Appendix A
Estimation of Parameters in Weibull Proportional Intensity Model

For continuous intensity nonhomogeneous Poisson processes, Cox and Lewis (1966) defined the probability, starting from time $t_j$, given that there has been no failure from time $t_j$ to time $t_{j+1}$, of the next failure in the interval $(t_{j+1}, t_{j+1} + \Delta t)$ as

$$
\lambda(t_{j+1}) \exp \left\{ - \int_{t_j}^{t_{j+1}} \lambda(u) \, du \right\} \Delta t + O(\Delta t)
$$

where $\lambda(t)$ is the intensity function of the process.

Thus, the total likelihood is expressed as:

$$
L = \prod_{j=1}^{n} \left[ \lambda(t_j) \exp \left\{ - \int_{t_{j-1}}^{t_j} \lambda(u) \, du \right\} \exp \left\{ - \int_{t_n}^{t_{n+1}} \lambda(u) \, du \right\} \right]
$$

where $t_j$ is the $j$th failure. The last term in this expression represents the likelihood of no failure from the time of the last repair, at time $t_n$, until censoring time $t_{n+1}$. Therefore, the likelihood function is expressed as

$$
L = \prod_{j=1}^{n} \left[ \lambda(t_j) \exp \left\{ - \sum_{j=1}^{n+1} \int_{t_{j-1}}^{t_j} \lambda(u) \, du \right\} \right]
$$

Considering the structure of proportional intensity model the likelihood becomes

$$
L = \left[ \prod_{j=1}^{n} \lambda_0(t_j) \right] \exp(\gamma^T z_{sum}) \times \exp \left\{ - \sum_{j=1}^{n+1} \exp(\gamma^T z_j) \int_{t_{j-1}}^{t_j} \lambda_0(u) \, du \right\}
$$

where

$$
z_{sum} = \sum_{j=1}^{n} z_j
$$

Thus, the log-likelihood function is

$$
1 = \sum_{j=1}^{n} \log(\lambda_0(t_j)) + \gamma^T z_{sum} - \sum_{j=1}^{n+1} \exp(\gamma^T z_j) \int_{t_{j-1}}^{t_j} \lambda_0(u) \, du
$$

Notice that

$$
\int_a^b \lambda_0(u) \, du = \left( \frac{b}{\eta} \right)^\beta - \left( \frac{a}{\eta} \right)^\beta
$$

and denote

$$
w_j = \int_{t_{j-1}}^{t_j} \lambda_0(u) \, du = \left( \frac{t_j}{\eta} \right)^\beta - \left( \frac{t_{j-1}}{\eta} \right)^\beta
$$

Using the relationship above, we obtain
\[l(\beta, \eta, \gamma, \delta) = n \log \left( \frac{\beta}{\eta} \right) + (\beta - 1) \sum_{j=1}^{n} \log \left( \frac{t_j}{\eta} \right) + \gamma^T z_{\text{sum}} - \sum_{j=1}^{n+1} \exp(\gamma^T z_j) w_j \]

where \( n \) is the total number of failures observed.

Optimal values of \( \beta, \eta \) and the covariate coefficients \( \gamma \) can be found using the Newton-Raphson algorithm, in which the \((m+1)\)th iteration is given by

\[
\begin{bmatrix}
\beta^{(m+1)} \\
\eta^{(m+1)} \\
\gamma^{(m+1)}
\end{bmatrix} = \begin{bmatrix}
\beta^{(m)} \\
\eta^{(m)} \\
\gamma^{(m)}
\end{bmatrix} - \left[ G^{(m)} \right]^{-1} g^{(m)}, m = 1, 2, \ldots
\]

where \( G^{(m)} \) represents the matrix of the second-order partial derivatives of \( l(\beta, \eta, \gamma) \) with respect to \( \beta, \eta \) and \( \gamma \). \( g^{(m)} \) is the vector of first-order partial derivatives of \( l(\beta, \eta, \gamma) \) with respect to \( \beta, \eta \) and \( \gamma \).

The first-order derivatives of \( l \) are, noting \( \frac{d}{d\chi}(\alpha^\chi) = \alpha^\chi \log(\alpha) \) as follows:

\[
\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{j=1}^{n} \log \left( \frac{t_j}{\eta} \right) - \sum_{j=1}^{n+1} \exp(\gamma^T z_{\text{sum}}) \left( \frac{t_j}{\eta} \right)^\beta \log \left( \frac{t_j}{\eta} \right)
\]

\[
\frac{\partial l}{\partial \eta} = -\frac{\beta}{\eta} \left\{ n - \sum_{j=1}^{n+1} \exp(\gamma^T z_{\text{sum}}) \left[ \left( \frac{t_j}{\eta} \right)^\beta - \left( \frac{t_{j-1}}{\eta} \right)^\beta \right] \right\}
\]

\[
\frac{\partial l}{\partial \gamma_k} = z_{\text{sum}k} - \sum_{j=1}^{n+1} z_{jk} \exp(\gamma^T z_{\text{sum}}) \left( \frac{t_j}{\eta} \right)^\beta \log \left( \frac{t_j}{\eta} \right) - \left( \frac{t_{j-1}}{\eta} \right)^\beta \log \left( \frac{t_{j-1}}{\eta} \right)
\]

\( k = 1, 2, \ldots, M \)

where \( z_{jk} \) is the value of the \( k \)th covariate observed at \( j \)th failure.

The second-order derivatives are derived as follows:

\[
\frac{\partial^2 l}{\partial \beta^2} = -\frac{n}{\beta^2} - \sum_{j=1}^{n+1} \exp(\gamma^T z_{\text{sum}}) \left[ \left( \frac{t_j}{\eta} \right)^\beta \log \left( \frac{t_j}{\eta} \right)^2 - \left( \frac{t_{j-1}}{\eta} \right)^\beta \log \left( \frac{t_{j-1}}{\eta} \right)^2 \right]
\]
\[
\frac{\partial^2 l}{\partial \beta \partial \eta} = -\frac{1}{\eta} \left\{ n - \sum_{j=1}^{n+1} \exp(\gamma^T z_j) \left[ \left( \frac{t_j}{\eta} \right)^\beta - \left( \frac{t_{j-1}}{\eta} \right)^\beta \right] \right\} \\
+ \frac{\beta}{\eta} \sum_{j=1}^{n+1} \exp(\gamma^T z_j) \left[ \left( \frac{t_j}{\eta} \right)^\beta \log \left( \frac{t_j}{\eta} \right) - \left( \frac{t_{j-1}}{\eta} \right)^\beta \log \left( \frac{t_{j-1}}{\eta} \right) \right]
\]

\[
\frac{\partial^2 l}{\partial \beta \partial \gamma_k} = -\sum_{j=1}^{n+1} z_{jk} \exp(\gamma^T z_j) \left[ \left( \frac{t_j}{\eta} \right)^\beta \log \left( \frac{t_j}{\eta} \right) - \left( \frac{t_{j-1}}{\eta} \right)^\beta \log \left( \frac{t_{j-1}}{\eta} \right) \right]
\]
k = 1, 2, \ldots, M

\[
\frac{\partial^2 l}{\partial \eta^2} = \frac{\beta}{\eta^2} n - \frac{\beta(\beta + 1)}{\eta^2} \sum_{j=1}^{n+1} \exp(\gamma^T z_j) \left[ \left( \frac{t_j}{\eta} \right)^\beta - \left( \frac{t_{j-1}}{\eta} \right)^\beta \right]
\]

\[
\frac{\partial^2 l}{\partial \eta \partial \gamma_k} = \frac{\beta}{\eta} \sum_{j=1}^{n+1} z_{jk} \exp(\gamma^T z_j) \left[ \left( \frac{t_j}{\eta} \right)^\beta - \left( \frac{t_{j-1}}{\eta} \right)^\beta \right]
\]
k = 1, 2, \ldots, M

\[
\frac{\partial^2 l}{\partial \gamma_{k1} \partial \gamma_{k2}} = \sum_{j=1}^{n} z_{jk1} z_{jk2} \exp(\gamma^T z_j) \left[ \left( \frac{t_j}{\eta} \right)^\beta - \left( \frac{t_{j-1}}{\eta} \right)^\beta \right]
\]
k_1, k_2 = 1, 2, \ldots, M