1, H_2, and H_3 are the metrics of the parallel surface, so that
\[ d\vec{R} = H_1 d\xi_1 \hat{t}_1 + H_2 d\xi_2 \hat{t}_2 + d\zeta \hat{n} \] \hspace{1cm} (2.18)
The magnitude of \( d\vec{R} \) is given by \( dS \), where
\[ dS^2 = H_1^2 d\xi_1^2 + H_2^2 d\xi_2^2 + d\zeta^2 \] \hspace{1cm} (2.19)
which concludes the introductory theory of surfaces.

**2.3 Kinematics of deformation & definition of strain measures**

Recall from the introductory section of this chapter that it is our intention to treat a shell as a small thickness envelope surrounding a reference surface. For our purposes, we will take the middle surface (MS) to be the reference surface. The shell will have a constant thickness of \( h \), with the bounding surfaces thus located at \( \zeta = h/2 \) and \( \zeta = -h/2 \). Furthermore, we are specifically interested here in “thin” shells, wherein \( h/R \ll 1 \), where \( R \) is the lesser of \( R_1 \) and \( R_2 \).
The system of coordinates $\xi_1, \xi_2$ and $\zeta$, along with the tangent vectors $\hat{t}_1, \hat{t}_2$ and $\hat{n}$ form an orthonormal basis for a three-dimensional body.

Consider that a material point $A$ is located by the vector $R$ in the undeformed body. A second point $B$ is located by $dR$ given by equation (2.18) relative to $A$. The points $A$ and $B$, along with the basis vectors, define a rectangular parallelepiped. Under deformation, the point $A$ moves through a displacement vector $U$ to $A^*$ located by

$$R^* = R + U \quad (2.20)$$

In the same deformation, the basis vectors are translated, rotated and stretched into new basis vectors $g_1, g_2, g_3$ for the deformed system, as illustrated in Fig. 2.7, and the deformed parallelepiped is not generally rectangular. That is, the deformed basis vectors do not form an orthonormal set.

Under deformation, the line element $dR$ becomes
\[ \vec{dR} = H_1 d\xi_1 \vec{g}_1 + H_2 d\xi_2 \vec{g}_2 + d\xi_3 \vec{g}_3. \] (2.21)

We may also write \( \vec{dR} \) as

\[ \vec{dR} = \vec{R}_1 d\xi_1 + \vec{R}_2 d\xi_2 + \vec{R}_3 d\xi_3. \] (2.22)

Comparison of (2.21) and (2.22), in view of (2.20) and (2.16) yields

\[ \begin{align*}
\vec{g}_1 &= \hat{t}_1 + \frac{1}{H_1} \vec{U}_1, \\
\vec{g}_2 &= \hat{t}_2 + \frac{1}{H_2} \vec{U}_2, \\
\vec{g}_3 &= \hat{n} + \vec{U}_\zeta.
\end{align*} \] (2.23)

The displacement gradients within (2.23) may be written in terms of the basis vectors of the undeformed system as

\[ \begin{align*}
\frac{1}{H_1} \vec{U}_1 &= \epsilon_{11} \hat{t}_1 + \epsilon_{12} \hat{t}_2 + \epsilon_{13} \hat{n} \\
\frac{1}{H_2} \vec{U}_2 &= \epsilon_{21} \hat{t}_1 + \epsilon_{22} \hat{t}_2 + \epsilon_{23} \hat{n} \\
\vec{U}_\zeta &= \epsilon_{31} \hat{t}_1 + \epsilon_{32} \hat{t}_2 + \epsilon_{33} \hat{n}
\end{align*} \] (2.24)

whence,

\[ \begin{align*}
\vec{g}_1 &= (1 + \epsilon_{11}) \hat{t}_1 + \epsilon_{12} \hat{t}_2 + \epsilon_{13} \hat{n} \\
\vec{g}_2 &= \epsilon_{21} \hat{t}_1 + (1 + \epsilon_{22}) \hat{t}_2 + \epsilon_{23} \hat{n} \\
\vec{g}_3 &= \epsilon_{31} \hat{t}_1 + \epsilon_{32} \hat{t}_2 + (1 + \epsilon_{33}) \hat{n}
\end{align*} \] (2.25)

We now make an assumption: straight line elements initially normal to the MS remain straight after deformation. This assumption implies that the basis vector \( \vec{g}_3 \) is independent of the thickness coordinate \( \zeta \). Then by the third of equations (2.23), \( \vec{U}_\zeta \) is also independent of \( \zeta \).
further assume vector $\mathbf{U}_\zeta$ is parallel to the tangent plane in the undeformed shell, or $\varepsilon_{33} = 0$.

See equation (2.24). These assumptions limit us to the First-order Transverse Shear Deformation Theory. For the displacement vector $\mathbf{U}$ given by

\[
\mathbf{U} = U_1 \hat{t}_1 + U_2 \hat{t}_2 + U_\zeta \hat{n}
\]  

(2.26)

the first-order transverse shear deformation assumptions result in

\[
\begin{align*}
U_1(\xi_1, \xi_2, \zeta) &= u(\xi_1, \xi_2) + \zeta \phi_1(\xi_1, \xi_2) \\
U_2(\xi_1, \xi_2, \zeta) &= v(\xi_1, \xi_2) + \zeta \phi_2(\xi_1, \xi_2) \\
U_3(\xi_1, \xi_2, \zeta) &= w(\xi_1, \xi_2)
\end{align*}
\]  

(2.27)

with $u, v, w, \phi_1, \phi_2$ as graphically depicted in Fig. 2.8; $u, v, w$ are translations, $\phi_1, \phi_2$ are rotations of the normal element.

Use of (2.26), (2.27) in (2.24), along with the Gauss-Weingarten relations (2.11) yields, after some algebra,
Equations (2.28), (2.29) allow us to write the displacement gradients of any point within the shell as a linear function of the thickness coordinate, $\zeta$. The superscript $^o$ on the terms above indicates a MS value; terms $\varepsilon_{ij}^o$ represent MS displacement gradients, and terms $\kappa_{ij}$ are gradients of the rotations, commonly referred to as curvatures.

Green’s strain measure is defined by the equation

$$ E_g = \frac{(dS^*)^2 - (dS)^2}{2(dS)^2} $$

(2.30)

where $dS^2 = \overrightarrow{\text{d}R} \cdot \overrightarrow{\text{d}R} = |\overrightarrow{\text{d}R}|^2$, and $dS^2 = \overrightarrow{\text{d}R} \cdot \overrightarrow{\text{d}R} = |\overrightarrow{\text{d}R}|^2$. We make the following definitions: $l_1 = \frac{H_1 d\xi_1}{dS}$, $l_2 = \frac{H_2 d\xi_2}{dS}$, $l_3 = \frac{d\zeta}{dS}$. Here, $l_1, l_2, l_3$ define the direction cosines of
the line element \(d\mathbf{R}\); equation (2.19) shows that \(l_1^2 + l_2^2 + l_3^2 = 1\). Then, using equations (2.18), (2.21), (2.25) in (2.30), we get

\[
E_g = E_{11}l_1^2 + \Gamma_{12}l_1l_2 + \Gamma_{13}l_1l_3 + E_{22}l_2^2 + \Gamma_{23}l_2l_3 + E_{33}l_3^2
\]

(2.31)

where

\[
E_{11} = \varepsilon_{11} + \frac{1}{2}(\varepsilon_{11}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2) \quad \Gamma_{13} = \varepsilon_{13} + \phi_1 + \varepsilon_{11}\phi_1 + \varepsilon_{12}\phi_2
\]

\[
E_{22} = \varepsilon_{22} + \frac{1}{2}(\varepsilon_{21}^2 + \varepsilon_{22}^2 + \varepsilon_{23}^2) \quad \Gamma_{23} = \varepsilon_{23} + \phi_2 + \varepsilon_{21}\phi_1 + \varepsilon_{22}\phi_2
\]

(2.32)

\[
E_{33} = \frac{1}{2}(\phi_1^2 + \phi_2^2) \quad \Gamma_{12} = \varepsilon_{12} + \varepsilon_{21} + \varepsilon_{11}\varepsilon_{21} + \varepsilon_{12}\varepsilon_{22} + \varepsilon_{13}\varepsilon_{23}
\]

The strain measures \(E_{ij}, \Gamma_{ij}\) represent the full nonlinear Green’s strains consistent with the first-order transverse shear deformation theory. We may further simplify the strain-displacement relations, under the assumption of small strains and moderate rotations.

Consider three infinitesimal line elements \((\hat{t}_1 d\xi_1, \hat{t}_2 d\xi_2, \hat{n} d\zeta)\) in the undeformed shell which displace, stretch and rotate to vectors \((\hat{g}_1 d\xi_1, \hat{g}_2 d\xi_2, \hat{g}_3 d\zeta)\), respectively, in the deformed shell. Omitting the displacement, equations (2.25) are used to depict the basis vectors in the
deformed shell. See Fig. 2.9. The displacement gradients $\varepsilon_{13}, \varepsilon_{23}$ represent the out-of-plane rotations of the elements originally in the tangent plane of the undeformed shell. For the purpose of this argument, we will refer to $\varepsilon_{13}, \varepsilon_{23},$ and the rotational displacements $\phi_1, \phi_2$ as “rotations”, and the remaining gradients as “strains”. Note that by equation (2.28), $\varepsilon_{31} = \phi_1,$ and $\varepsilon_{32} = \phi_2.$

Under the assumption that all strains and all rotations are small, we are able to simplify the strain measures of equations (2.32) by simply retaining only the lowest-ordered terms of each strain measure. This leads us to the linear strain measures:

$$
\begin{align*}
E_{11}^L &= \varepsilon_{11} \\
E_{22}^L &= \varepsilon_{22} \\
\Gamma_{12}^L &= \varepsilon_{12} + \varepsilon_{21} \\
\Gamma_{13}^L &= \varepsilon_{13} + \phi_1 \\
\Gamma_{23}^L &= \varepsilon_{23} + \phi_2
\end{align*}
$$

along with a nonlinear equation for $E_{33}$:

$$
E_{33} = \frac{1}{2}(\phi_1^2 + \phi_2^2)
$$

The linearized strain-displacement relations may be made explicit in the thickness coordinate by use of equations (2.28), (2.29).

In order to account for some geometric nonlinearity, we may assume the strains to be small, and rotations to be of moderate size. That is, we assume the strains to be on the order of $\mu$, $0 < \mu \ll 1$, and assume rotations $\varepsilon_{13}, \varepsilon_{23}, \phi_1, \phi_2$ to be on the order of $\mu^{1/2}$. These assumptions are justified because we are interested in developing a shell theory, wherein we expect the shell to have large stiffness in-plane, relative to out-of-plane stiffness. Keeping terms to the lowest order only from equation (2.32) thus leads to the strain measures

$$
\begin{align*}
E_{11} &= \varepsilon_{11} + \frac{1}{2}\varepsilon_{13}^2 \\
E_{22} &= \varepsilon_{22} + \frac{1}{2}\varepsilon_{23}^2 \\
E_{33} &= \frac{1}{2}(\phi_1^2 + \phi_2^2) \\
\Gamma_{13} &= \gamma_{13} + \phi_1 \\
\Gamma_{23} &= \gamma_{23} + \phi_2 \\
\Gamma_{12} &= \gamma_{12} + \varepsilon_{12} + \varepsilon_{21} + \varepsilon_{13}\varepsilon_{23}
\end{align*}
$$

Note that the normal strain in the thickness direction, $E_{33}$, is non-zero, and is a nonlinear function of the rotations. This issue of non-zero $E_{33}$ will be dealt with in the discussion of the material law. We may write the strain measures (2.34) explicitly in $\zeta$ by the following process: use equa-
tions (2.28) to substitute for $\varepsilon_{11}$, etc.; factor out a common denominator from each strain measure; in the numerator, approximate the terms $\frac{1}{1 + \zeta/R}$ by the first term of their series expansions:

$$\frac{1}{1 + \zeta/R} = 1 - \frac{\zeta}{R}, \text{ for } R = R_1, R_2,$$

which is an appropriate approximation for thin shells; linearize the numerators with respect to $\frac{\zeta}{R_1}, \frac{\zeta}{R_2}$. We finally arrive at the terms

$$E_{11} = \frac{1}{1 + \zeta/R_1}(E_{11}^\rho + \zeta\chi_{11}) \quad E_{22} = \frac{1}{1 + \zeta/R_2}(E_{22}^\rho + \zeta\chi_{22})$$

$$E_{33} = \frac{1}{2}(\phi_1^2 + \phi_2^2) \quad \Gamma_{12} = \frac{1}{(1 + \zeta/R_1)(1 + \zeta/R_2)}(\Gamma_{12}^\rho + \zeta\chi_{12})$$

(2.35)

with

$$E_{11}^\rho = \varepsilon_{11} + \frac{1}{2}(\varepsilon_{13})^2 \quad E_{22}^\rho = \varepsilon_{22} + \frac{1}{2}(\varepsilon_{23})^2$$

(2.36)

$$\Gamma_{12}^\rho = \varepsilon_{12} + \varepsilon_{21} + \varepsilon_{13}\varepsilon_{23} \quad \Gamma_{13}^\rho = \varepsilon_{13} + \phi_1 \quad \Gamma_{23}^\rho = \varepsilon_{23} + \phi_2$$

$$\chi_{11} = \kappa_{11} - \frac{\varepsilon_{13}^2}{2R_1} - \frac{\varepsilon_{13}\phi_1}{R_1} \quad \chi_{22} = \kappa_{22} - \frac{\varepsilon_{23}^2}{2R_2} - \frac{\varepsilon_{23}\phi_2}{R_2}$$

$$\chi_{12} = \kappa_{12} + \kappa_{21} + \frac{\varepsilon_{12}}{R_2} + \frac{\varepsilon_{21}}{R_1} - \frac{\varepsilon_{13}\phi_2}{R_2} - \frac{\varepsilon_{23}\phi_1}{R_1}$$

2.4 Material law

We will assume linear elastic behavior, with the shell constructed of a number of laminae of monoclinic materials, each lamina having a plane of symmetry parallel to the shell MS. In such a case, the generalized Hooke’s Law for each lamina is given by
where \( S_{ij} \) are “stresses” of the second Piola-Kirchhoff type. That is, they measure force on a small element in the deformed state, divided by the undeformed area of the element. A reduced material law is found by assuming \( S_{33} \) to be negligible compared to the in-plane stresses \( S_{11}, S_{22}, \) and \( S_{12} \). This allows us to take \( S_{33} = 0 \). We then solve for \( E_{33} \) in terms of the other stresses, substitute back into the material law and get

\[
\begin{bmatrix}
S_{11} \\
S_{22} \\
S_{33} \\
S_{23} \\
S_{13} \\
S_{12}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\
0 & 0 & 0 & C_{44} & C_{45} & 0 \\
0 & 0 & 0 & C_{45} & C_{55} & 0 \\
C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
E_{11} \\
E_{22} \\
E_{33} \\
\Gamma_{23} \\
\Gamma_{13} \\
\Gamma_{12}
\end{bmatrix}
\]

(2.37)

where

\[
\widetilde{Q}_{ij} = C_{ij} - (C_{i3}C_{j3})/C_{33} \quad i, j = 1, 2, 6
\]

\[
\widetilde{Q}_{ij} = C_{ij} \quad i, j = 4, 5
\]

Note that the reduction has effectively removed normal strain \( E_{33} \) from the problem. The troublesome condition of having a nonlinear expression for \( E_{33} \) in the small strain, moderate rotation theory is thus circumvented. See the third of equations (2.34).
2.5 Equilibrium

2.5.1 Internal virtual work

From the theory of elasticity, we know that the internal virtual work of a body is given by

\[ \delta W_{int} = \int \int \int_{Vol} [S_1 \cdot \delta \varepsilon_1 + S_2 \cdot \delta \varepsilon_2 + S_3 \cdot \delta \varepsilon_3] H_1 H_2 d\xi_1 d\xi_2 d\zeta \] (2.38)

where the volume element \( dV \) is equal to \( H_1 H_2 d\xi_1 d\xi_2 d\zeta \). The integral expression in (2.38) represents the application of a small virtual displacement to each body element in the deformed state.

The stress vectors \( \vec{S}_1, \vec{S}_2, \vec{S}_3 \) may be chosen as referring to the undeformed body axes and the undeformed areas of the element (first Piola-Kirchhoff stresses or P-K-1), to the deformed body axes and undeformed areas (second Piola-Kirchhoff stresses or P-K-2) or to the deformed body axes and deformed area (Cauchy-Lagrange stresses). In terms of P-K-1, we write

\[
\begin{align*}
\vec{S}_1 &= T_{11} \hat{t}_1 + T_{12} \hat{t}_2 + T_{13} \hat{n} \\
\vec{S}_2 &= T_{21} \hat{t}_1 + T_{22} \hat{t}_2 + T_{23} \hat{n} \\
\vec{S}_3 &= T_{31} \hat{t}_1 + T_{32} \hat{t}_2 + T_{33} \hat{n}
\end{align*}
\]

We get variations of the lattice vectors from equation (2.25) as

\[
\begin{align*}
\delta g_1 &= \delta \varepsilon_{11} \hat{t}_1 + \delta \varepsilon_{12} \hat{t}_2 + \delta \varepsilon_{13} \hat{n} \\
\delta g_2 &= \delta \varepsilon_{21} \hat{t}_1 + \delta \varepsilon_{22} \hat{t}_2 + \delta \varepsilon_{23} \hat{n} \\
\delta g_3 &= \delta \phi_1 \hat{t}_1 + \delta \phi_2 \hat{t}_2
\end{align*}
\] (2.39)

Then we get from (2.38)

\[
\delta W_{int} = \int \int \int_{Vol} [T_{11} \delta \varepsilon_{11} + T_{12} \delta \varepsilon_{12} + T_{13} \delta \varepsilon_{13} + T_{21} \delta \varepsilon_{21} + T_{22} \delta \varepsilon_{22} + T_{23} \delta \varepsilon_{23} + T_{31} \delta \phi_1 + T_{32} \delta \phi_2] H_1 H_2 d\xi_1 d\xi_2 d\zeta \] (2.40)

The displacement gradients may be made explicit in \( \zeta \) by equations (2.28), (2.29), and integration.
through the thickness may be performed to get

\[ \delta W_{int} = \int \int_{\text{Area}} \left[ N_{11} \delta e_{11}^0 + N_{12} \delta e_{12}^0 + N_{21} \delta e_{21}^0 + N_{22} \delta e_{22}^0 + M_{11} \delta \kappa_{11} + M_{12} \delta \kappa_{12} + M_{21} \delta \kappa_{21} + M_{22} \delta \kappa_{22} + (Q_1 \delta \epsilon_{13} + S_1 \delta \phi_1) \right. \\
\left. + (Q_2 \delta \epsilon_{23} + S_2 \delta \phi_2) \right] A_1 A_2 d\xi_1 d\xi_2 \]

where the implicit definitions have been made:

\[ (N_{11}, M_{11}) = \int_{-h/2}^{h/2} T_{11}(1, \zeta)(1 + \zeta/R_2) d\zeta \]

\[ (N_{12}, M_{12}) = \int_{-h/2}^{h/2} T_{12}(1, \zeta)(1 + \zeta/R_2) d\zeta \]

\[ (N_{21}, M_{21}) = \int_{-h/2}^{h/2} T_{21}(1, \zeta)(1 + \zeta/R_1) d\zeta \]

\[ (N_{22}, M_{22}) = \int_{-h/2}^{h/2} T_{22}(1, \zeta)(1 + \zeta/R_1) d\zeta \]

\[ Q_1 = \int_{-h/2}^{h/2} T_{13}(1 + \zeta/R_2) d\zeta \]

\[ Q_2 = \int_{-h/2}^{h/2} T_{23}(1 + \zeta/R_1) d\zeta \]

\[ S_1 = \int_{-h/2}^{h/2} [T_{31}(1 + \zeta/R_1)(1 + \zeta/R_2) - T_{13}(1 + \zeta/R_2)(\zeta/R_1)] d\zeta \]

\[ S_2 = \int_{-h/2}^{h/2} [T_{32}(1 + \zeta/R_1)(1 + \zeta/R_2) - T_{23}(1 + \zeta/R_1)(\zeta/R_2)] d\zeta \]

Here we have looked at the case of geometric nonlinearity by considering virtual work on the deformed body. For a geometrically linear analysis, we allow equilibrium stresses to exist on an undeformed body. Moment equilibrium then leads to the conclusions \((T_{13} = T_{31})\), \((T_{23} = T_{32})\) and \((T_{12} = T_{21})\). This symmetry of stresses leads to a different expression for \(\delta W_{int}\).
\[ \delta W_{int}^L = \iint_{Area} \left[ N_{11} \delta \varepsilon_{11}^o + N_{12} \delta \varepsilon_{12}^o + N_{21} \delta \varepsilon_{21}^o + N_{22} \delta \varepsilon_{22}^o + M_{11} \delta \kappa_{11} \ight. \\
+ M_{12} \delta \kappa_{12} + M_{21} \delta \kappa_{21} + M_{22} \delta \kappa_{22} + Q_1 \delta \gamma_{13}^o + Q_2 \delta \gamma_{23}^o \big] A_1 A_2 d\xi_1 d\xi_2 \]

Note that in this latest expression, the shear stress couple terms \( S_1, S_2 \) do not appear, and the shear stress resultants \( Q_1 \) and \( Q_2 \) multiply different strain measures than in the nonlinear case. These new strain measures are defined by the equations \( \gamma_{13}^o = \varepsilon_{13}^o + \phi_1, \gamma_{23}^o = \varepsilon_{23}^o + \phi_2 \). We also define \( \gamma_{12}^o = \varepsilon_{12}^o + \varepsilon_{23}^o \).

The internal virtual work may also be written in terms of P-K-2 stresses, wherein

\[
\begin{align*}
\rightarrow S_1 &= S_{11} g_1 + S_{12} g_2 + S_{13} g_3 \\
\rightarrow S_2 &= S_{21} g_1 + S_{22} g_2 + S_{23} g_3 \\
\rightarrow S_3 &= S_{31} g_1 + S_{32} g_2 + S_{33} g_3
\end{align*}
\]

These expansions for the stress vectors are substituted into equation (2.38) along with (2.25) and (2.39) and the multiplication is carried out, recalling that we have assumed in the material law that \( S_{33} = 0 \). Moment equilibrium on the deformed body leads to the conclusion that \( S_{ij} \) is symmetric. In view of equations (2.32) we are thus able to write

\[ \delta W_{int} \]

\[
= \iint_{\nu_{,\nu}} \left[ S_{11} \delta E_{11} + S_{12} \delta \Gamma_{12} + S_{13} \delta \Gamma_{13} + S_{22} \delta E_{22} + S_{23} \delta \Gamma_{23} \right] H_1 H_2 d\zeta d\xi_1 d\xi_2
\]

which, in terms of the small strain, moderate rotation approximations (2.44) becomes
Comparing equations (2.46) and (2.40), we see a correspondence between stresses in the P-K-1 form and the P-K-2 form:

\[
\begin{align*}
T_{11} &= S_{11} & T_{12} &= S_{12} & T_{13} &= S_{13} + S_{11}\epsilon_{13} + S_{12}\epsilon_{23} \\
T_{21} &= S_{21} & T_{22} &= S_{22} & T_{23} &= S_{23} + S_{12}\epsilon_{13} + S_{22}\epsilon_{23} \\
T_{31} &= S_{31} & T_{32} &= S_{32}
\end{align*}
\]  

(2.47)

We note from (2.33) and (2.40) that the P-K-1 stresses are conjugate to the displacement gradients, and from (2.45) that the P-K-2 stresses are conjugate to the Green’s strains.

Use of the equations (2.35) in (2.45) and integration through the thickness results in the expression for internal virtual work

\[
\delta W_{int} = \int \int_{Vol} \left[ S_{11}\delta\epsilon_{11} + S_{12}\delta\epsilon_{12} + (S_{13} + S_{11}\epsilon_{13} + S_{12}\epsilon_{23})\delta\epsilon_{13} + S_{12}\delta\epsilon_{21} + S_{22}\delta\epsilon_{22} + (S_{23} + S_{12}\epsilon_{13} + S_{22}\epsilon_{23})\delta\epsilon_{23} + S_{13}\delta\phi_1 + S_{23}\delta\phi_2 \right] H_1 H_2 d\zeta d\xi_1 d\xi_2
\]

(2.46)

We note from (2.33) and (2.40) that the P-K-1 stresses are conjugate to the displacement gradients, and from (2.45) that the P-K-2 stresses are conjugate to the Green’s strains.

Use of the equations (2.35) in (2.45) and integration through the thickness results in the expression for internal virtual work

\[
\delta W_{int} = \int \int_{Area} \left[ \bar{N}_{11}\delta\epsilon_{11} + \bar{N}_{12}\delta\epsilon_{12} + \bar{N}_{22}\delta\epsilon_{22} + \bar{M}_{11}\delta\chi_{11} + \bar{M}_{12}\delta\chi_{12} + \bar{M}_{22}\delta\chi_{22} + \bar{Q}_1\delta\Gamma_{13} + \bar{Q}_2\delta\Gamma_{23} \right] A_1 A_2 d\xi_1 d\xi_2
\]

(2.48)

where

\[
\begin{align*}
\bar{N}_{11}, \bar{M}_{11} &= \int_{-h/2}^{h/2} S_{11}(1, \zeta)(1 + \zeta/R_2) d\zeta \\
\bar{N}_{22}, \bar{M}_{22} &= \int_{-h/2}^{h/2} S_{22}(1, \zeta)(1 + \zeta/R_1) d\zeta \\
\bar{N}_{12}, \bar{M}_{12} &= \int_{-h/2}^{h/2} S_{12}(1, \zeta) d\zeta \\
\bar{Q}_1 &= \int_{-h/2}^{h/2} S_{13}(1 + \zeta/R_2) d\zeta \\
\bar{Q}_2 &= \int_{-h/2}^{h/2} S_{23}(1 + \zeta/R_1) d\zeta
\end{align*}
\]  

(2.49)

It may be somewhat interesting to note that symmetry of the P-K-2 stresses leads to sym-
metry of the stress resultants ($\overline{N}_{12} = \overline{N}_{21}$ and $\overline{M}_{12} = \overline{M}_{21}$), regardless of the geometry of the shell. No such symmetry exists when P-K-1 stresses are used. As a result, the geometrically linear analysis has 10 stress resultants and 10 strain measures, where the geometrically nonlinear analysis has only eight. Further, use of P-K-1 stresses in the geometrically nonlinear analysis produces 12 stress resultants and couples.

In consideration of the correspondence between P-K-1 stresses and P-K-2 stresses shown in equations (2.47), we may compare the equations (2.42) and (2.49) to get a correspondence between stress resultants and stress couples in the two systems. Beginning, for example, with the first of (2.42)

$$ (N_{11}, M_{11}) = \int_{-h/2}^{h/2} T_{11}(1, \zeta)(1 + \zeta/R_2) d\zeta $$

using the first of equations (2.47)

$$ (N_{11}, M_{11}) = \int_{-h/2}^{h/2} S_{11}(1, \zeta)(1 + \zeta/R_2) d\zeta $$

and finally comparing directly to the first of (2.49), leads to the conclusion

$$ (N_{11}, M_{11}) = (\overline{N}_{11}, \overline{M}_{11}) $$

Similar comparisons may be made among all the stress resultants and stress couples, using equations (2.28) as necessary to yield

$$ (N_{11}, M_{11}) = (\overline{N}_{11}, \overline{M}_{11}) \quad (N_{22}, M_{22}) = (\overline{N}_{22}, \overline{M}_{22}) \quad \text{(2.50)} $$

$$ Q_1 = \overline{Q}_1 + \epsilon_{13}^o \overline{N}_{11} + \epsilon_{23}^o \overline{N}_{12} - \frac{\gamma_{13}^o}{R_1} \overline{M}_{11} - \frac{\phi_2}{R_2} \overline{M}_{12} $$

$$ Q_2 = \overline{Q}_2 + \epsilon_{13}^o \overline{N}_{12} + \epsilon_{23}^o \overline{N}_{22} - \frac{\phi_1}{R_1} \overline{M}_{12} - \frac{\gamma_{23}^o}{R_2} \overline{M}_{22} $$

$$ N_{12} = \overline{N}_{12} + \frac{1}{R_2} \overline{M}_{12} \quad N_{21} = \overline{N}_{21} + \frac{1}{R_1} \overline{M}_{21} $$

$$ M_{12} = \overline{M}_{12} \quad M_{21} = \overline{M}_{21} $$
The correlations of equation (2.50) are approximate -- in addition to the thin shell assumption, higher-order resultants have been discarded in the relations for $M_{12}$, $M_{21}$, $S_1$ and $S_2$. For example, in the exact correlation for $M_{12}$, we have

$$M_{12} = \bar{M}_{12} + \frac{1}{R_2} L_{12}$$

with

$$L_{12} = \int_{-h/2}^{h/2} S_{12} r^2 d\zeta$$

We have chosen to discard the higher-order resultants like $\bar{L}_{12}$ because their retention complicates the equilibrium equations, to be introduced in the next section. We rationalize this decision on the basis of smallness: magnitudes of the resultants tend to diminish as the order of the thickness coordinate increases within the integrals which define the resultants. We note, however, that the simplification leads to the erroneous conclusion $(M_{12} = M_{21})$, contrary to the definitions of (2.42). We thus accept a certain loss of accuracy in exchange for a set of equations which will be more readily solvable.

### 2.5.2 External virtual work

We consider that the shell is loaded under hydrostatic pressure only, except at the edges; the loading remains normal to the shell middle surface under deformation. It is assumed that the pressure remains constant as the structure responds.

On the deformed MS, the force due to pressure loading on an infinitesimal area is given by

$$\tilde{f} = pdA^* \hat{n}^*$$

The term $dA^* \hat{n}^*$ is found by
\[ \mathbf{n}^* \, dA^* = g_1^o \mathbf{A}_1 d\xi_1 \times g_2^o \mathbf{A}_2 d\xi_2 \]

with \( g_1^o \), \( g_2^o \) given by (2.25), applied at the MS. External virtual work is found by applying a small (virtual) displacement and integrating over the deformed area of the shell. This gives

\[ \delta W_{ext} = p \int \int_{\text{Area}} (g_1^o \times g_2^o \cdot \mathbf{\delta U}) A_1 A_2 d\xi_1 d\xi_2 \]

If the multiplication is carried out, the expression becomes

\[ \delta W_{ext} = p \int \int_{\text{Area}} \{[\varepsilon_{12}^o \varepsilon_{23}^o - (1 + \varepsilon_{22}^o) \varepsilon_{13}^o] \delta u + [\varepsilon_{21}^o \varepsilon_{13}^o - (1 + \varepsilon_{11}^o) \varepsilon_{23}^o] \delta v \\
+ [(1 + \varepsilon_{11}^o)(1 + \varepsilon_{22}^o) - \varepsilon_{12}^o \varepsilon_{21}^o] \delta w \} A_1 A_2 d\xi_1 d\xi_2 \]

Now applying the small strain, moderate rotation assumptions and keeping terms to order \( \mu \), we get

\[ \delta W_{ext} = p \int \int_{\text{Area}} \{-\varepsilon_{13}^o \delta u - \varepsilon_{23}^o \delta v + (1 + \varepsilon_{11}^o + \varepsilon_{22}^o) \delta w \} A_1 A_2 d\xi_1 d\xi_2 \]

Finally, if the displacement gradients are written in terms of displacements according to equations (2.29) and the necessary integrations are performed, we get

\[ \delta W_{ext} = p \delta \left\{ \int \int_A \left[ w + \frac{u^2}{2R_1} + \frac{v^2}{2R_2} + \frac{1}{R_1} + \frac{1}{R_2} \right] w^2 - u A_1 w_1 - v A_2 w_2 \right\} dA \right\} \tag{2.51} \]

\[ + p \int_C A d\xi_2 \bigg|_{\xi_1=b}^{\xi_1=a} \]

where \( A \) is the area of the surface; \( dA = A_1 A_2 d\xi_1 d\xi_2 \) and \( C \) is the bounding curve defining the shell edge.

For the geometrically linear analysis, we assume application of the load on an undeformed surface, and simply drop all of the nonlinear terms of equation (2.51) to get
\[ \delta W_{\text{ext}}^L = p \delta \int \int_{\text{Area}} w A_1 A_2 d\xi_1 d\xi_2 + p \int_C \left[ u A_2 \delta w \right]_{\xi_1=0}^{\xi_1=b} d\xi_2. \] (2.52)

2.5.3 Principle of virtual work

The principle of virtual work states that the shell is in a state of equilibrium if and only if the virtual work of internal forces and moments exactly balances the virtual work of external forces and moments, for all kinematically admissible virtual displacements. Mathematically, if

\[ \delta W_{\text{int}} = \delta W_{\text{ext}} \]

for every kinematically admissible displacement, then the body is in equilibrium. We have internal virtual work in terms of P-K-L stresses from (2.41) and external virtual work from (2.51). We write the variations of the displacement gradients in terms of displacements with the aid of (2.28), (2.29) and integrate by parts as necessary to get

\[
\int \int_{\text{Area}} \left\{ \left[ -\frac{1}{A_1 A_2} (A_2 N_{11} \delta u)_{\xi_1=a}^{\xi_1=b} - \frac{1}{A_1 A_2} (A_1 N_{21} \delta v)_{\xi_1=a}^{\xi_1=b} \right] + \left[ -\frac{1}{A_1 A_2} (A_2 Q_{11} \delta w)_{\xi_1=a}^{\xi_1=b} - \frac{1}{A_1 A_2} (A_1 Q_{21} \delta \phi_{11})_{\xi_1=a}^{\xi_1=b} \right] + \left[ -\frac{1}{A_1 A_2} (A_2 M_{11} \delta \phi_{12})_{\xi_1=a}^{\xi_1=b} - \frac{1}{A_1 A_2} (A_1 M_{21} \delta \phi_{22})_{\xi_1=a}^{\xi_1=b} \right] + \left[ -\frac{1}{A_1 A_2} (A_2 N_{12} \delta u)_{\xi_1=a}^{\xi_1=b} - \frac{1}{A_1 A_2} (A_1 N_{22} \delta v)_{\xi_1=a}^{\xi_1=b} \right] \right\} dA
\]

\[
+ \left[ A_2 N_{11} \delta u \right]_{\xi_1=a}^{\xi_1=b} + \left[ A_2 N_{12} \delta v \right]_{\xi_1=a}^{\xi_1=b} + \left[ A_2 Q_1 \delta w \right]_{\xi_1=a}^{\xi_1=b} + \left[ A_2 M_{11} \delta \phi_{11} \right]_{\xi_1=a}^{\xi_1=b} + \left[ A_2 M_{12} \delta \phi_{21} \right]_{\xi_1=a}^{\xi_1=b} dA
\]

\[
+ \left[ A_2 M_{12} \delta \phi_{21} \right]_{\xi_1=a}^{\xi_1=b} d\xi_2 = \int \int_{\text{Area}} \left\{ \left[ -p \left( \frac{1}{A_1} \frac{u}{R_1} - \frac{v}{R_2} \right) \delta u - p \left( \frac{1}{A_2} \frac{w}{R_1} - \frac{v}{R_2} \right) \delta v \right] + p \left( 1 + \frac{1}{A_1} \frac{u}{R_1} + \frac{1}{A_1} \frac{v}{R_2} \right) w + p \delta w \right\} dA
\]
Noting that the virtual displacements are arbitrary and independent, we see that the following equilibrium conditions must hold:

\[
\begin{align*}
\left[ \frac{1}{A_1A_2}(A_2N_{11})_1 + \frac{1}{A_1A_2}(A_1N_{21})_2 + \frac{A_{1,2}}{A_1A_2}N_{12} - \frac{A_{2,1}}{A_1A_2}N_{22} + \frac{1}{R_1}Q_1 \right] &= p\left(\frac{1}{A_1}w_{,1} - \frac{u}{R_1}\right) \\
\left[ \frac{1}{A_1A_2}(A_2N_{12})_1 + \frac{1}{A_1A_2}(A_1N_{22})_2 + \frac{A_{2,1}}{A_1A_2}N_{21} - \frac{A_{1,2}}{A_1A_2}N_{11} + \frac{1}{R_2}Q_2 \right] &= p\left(\frac{1}{A_2}w_{,2} - v\right)
\end{align*}
\]

Boundary conditions are such that one element from each of the following pairs must be prescribed at the edges \(\xi_1 = a, \xi_1 = b\):

\[
(N_{11}, u), (N_{12}, v), (Q_1, w), (M_{11}, \phi_1), (M_{12}, \phi_2).
\]  

The entire set of equilibrium equations and boundary conditions may be recast in terms of P-K-2 stress resultants and stress couples by use of equations (2.50). It is in this conversion that we justify the decision to neglect higher-order resultants such as \(\tilde{L}_{12}\).

### 2.6 Constitutive law

A relationship between the stress resultants and the middle surface strains is found by combining the definitions (2.49) and the material law (2.37). Specifically, for a laminate of plies of monoclinic material, the material law is applicable to each lamina; it is assumed that each ply is homogeneous.
The integrals defining the stress resultants and stress couples may be viewed as sums of integrals over the ply thicknesses so that, for example,

\[
\bar{N}_{11} = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} S_{11}(1 + \zeta/R_2) d\zeta = N \int_{\zeta_0}^{\zeta_N} S_{11}(1 + \zeta/R_2) d\zeta
\]

(2.55)

where there are \( N \) plies in the laminate, \( \zeta_0 = -h/2 \), \( \zeta_k = \zeta_{k-1} + t_k \) and \( t_k \) is the thickness of the \( k^{th} \) ply. See Fig. 2.10. The superscript \( (k) \) indicates the \( k^{th} \) ply.

Use of the material law (2.37) in (2.55) leads to

\[
\bar{N}_{11} = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} [Q_{11} E_{11} + Q_{12} E_{22} + Q_{16} \Gamma_{12}](1 + \zeta/R_2) d\zeta
\]

Use equations (2.35) to get relations of the form

\[
\bar{N}_{11} = A_{11} E_{11} + A_{12} E_{22} + A_{16} \Gamma_{12} + B_{11} \chi_{11} + B_{12} \chi_{22} + B_{16} \chi_{12}
\]

for all of the stress resultants and stress couples. The process yields the constitutive law:
The coefficients of equation (2.56) are defined by

\[
\begin{bmatrix}
\bar N_{11} \\
\bar N_{22} \\
\bar Q_2 \\
\bar Q_1 \\
\bar N_{12} \\
\bar M_{11} \\
\bar M_{22} \\
\bar M_{12}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & 0 & 0 & A_{26} & B_{12} & B_{22} & B_{26} \\
0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 & 0 \\
A_{16} & A_{26} & 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & 0 & 0 & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & 0 & 0 & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & 0 & 0 & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\bar E_{11} \\
\bar E_{22} \\
\bar \Gamma_{13} \\
\bar \Gamma_{12} \\
\bar \chi_{11} \\
\bar \chi_{22} \\
\bar \chi_{12}
\end{bmatrix}
\]

(2.56)

The coefficients of equation (2.56) are defined by

\[
(A_{11}, B_{11}, D_{11}) = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{11}^{(k)} (1, \zeta, \zeta^2) \frac{1 + \zeta / R_2}{1 + \zeta / R_1} d\zeta
\]

\[
(A_{12}, B_{12}, D_{12}) = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{12}^{(k)} (1, \zeta, \zeta^2) d\zeta
\]

\[
(A_{22}, B_{22}, D_{22}) = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{22}^{(k)} (1, \zeta, \zeta^2) \frac{1 + \zeta / R_1}{1 + \zeta / R_2} d\zeta
\]

\[
(A_{16}, B_{16}, D_{16}) = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{16}^{(k)} (1, \zeta, \zeta^2) \frac{1}{1 + \zeta / R_1} d\zeta
\]

\[
(A_{26}, B_{26}, D_{26}) = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{26}^{(k)} (1, \zeta, \zeta^2) \frac{1}{1 + \zeta / R_2} d\zeta
\]

\[
(A_{66}, B_{66}, D_{66}) = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{66}^{(k)} \left(1 + \frac{\zeta / R_1}{1 + \zeta / R_2}\right) d\zeta
\]

\[
A_{44} = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{44}^{(k)} \frac{1 + \zeta / R_1}{1 + \zeta / R_2} d\zeta
\]

\[
A_{45} = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{45}^{(k)} d\zeta
\]

\[
A_{55} = \sum \int_{\zeta_{k-1}}^{\bar \zeta_k} \bar Q_{55} \left(1 + \frac{\zeta / R_2}{1 + \zeta / R_1}\right) d\zeta
\]

where the sums are taken over the plies: \( k = 1 \rightarrow N \).
2.7 Summary of equations

The preceding sections of this chapter give a somewhat detailed development of the theory of shells to be used in this work. As such, there are many equations presented which are important for the derivation, but not for the analysis. This section recaps the relevant equations for stress analysis. These equations are for a thin shell of revolution, in lines of curvature coordinates, and it is assumed that the shape of the meridian is known; that is, the vector $\hat{r}$ which describes the position of the surface relative to a fixed global coordinate system is known as a function of the global coordinates and/or the Gaussian coordinates.

**Geometry:** The metrics of the middle surface are found by equations (2.5), and metrics of a parallel surface within the shell are found by equations (2.17). The radii of curvature of the middle surface are given by equations (2.15).

**Displacement gradients:** Displacement gradients which vary through the thickness are described in terms of their values on the middle surface, and in terms of MS displacements by equations (2.28), (2.29). These displacement gradients form the building blocks of the strain measures, which are derived depending upon the assumptions used.

**Strain measures:** In the preceding derivation, two sets of strain measures were developed -- one for small strains and rotations (i.e., a linear theory), and one set for small strains and moderate out-of-plane rotations.

For the **linear** theory, the strain measures are as given in equation (2.33), using the displacement gradients found previously. Note that for the linear theory, the displacement gradients are the same as the strains, with the exception of the (negligible) strain $\varepsilon_{33}$. Thus, the linear strains in terms of MS values are found from equations (2.28), (2.29).

For the **nonlinear** (i.e., small strain, moderate rotations) theory, the strain measures are found by equation (2.34). In terms of MS values, the nonlinear strains are expressed as in equations (2.35), (2.36).

**Internal virtual work:** The integral expression for internal virtual work is given in equation (2.43) for the linear theory, in terms of P-K-1 stresses. The stress resultants in terms of P-K-
I are as given by equations (2.42). For the nonlinear theory, there are two expressions for internal 
virtual work: in terms of P-K-1 resultants in equation (2.41) with (2.42), and in terms of P-K-2 in 
equation (2.48) with (2.49). Correlation of P-K-1 stresses with P-K-2 stresses is shown in equa-
tion (2.47), and correlation of resultants between the two systems is as given in equation (2.50).

**External virtual work:** The integral expression for incremental work of external loading 
is given in equation (2.52) for the linear response case. For the geometrically nonlinear response, 
the external virtual work is given by equation (2.51).

**Equilibrium:** Equilibrium equations are here shown only in terms of P-K-1 stress result-
ants and for the geometrically nonlinear response (equations (2.53), with boundary conditions 
(2.54)). If the equations of equilibrium are desired in their linear form, these may be found by use 
of the virtual work principle, using the linear equations for internal and external virtual work 
already discussed, and using the linear strain-displacement relations. If the equations are desired 
in terms of P-K-2 resultants, they may be found by transformation of (2.53) and (2.54) using the 
correlation (2.50).

** Constitutive law:** The constitutive law for a composite laminate shell in terms of P-K-2 
stress resultants is given by equations (2.55), (2.56), with the lamina material properties given in 
equation (2.37). If a theory is desired for linear response, it is common to use the material law 
(2.37), disregarding the differences between P-K-1 and P-K-2 stresses, and also disregarding the 
difference between the linear and nonlinear strains. That is, we simply replace the P-K-2 stress 
terms $S_{ij}$ with the P-K-1 stress terms $T_{ij}$, and replace the strains $E_{ij}$, $\Gamma_{ij}$ with the linear strains $\varepsilon_{ij}$, 
and $\gamma_{ij}$, all of which is consistent with the assumption of small strains and small rotations.