Multidimensional Behavioral Complexes

Grant Michael Boquet

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J. A. Ball, Chair
P. A. Linnell
P. E. Haskell

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In [6], chains and boundary maps were defined for 2-D discrete behavioral systems. The corresponding homology groups were studied and tied to trajectory properties. Indeed, the homology groups encapsulated the concepts of autonomy, controllability, and signal restriction.

We shall present an extension of their work to n-D discrete behavioral systems. In particular, we shall streamline the construction of the chain groups, the boundary maps between chains, and the study of the resultant homology groups. While constructing this machinery, we shall point out intrinsic flaws in their approach that make extension of their results less systematic. Finishing remarks shall be made on using the homology groups to determine system properties and potentially classify forms of controllability.
Preface

This thesis is intended to serve as an introduction to multidimensional behavioral systems theory. This, however, is a rather difficult task to accomplish because of both the vast amount of literature in the area and the lack of an “approach” that everyone can agree on. Indeed, each mathematician working in the area has his/her own take on how things should be and this can lead to a very confusing smorgasbord of ideas. This is the frustration of writing a text on a developing area of mathematics!

In the author’s opinion, the best way to deal with the variety of material is to try to unify the ideas that each person presents, allowing one to see the connections in the work of various mathematicians. We offer a presentation that heavily focuses on the controllability of discrete dynamical systems. In order to make this presentation complete, we must also introduce a vast amount of prerequisite material. This area of mathematics, to the author’s surprise, encompasses most areas of mathematics. The layout is as follows:

Chapter 1. A quick brush is passed over the latter chapters. The reader is led through the concept of a behavior and examples of controllability and multidimensional extensions. The chapter concludes with material on behavioral complexes.

Chapter 2. Behavioral systems are defined and studied in detail. In particular, we address latent variables, system memory, controllability, autonomy, and some finer points that lie in between. We focus on discrete systems throughout this chapter and then motivate why it is reasonable to focus on autoregressive systems.

Chapter 3. Multidimensional systems theory is studied in this chapter. We begin with Rocha’s work in [11], only covering the points that are related to autoregressive systems. After a description of controllability and autonomy are provided, we move into n-dimensional discrete behavioral systems. The extension of controllability into n dimensions proves to be more work than originally anticipated. Concepts that motivate our definition of controllability are tied to the controllable-autonomous decomposition $B = B^a \oplus B^c$.

Chapter 4. Factorization of behavioral equations is considered in detail. This begins the study of dimension-invariant properties that are used to lift definitions into higher dimension. For the one and two dimensional situation, the factorization results are reasonable; in the n-dimensional situation, more sophisticated approaches are necessary.
Chapter 5. Chains, boundary maps, and homology are defined for discrete behavioral systems; this approach is motivated by the preprint [6]. We extend the results from [6] into $n$ dimensions. Homology groups are then computed and characterized to demonstrate controllability, autonomy, and signal restriction. Although further applications are not provided, the consequences of V. Lomadze’s paper [7] suggest that there may be significant uses for the results we present here.

The material presented here, excluding Chapter 5, is mostly expository. I tried to provide references where applicable. In general, if a definition, theorem, etc. is cited then the material is ‘very close’ to the presentation in that source. If the result is not cited, then the author most likely significantly changed the result or it is original.

It is hard to acknowledge each person that has influenced my study of mathematics and this text. I am indebted to my advisor J. A. Ball for his understanding of certain non-mathematical predicaments in my life and for his lively presentation of ideas. I find from my conversations with him I was able to appreciate mathematics as a collective entity. I am also grateful for the time P. E. Haskell has spent teaching me, answering my random questions, and reminding me that mathematicians are humans too.

G. M. Boquet

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Chapter 1

Introduction

The behavioral definition of controllability has been around for about 20 years. It has essentially nowhere been adopted in teaching. It is hard to comprehend this. The cause must be sociological. It cannot be pedagogical or scientific.

- Jan C. Willems

1.1 Systems

This entire text is based on the study of discrete dynamical systems. However, we shall start our adventure at an unusual starting place; we shall take the behavioral approach to defining and studying dynamical systems. The more conventional method of systems would be the input-state-output definition as in [3]. Our approach will not focus on whether something is an input or output; nor will it be concerned with the past or present. Only in this chapter will we compare of these two definitions.

1.1.1 Kalman & Co.

In [3], Kalman defines a dynamical system as follows.

Definition 1.1.1. [3] A dynamical system $\Sigma$ is a composite mathematical concept defined by the following axioms.

1. There is a given time set $T$, a state set $X$, a set of input values $U$, a set of acceptable input functions $\Omega = \{w : T \to U\}$, a set of output values $Y$, and a set of output functions $\Gamma = \{\gamma : T \to Y\}$. 
2. (Direction of time). $T$ is an ordered subset of $\mathbb{R}$.

3. The input space $\Omega$ satisfies the following conditions:
   
   (a) (Nontriviality). $\Omega$ is nonempty
   
   (b) (Concatenation of inputs). An input segment $w_{[t_1, t_2]}$ is $w \in \Omega$ restricted to $(t_1, t_2] \cap T$. If $w, w' \in \Omega$ and $t_1 < t_2 < t_3$, there is $w'' \in \Omega$ such that $w''_{[t_1, t_2]} = w_{[t_1, t_2]}$ and $w''_{[t_2, t_3]} = w'_{[t_2, t_3]}$.

4. There is given a state-transition function
   
   $\phi : T \times T \times X \times \Omega \rightarrow X$

   whose value is the state $x(t) = \phi(t; \tau, x, w) \in X$ resulting at time $t \in T$ from the initial state $x = x(\tau) \in X$ at initial time $\tau \in T$ under the action of the input $x \in \Omega$. $\phi$ has the following properties:

   (a) (Direction of time). $\phi$ is defined for all $t \geq \tau$, but not necessarily for all $t < \tau$.
   
   (b) (Consistency). $\phi(t; t, x, w) = x$ for all $t \in T$, all $x \in X$, and all $w \in \Omega$.
   
   (c) (Composition property). For any $t_1 < t_2 < t_3$ we have
       
       $\phi(t_3; t_1, x, w) = \phi(t_3; t_2, \phi(t_2; t_1, x, w), w)$

   for all $x \in X$ and all $w \in \Omega$.
   
   (d) (Causality). If $w, w' \in \Omega$ and $w_{[\tau, t]} = w'_{[\tau, t]}$, then
       
       $\phi(t; \tau, x, w) = \phi(t; \tau, x, w')$.

5. There is given a readout map $\eta : T \times X \rightarrow Y$ which defines the output $y(t) = \eta(t, x(t))$.

   The map $(\tau, t) \rightarrow Y$ given by $\sigma \mapsto \eta(\sigma, \phi(\sigma; \tau, x, w))$, $\sigma \in ([\tau, t])$, is an output segment, that is, the restriction $\gamma_{[\tau, t]}$ of some $\gamma \in \Gamma$ to $(\tau, t)$.

   This definition states that there is a notion of input and output of a system. Moreover, the concept of past behavior is brought up. In the behavioral setting, none of these concepts are required. We make no distinction between input or output.

### 1.1.2 Willems & Co.

The definition of a one dimensional discrete behavioral dynamical system is the triple

$\Sigma = (T, W, B)$
where $T = \mathbb{Z}$, $W = \mathbb{R}^2$, and $B \subset W^T$. We shall impose conditions on such systems throughout early on in our hunt. The foremost condition is that the systems will be autoregressive, which states that we can write $B$ as the signals that satisfy some behavioral equations. Even under these assumptions, we never label inputs or outputs. Consider the case where $W = \mathbb{R}^2$, $T = \mathbb{Z}$ and we are modeling the relationship between force and mass under gravity. For some point $t \in \mathbb{Z}$, we have

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

where $w_1(t)$ and $w_2(t)$ are respectively the force and mass at point $t$. That $w \in W^T$ is admissible requires that, for $g$ the gravity on earth,

$$w_1(t) = g \cdot w_2(t) \quad t \in \mathbb{Z}.$$ 

It is irrelevant to consider whether force or mass is the input and the same for output. The only important property is that it satisfies the governing laws of the system. The behavior in this situation is set

$$B = \left\{ \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \in W^T : w_1(t) - g \cdot w_2(t) = 0 \right\}$$

and our system is $\Sigma = (\mathbb{Z}, \mathbb{R}^2, B)$.

### 1.2 Controllability and Autonomy

When we considered the force-mass concept above, notice that we only needed $w(t)$ to determine if a signal was admissible at $t \in \mathbb{Z}$. The behavioral equations did not constrain $w(\tilde{t})$ for $\tilde{t} \neq t$. This turns out to be an important property when we state it in general. In the mean time, we shed some light on this concept by developing the concept around the force-mass problem.

Let $T_1, T_2 \subset \mathbb{Z}$ be given such that $T_1 \cap T_2 = \emptyset$. Moreover, let $w_1, w_2 \in B$ be two given signals. We can construct a new signal $w$ defined as

$$w(t) = \begin{cases} w_1(t) & t \in \mathbb{Z} \setminus T_2 \\ w_2(t) & t \in T_2 \end{cases}$$

Through simple inspection we see that $w \in B$ and $w|_{T_i} = w_i|_{T_i}$, $i = 1, 2$. This property is known as controllability of our system.

The concept of controllability is prevalent in our day-to-day lives; we expect our cars to controllable, our air conditioning to be controllable, and our ovens to be controllable. For this reason the Willems definition of controllability makes sense intuitively.
Autonomy is the opposite of controllability; it is the interwoven connection of the signal domain and its range. Examples are not as trivial to construct, however, one that comes to mind is analytic functions. If they are specified on a set with a limit point, they are specified everywhere. As one would expect, Autonomous systems are a little bit harder to characterize.

1.3 Multidimensional Generalizations

If we extend our signal domain to multiple dimensions, i.e. $T = \mathbb{Z}^n$, then we run into the issues of extending our definition of both systems and system properties. This is no small task since many of the tools used in one dimension (such as a Smith form) are no longer accessible. Furthermore, we have to use the implications of controllability and autonomy to find dimension-invariant equivalencies that carry over.

In [11], Rocha defines a 2-D behavioral system as the triple

$$\Sigma = (T, W, \mathcal{B})$$

where $T = \mathbb{Z}^2$, $W = \mathbb{R}^q$, and $\mathcal{B} \subset W^T$. We follow this construction, focusing on autoregressive systems. Once we have established the structure of autoregressive systems, we move into extending the definitions into $n$-dimensions.

Questions that arise in the $n$-dimensional generalization is the connection between system structure and the signal domain. It turns out that when we consider systems over $\mathbb{N}^n$, the results are significantly different. In particular, autonomous-controllable decompositions become less trivial. In [18], some of these issues are addressed. We shall give a bird’s eye view of the results related to controllability and autonomy and the progress they have made.

1.4 Factorization

It turns out that we can avoid looking at signals to determine if a system is controllable or autonomous. This is somewhat apparent in the equivalencies of controllability and autonomy that deal with the structure of governing laws. We can study these ideas further in the factorization of the governing laws. When we study their algebraic varieties and factorization properties we build insight on system properties. This understanding helps extend many ideas into multiple dimensions when we expect the algebraic structure to carry over. Indeed, it is in this point of view that we begin establishing algebraic invariants as a basis for our extension results.
1.5 Homology

In [6] discrete 2-D behaviors are generalized from the approach presented in [11]. We shall spend some time discussing the results in 2-D and then provide a natural extension to n-D. Instead of considering a signal space to be $\mathbb{R}^{Z^2}$ or $\mathbb{R}^{N^2}$, half-planes and quarter-planes are allowed. This not only allows us to use the definition of controllability for discrete systems on $Z^a \times N^b$, $a, b \in \mathbb{N}$, as in [20] but allows for a new path in the search of algebraic invariants.

As we develop the machinery for the n-dimensional extension, we shall notice that the development in [6] is unnatural; furthermore, the computational difficulty in extending by their method results is a byproduct of their view taken towards the problem. Our presentation is so that the chain complex associated to behaviors is streamlined. Indeed, the entire process is just as algorithmic as, say, simplicial homology.
Chapter 2

One Dimensional Behaviors

What really constitutes a dynamical system? How should one conceptualize it? What are the essential common features in the mathematical models for dynamical phenomena? What is a suitable paradigm on which we can base our definitions and, from there, our problem formulations?

- Jan C. Willems

This section will introduce the origin and development of Behavioral Systems Theory as initiated by Willems in [15] and developed by him and his associates in subsequent papers. In the next chapter we shall extend the results to multiple dimensions. We shall take an “all encompassing” approach to this, considering most of the angles that researchers have developed when considering the n-D generalization.

2.1 Behaviors

We begin with the definition of a dynamical system, a rather abstract yet encapsulating definition from which the material in this chapter is developed. The approach we take is behavioral in the sense that there are governing laws, yet, the notion of “input” or “output” is not considered.

Definition 2.1.1. We shall call a dynamical system the triple \(\Sigma = (T, W, B)\) where \(T \subset \mathbb{R}\) is the time axis, \(W\) the signal-value set, and \(B \subset W^T\) is the set of admissible trajectories\(^1\)

\[
B = \{w : T \to W : w \text{ satisfies the system law}\}.
\]

\(^1\)For clarity, we note that \(W^T\) is the set of maps from \(T\) to \(W\).
We shall consider
\[ B = \{ w \in W : Rw = 0 \} \]
for some map \( R : W^T \to \{0, 1\} \) also known as the behavioral equation\(^2\). Note in the case that we are observing \( B \) as the kernel of \( R \), we shall say that \( R \) is the kernel representation of \( B \). This is quite reasonable because ‘0’ stands for “obey’s the governing laws” while ‘1’ stands for “does not obey governing laws.” In general there may be an infinite number of ways of describing a system, and thus we are only concerned with the “overall behavior” of the system; when we study the module \( B \), we can study it outside of the governing equations of the system. This also will allow us to use the framework of both commutative algebra and homological algebra, which were not as accessible in the classical theory. Before moving onto the extension to multiple dimensions, we shall spend a bit of time motivating our intuition through single variable theory.

### 2.2 Examples of Systems and Behaviors

**Example 2.2.1.** [15] Consider the standard pendulum problem. We have \( T = \mathbb{R} \) and \( W = \mathbb{R}^3 \times \mathbb{R}^3 \) since the signal values will be the two position vectors at time \( t \). To write down the behavior we must construct a set of behavioral equations.

\[
m \frac{d^2 \vec{w}_1}{dt^2} = mg \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \vec{F} \quad ||\vec{w}_1 - \vec{w}_2|| = L \quad \vec{F} = a(\vec{w}_1 - \vec{w}_2) \quad (2.1)
\]

where \( m \) denotes the mass of the pendulum with length \( L \), \( g \) the gravitational constant, \( a \) is the proportionality factor between \( \vec{F} \) and \( \vec{w}_1 - \vec{w}_2 \). We thus have

\[ B = \{ (\vec{w}_1, \vec{w}_2) : \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 : \exists \vec{F} : \mathbb{R} \to \mathbb{R}^3 \text{ and } a : \mathbb{R} \to \mathbb{R}^3 \text{ s.t. } (2.1) \text{ is satisfied} \} \].

**Example 2.2.2.** Consider *Dempster’s Combination Rule* from Dempster-Shafer Theory (see [14].) Intuitively, the definition of a belief is equivalent to that of an outer measure; the requirement of additivity of a probability measure is reduced to subadditivity. Denote by \( \chi_\mu \) the set of non-negative, unitary, and consonant beliefs. Dempster’s Combination Rule is a binary map

\[
\cdot \otimes \cdot : \chi_\mu \times \chi_\mu \longrightarrow \chi_\mu.
\]

When a sequence of observations over time are made, they are combined to form a new belief over the sample space. Let \( t \in \mathbb{N} \) be a given observation. We define the system as \( W = \chi^3_\mu \), \( T = \mathbb{N} \), and

\[ B = \{ (w_{prior}, w_{observation}, w_{posterior}) \in W^T : w_{prior}(t + 1) = w_{posterior}(t) = w_{prior}(t) \otimes w_{observation}(t) \quad \forall \ t \in T \}. \]

\(^2\)Later on we shall discard the image of \( R \) as \{0, 1\} and instead choose a more natural image.
In [16], Dempster’s Combination Rule is compared to product measures. This is done in hopes of reinterpreting Bayes Rule by a symmetric monoidal structure. Behavioral equivalence may be the answer to this problem.

2.3 Latent Variables and Extended Behaviors

In Example 2.2.1, notice that we have the pair \((\vec{w}_1, \vec{w}_2)\) as the trajectory, however, \(\vec{F}\) and \(a\) are required to exist to satisfy the system laws. We don’t care what \(\vec{F}\) and \(a\) are, we only require that they satisfy the governing physics laws in tandem with \((\vec{w}_1, \vec{w}_2)\). In [15], Willems states the following about this observation.

Willems states the following about this observation.

...models which we write down from first principles will invariably involve, in addition to the basic variables which we are trying to describe, auxiliary variables. We will call such variables *latent variables*. These latent variables could be introduced, if for no other reason, because they make it more convenient to write down the equations of motion, or because they are essential in order to express the constitutive laws or the conservation laws defining the system’s behavior.

To bring light to this quote, let \(R\) be such that

\[
B_1 = \{ (\vec{w}_1, \vec{w}_2) : \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 : \exists \vec{F} : \mathbb{R} \to \mathbb{R}^3 \text{ and } a : \mathbb{R} \to \mathbb{R}^3 \text{ s.t. (2.1) is satisfied} \}
\]

and

\[
B_2 = \{ (\vec{w}_1, \vec{w}_2) : \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 : R(\vec{w}_1, \vec{w}_2) = 0 \}
\]

are equal. Indeed, these are two different ways of representing the behavior. On one hand we have that \(B_2\) is determined by \(\ker(R)\) and on the other we have that \(B_1\) is determined by “tearing”, i.e. constructing the system through concatenating physical laws and restrictions. In each of the above representations we see that there are benefits and hindrances.

When we defined \(B_1\) we used auxiliary variables, namely \(\vec{F}\) and \(a\), to define the behavior. Such variables we shall call *latent variables*. The introduction of latent variables in this example is solely a matter of convenience and is usually unavoidable when we construct a model based through concatenating subsystems.

At first, the consideration of latent variables seems quite superficial. However, they will have far reaching theoretical consequences. In particular, they play an important role in understanding *free variables* in regards to autonomous systems (see [19].) Before we introduce a formal definition for dynamical systems with latent variables, there is yet another quote from Willems in [15] on the matter of latent variables.

---

3This introduction of latent variables is quite vague. The reader should rest assured that a formal definition will be provided later.
In thinking about the difference between signal variables and latent variables it is helpful in first instance to think of the signal variables being directly observable: they are explicit, while the latent variables are not: they are implicit. Examples: in pedagogy, scores of tests can be viewed as the signal, and native intelligence can be viewed as a latent variable aimed at explaining these scores.

At this point, we finally state the definition of a system with latent variables.

**Definition 2.3.1.** A dynamical system with latent variables is the quadruple

$$\Sigma_\ell = (T, W, X, B_\ell)$$

with $T$ the time set, $W$ the signal value set, $X$ the latent variable value set, and $B_\ell \subset (W \times X)^T$ the extended behavior.

It may be convenient to “remove” the latent variables from a dynamical system. The obvious way of doing this is through a canonical projection, namely

$$\pi_W : W \times X \to W \quad \pi_W(w, a) = w \quad w \in W, a \in X$$

Notational Remark. Because we switch between dynamical systems with and without latent variables through projection, the notation $\Sigma_\ell$ was chosen to prevent confusion since ‘$\ell$’ was selected for ‘latent.’

We can clearly extend this to signals through projection of the image: $\pi_W(w(t), a(t)) = w(t)$. Naturally, this notation can be even further extended to $\pi_W B_\ell$ as one would expect. The reason for this (maybe abusive) notation is that we can consider the induced dynamical system

$$\Sigma = (T, W, \pi_W B_\ell).$$

To be consistent with the terminology of [15], we shall refer to $B_\ell$ as the internal behavior and $\pi_W B_\ell$ as the external behavior. For a pictorial view of this, see Figure 2.1. We now specify some properties that we will desire in our systems.
2.4 Special Types of Systems

In most areas of mathematics a general class of problems, functions, etc. are defined; however, the vast nature of the definition prevents a fruitful theory to be developed. It is usually in the special cases (linear operators, elliptic equations, etc.) that we can derive information about not only special cases of the theory, but usually situations that can be approximated by the special cases. We have reached the point in our journey in which we shall start placing restrictions on our systems.

**Definition 2.4.1.** We shall call a dynamical system *linear* if $W$ is a vector space and $B$ is a linear subspace of $W^T$.

**Example 2.4.1.** Let $N = w \cdot h$, where $w$ and $h$ are respectively the width and height of a 2-D bitmap grayscale image. Let $T = N$ and $W = \mathbb{R}^N$. An image filter is commonly applied to an image prior to using a classifier. Usually, the filter helps bring out characteristics the classifier uses to distinguish between images. For linear-discriminant classifiers, approximating subspaces of $\mathbb{R}^n$ are computed in which projection into the subspaces limits the image to the “features” of the class associated to the subspace (i.e. there is a subspace for circles and a subspace for triangles.) Consider the Gaussian filter

$$f(x, y) = \sqrt{\frac{a}{\pi}} e^{-a(x+y)^2}.$$  

The image is convolved with the filter to produce the filtered image, i.e.

$$\mathcal{B} = \{(i_1 : T \to W, i_2 : T \to W) \ : \ i_2(t) = i_1 * f\}.$$  

In this example, $W$ is a vector space and $\mathcal{B}$ is a vector subspace.

**Definition 2.4.2.** We shall call a dynamical system *time invariant* if $T$ is an additive semigroup and for every signal and $t' \in T$, $w \in B$, $\hat{w}(t) = w(t + t') \in B$.

**Remark 2.4.1.** Let $\sigma^t$ be the left-shift operator, $\sigma^t(w) = w(\cdot + t)$. Then we can see that Definition 2.4.2 implies that $\sigma^t\mathcal{B} \subset \mathcal{B}$ and thus is an “invariant subspace.” Of course, we are able to remove the quotations when it is indeed a subspace.

2.5 Concatenation and Anticipation

Before we can define complete systems and talk of their memory, we must define the concatenation of signals. The definition is as one would anticipate. The notation we introduce is from [15] and is quite self explanatory. First we define some restriction notation for signals.

$$w^- = w|_{T \cap (-\infty, 0)} \quad w^0 = w|_{T \cap (-\infty, 0]} \quad w^+ = w|_{T \cap (0, \infty)} \quad w^0+ = w|_{T \cap [0, \infty)}.$$  

In the above notation, we extend by zero. That is, $w^-(t) = 0$ for $t \geq 0$. From here we may introduce the concatenation of signals.
Definition 2.5.1. Let \( w_1, w_2 : T \to W \) be two signals and \( t \in T \). We define the concatenation of \( w_1 \) and \( w_2 \) at \( t \) to be one of the following

\[
(w_1 \Lambda_{t-} w_2) (t') = \begin{cases} 
  w_1(t') & t' < t \\
  w_2(t') & t' \geq t
\end{cases}
\]

\[
(w_1 \Lambda_{t+} w_2) (t') = \begin{cases} 
  w_1(t') & t' \leq t \\
  w_2(t') & t' > t
\end{cases}
\]

This notation, of course, can and shall be extended naturally to behaviors. That is, for \( B_1, B_2 \subset W^T \), we may write \( B_1 \Lambda_{t-} B_2 \) and \( B_1 \Lambda_{t+} B_2 \).

Definition 2.5.2. Let \( T \) be specified and \( W_1, W_2 \) be two given signal value sets where \( B_1 \subset W_1^T \), \( B_2 \subset W_2^T \). For an operator \( F : B_1 \to B_2 \) we shall call \( F \) non-anticipating if for any \( t \in T \) and \( w_1, w_2 \in B_1 \),

\[
w_1(t') = w_2(t') \quad t' \leq t
\]

implies that

\[
F(w_1)(t') = F(w_2)(t') \quad t' \leq t.
\]

We shall call \( F \) strictly non-anticipating if \( w_1(t') = w_2(t') \) for \( t' < t \) implies \( F(w_1)(t') = F(w_2)(t') \) for \( t' < t \).

The following example distinguishes non-anticipating and strictly non-anticipating operators.

Example 2.5.1. [15] Let \( W = \mathbb{R} \) and \( T = \mathbb{R} \). Define

\[
\mathcal{B} = \{ w : \mathbb{R} \to \mathbb{R} : w \text{ is bounded, piecewise continuous, and right continuous} \}.
\]

Define the map \( F : \mathcal{B} \to W^T \) as

\[
F(w)(t) = \lim_{t' \downarrow t} w(t').
\]

For two signals \( w_1, w_2 \in \mathcal{B} \) and some \( t \in T \) assume that

\[
w_1(t') = w_2(t') \quad t' \leq t, \quad t' \in T.
\]

By the definition of \( \mathcal{B} \) we know that \( F(w_1)(t) = F(w_2)(t) \) is not true in general. It follows that \( F \) is not a non-anticipating operator.

2.6 Memory

We have introduced enough concepts in dynamical systems to now consider deeper issues. In particular, how does the future of a signal depend on the signal’s past? This very question is what separates dynamical systems from function spaces and allows us to develop a hearty theory. We begin with the idea of determining if a signal is contained in the behavior by only looking at pieces of the signal.
Definition 2.6.1. Let $\Sigma = (T, W, B)$ be a given dynamical system. We shall call $\Sigma$ complete if for any given signal $w \in W^T$ that $w \in B$ if and only if $w|_{[t_0, t_1]} \in B|_{[t_0, t_1]}$ for all $t_0 \leq t_1 \in T$. We shall call $\Sigma$ $L$-complete if $w \in B$ if and only if $w|_{[t, t+L]} \in B|_{[t, t+L]}$ for all $t \in T$.

When a dynamical system is $L$-complete for all $L > 0$ then we shall call it locally specified. If $L = 0$ then it is instantly specified since the signal is not governed by the dynamics of the signal. We shall call $L$ the lag of the system.

Now we shall diverge for a bit to see why this definition is of importance.

Notice that completeness is, essentially, asking how little information do we need to determine if a signal is admissible (i.e. is an element of the behavior $B$.) In particular, we do not need to look at the steady-state behavior of a signal. Now we shall try to move a bit out some.

Now we shall ask “how does the signal propagate?” For instance, with analytic functions, the definition of a function on an accumulation point (and thus any open set) will uniquely define the function on the entire plane. Is it possible that there is only a finite amount of time necessary to transition to another given state? What do we need so that this transition can occur?

The idea of transitioning from one signal to another is the starting point for the behavioral point of view of controllability. First, however, we must define the memory of a system.

Definition 2.6.2. We shall say that $\Sigma = (T, W, B)$ has $\nabla$-finite memory if for $w_1, w_2 \in B$ where $w_1|_{[0, \nabla)} = w_2|_{[0, \nabla)}$ implies that $w_1 \Lambda_{0-w_2} \in B$. If $\Sigma$ is $\nabla$-finite for all $\nabla > 0$ we shall say it has a local memory.

Notice that in Definition 2.6.2 we are introducing the idea of transitioning between trajectories provided they equal each other on a particular set. We are also looking at how local the information in a signal lingers on. For dynamical systems with local memory, this transition time is arbitrary and thus for local memory systems we truly have that the dynamics are local. We now define some systems with special types of memory.

Definition 2.6.3. We shall call $\Sigma = (T, W, B)$ Markovian if for $w_1, w_2 \in B$, $w_1(0) = w_2(0)$ implies that $w_1 \Lambda_{0-w_2} \in B$.

Definition 2.6.4. We shall call $\Sigma = (T, W, B)$ memory-less if $B$ is closed under concatenation at zero.

Now that we have considered both complete systems and the memory of systems, we can ask how the two concepts are related. Indeed, it appears obvious that they both address the same issue; however, each has its own way of approaching the concept of a system’s lag.
2.7 Splitting the External Behavior

So far we have considered the memory of signals for dynamical systems without latent variables. When we introduce latent variables it becomes a question of how the latent variables interact with the memory of the dynamical system.

Definition 2.7.1. Let \( \Sigma = (T, W, X, \mathcal{B}) \) be a dynamical system with latent variables. We shall say that the latent variable \( \) splits the external behavior if for \((w_1, x_1), (w_2, x_2) \in \mathcal{B} \) \( x_1(0) = x_2(0) \) implies that \( w_1 \Lambda_0 - w_2 \in \pi_W \mathcal{B} \).

Notice though that we are not asking \((w_1, x_1) \Lambda_0 - (w_2, x_2) \in \mathcal{B} \), only that in the projection the concatenation is admissible. For this reason we introduce the following definition.

Definition 2.7.2. Let \( \Sigma = (T, W, X, \mathcal{B}_w) \) be a dynamical system with latent variables. We shall call \( \Sigma \) a dynamical system in the state space form if for \((w_1, x_1), (w_2, x_2) \in \mathcal{B}_s \) \( x_1(0) = x_2(0) \) implies that \((w_1, x_1) \Lambda_0 - (w_2, x_2) \in \mathcal{B}_s \).

2.8 Linear Time-Invariant Systems

2.8.1 Laurent Polynomial Operators in the Shift

An important class of dynamical systems will have behavioral equations defined through Laurent polynomials. We begin by defining these polynomials.

Definition 2.8.1. Let \( R \) be a ring with identity. We define a Laurent polynomial ring as the polynomial ring \( R[s, s^{-1}] \) where \( s \cdot s^{-1} = 1_R \).

When the domain of the input signal is \( \mathbb{N} \) or \( \mathbb{Z} \), we can define the shift operator. That is

Definition 2.8.2. Let \( T = \mathbb{N} \) or \( T = \mathbb{Z} \) and \( w \) be a map over \( T \). We define the left shift operator, \( \sigma \), on \( w \) as

\[
\sigma(w(t)) = w(t + 1) \quad t \in T.
\]

When \( T = \mathbb{Z} \) we can define the right shift operator, \( \sigma^{-1} \) on \( w \) as

\[
\sigma^{-1}(w(t)) = w(t - 1) \quad t \in T.
\]

Properties of \( \sigma \) and \( \sigma^{-1} \) are

\[
\sigma(\sigma^{-1}(w(t))) = w(t) \quad \sigma(\sigma^n(w(t))) = \sigma^{n+1}(w(t)) \quad \sigma^{-1}(\sigma^{-n}(w(t))) = \sigma^{-n-1}(w(t)).
\]
That is, the polynomial ring $R[\sigma, \sigma^{-1}]$ contains operators on maps with values for some time domain $T$. For convenience, we introduce the notation $\mathcal{A} = R[\sigma, \sigma^{-1}]$ and consider $\mathcal{A}^{m\times n}$, $m, n \in \mathbb{N}$, as the set of $m \times n$ matrices with entries in $\mathcal{A}$.

We shall use the following terminology when referring to $\mathcal{A}$ and $\mathcal{A}$-modules such as $\mathcal{A}^{m\times n}$:

1. We call $p \in R[s]$ unimodular if $\deg(p) = 0$.
2. We call $p \in R[s, s^{-1}]$ unimodular if $p = \alpha s^d$ where $p \neq 0$ and $d \in \mathbb{Z}$.

### 2.8.2 Fréchet Coordinate Space

The following definition will move to the center stage the signal spaces we work with. After we have a collection of spaces, we will develop some results on them that characterize their properties as behavioral systems. In particular, we will construct results that characterize their kernel representation.

**Definition 2.8.3.** Define $\mathbb{L}^q$ as the space of vector sequences $(\mathbb{R}^q)^\mathbb{Z}$ equipped with the topology of pointwise convergence. Furthermore, we define $\mathcal{L}^q$ as the class of all linear, shift-invariant, and closed subspaces of $\mathbb{L}^q$.

It is important to note that both $\sigma$ and $\sigma^{-1}$ are defined on $\mathbb{L}^q$. If, however, we restrict the signal domain to $\mathbb{N}$, we no longer have the luxury of inverting $\sigma$. We wish to demonstrate that the behaviors contained in $\mathcal{L}^q$ are specifically the ones represented by kernels of operators in $\mathbb{R}[\sigma, \sigma^{-1}]^{q\times q}$. However, we must first present some notation and results.

### Constructing the Continuous Dual

**Lemma 2.8.1** (J.A. Ball). The continuous dual space of $\mathbb{L}^q$, which we denote by $(\mathbb{L}^q)^*$, is comprised of finitely supported sequences of vectors, i.e.

$$(\mathbb{L}^q)^* = \{w \in \mathbb{L}^{1\times q} : \text{there exists } N \in \mathbb{N} \text{ such that } w_{|i|} = 0 \text{ for all } |i| > N, i \in \mathbb{Z}\}.$$ 

**Proof.** We first show this result for $q = 1$. Let $\ell \in \mathbb{L}^*$ be given and define

$$w = \{w_k\}_{k \in \mathbb{Z}} \quad \text{where} \quad w_k = \ell(\delta_k).$$

In the above, $\delta_k$ is the Dirac delta, $\delta_k : \mathbb{Z} \to \mathbb{R}$, supported at $k \in \mathbb{R}$. Now let $v \in \mathbb{L}$ be an arbitrary element. We may write

$$v = \lim_{N \to \infty} \sum_{n=-N}^{N} v_n \delta_n, \quad \ell(v) = \lim_{N \to \infty} \sum_{n=-N}^{N} v_n w_n,$$

\footnote{When we speak of limits, it is under the auspices of point-wise operations.}
We have, up to this point, broken down $\ell$ into its action on the Schauder basis of $\mathbb{L}$. We wish to use this reinterpretation to demonstrate $\ell$ has finite support; assume otherwise and define

$$v_k = \begin{cases} \frac{1}{w_k} & w_k \neq 0 \\ 0 & w_k = 0 \end{cases} \quad (2.2)$$

From here we construct

$$v = \lim_{N \to \infty} \sum_{n=-N}^{N} v_k \delta_k,$$

only to see that

$$\ell(v) = \ell \left( \lim_{N \to \infty} \sum_{n=-N}^{N} v_k \delta_k \right) = \lim_{N \to \infty} \sum_{n=-N}^{N} v_k w_k.$$

However, the right side does not converge in $\mathbb{R}$. It follows that $\ell \notin \mathbb{L}^*$ and thus elements from $\mathbb{L}^*$ must have finite support. We may now extend this approach for $q \in \mathbb{N}$. This, requires using real vector valued sequences instead of scalar sequences. Up until (2.2), this is not an issue; we instead use

$$v_k = \begin{cases} \frac{1}{\|w_k\|^2} w_k & w_k \neq 0 \\ 0 & w_k = 0 \end{cases}$$

From this point the calculations follow naturally and we reach that the continuous dual consists of vector valued sequences with finite support.

Conversely, if $w \in \mathbb{L}^{1 \times q}$ has finite support, it is clear that it is a continuous linear functional.

Now that we have identified the space $(\mathbb{L}^q)^*$, we may construct an $\mathbb{R}$-module isomorphism to represent its elements as polynomials.

**Corollary 2.8.2.** For the map $\phi: (\mathbb{L}^q)^* \to \mathbb{R}^{1 \times q}[x,x^{-1}]$ defined as

$$\phi(w) = \sum_k w_k x^k \quad w \in (\mathbb{L}^q)^*$$

is an $\mathbb{R}$-module isomorphism.

**Remark 2.8.1.** An important observation to make is that the continuous dual, $(\mathbb{L}^q)^*$, is not the same as the algebraic dual. Indeed, we have the inclusion

$$(\mathbb{L}^q)^* \subset \text{hom}(\mathbb{L}^q, \mathbb{R})$$

but not equality.
Now that we have identified the dual space, we can discuss certain properties of the functionals. Before delving into such result, we define the orthogonal sets of \( \mathbb{L}^q \) and of \((\mathbb{L}^q)^*\).

**Definition 2.8.4.** Let \( B \subseteq \mathbb{L}^q \) be given. We define

\[
B^\perp = \{ \ell \in (\mathbb{L}^q)^* : \ell(w) = 0 \text{ for all } w \in B \}.
\]

Furthermore, for \( N \in (\mathbb{L}^q)^* \) we define

\[
N^\perp = \{ w \in \mathbb{L}^q : \ell(w) = 0 \text{ for all } \ell \in N \}.
\]

**Lemma 2.8.3.** Let \( B \in \mathbb{L}^q \) be a given behavior. It follows that \( B = (B^\perp)^\perp \).

**Proof.** For free we get the inclusion \( B \subseteq (B^\perp)^\perp \). For the reverse inclusion, assume that it is not true; that is, there exists \( w \in (B^\perp)^\perp \) such that \( w \notin B \). Define \( A = \{ w \} \). We have that \( A \) and \( B \) are disjoint, nonempty, convex sets in \( W^T \). By the Hahn-Banach theorem (see [13]), there exists \( \ell \in (\mathbb{L}^q)^* \) and \( \gamma \in \mathbb{R} \) such that

\[
\ell(b) < \gamma < \ell(w) \quad b \in B.
\]  

(2.3)

It follows that \( \ell(B) \cap \ell(w) = \emptyset \) and thus \( \ell(B) \neq \mathbb{R} \). Linearity of \( B \) thus forces \( \ell(B) = \{0\} \) and (2.3) gives us that \( \ell(w) > 0 \). Note, however, that \( \ell \in B^\perp \) and thus \( w \notin (B^\perp)^\perp \). We conclude that such a \( w \) cannot exist. \( \square \)

**Characterization of Autoregressive Systems**

In [15], Willems demonstrates that autoregressive systems have desirable system theoretic properties.

**Proposition 2.8.4.** [15] Let \( B \) be the behavior for a dynamical system \( \Sigma \). It follows that \( B \in \mathbb{L}^q \) if and only if there exists \( M \in \mathbb{A}^{q \times q} \) such that \( B = \ker M \). Moreover, we may assume \( 0 \leq g \leq q \) and assume \( M \in \mathbb{R}[\sigma]^{q \times q} \). If \( g = 0 \), \( B = \mathbb{L}^q \).

**Proof.** We have previously demonstrated

- \((\mathbb{L}^q)^* \rightarrow \mathbb{R}^{1 \times q}[x, x^{-1}]
- B = (B^\perp)^\perp

By definition, \( \mathbb{R}^{1 \times q}[x, x^{-1}] \) is free over \( \mathbb{R}[x, x^{-1}] \); because \( \mathbb{R}[x, x^{-1}] \) is a PID, every \( \mathbb{R}[x, x^{-1}] \)-submodule of \( \mathbb{R}^{1 \times q}[x, x^{-1}] \) is finitely generated. For our assumed \( B \), we have \( B^\perp \) is an
\( \mathbb{R}[x, x^{-1}] \)-submodule and thus is finitely generated. Let \( r_1(x, x^{-1}), \ldots, r_g(x, x^{-1}) \in \mathbb{R}[x, x^{-1}] \) be such that

\[
\mathcal{B}^\perp = \mathbb{R}[x, x^{-1}]r_1(x, x^{-1}) + \mathbb{R}[x, x^{-1}]r_2(x, x^{-1}) + \ldots + \mathbb{R}[x, x^{-1}]r_g(x, x^{-1}).
\]

Define

\[
V = \{ w \in \mathbb{L}^q : r_i(\sigma, \sigma^{-1})(w) = 0, \ i = 1, \ldots, g \}.
\]

By Lemma 2.8.3, \( \mathcal{B} = V \). Define

\[
R(s, s^{-1}) = \begin{bmatrix}
    r_1(s, s^{-1}) & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & r_g(s, s^{-1})
\end{bmatrix}.
\]

We have constructed an \( R \) such that \( \mathcal{B} = \ker R(\sigma, \sigma^{-1}) \). Because \( \mathcal{B}^\perp \) is a submodule of \( \mathbb{R}^{1 \times q}[x, x^{-1}] \), we have that \( 0 \leq g \leq q \). Furthermore, that we can take \( R \) to have polynomial entries follows from shift invariance of \( \mathcal{B} \).

The reverse direction of the proof is a straight-forward verification: \( \ker R(\sigma, \sigma^{-1}) \) is linear, shift invariant, and closed.

Remark 2.8.2. It is quite important to note that Willems uses \( \mathbb{R}[\sigma, \sigma^{-1}] \) is a PID to obtain a finite generating set for \( \mathcal{B}^\perp \). When we observe the extension to signals with domains in more than one dimension, we notice that this is no longer an option. Thankfully, we do not need to work with free modules over a PID, instead using free modules over a Noetherian polynomial ring. This, naturally, gives us that every submodule is finitely generated.

2.9 Autonomous Systems

Before reaching the concept of controllability, we must discuss systems that are the opposite of controllable; these systems are called autonomous. The definition of such systems is a reasonable starting point for their analysis.

Definition 2.9.1. Let \( \Sigma = (T, W, \mathcal{B}) \) be a time-invariant system. We shall call \( \Sigma \) autonomous if there exists a map \( f : \mathcal{B}^\perp \to \mathcal{B}^+ \) and \( L \in T \) such that for all \( w \in W^T, w \in \mathcal{B} \) if and only if \( w|_{(-\infty, L) \cap T} \in \mathcal{B}|_{(-\infty, L) \cap T} \) and \( w|_{[L, \infty)} = f(w|_{(-\infty, L)}) \). That is, the future is uniquely determined by the past.

Consider the system \( \Sigma = (\mathbb{N}, \mathbb{R}, \mathcal{B} = \ker(R)) \) where \( R \in \mathbb{R}[s] \); that is,

\[
R(s) = \alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n \quad \alpha_n \neq 0.
\]

\( ^5 \) Recall that \( \mathcal{B} \) must be shift-invariant and this carries to \( \mathcal{B}^\perp \).
Let $w \in \mathcal{B}$ be given. Then $R(\sigma)(w)(t) = 0$ for all $t \in \mathbb{N}$ implies that

$$R(\sigma)(w)(t) = \alpha_0 w(t) + \alpha_1 w(t + 1) + \cdots + \alpha_n w(t + n) = 0.$$  

If we let $t = 0$, then

$$-\frac{1}{\alpha_n} (\alpha_0 w(0) + \alpha_1 w(1) + \cdots + \alpha_{n-1} w(n-1)) = w(n).$$

That is, $\mathcal{B}_{|[0,n)} = \mathbb{R}^n$ since we can arbitrarily choose the first $n$ terms of the sequence; this leaves $w(n)$ determined by the earlier elements in the sequence. Now consider $t = 1$.

$$-\frac{1}{\alpha_n} (\alpha_0 w(1) + \alpha_1 w(2) + \cdots + \alpha_n w(n)) = w(n + 1).$$

By induction, this process can continue on so that for $t' \geq n$, $w(t')$ is determined. This leaves us with the following result.

**Proposition 2.9.1.** Let $\Sigma = (\mathbb{N}, \mathbb{R}, \mathcal{B} = \ker(R))$ where $R \in \mathbb{R}[s]$. It follows that $\Sigma$ is autonomous with $L = \deg(R)$. Moreover, we have the $\mathbb{R}$-module isomorphism $\mathcal{B} \cong \mathbb{R}^n$.

This leads to an interesting implication of autonomy to the decomposition of the signal space as seen by the following proposition.

**Proposition 2.9.2.** Let $\Sigma = (\mathbb{N}, \mathcal{W} = \mathbb{R}, \mathcal{B} = \ker(R))$ be a time-invariant system where $R \in \mathbb{R}[z]$. If $\Sigma$ is autonomous then there exists the decomposition $\mathbb{R}^N = \mathcal{B}_{|[0,L)} \oplus \mathbb{R}^{[L,\infty)}$.

**Proof.** First we demonstrate that the decomposition is between complementing spaces. Let $w \in \mathcal{B} \cap \mathbb{R}^{[L,\infty)} \subset \mathbb{T}$. Then $Rw = 0$ and $w\big|_{[0,L)} = 0$ and thus $w = 0$ by uniqueness of the extension. Let $w \in \mathbb{R}^{[0,L)}$ be given, noting that it has a unique extension $\hat{w} \in \mathcal{W}^T$ such that $\hat{w}\big|_{[0,L)} = w$. Write $w = \hat{w} + (w - \hat{w})$. This yields

$$(w - \hat{w})\big|_{[0,L)} = w - \hat{w}\big|_{[0,L)} = 0.$$

Therefore $x - y \in \mathbb{R}^{[L,\infty)}$ since it consists precisely of the elements which have no support on $[0, L)$. By the decomposition we see that

$$w \in \mathcal{B} + \mathbb{R}^{[L,\infty)}$$

and thus $\mathbb{R}^{[0,L)} \subset \mathcal{B} + \mathbb{R}^{[L,\infty]}$. Because $\mathcal{B} + \mathbb{R}^{[L,\infty)} \subset \mathbb{T}$ and $\mathbb{T} = \mathbb{R}^{[0,L)} + \mathbb{R}^{[L,\infty]}$, $\mathcal{W}^T = \mathbb{R}^{[0,L)} + \mathbb{R}^{[L,\infty)} = \mathcal{B} + \mathbb{R}^{[L,\infty)} = \mathcal{B} \oplus \mathbb{R}^{[L,\infty)}$.

A natural extension of the recent results is to the situation where $T = \mathbb{Z}$ and we consider $R \in \mathbb{R}[z, z^{-1}]$; by the same methods used above, we reach the following result.
Corollary 2.9.3. Let $\Sigma = (\mathbb{Z}, \mathbb{R}, \mathcal{B} = \ker(R))$ where $R \in \mathbb{R}[s, s^{-1}]$. It follows that $\Sigma$ is autonomous with $L = n_1 + n_2$ where

$$R = \alpha_{-n_1} s^{-n_1} + \alpha_{n_1+1} s^{-n_1+1} + \cdots + \alpha_{n_2} s^{n_2} \quad \alpha_{-n_1} \neq 0, \alpha_{n_2} \neq 0.$$ 

Moreover, we have the $\mathbb{R}$-module isomorphism $\mathcal{B} \cong \mathbb{R}^L$.

A natural conclusion to reach is that the $L$ used in the definition of autonomy is tied to the degree of the polynomial in the behavioral equation. Of course, when $W = \mathbb{R}^q$ the above is in general false. The reasons for this lie in the following theorem.

Theorem 2.9.4. [15] Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$ be a dynamical system with $\mathcal{B} \in \mathcal{L}^q$. Then the following conditions are equivalent:

1. $\Sigma$ is autonomous.
2. $\mathcal{B}$ is finite dimensional.
3. There exists an $R \in \mathbb{R}^{q \times q}[s, s^{-1}]$ with $\det R \neq 0$ such that $\mathcal{B} = \ker(R(\sigma, \sigma^{-1}))$.
4. There exists $t \in \mathbb{N}$ and a linear map $f : \mathcal{B}_{[0,t)} \to W^T$ such that $w = f(w|_{[0,t)})$ if and only if $w \in \mathcal{B}$.

Notice that items 2 and 4 have very strong ties to the results we have derived for $W = \mathbb{R}$; however, it is item 3 that is new in our journey. It states that a system is autonomous if and only if $R$ has full column rank. Notice that in the case $W = \mathbb{R}$ this is the case for any nonzero polynomial. Item 4 is referred to as an image representation, albeit not as noticeable in its current form.

2.10 Controllable Systems

We now move to the idea of a controllable behavior. This is one of the key issues addressed in this section. Willems defines controllability of a behavior as the following.

Definition 2.10.1. [9] Let $\mathcal{B}$ be a behavior of a time-invariant dynamical system. This system is called controllable if there exists $t_1 \in T$ such that for any two trajectories $w_1, w_2 \in \mathcal{B}$ there exists a trajectory $w \in \mathcal{B}$ with the property

$$w(t) = \begin{cases} w_1(t) & t \leq 0 \\ w_2(t-t_1) & t \geq t_1. \end{cases}$$

Notice that items 2 and 4 have very strong ties to the results we have derived for $W = \mathbb{R}$; however, it is item 3 that is new in our journey. It states that a system is autonomous if and only if $R$ has full column rank. Notice that in the case $W = \mathbb{R}$ this is the case for any nonzero polynomial. Item 4 is referred to as an image representation, albeit not as noticeable in its current form.
In Definition 2.10.1, the systems memory range \((t_1)\) is not specified. This implies that there needs to be an ample amount of “transition time” between trajectories in order for concatenation to occur. The main observation that should be made about a controllable system is that given any signal, we can “steer” the signal so that a desired future shall occur.

**Example 2.10.1.** Consider the thermostat in a house. We specify a desired temperature and the cooling/heating system in the house adjusts the airflow temperature so that eventually a given temperature is reached. Of course, this assumes that the house is insulated well enough so that this is possible!

We have thus far introduced two fundamental concepts to systems theory: controllability and autonomy. Notice that for an autonomous system, given a signal’s past, we can produce a unique future. Conversely, for controllable systems, we can specify both a signal’s past and future. Of course, an ample amount of transition time is necessary for the desired future to be reached - this should correlate to the reader’s intuition on, say, cooling a building. We expect that these two ideas are the two extreme situations - either the future is uniquely determined or it can be freely prescribed independently of the past.
Chapter 3

Multidimensional Behaviors

Dynamic systems evolve in time, that is, they depend on one free parameter. The world, however, is not one-dimensional.

- Eva Zerz

3.1 Introduction to 2D Behaviors

In [11] Rocha introduces the definition of a 2-D behavioral system and defines controllability for 2-D systems. We shall only retain her definition of controllability for systems over $\mathbb{Z}^2$. However, Rocha’s definition of a 2-D dynamical system is what we shall use as the basis for the generalization to multiple dimensions. As the story unfolds we shall discover that controllability has deeper implications than immediately apparent. We begin with the definition of a 2-D dynamical system.

Definition 3.1.1. [11] A 2-D system is characterized by an index set $T \subset \mathbb{R}^2$, a signal value space $W$, and a subset $B$ of $W^T$ (the set of all functions $T \rightarrow W$), called the behavior of the system. The system $\Sigma$ defined by $T$, $W$, and $B$ will be denoted by $\Sigma := (T, W, B)$.

As in the single variable situation, the signals $w : T \rightarrow W$ contained in $B$ will satisfy the governing laws of some phenomena; however, we are no longer limited to “time” as our signal domain. Indeed, we are opening up the door for multidimensional inputs. In [11] and [12] signals were assumed to have $\mathbb{Z}^2$ as their domain and thus for this section we shall make the same assumption. If we consider $T = \mathbb{N}^n$ or $\mathbb{R}^n$ matters shall change significantly.

Example 3.1.1. In digital image processing, a convolution filter is used to enhance some features that are used in the application of images. One very common filter is the normalized
blur filter,

\[ F = \begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}. \]

Given a digital image, \( I \), the blurred image is computed as

\[ \hat{I}(x, y) = \sum_{v, w} I(x - v, y - w) F(v, w). \]

In this situation, we have the map \( R(I) = \hat{I} \) as the behavioral equation and our behavior is defined as

\[ B = \{ (I, \hat{I}) : \hat{I} = R(I) \}. \]

We can see that \( \Sigma = (\mathbb{Z}^2, \mathbb{R}^2, B) \).

### 3.2 Results on Bivariate Laurent Operator Rings

To prevent notational clutter, we define \( D = \mathbb{R}[s_1, s_2, s_1^{-1}, s_2^{-1}] \) and call this ring a *Laurent operator ring*. Let \( T = \mathbb{Z}^2 \) and \( W = \mathbb{R}^q \); it follows that the signal space \( W^T \) is a vector-valued doubly index sequence. One natural operation that we shall consider on this space of functions is the shift-operator given by the following definition.

**Definition 3.2.1.** Let \( w : T \rightarrow W \) be a given signal. We define the *shift-operators* on \( W^T \) as

\[ \begin{align*}
(\sigma_1 w)(t_1, t_2) &= w(t_1 + 1, t_2) \\
(\sigma_1^{-1} w)(t_1, t_2) &= w(t_1 - 1, t_2)
\end{align*} \]

Furthermore, we define recursively \( \sigma_i^n w = \sigma_i^{n-1}(\sigma_i w), i = 1, 2 \).

**Remark 3.2.1.** One might ask, “Why are shift-operators important?” The natural explanation is that they allow one to look at a collection of terms from a sequence and determine if they satisfy a specified condition. Moreover, they have uses in discretized partial differential equations. One needs only to look at their applications in the 1-D situation to understand their importance in 2-D.

**Definition 3.2.2.** Let \( R \in D^{p \times q} \) be given. For the specified \( R \), we define \( R(\sigma) \) where the indeterminates \( s_i, s_i^{-1} \) are replaced, respectively, with \( \sigma_i, \sigma_i^{-1}, i = 1, 2 \). We call \( \Sigma = (\mathbb{Z}^2, \mathbb{R}^q, B = \ker(R(\sigma))) \) an *autoregressive system*. 
Remark 3.2.2. It is important that the reader note when we talk of $R(\sigma)(w) = 0$ for $w \in W^T$ that $(R(\sigma)(w))(t) = 0$ for all $t \in T$. This follows from $W^T$ being an $\mathbb{R}$-module with the zero sequence as identity.

We shall spend some time looking at the behavioral structure of autoregressive (AR) systems. These, as we will demonstrate have a connection to the topology of the Fréchet Coordinate Space $L^q \times L^q$. As one would expect, we must introduce numerous definitions before getting to such results.

### 3.2.1 Autoregressive 2-D Systems

Before the onslaught of definitions, we will consider a somewhat trivial example to motivate our work.

**Example 3.2.1.** Consider the system $\Sigma = (\mathbb{Z}^2, \mathbb{R}, \mathcal{B})$ where the behavior is defined as

$$\mathcal{B} = \ker (1 - \sigma_1 \sigma_2).$$

Equivalently, for $w \in \mathbb{R}^{\mathbb{Z}^2}$ we have that it is admissible (i.e. $w \in \mathcal{B}$) if

$$w(t_1, t_2) = w(t_1 + 1, t_2 + 1) \quad t_1, t_2 \in \mathbb{Z}.$$

First, let us take $w_1, w_2 \in \mathcal{B}$ and try to add them. We easily see

$$w_1(t_1, t_2) + w_2(t_1, t_2) = w_1(t_1 + 1, t_2 + 1) + w_2(t_1 + 1, t_2 + 1) \quad t_1, t_2 \in \mathbb{Z}.$$

Moreover, for $\alpha \in \mathbb{R}$ and $w \in \mathcal{B}$ we have

$$\alpha w(t_1, t_2) = \alpha w(t_1 + 1, t_2 + 1) \quad t_1, t_2 \in \mathbb{Z}.$$

We can therefore say that $\mathcal{B}$ is an $\mathbb{R}$-module under addition. Now that we have one nice result, let us go for another: how does $\sigma_1$ act on $\mathcal{B}$? Let $w \in \mathcal{B}$ be given. Then

$$(\sigma_1 w) = w(t_1 + 1, t_2) = w(t_1 + 2, t_2 + 1)$$

so $(\sigma_1 w) \in \mathcal{B}$. By the same argument, we can state $\sigma_1^{-1} \mathcal{B} \subset \mathcal{B}$. As a result, $\sigma_1 \mathcal{B} = \mathcal{B}$.

We have pulled out two concepts from this example that we shall state in general: shift-invariance and linearity.

**Definition 3.2.3.** The system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B})$ is said to be linear if $\mathcal{B}$ is a linear subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$. This is a natural extension of Definition 3.3.7 where $T = \mathbb{Z}^2$ and $W = \mathbb{R}^q$.

**Remark 3.2.3.** When we speak of $\mathcal{B}$ being a linear space, we mean that it is an $\mathbb{R}$-module. Multiplication is defined as scalar multiplication of the image; in other words, $\alpha w(t)$ satisfies the system laws for all $t \in T$. Pointwise operations come from the vector space structure of $W = \mathbb{R}^q$. 
Definition 3.2.4. [11] We shall call $\Sigma = (Z^2, \mathbb{R}^q, B)$ shift-invariant if both $\sigma_i B = B$ and $\sigma_i^{-1} B = B$ for $i = 1, 2$.

In the one dimensional theory, the definition of shift-invariant behaviors only was the inclusion $\sigma B \subset B$. However, we are only considering $Z^2$ as the signal space domain and thus we have that shifts are invertible.

The reader should note that the above definitions are “natural” extensions to the ones specified by Willems in [15]; the following example should make this connection apparent.

Example 3.2.2. Let $W = \mathbb{R}$ and $T = Z^2$ and consider

$$R(\sigma) = \begin{bmatrix} \sigma_1 - 1 \\ \sigma_2 - 1 \end{bmatrix}.$$ 

If $w \in B = \ker(R(\sigma))$ we have that $w$ must be a constant sequence. To see this, let $w : Z^w \to \mathbb{R}$ be an arbitrary signal. If it is admissible, then for all $s, t \in \mathbb{Z}$,

$$w(t + 1, s) - w(t, s) = 0 \quad w(t, s + 1) - w(t, s) = 0.$$

A natural observation of $B$ is that this system is both shift-invariant and linear; this can be realized by either looking at $R$ or $B$.

As in the one dimensional case, shift operators are linear. However, we must note that they commute with each other; this is a concept that was nonexistent in the one dimensional theory. The following stream of definitions and lemmas shall be used to characterize autoregressive systems; like in the one dimensional theory, they shall be the linear, shift-invariant, complete systems.

Lemma 3.2.1. The shift operators are linear and thus for all $R \in D^{p \times q}$, $R(\sigma)$ is a linear operator on $W^T$. Moreover, the shift operators commute with each other.

Proof. First we note that for $\alpha \in \mathbb{R}$, $w_1, w_2 : T \to W$,

$$(\sigma_1(\alpha(w_1 + w_2))(t_1, t_2) = \alpha(w_1(t_1 + 1, t_2) + w_2(t_1 + 1, t_2)) = \alpha(\sigma_1 w_1 + \sigma_1 w_2)(t_1, t_2).$$

Clearly this holds for $\sigma_2$, $\sigma_1^{-1}$, and $\sigma_2^{-1}$ as well. Composition of linear operators is linear as well and thus we have our result. Commutativity is trivial.

Lemma 3.2.2. Let $\Sigma$ be a 2-D autoregressive system with $B = \ker(R(\sigma))$ for $R \in D^{p \times q}$. It follows that $B$ is a shift-invariant subspace of $W^T$ in the sense of $\sigma_i B = B$.

Proof. Let $w \in B$ be given. Then $R(\sigma)(w) = 0$. It follows from commutativity and associativity that

$$R(\sigma)(\sigma_i w) = \sigma_i R(\sigma)(w) = \sigma_i 0 = 0.$$
We thus have $\sigma_1B \subset B$. Conversely, let $w \in B$ be given. Then there exists $\hat{w}$ such that $w = \sigma_1\hat{w}$. Namely, $\hat{w} = \sigma_1^{-1}w$. We conclude with the desired result\textsuperscript{1}.

[72x71]Autoregressive systems are defined by polynomials and thus only a finite number of points around a given location, $(t_1, t_2)$, are required to check whether or not the signal is admissible at $(t_1, t_2)$. We would like to encapsulate this concept in a more general definition.

**Definition 3.2.5.** Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}^2$ be specified. We shall call

$$I = \{(a, b) \in \mathbb{Z}^2 : a_1 \leq a \leq a_2, b_1 \leq b \leq b_2\}$$

an *interval*. Provided that $a_1, a_2, b_1, b_2$ are finite, we shall refer to $I$ as a *finite interval*.

**Definition 3.2.6.** \textsuperscript{[11]} We shall call $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, B)$ complete if $w \in B$ if and only if $w|_I \in B|_I$ for all finite intervals $I$.

Completeness has many intuitive implications. The most superficial is that determination of whether a signal is contained in $B$ (i.e. admissible) requires only looking on finite intervals. It is in this perspective that we see admissibility is a local property for complete systems. Another way we can interpret completeness is that, for autoregressive systems, determination of whether a signal is admissible requires checking intervals with sides of length up to the largest degree of the indeterminates of $R$ (where $R \in \mathbb{R}^{q \times q}[s_1, s_2, s_1^{-1}, s_2^{-1}]$). We have introduced enough concepts to make the connection between complete systems and their topological consequences.

**Lemma 3.2.3.** \textsuperscript{[11]} Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, B)$ be linear and shift-invariant. It follows that $\Sigma$ is complete if and only if $B$ is a closed subspace of $W^T$.

*Proof. (⇒) Let $w_n \in B$ be a sequence and $w \in W^T$ be its limit. We shall demonstrate that $w \in B$ by demonstrating it is admissible locally (i.e. the restriction to $I$ is contained in $B|_I$.) Let $I$ be a finite interval. Clearly $\lim w_n = w$ implies that $\lim w_n|_I = w|_I$ by pointwise convergence. Moreover, since $w_n|_I \in B|_I$ for all $n \in \mathbb{N}$, we have that $w|_I \in B|_I$ since $W^T$ is finite dimensional linear space and thus closed. That is, $w|_I \in B|_I$ for all finite intervals $I$. We conclude that $w \in B$ and thus $B$ is a closed subspace of $W^T$.

(⇐) Let $w \in W^T$ be given and assume that, for all finite intervals $I$, $w|_I \in B|_I$. For each $n \in \mathbb{N}$ define

$$I_n = \{(a, b) \in \mathbb{Z}^2 : -n \leq a, b \leq n\}.$$ 

Each $I_n$ is a finite interval and thus $w|_{I_n} \in B|_{I_n}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $w_n \in B$ be such that $w_n|_{I_n} = w|_{I_n}$; since $w_n|_{I_n} \in B|_{I_n}$ such a $w_n$ must exist. By hypothesis, $\lim w_n = \hat{w} \in B$, we must only demonstrate that $w = \hat{w}$; this, however, follows since $\hat{w}|_{I_n} = w|_{I_n}$ for all $n \in \mathbb{N}$. We conclude with the desired result.*

\textsuperscript{1}If we are not working with a Laurent operator ring, notice that the inclusion $B \subset \sigma_1B$ cannot follow from this same argument. Indeed, we do not have an inverse operator to use!
We shall now make a connection between the closedness of \( B \) as a subspace of \( W^T \) and how \( B \) is represented. Define \( \mathcal{L} = W^T \) and

\[
\mathcal{L}^* = \{ v^* \in \left( \mathbb{R}^{1 \times q} \right)^T : v^* \text{ has compact support} \}.
\]

Note that \( \mathcal{L}^* \) is clearly an \( \mathbb{R} \)-module under the obvious operations. As in the one dimensional case, we have that \( \mathcal{L}^* \) is the continuous dual of \( \mathcal{L} \); the proof for this is almost identical to that of Lemma 2.8.1.

**Lemma 3.2.4.** \( \mathcal{L}^* \) is isomorphic to \( \mathbb{R}^{1 \times q}[s_1, s_2, s_1^{-1}, s_2^{-1}] \) as an \( \mathbb{R} \)-module. When we observe this isomorphism we shall refer to it as \( \mathcal{L}^*_1 \).

**Proof.** Let us first define the bijection. Define for \( f \in \mathcal{L}^* \)

\[
\phi(f) = \sum_{(i,j) \in \mathbb{Z}^2} f_{i,j} s_1^i s_2^j \quad f_{i,j} = f(i, j).
\]  (3.1)

Notice that for \( \alpha \in \mathbb{R}, f, g \in \mathcal{L}^* \) we have that

\[
\phi(\alpha f) = \alpha \phi(f) \quad \phi(f + g) = \phi(f) + \phi(g).
\]

This gives us that \( \phi \) is an \( \mathbb{R} \)-homomorphism. To establish that \( \phi \) is an epimorphism, let \( g \in \mathbb{R}^{1 \times q}[s_1, s_2, s_1^{-1}, s_2^{-1}] \) be given so that

\[
g = \sum_{i,j} g_{i,j} s_1^i s_2^j.
\]

If we define

\[
\hat{g}(i,j) = g_{i,j} \quad i, j \in \mathbb{Z}
\]

then we have a function \( \hat{g} \in \mathcal{L}^* \) such that \( \phi(\hat{g}) = g \). That \( \phi \) is a monomorphism is trivial since functions in \( \mathcal{L}^* \) have compact support.

We now define the bilinear map (i.e. pairing) \( \langle \cdot, \cdot \rangle : \mathcal{L}^* \times \mathcal{L} \to \mathbb{R} \). Let \( f \in \mathcal{L}^* \) and \( v \in \mathcal{L} \) be given and define

\[
\langle f, v \rangle = \sum_{(i,j) \in \mathbb{Z}^2} f_{i,j} \cdot v_{i,j}.
\]

We may now consider the rather interesting observation

**Lemma 3.2.5.** For \( k = 1, 2 \) it follows that \( \langle s_k f, v \rangle = \langle f, \sigma_k v \rangle \).
Proof. Without loss of generality, assume that \( k = 1 \). First note that
\[
s_1 f = \sum_{(i,j) \in \mathbb{Z}^2} f_{i,j} s_1^{i+1}s_2^j.
\]
It follows that
\[
\langle s_1 f, v \rangle = \sum_{(i,j) \in \mathbb{Z}^2} f_{i,j} v_{i+1,j} = \langle f, \sigma_1 v \rangle.
\]

Let \( \mathcal{N} \subset \mathcal{L} \) and \( \mathcal{N}^* \subset \mathcal{L}^* \) be given. We define
\[
\mathcal{N}^\perp = \{ f \in \mathcal{L}^*: \langle f, v \rangle = 0 \ \forall v \in \mathcal{N} \}
\]
\[
\mathcal{N}^{*\perp} = \{ v \in \mathcal{L} : \langle f, v \rangle = 0 \ \forall f \in \mathcal{N}^* \}.
\]
We are now ready to state and prove the following result.

**Theorem 3.2.6.** A 2-D system \( \Sigma = (\mathbb{Z}^2, \mathbb{R}^q, B) \) is linear, shift-invariant, and complete if and only if it is autoregressive.

Proof. (\( \Rightarrow \)) From Lemma 3.2.3 it follows that \( B \) is a closed subspace of \( \mathcal{L} \). By Lemma 3.2.4, \( \mathcal{L}^* \cong \mathbb{R}^{1 \times q}[s_1, s_2, s_1^{-1}, s_2^{-1}] \). The isomorphism yields for \( f \in \mathcal{L}^* \) something that looks like 3.1 and thus is polynomial with vector coefficients. Clearly we have the \( \mathcal{D} \)-isomorphism
\[
\mathbb{R}^{1 \times q}[s_1, s_2, s_1^{-1}, s_2^{-1}] \cong \mathcal{D}^{1 \times q}
\]
It follows that \( \mathcal{L}^* \) is a finitely generated free module over the Noetherian ring \( \mathcal{D} \). Since \( B^\perp \) is a \( \mathcal{D} \)-submodule of \( \mathcal{L}^* \), it is finitely generated and thus there exists a finite natural number, say \( \ell \), of \( r_i \in B^\perp \) such that
\[
B^\perp \cong \bigoplus_{i=1}^{\ell} \mathcal{D}r_i.
\]
That is, we have a generating set for \( B^\perp \). From here, let us define
\[
\mathcal{V} = \{ w \in \mathcal{L} : \langle r_i, w \rangle = 0 \ \ i = 1, \ldots, \ell \}.
\]
Because we pinned down the generating set for \( B^\perp \), it makes sense that \( (B^\perp)^\perp = \mathcal{V} \). If we can show this, then we have some way of manually constructing \( B \) from a finite number of polynomials and thus can construct the necessary matrix for autoregressive systems. Clearly we get the inclusion
\[
(B^\perp)^\perp \subset \mathcal{V}
\]
since the \( r_i \) can only kill less elements than that of what's in \( (B^⊥)^⊥ \) already. To get the other inclusion, we have to work a bit more. Take \( w \in \mathcal{V} \) and \( a \in B^⊥ \). We have shown that there exists some \( a^1, \ldots, a^\ell \in \mathcal{D} \) such that

\[
a = a^1 r_1 + \cdots + a^\ell r_\ell.
\]

Because each \( a^i \in \mathcal{D} \) we can say that

\[
a^i = \sum_{j,k} a^i_{j,k} s^j_1 s^k_2
\]

and thus

\[
\langle a, w \rangle = \sum_{i=1}^\ell \langle a^i r_i, w \rangle = \sum_{i=1}^\ell \sum_{j,k} \langle a^i_{j,k} r_i s^j_1 s^k_2, w \rangle = \sum_{i=1}^\ell \sum_{j,k} a^i_{j,k} \langle r_i, \sigma^j_1 \sigma^k_2 w \rangle.
\]

From shift-invariance of \( B \) (see Lemma 3.2.5) we have that

\[
\langle r_i, \sigma^j_1 \sigma^k_2 w \rangle = 0 \quad i = 1, \ldots, \ell
\]

and thus \( \langle a, w \rangle = 0 \); by definition this implies that \( w \in (B^⊥)^⊥ \). By Lemma 2.8.3 (it still holds for \( T = \mathbb{Z}^2 \)), we have that \( B = (B^⊥)^⊥ \) and thus \( \mathcal{V} = B \). We conclude that for the matrix

\[
R = \text{col} \left[ r_1 \cdots r_\ell \right]
\]

that \( B = \ker R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) \).

The converse direction of the proof is straight forward. \( \square \)

This result gives us a way of determining all autoregressive systems by properties that are not related to the matrix. Indeed, autoregressive systems can be determined entirely by the structure of their behavior. One interesting and useful corollary of this result is the following.

**Corollary 3.2.7.** A 2-D system \( \Sigma = (\mathbb{Z}^2, \mathbb{R}^q, B) \) is linear, shift-invariant, and complete if and only if it is autoregressive with \( R \in \mathbb{R}^{q \times q}[s_1, s_2] \).

**Proof.** This follows from the fact that for \( d_1, d_2 \in \mathbb{N} \),

\[
\ker R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) = \sigma_1^{d_1} \sigma_2^{d_2} \ker R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) = \ker \sigma_1^{d_1} \sigma_2^{d_2} R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}).
\]

We can select \( d_1 \) and \( d_2 \) large enough so that \( R \in \mathbb{R}^{q \times q}[s_1, s_2] \). \( \square \)
3.2.2 Memory and Controllability

As in the 1-D scenario, controllability of a 2-D system is related to the memory of a system; however, we must consider a different notion of lag since we have two dimensions in our signal domain. We will start with the concatenation of two signals.

**Definition 3.2.7.** Let $T_1, T_2 \subseteq T$ be given as well as $w_1, w_2 \in W^T$. Provided that $w_1|_{T_1 \cap T_2} = w_2|_{T_1 \cap T_2}$, we define the concatenation $w_1 \wedge w_2|_{T_1}$ as $w \in W^{T_1 \cup T_2}$ such that $w|_{T_1} = w_1|_{T_1}$ and $w|_{T_2} = w_2|_{T_2}$.

A reasonable question is: *how do we know that such a $w$ is admissible?* Observant readers will notice that existence is not necessary since we are not talking about admissible signals. However, it is an important concept that we note in the following definition.

**Definition 3.2.8.** For a system $\Sigma$, let $T_1, T_2 \subseteq T$ be given as well as $w_1, w_2 \in B$. We say that $w_1$ and $w_2$ are *concatenable* over $T_1$ and $T_2$ if there exists $w \in B$ such that $w|_{T_1} = w_1|_{T_1}$ and $w|_{T_2} = w_2|_{T_2}$.

The concept of an admissible signal that is split between two possibly different signals allows us to consider the definition of independence of behaviors in the following definition.

**Definition 3.2.9.** [11] Let $\Sigma = (T, W, B)$ be a 2-D system and let $T_1, T_2 \subseteq T$ be such that $T_1 \cap T_2 = \emptyset$. Then the restricted behaviors $B|_{T_1}$ and $B|_{T_2}$ are called *independent* on $T_1$ and $T_2$ if $B|_{T_1 \cup T_2} \cong B|_{T_1} \times B|_{T_2}$.

This states that, for the given two sets $T_1$ and $T_2$, two arbitrary signals are concatenable. Moreover, with insights from the 1-D development, we should end up with results from this idea that lead us to a definition of controllability.

**Definition 3.2.10.** [12] A 2-D system $\Sigma = (T, W, B)$ is said to be *controllable* if there exists a positive real number $\rho$ such that for all $T_1, T_2 \subseteq T$ where $d(T_1, T_2) \geq \rho$ and $w_1, w_2 \in B$ there exists $w \in B$ such that $w|_{T_1} = w_1|_{T_1}$ and $w|_{T_2} = w_2|_{T_2}$.

In the above definition, $\rho$ will serve as the memory of the system. Indeed, it says that $B|_{T_1}$ and $B|_{T_2}$ are independent provided they are sufficiently far apart for any arbitrary sets $T_1$ and $T_2$. At first glance, it appears that this definition feels like the natural extension to the 1-D definition in [15]. The question becomes: *do we get the same structural equivalences?* Before answering this question, we have to define some notation and terminology.
Definition 3.2.11. For a given behavior $B \subset W^T$, we define the subspace of compactly supported signals of $B$ as

$$B^{\text{compact}} = \{ w \in B : \text{there exists a finite interval } I \subset \mathbb{Z}^2 \text{ such that } w|_{\mathbb{Z}^2 \setminus I} = 0 \}.$$ 

When we talk of the closure of the module $B$, it is in the sense of the topology of pointwise convergence.

Theorem 3.2.8. [11] Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, B)$ be an autoregressive system. The following are equivalent:

1. $B$ controllable.
2. $B = B^{\text{compact}}$.
3. There exists a positive integer $l$ and a $q \times l$ polynomial matrix $M(s_1, s_2)$ such that $B = \text{im}M(\sigma_1, \sigma_2)$.

The last statement in the above theorem is the image representation condition of controllability. It happens that this will be the definition that is the driving force behind higher dimensional generalizations.

3.3 Multidimensional Behaviors

3.3.1 Introduction

In the multivariable generalization a system, $\Sigma = (T, W, B)$, is the expected $n$-dimensional lift of the two-dimensional case; $T \subset \mathbb{Z}^n$ will be the set of independent variables and $W \subset \mathbb{R}^q$ will be the signal value set. We define a $n$-dimensional dynamical system as follows.

Definition 3.3.1. An $n$-D system is characterized by an index set $T \subset \mathbb{Z}^n$, a signal value space $W$, and a subset $B$ of $W^T$ (the set of all functions $T \to W$), called the behavior of the system. The system $\Sigma$ defined by $T$, $W$, and $B$ will be denoted by $\Sigma := (T, W, B)$.

3.3.2 $n$-D Autoregressive Systems

As in the lower dimension situations, we shall focus on autoregressive systems and thus we consider a kernel representation. Let $\mathcal{D} = \mathbb{R}[s_1, \ldots, s_n]$ or $\mathcal{D} = \mathbb{R}[s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}]$ denote a polynomial ring in $n$ variables (the difference will be obvious based on $T$.) We shall make the assumption that our signal space $A$ is a left $\mathcal{D}$-module. A matrix $R \in \mathcal{D}^{p \times q}$
will be written as $R(\sigma)$ to represent replacing the indeterminate in the polynomial with the shift-operator. The notation will be the same as in the two dimensional situation, namely $\sigma_i$ will correspond to the shift in the $i$-th coordinate; that is, for $w \in \mathcal{B}$,

$$(\sigma_i w)(t_1, \ldots, t_n) = w(t_1, \ldots, t_i + 1, t_{i+1}, \ldots, t_n).$$

The effect of $\sigma_i^{-1}$, when defined, is as expected. The behavior of the system can then be defined as

$$\mathcal{B} = \ker A(R) = \{ w \in W^T : Rw = 0 \}.$$

The characterization of autoregressive (AR) systems is the same as in the one-dimensional and two-dimensional situations. Namely, they are the behaviors which are linear, closed, shift-invariant subspaces of $\mathbb{R}^q$. This is obtained through the same proof as in the two-dimensional case using the results characterizing the dual space of $(\mathbb{R}^q)^{\mathbb{Z}^n}$ as Laurent polynomials in $n$-variables. From here we use that the behavior is a finitely generated module over $\mathbb{R}[s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}]$ to reconstruct the behavior.

**Theorem 3.3.1.** An $n$-D system $\Sigma = (\mathbb{Z}^n, \mathbb{R}^q, \mathcal{B})$ is linear, shift-invariant, and complete if and only if it is autoregressive.

Moreover, we have, through no significant modification of the two-dimensional result,

**Corollary 3.3.2.** An $n$-D system $\Sigma = (\mathbb{Z}^n, \mathbb{R}^q, \mathcal{B})$ is linear, shift-invariant, and complete if and only if it is autoregressive with $R \in \mathbb{R}^{q \times q}[s_1, \ldots, s_n]$.

An important note from J. Wood and D. Owens in [17] that may have been brushed over is that the behavior itself is not finitely generated as a module over $\mathbb{R}[s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}]$. Indeed, the dual of the behavior, $\mathcal{B}^\perp$ is finitely generated and we use this to construct a kernel representation. This has no connection to whether or not $\mathcal{B}$ is finitely generated.

### 3.3.3 Autonomous Systems

For a given autoregressive system, $\Sigma$, we have that $\mathcal{B} \subset W^T$ where $W = \mathbb{R}^q$ and $T = \mathbb{Z}^n$. That is, for a given signal, $w$,

$$w(t) = (w_1(t), \ldots, w_q(t)) \quad t \in T.$$

It is convenient to consider $w \in (\mathbb{R}^T)^q$ in this case; define $A = \mathbb{R}^T$ and then we can say that $\mathcal{B} \subset A^q$. Now we begin with the notion of a free variable.

**Definition 3.3.2.** For a behavior $\mathcal{B}$, the $i^{th}$ component $w_i$ of the signal vector $w = (w_1, \ldots, w_q)^T$ is called a free variable or input if the canonical $\mathcal{D}$-morphism $\pi_i : \mathcal{B} \rightarrow A$ defined as $w \mapsto w_i$ is an epimorphism.
This definition is motivated by the idea that a free variable allows one to construct a trajectory from any given \( w_i \in \mathcal{A} \). We shall start our journey with the following definition.

**Definition 3.3.3.** A behavior \( \mathcal{B} \) is said to be *autonomous* if it has no free variables.

There are situations where it will be favorable to have a different definition of autonomy mainly to alter our intuition. Recall Definition 2.9.1 and how autonomy involved the unique extension of a signal. This is what we should keep in mind when looking at autonomous systems! We are given some signal restricted to some set and want to see if the extension of the signal is uniquely defined off the set. First consider \( T = \mathbb{N}^n \).

**Definition 3.3.4.** \(^2\) A *cw-ideal* is a non-empty subset \( I \subset \mathbb{N}^n \) such that for \( \ell \in I \), \( \ell + m \in I \) for all \( m \in \mathbb{N}^n \).

A good example for a cw-ideal in \( \mathbb{Z}^2 \) is the idea of a right-angled wedge in the plane.

**Lemma 3.3.3.** \(^{[20]}\) Let \( \Sigma = (T \subset \mathbb{N}^n, W, \mathcal{B}) \). A behavior \( \mathcal{B} \) is autonomous if and only if there exists \( J \subset \mathbb{N}^n \) such that \( \mathbb{N}^n \setminus J \) is a cw-ideal and for all \( w \in \mathcal{B} \), \( w|_J = 0 \) implies that \( w = 0 \).

It happens that there is a relationship between a behavior’s autonomy and the torsion of the the kernel representation’s cokernel. Before stating this connection, we have the following definition.

**Definition 3.3.5.** \(^{[5]}\) Let \( R \) be a ring and \( I \) an \( R \)-module. We say that \( I \) is an injective \( R \)-module if for any given short exact sequence of \( R \)-modules,

\[
0 \to M \to N,
\]

the induced sequence

\[
\text{hom}_R(N, I) \to \text{hom}_R(M, I) \to 0
\]

is also exact.

**Definition 3.3.6.** Let \( R \) be a ring and \( I \) an \( R \)-module. We say that \( I \) is an injective cogenerator if for any given short sequence of \( R \)-modules,

\[
M \to N \to P,
\]

is exact if and only if the induced sequence

\[
\text{hom}_R(P, I) \to \text{hom}_R(N, I) \to \text{hom}_R(M, I)
\]

is exact.

\(^2\)A cw-ideal is also known as a shift-invariant sublattice (see \([1] \).)
Now we are able to state our result.

**Lemma 3.3.4.** [21] Assume that $A$ is an injective cogenerator $\mathcal{D}$-module. Then $B$ is autonomous if and only if $M = \mathcal{D}^q/\text{im}(R^T)$ is a torsion $\mathcal{D}$-module.

**Proof.** ($\Rightarrow$) Assume that $M$ is a not a torsion $\mathcal{D}$-module. Recalling that $M = \mathcal{D}^q/\text{im}(R^T)$, we see that this implies there exists some $[m] \in M$ such that $d[m] = [dm] \neq 0$ for all $d \in \mathcal{D} \setminus \{0\}$. Define the map $\iota: \mathcal{D} \rightarrow M$ as $d \mapsto d[m]$ and thus we have the exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow M.$$

Applying $\text{hom}(-, A)$ to this sequence\(^3\) yields the exact sequence

$$B \xrightarrow{\pi_i} A \longrightarrow 0.$$

This implies that $\pi_i$ is onto and thus $B$ is not autonomous.

($\Leftarrow$) Assume that there exists a free variable. Then we have the exact sequence

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\pi_i} & \mathcal{A} \\
\downarrow & & \downarrow \\
\text{hom}(M, A) & \longrightarrow & \text{hom}(\mathcal{D}, A) \\
\end{array} \longrightarrow 0.$$

Because $A$ is an injective cogenerator, the following sequence is exact.

$$0 \longrightarrow \mathcal{D} \longrightarrow M,$$

where $\iota$ is injection. That is $\iota(\text{id}_\mathcal{D})$ is not a torsion element and thus $M$ is not a torsion $\mathcal{D}$-module. \hfill \Box

This idea, however, also connects to the kernel representation through the following lemma.

**Lemma 3.3.5.** [20] Assume that $A$ is an injective $\mathcal{D}$-module. Then $B$ is autonomous if and only if $R$ has full column rank.

We sum our results on autonomous systems in the following theorem.

**Theorem 3.3.6.** [17] Let $B$ be an autoregressive behavior over $\mathbb{R}[s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}]$. The following are equivalent.

1. $B$ is autonomous.

\(^3\)See [21] for a proof of $\text{hom}(M, A) \cong B$; the main ingredient of the proof is the left exactness of the contravariant functor $\text{hom}(-, A)$. 
2. Any kernel representation of $B$ has full column rank.

3. There exists a region $T \subset \mathbb{Z}^n$ such that $T^c$ is nD unbounded and for any $w_1, w_2 \in B$,

$$w_1|_T = w_2|_T \Rightarrow w_1 = w_2.$$ 

4. For a kernel representation $R$ we have that $\mathcal{M} = \mathcal{D}^q/\text{im}(R^T)$ is a torsion module.

### 3.3.4 Controllable-Autonomous Decomposition

In the one-dimensional theory, we pointed out that controllable was “not autonomous.” This allowed the decomposition

$$B = B^c \oplus B^a$$

where $B^c$ and $B^a$ were, respectively, the controllable and autonomous signals of the behavior. It is important to note that there were no systems that were both controllable and autonomous; indeed, by definition this would be absurd. In the $n$-dimensional theory, the definition of controllability must be picked so that we do not have behaviors that are both controllable and autonomous.

The definitions that we first introduce are the obvious extension to Rocha’s in [12]. We shall soon observe that these are not consistent with the necessary\(^4\) implications when $T \neq \mathbb{Z}^n$; that is, they yield both controllable and autonomous systems. Let us consider the “natural” extension into $n$-dimensions.

**Definition 3.3.7.** An $n$-D system $\Sigma = (T, W, B)$ is said to be **controllable** if there exists a positive real number $\rho$ such that for all $T_1, T_2 \subset T$ where $\text{dist}(T_1, T_2) \geq \rho$ we have

$$B|_{T_1 \cup T_2} \cong B|_{T_1} \times B|_{T_2}.$$ 

Definition 3.3.7 yields both autonomous and controllable systems when $T = \mathbb{N}$ via the following result.

**Proposition 3.3.7.** There exists a behavior $B$ which is both controllable in the sense of Definition 3.3.7 and autonomous

**Proof.** Let us consider the one-dimensional behavior\(^5\) where $T = \mathbb{N}$ and $W = \mathbb{R}^2$.

$$B = \ker \begin{bmatrix} \sigma + 1 & -1 \\ 1 & \sigma - 1 \end{bmatrix}.$$ 

\(^4\)Although not logically necessary, they are necessary for applications to make sense.

\(^5\)This behavior was found in [18].
Let us take \( w = (w_1, w_2) \in B \). It follows from \( w = \ker R(\sigma) \) that for all \( t \in \mathbb{N} \),

\[
\begin{align*}
  w_1(t + 1) + w_1(t) &= w_2(t) \quad \Rightarrow \quad w_1(t + 1) = w_2(t) - w_1(t) \\
  w_1(t) &= w_2(t) - w_2(t + 1) \quad \Rightarrow \quad w_2(t + 1) = w_2(t) - w_1(t).
\end{align*}
\]

This implies that \( w(t) \) is zero unless \( t = 0, 1 \). At \( t = 0 \) we are free to have anything we wish, at \( t = 1 \) we have equality in both components, and at \( t > 1 \) we have the signal is zero. This is one way to find \( B \) is autonomous; the other way is to use that \( R \) has full column rank. Note, however, that \( B \) is controllable. Let \( T_1, T_2 \subset \mathbb{N} \) be two intervals and \( v_1, v_2 \in B \) be two trajectories. Furthermore, assume that \( T_1 \cap T_2 = \emptyset \) and \( \dist(T_1, T_2) > 1 \). It follows that

\[
v(t) = \begin{cases} v_1(t) & t \in T \setminus T_2 \\ v_2(t) & t \in T_2 \end{cases}
\]

is contained in \( B \). This should be to no surprise if neither sets contain 0 or 1 since those are the only non-zero points in the trajectory. By inspection, we have \( v \in B \) and thus \( B \) is controllable. This yields a system that is both autonomous and controllable. □

We shall move towards defining controllability so that such anomalies will no longer occur.

### 3.3.5 Controllability as Concatenation of Trajectories

J. Wood, E. Zerz, and P. Rocha laid the necessary groundwork for a multidimensional extension to the definition of controllability. Now we shall introduce a new definition of controllability that is equivalent to the definition in [17] when \( T = \mathbb{Z}^n \), yet, prevents the problems we have seen in the previous section. To prevent any confusion in our definition of controllability, we present the following definitions.

**Definition 3.3.8.** Let \( a \in \mathbb{N}^{r_1} \oplus \mathbb{Z}^{r_2} \) and \( T_1 \subset T = \mathbb{N}^{r_1} \oplus \mathbb{Z}^{r_2} \). We define the shift-operator on a subset as

\[
\sigma^a T_1 = \{ t \in T : t + a \in T_1 \}
\]

**Definition 3.3.9.** The diameter of a bounded set \( T_1 \subset T = \mathbb{N}^{r_1} \oplus \mathbb{Z}^{r_2} \) is

\[
p(T_1) = \max\{|t - t'| : t, t' \in T_1\}
\]

**Definition 3.3.10.** Let \( M \in D^{p \times q} \), \( M = \sum_{a \in \mathbb{N}^r} M_a z^a \) where \( M_a \) are coefficient matrices over \( F \), be given. We define its support to be

\[
\text{supp}(M) = \{ a \in \mathbb{N}^r : M_a \neq 0 \}
\]

Now we are able to introduce the definition of a controllable behavior for discrete systems.
Definition 3.3.11. [20],[18] Let $\mathcal{B}$ be an $r$-dimensional behavior with signal domain $T = \mathbb{N}^{r_1} \oplus \mathbb{Z}^{r_2}$. Then $\mathcal{B}$ is said to be controllable if there exists $\rho \geq 0$ such that for all $T_1, T_2 \subset T$ with $d(T_1, T_2) > \rho$, for all $w_1, w_2 \in \mathcal{B}$, and for all $b_1, b_2 \in T$, there exists $w \in \mathcal{B}$ such that

$$(\sigma^{b_1}w)|_{\sigma^{b_1}T_1} = (w_1)|_{\sigma^{b_1}T_1} \quad (\sigma^{b_2}w)|_{\sigma^{b_2}T_2} = (w_2)|_{\sigma^{b_2}T_2}.$$ 

We shall say that $\mathcal{B}$ is controllable with separation distance $\rho$.

This definition of controllability is equivalent to the past definitions introduced when we make the necessary restrictions. Moreover, we have the following result.

Theorem 3.3.8. [20] Let $\Sigma = (T, W, \mathcal{B})$ be an $n$-dimensional behavioral system where $T = \mathbb{N}^{r_1} \oplus \mathbb{Z}^{r_2}$ where $r_1 + r_2 = n$ and $r_1, r_2 \in \mathbb{N}$. It follows that the following are equivalent:

1. $\mathcal{B}$ is controllable.

2. $\mathcal{B}$ has an image representation.

3. If $\mathcal{B}$ is controllable and autonomous, then $\mathcal{B}$ is trivial.
Chapter 4

Divisibility of Polynomial Matrices

The notions of observability and controllability lie at the very heart of systems theory, and they are closely related to questions of primeness.

- Eva Zerz

4.1 Introduction

The definitions of controllability that we have considered are based on Willems’ trajectory approach to controllability. These definitions consider the signals in the behavior instead of the kernel representation. We would like to find a characterization of controllability in terms of a polynomial matrix $R$ whose kernel gives $B$. We will begin with a brief run through of the two dimensional results then slide into the $n$-dimensional results.

4.2 Two Dimensional Results

Definition 4.2.1. Let $R$ be a matrix with entries in $D$. We call $R$ factor left prime if it has full row rank and for any square left factorization

$$R = DR_1$$

we have that det$(D)$ is a non-zero constant.

Proposition 4.2.1. Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, B)$ be an autoregressive system. It follows that $B$ is controllable if and only if any kernel representation is factor left prime.
4.3 Primeness of Multivariate Polynomial Matrices

In [20], Zerz has several versions of primeness for polynomial matrices. We shall discuss and compare these definitions to those in [10].

4.3.1 Zero and Minor Primeness

For this section \( D = \mathbb{R}[z_1, \ldots, z_n] \) shall denote the polynomial ring over \( \mathbb{R} \) in \( r \) indeterminates \( z_i \). We shall denote its quotient field, the field of rational functions, by \( K = \mathbb{R}(z_1, \ldots, z_n) \).

We shall let \( R \) be a matrix with entries in \( D \), \( R \in D^{g \times q} \) where \( g \leq q \). The \( g \times g \) minors of \( R \) shall be denoted by \( m_1, \ldots, m_k \in D \). These generate an ideal that we shall denote by \( I = \langle m_1, \ldots, m_k \rangle \).

**Definition 4.3.1.** [20] A matrix \( R \) is called

1. zero left prime (ZLP) if its minors possess no common zero in \( \mathbb{C}^r \),
2. weakly zero left prime (WLP) if the minors have only a finite number of common zeros in \( \mathbb{C}^r \), and
3. minor left prime (MLP) if the minors are devoid of a non-trivial common factor.

For a polynomial matrix \( R \) of full row rank\(^1\), the algebraic set of rank singularities is defined as

\[
\mathcal{V}(R) = \{ \xi \in \mathbb{C}^r : \text{rank}(R(\xi)) < \text{rank}(R) \} = \{ \xi \in \mathbb{C}^r : m_1(\xi) = \ldots = m_k(\xi) = 0 \} = \mathcal{V}(I).
\]

Note that by \( \text{rank}(R(\xi)) \) we mean the number of linearly independent row vectors formed through evaluation of the matrix at \( \xi \). This means that the algebraic set \( \mathcal{V}(R) \) contains elements which introduce rank-deficiencies to \( R \).

**Lemma 4.3.1.** Let \( R \) have full row rank. Then \( \dim \mathcal{V}(R) \leq n - 1 \).

**Proof.** If \( \dim \mathcal{V}(R) = r \) then for all \( \xi \in \mathbb{C}^r \) we have \( \text{rank}(R(\xi)) < \text{rank}(R) \) and thus \( R \) cannot have full row rank.

The definition of an algebraic set of rank singularities leads to the following lemma which relates the various notions of primeness to the algebraic set of a polynomial matrix.

**Lemma 4.3.2.** [20] The matrix \( R \) is

\(^1\)When we discuss rank it is as a matrix over the rational field.
1. ZLP iff \( \mathcal{V}(R) \) is empty (dim \( \mathcal{V}(R) = -1 \)),
2. WLP iff \( \mathcal{V}(R) \) is finite (dim \( \mathcal{V}(R) \leq 0 \), and
3. MLP iff \( \mathcal{V}(R) \) contains no algebraic hyper-surface (dim \( \mathcal{V}(R) \leq n - 2 \)).

Proof. Assume that \( R \) is ZLP. Then there exists no elements \( \xi \) for which rank(\( R(\xi) \)) < rank(\( R \)). The converse follows from definition.

The second result follows from definition.

Assume that the minors of \( R \) contain a non-constant common factor, \( h \); it follows that \( \mathcal{V}(h) \subset \mathcal{V}(R) \). In particular, dim \( \mathcal{V}(R) \geq \text{dim} \mathcal{V}(h) = n - 1 \). If dim \( \mathcal{V}(R) = n \) then the rank condition for \( \tilde{R} \) is violated. Now assume that dim \( \mathcal{V}(R) = n - 1 \). Then \( \mathcal{V}(R) \) contains some algebraic hyper-surface, say \( h \). Note that \( h \) is a radical ideal since \( \text{rad}(\langle h \rangle) = \langle h \rangle \). By Hilbert’s Nullstellensatz,

\[
\langle h \rangle = \text{rad}(\langle h \rangle) = \mathcal{J}(\langle h \rangle) \supset \mathcal{J}(\mathcal{V}(R)) = \mathcal{J}(\mathcal{V}(I)) = \text{rad}(I) \supset I.
\]

This implies that \( h \) divides each minor of \( R \), and thus is not MLP.

\[\Box\]

### 4.3.2 Factor Primeness

**Definition 4.3.2.** [20] A full row rank matrix \( R \) is factor left prime (FLP) if any square left factor is unimodular. This is an \( n \)-dimensional analogue of Definition 4.2.1.

Let \( m_i \) be a minor of \( R \) and \( \hat{m}_i \) be the corresponding minor of \( R_1 \) for \( R = DR_1 \). Then \( m_i = \det(D)\hat{m}_i \) and thus \( \det(D)|m_i \). This holds for all the minors of \( R \). As a result, if \( \tilde{R} \) is MLP (there are no non-trivial common factors of the minors) then we see it is also FLP. Consider the following example.

**Example 4.3.1.** [20] The non-zero minors of both the matrices

\[
R = \begin{bmatrix} z_1 & 0 & -z_2 \\ 0 & z_1 & z_3 \end{bmatrix} \quad \tilde{R} = \begin{bmatrix} z_1 & 0 & -z_2 & -z_3 \\ 0 & z_1 & 0 & 0 \end{bmatrix}
\]

are \( z_1^2, z_1z_2, z_1z_3 \). As a result, they share the same algebraic set. However, we can factor \( \tilde{R} \) as

\[
\tilde{R} = \begin{bmatrix} 1 & 0 \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} z_1 & 0 & -z_2 & -z_3 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\]

This implies that \( \tilde{R} \) is not FLP. Assume there is a factorization \( R = DR_1 \). Because the kernel must be preserved (recall that \( \det(D) \) is a non-zero constant), this requires that \( R_1\zeta = 0 \),
where $\zeta = (z_2, -z_3, z_1)^T$. However, each entry in this kernel element consists of a different indeterminate and thus the constant coefficients of each entry of $R_1$ must be zero. That is, each entry is contained in the maximal ideal $\mathcal{J} = \langle z_1, z_2, z_3 \rangle$. However, the minors of $R_1$ must be contained in $\mathcal{J}^2$ if this is the case (each entry is multiplied in the determinant). It follows from this that none of the minors of $R_1$ are linear (the degree of the minors will be $\geq 2$). This means that $\det(D) = z_1$ is not an option if we wish to match the minors. That is, $\det(D)$ must be constant.

The above example implies that we are not able to determine if a matrix is FLP from the algebraic set $\mathcal{V}(R)$ alone, as we had $\mathcal{V}(R) = \mathcal{V}(R_1)$. Moreover, FLP does not imply MLP. Instead, we shall look at the projective dimension of a system representation, i.e. its cokernel.

We now present some definitions that we shall use in our discussion of projective resolutions.

**Definition 4.3.3.** [20] A $\mathcal{D}$-matrix, $Q$, is a minimal left annihilator (MLA) of $R$ if and only if

1. $QR = 0$
2. For any $\tilde{Q}$ such that $\tilde{Q}R = 0$ we have $\tilde{Q} = XQ$ for some $\mathcal{D}$-matrix $X$.

**Definition 4.3.4.** [2] A projective resolution of an $R$-module $M$ is an exact sequence

$$\cdots \rightarrow F_n \xrightarrow{\phi_n} \cdots \rightarrow F_1 \xrightarrow{\phi_1} F_0 \rightarrow M \rightarrow 0$$

of projective $R$-modules. If for some $n < \infty$ we have $F_{n+1} = 0$ but $F_i \neq 0$ for $0 \leq i \leq n$, then we shall say that the resolution has length $n$.

**Definition 4.3.5.** The minimum length of all projective resolutions of an $R$-module $M$ is called the projective dimension of $M$. This is written as $\text{pd}(M)$.

The following lemma begins our study on projective resolutions.

**Lemma 4.3.3.** Let $\mathcal{M} = \mathcal{D}^{1 \times q} / \mathcal{D}^{1 \times q} R$ be finitely generated where $R \in \mathcal{D}^{q \times q}$. $R$ has full row rank if and only if the projective dimension of $\mathcal{M}$ is less than or equal to one, i.e. $\text{pd}(\mathcal{M}) \leq 1$.

**Proof.** By hypothesis, we have the following short exact sequence

$$0 \rightarrow \mathcal{D}^{1 \times q} \xrightarrow{R} \mathcal{D}^{1 \times q} \xrightarrow{\phi} \mathcal{M} \rightarrow 0.$$

Notice that since $R$ has full row rank, the left operation is injective. Because $\ker(\phi) = \text{im}(R)$, the above sequence is a projective resolution of length one. It follows that $\text{pd}(\mathcal{M}) \leq 1$. Note that if $\mathcal{M}$ is projective then $\text{pd}(\mathcal{M}) = 0$.

---

Conversely, if \( \text{pd}(\mathcal{M}) = 1 \) then there exists the following short exact sequence

\[
0 \rightarrow P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} \mathcal{M} \rightarrow 0.
\]

By the Quillen-Suslin Theorem\(^3\), \( P_1 \) and \( P_0 \) are free and so we have

\[
0 \rightarrow D^{1 \times n_1} \xrightarrow{\phi_1} D^{1 \times n_0} \xrightarrow{\phi_0} \mathcal{M} \rightarrow 0.
\]

where \( \phi_1 \) is a full row rank \( D \)-matrix and \( n_0 \geq g \). Let \( \phi_1 = [R_1, \ldots, R_{n_0}]^T \). Since \( \text{ker}(\phi_0) = \text{im}(\phi_1) \) we have that \( \phi_1 \) is a permutation of the rows of \([R : D]^T\) where \( D \) has full row rank. Because \( \phi_1 \) has full row rank, \( R \) does as well. If \( \text{pd}(\mathcal{M}) = 0 \) then \( \mathcal{M} \) is projective and thus by the Quillen-Suslin Theorem is free. It follows that \( R \) has full row rank since \( D^{1 \times g} R \cong D^{1 \times m} \) for some \( m \in \mathbb{N} \), \( m \leq q \).

As a result of the above lemma, if \( R \) does not have full row rank, then we are in a situation where \( \text{pd}(\mathcal{M}) > 1 \). In particular, we have that there is a nontrivial annihilator on the left of \( R \). Now we shall provide a general method for constructing free resolutions for \( \mathcal{M} \).

Let \( k \in \mathbb{N} \) be given. We shall construct an exact sequence of length \( k \). We can trivially construct a resolution

\[
0 \rightarrow D^{1 \times g} R \hookrightarrow D^{1 \times q} \rightarrow \mathcal{M} \rightarrow 0.
\]

There is also the short exact sequence

\[
0 \rightarrow \ker(R) \hookrightarrow D^{1 \times g} R \xrightarrow{R} D^{1 \times q} \rightarrow \mathcal{M} \rightarrow 0.
\]

We can combine these two sequences to get

\[
0 \rightarrow \ker(R) \hookrightarrow D^{1 \times g} \xrightarrow{R} D^{1 \times q} \rightarrow \mathcal{M} \rightarrow 0.
\]  \( (4.1) \)

Let \( Q \) be a minimal left annihilator of \( R \). Then the rows of \( Q \) will generate \( \ker(R) \). In particular, \( \ker(R) = D^{1 \times m} Q \) and thus from the rank-nullity theorem, \( \text{rank}(Q) + \text{rank}(R) = g \). This allows us to construct the short exact sequence

\[
0 \rightarrow \ker(Q) \hookrightarrow D^{1 \times m} \xrightarrow{Q} D^{1 \times m} Q \rightarrow 0.
\]

that we can append to \((4.1)\) to get

\[
0 \rightarrow \ker(Q) \hookrightarrow D^{1 \times m} \xrightarrow{Q} D^{1 \times g} \xrightarrow{R} D^{1 \times q} \rightarrow \mathcal{M} \rightarrow 0.
\]

Inductively, we can compute a resolution off of the MLAs to obtain a sequence of arbitrary length \( k \). Note that different kernel representations of \( \mathcal{B} \) will result in the same cokernel and thus we have the projective dimension of \( \mathcal{M} \) is well-defined.

\(^3\)See \([4]\) for this result.
Example 4.3.2. [20] Consider the behavior
\[ B = \{ w : \mathbb{N}^3 \to \mathbb{R}^3 : \text{there exists } l : \mathbb{N}^3 \to \mathbb{R}, w_i = \sigma_i l, i = 1, 2, 3 \} \]
where \( \sigma_i \) is the unity-shift function. We are able to construct the matrix \( R \) such that
\[
B = \left\{ w : \mathbb{N}^3 \to \mathbb{R}^3 : \begin{bmatrix} 0 & -\sigma_3 & \sigma_2 \\ \sigma_3 & 0 & -\sigma_1 \\ -\sigma_2 & \sigma_1 & 0 \end{bmatrix} w = 0 \right\}.
\]
This is a discrete version of the idea that a vector field \( w \) is derivable from a scalar potential if and only if it has a vanishing curl. Notice that the rank of \( R \) is two since
\[
\sigma_1 \begin{bmatrix} 0 \\ -\sigma_3 \\ \sigma_2 \end{bmatrix} + \sigma_2 \begin{bmatrix} \sigma_3 \\ 0 \\ -\sigma_1 \end{bmatrix} + \sigma_3 \begin{bmatrix} -\sigma_2 \\ \sigma_1 \\ 0 \end{bmatrix} = 0.
\]
As a result we can see that the \( \text{pd}(\mathcal{M}) > 1 \) and in particular that \( \text{pd}(\mathcal{M}) = 2 \) since we have for \( \psi = (\sigma_1, \sigma_2, \sigma_3) \)
\[
0 \longrightarrow \mathcal{D} \xrightarrow{\psi} \mathcal{D}^{1 \times 3} \xrightarrow{R} \mathcal{D}^{1 \times 3} \longrightarrow \mathcal{M} \longrightarrow 0.
\]
One interesting observation is that even though \( R \) has rank two, \( B \) needs three equations to be constructed. This purely follows because the right-kernel of \( R \) has dimension one.

Definition 4.3.6. [20] A matrix \( R \) is called \textit{factor left prime in the generalized sense (GFLP)} if the existence of a factorization \( R = DR_1 \) \((D \text{ not necessarily square})\) with \( \text{rank}(R) = \text{rank}(R_1) \) implies the existence of a polynomial matrix \( E \) such that \( R_1 = ER \).

Remark 4.3.1. Notice that the above definition implies that, if there is a generalized factorization, we have that if \( R \) and \( R_1 \) have the same rank then there is a polynomial left-inverse of \( D \); this inverse is \( E \):
\[
R = DR_1 = EDR.
\]

Now we introduce an algorithm for determining if a matrix is GFLP. Note that the notation contained in this algorithm shall be used in subsequent remarks. In particular, the matrices introduced shall be heavily used.

Algorithm 4.3.1. 1. Solve the linear system \( R\zeta = 0 \) over the polynomial ring. In other words, find an integer \( m \) and a matrix \( M \in \mathcal{D}^{q \times m} \) whose columns are generators of the syzygy module of \( R \),
\[
\{ \zeta \in \mathcal{D}^q, R\zeta = 0 \} = M\mathcal{D}^m = \text{CM}(M)
\]
Here we use the notation \( \text{CM}(M) \) as the column-generated-module of \( M \). Notice that not only are we making the requirement that \( RM = 0 \) but we are also requiring that the following sequence is exact
\[
\mathcal{D}^m \xrightarrow{M} \mathcal{D}^q \xrightarrow{R} \mathcal{D}^g.
\]
2. Now find a minimal left annihilator of $M$, that is, a matrix $R^c \in D^{q \times g}$ such that

$$\{\eta \in D^{1 \times q}, \eta M = 0\} = RM(R^c).$$

Here we use the notation $RM(R^c)$ as the row-generated-module of $R^c$. Note that by using generators of the left kernel of $M$ we can achieve this. Apply $\text{hom}(-, D)$ to (4.2) to get the complex

$$D^{1 \times g} \xrightarrow{R} D^{1 \times q} \xrightarrow{M} D^{1 \times m}.$$

Notice that for $\xi \in D^{1 \times g}$, $\xi RM = 0$ since the columns of $M$ generate the right kernel of $R$ and thus it is a complex. However, for $\xi \in D^{1 \times q}$ such that $\xi M = 0$ we cannot require that there exists $w \in D^{1 \times g}$ such that $wR = \xi$ unless $R$ is onto. Therefore we cannot say that the sequence above is exact. However, we can say that

$$D^{1 \times g} \xrightarrow{R^c} D^{1 \times q} \xrightarrow{M} D^{1 \times m}$$

is exact. This follows because $R^c$ was produced to generate the left kernel of $M$.

3. Check to see if the rows of $R$ and $R^c$ generate the same module. If they do then we have that $\ker(M) = \text{im}(R)$. By construction, $RM = 0$. Moreover, since the rows of $R$ and $R^c$ generate the same module, we have that $\text{rank}(R) = \text{rank}(R^c)$. $R^c$ is the MLA of $M$ and thus by definition there exists a polynomial matrix $D$ such that $R = DR^c$. If $R$ is GFLP, then there exists an $E$ such that $R^c = ER$, hence $RM(R) = RM(R^c)$. In fact, equality of the modules implies that $R$ is GFLP.

### 4.3.3 Relations Between Primeness

We introduce several results that provide connections between the various definitions of primeness that have been introduced.

**Lemma 4.3.4.** [20] If $R$ is ZLP then it has a polynomial right inverse.

**Proof.** We can construct a permutation matrix $\Pi_j$ such that

$$R\Pi_j = \begin{bmatrix} R_j^{(1)} & R_j^{(2)} \end{bmatrix}$$

where $R_j^{(1)}$ is square and $\det(R_j^{(1)}) = m_j$. Furthermore, we have the equality

$$R\Pi_j \begin{bmatrix} \text{adj}(R_j^{(1)}) \\ 0 \end{bmatrix} = m_jI_g.$$
Because $R$ is ZLP, we have that $1 \in \mathcal{I}$ (the ideal generated by the minor polynomials.) A consequence of this observation is that there exists polynomials $p_j$ such that

$$\sum_j p_j m_j = 1.$$  

It follows that

$$\sum_j R\Pi_j \begin{bmatrix} \text{adj}(R_j^{(1)}) \\ 0 \end{bmatrix} p_j = I_g.$$  

The polynomial right inverse is

$$Z = \sum_j \sum_j \Pi_j \begin{bmatrix} \text{adj}(R_j^{(1)}) \\ 0 \end{bmatrix} p_j.$$  

**Proposition 4.3.5.** ZLP implies GFLP.

**Proof.** Assume that $R$ is ZLP and let there be a factorization $R = DR_1$ with $\text{rank}(R) = \text{rank}(R_1)$. Notice that we can construct a rational matrix $E$ such that $R_1 = ER$ by inverting $D$ in the rational field. By Lemma 4.3.4, $R$ is ZLP and thus has a polynomial right inverse; let $X$ be the polynomial right inverse of $R$. We conclude $E = R_1X$ and $E$ is a polynomial matrix.  

**Proposition 4.3.6.** If $R$ has full row rank then GFLP implies FLP; however, the converse is not true.

**Proof.** If $R$ has full row rank and is GFLP then $R = DR_1$ and $\text{rank}(R) = \text{rank}(R_1)$ implies the existence of $E$ such that $R_1 = ER$. Furthermore, since $R$ has full row rank, $D$ is square. That is, $E$ is a polynomial inverse of $D$ since $R_1$ has full row rank. It follows that $D$ is unimodular.

For the other part we introduce the following example. Consider

$$R = \begin{bmatrix} z_1 & 0 & -z_2 \\ 0 & z_1 & z_3 \end{bmatrix}.$$  

Recall we demonstrated this was FLP. There is a minimal left annihilator of $R$, namely $M = (z_2, -z_3, z_1)^T$. An MLA of $M$ is

$$R^c = \begin{bmatrix} z_1 & 0 & -z_2 \\ 0 & z_1 & z_3 \\ z_3 & z_2 & 0 \end{bmatrix}.$$  

Notice that the submodule generated by the rows of $R^c$ is larger than that of $R$. 

\[\square\]
4.3.4 Controllability

Consider the following short exact sequence

\[ 0 \rightarrow \mathcal{D}^m \xrightarrow{M} \mathcal{D}^q \xrightarrow{R} \mathcal{D}^g \rightarrow 0 \]

where \( R \in \mathcal{D}^{g \times q} \) and \( M \in \mathcal{D}^{q \times m} \). Notice that \( \ker(R) = \text{im}(M) \). Consider \( M(\sigma)l = w \) for some given forcing vector \( w \) and solution \( l \). Since \( w \) lies in the image of \( M \), we have that \( R(\sigma)w = 0 \). This allows us to consider what is called a *image representation* of a behavior. Before we have only considered the *kernel representation* by \( R \). What properties does \( B \) need in order to provide such a representation?

If \( R \) is a minimal left annihilator of \( M \), then the row vectors of \( R \) generate the left kernel of \( M \). Thus if \( R \) is a MLA of \( M \) and we have

\[ B = \ker(R) = \text{im}(M) = \{ w \in \mathcal{A}^q : \text{there exists } l \in \mathcal{A}^m \text{ such that } w = M(\partial)l \} \]

**Lemma 4.3.7.** [8] Let \( Q \in \mathcal{D}^{k,l} \) and \( P \in \mathcal{D}^{l,m} \) be two matrices. The following are equivalent.

1. The sequence

\[ \mathcal{D}^k \xrightarrow{Q^T} \mathcal{D}^l \xrightarrow{P^T} \mathcal{D}^m \]

is exact.

2. \( QP = 0 \) and if \( XP = 0 \) for some other matrix \( X \) then there exists matrix \( Y \) such that \( X = YQ \).

**Theorem 4.3.8.** [20] If \( R \) is GFLP, then there exists an image representation \( B = \text{im}(M) \) where the polynomial matrix \( M \) is constructed by Algorithm 4.3.1. Conversely, if \( B \) possesses an image representation, any kernel representation matrix of \( B \) is GFLP.

*Proof.* \(( \Rightarrow \) Recall that \( R^c \) was constructed so that \( RM(R) = RM(R^c) \). Because \( R \) is GFLP, \( \ker(R) = \ker(R^c) \). From exactness of \( \text{hom}(\mathcal{D}, \mathcal{A}) \) we have that

\[ \mathcal{D}^{1 \times q} \xrightarrow{R^c} \mathcal{D}^{1 \times q} \xrightarrow{M} \mathcal{D}^{1 \times m} \]

is exact and thus

\[ \mathcal{D}^m \xrightarrow{M} \mathcal{D}^q \xrightarrow{R^c} \mathcal{D}^{q \times c} \]

is exact. That is \( \ker(R) = \ker(R^c) = \text{im}(M) \).

\(( \Leftarrow \) Let \( M' \in \mathcal{D}^{q \times \ell} \) be such that \( \text{im}(M') = \ker(R) \). Then by Lemma 4.3.7, \( R \) is an MLA of \( M' \) with the sequence

\[ \mathcal{D}^l \xrightarrow{M'} \mathcal{D}^q \xrightarrow{R} \mathcal{D}^g \]
Let $M$ be from Algorithm 4.3.1 so we have the sequence

$$
\mathcal{D}^m \xrightarrow{M} \mathcal{D}^q \xrightarrow{R} \mathcal{D}^g.
$$

This implies that $M' = MX$ for some polynomial matrix $X$ because $M$ is a MRA of $R$. □

**Proposition 4.3.9.** [20] A behavior $\mathcal{B}$ is controllable as in Definition 3.3.11 if and only if it admits a kernel representation that is GFLP.
Chapter 5

Behavioral Complexes

Most mathematicians, including me, lie somewhere in the middle of the spectrum when it comes to our attitude to applications. We would be delighted if we proved a theorem that was found to be useful outside of mathematics, but we do not actually seek to do so. Given the choice between an interesting but purely mathematical problem and an uninteresting problem of potential benefit to computer scientists, physicists or engineers, we will opt for the former.

- Timothy Gowers

5.1 Introduction

In [6] discrete 2-D behaviors are generalized from the approach presented in [11]. We shall spend some time discussing the results in 2-D and then provide a natural extension to n-D. Instead of considering a signal space to be $\mathbb{R}^{Z^2}$ or $\mathbb{R}^{N^2}$, half-planes and quarter-planes are allowed. This not only allows us to use the definition of controllability for discrete systems on $Z^a \times N^b$, $a, b \in N$, as in [20] but allows a rich duality theory to ensue.

As we develop the machinery for the n-dimensional extension, we shall notice that the development in [6] is unnatural; furthermore, the computational difficulty in extending by their method is a byproduct of their view taken towards the problem. Our presentation is so that the chain complex associated to behaviors is streamlined. Indeed, the entire process is just as algorithmic as, say, simplicial homology.

In this chapter signals will have values in the vector space $F^q$ where $F$ is a given field and $q \in N$. We shall consider signals over $Z^n$, noting that this work should extend in this direction in a straightforward manner.
5.2 Two Dimensional Behavior and Homology

5.2.1 The Category $\mathbb{T}$

In [6], a category of “nice” subsets of $\mathbb{Z}^2$ is defined.

For all $a \in \mathbb{Z}$, define the sets

\begin{align*}
E(a) &= [a, \infty) \times \mathbb{Z} \\
W(a) &= (-\infty, a] \times \mathbb{Z} \\
N(a) &= \mathbb{Z} \times [a, \infty) \\
S(a) &= \mathbb{Z} \times (-\infty, a].
\end{align*}

The above sets correspond to the half-planes in $\mathbb{Z}^2$. For $a, b \in \mathbb{Z}$, we recover quarter-planes if we intersect, say, $N(a) \cap E(b)$. We collect all of the quarter-planes, half-planes, and $\mathbb{Z}^2$ and denote the set by $\mathbb{T}$. These are the signal domains that we shall consider in the upcoming work.

We may construct the category $\mathbb{T}$ where the objects are the sets we have just defined. Let $A, B$ be two objects; we define $\text{hom}(A, B)$ to consist of only the restriction map, i.e.

\[
\text{hom}(A, B) = \begin{cases} 
\{\pi_{A \cap B}\} & A \cap B \in \text{obj}(\mathbb{T}) \\
\emptyset & \text{otherwise}
\end{cases}
\]

At first glance, this category seems rather uninteresting. Let us motivate its construction by what it implies in a system theoretic sense.

**Example 5.2.1.** Let us consider for $a, b \in \mathbb{Z}$ the two sets $N(a)$ and $S(b)$. On $N(a)$, we may talk about kernel representations $R \in \mathbb{F}[x_1, x_1^{-1}, x_2]$. For instance, take $R = 1 - (x_1 + x_2 + x_1^{-1})$. For any $w \in \mathbb{F}^{N(a)}$, $R(\sigma_1, \sigma_1^{-1}, \sigma_2)$ is defined. That is, for all $(t_1, t_2) \in N(a)$,

\[
R(\sigma_1, \sigma_1^{-1}, \sigma_2)(w) = w(t_1, t_2) - (w(t_1 + 1, t_2) + w(t_1, t_2 + 1) + w(t_1 - 1, t_2)).
\]

Similarly, we can talk about kernel representations $\tilde{R} \in \mathbb{F}[x_1, x_1^{-1}, x_2^{-1}]$.

If we consider the intersection $N(a) \cap S(b)$, it may be non-zero; however, we are not able to discuss suitable shift operations on $\mathbb{F}^{N(a) \cap S(b)}$. 
5.2.2 \( B \) Functor

We will interpret a behavior as a functor in this section. In order to develop the material in a manageable manner, we restrict our study to autoregressive systems.

Denote by \( \mathbf{FK}(\mathbb{F}^q) \) the category of Frechet Coordinate Spaces over the vector space \( \mathbb{F}^q \). The category is comprised of

- (Objects). \( (\mathbb{F}^q)^I, I \in \text{obj}(\mathbb{T}) \).
- (Morphisms). For \( (\mathbb{F}^q)^I, (\mathbb{F}^q)^J \), the morphisms between these two spaces is induced from \( \mathbb{T} \), i.e.

\[
\text{hom}((\mathbb{F}^q)^I, (\mathbb{F}^q)^J) = \begin{cases} 
\{ \pi_{(\mathbb{F}^q)^I \cap J} \} & I \cap J \in \text{obj}(\mathbb{T}) \\
\emptyset & \text{otherwise}
\end{cases}
\]

Recall that for an autoregressive system over \( \mathbb{Z}^2 \), that the behavior is defined as the kernel of a Laurent polynomial matrix. Such behaviors were closed, shift-invariant, and linear subspaces of \( \mathbb{L}^q = (\mathbb{F}^q)^{\mathbb{Z}^2} \). Let \( \mathcal{M} \) be such a subspace. We define the category of restrictions of a given closed, shift-invariant, and linear subspace of \( \mathbb{L}^q \), denote by \( \mathbf{AR}(\mathcal{M}) \), as

- (Objects). \( \mathcal{M}|_I \) for \( I \in \text{obj}(\mathbb{T}) \)
- (Morphisms). The induced restriction morphism from \( \mathbb{T} \).

This entire setup is rigged so that we can define a behavior functor, \( \mathcal{B} \), that maps from \( \mathbb{T} \) to \( \mathbf{AR}(\mathcal{M}) \). Let \( I \in \text{obj}(\mathbb{T}) \) be given. We define

\[
\mathcal{B}(I) = \mathcal{M}_I.
\]

We define \( \mathcal{B} \) on the morphisms in the same manner.

**Observation 5.2.1.** Let \( (a,b) \in \mathbb{Z}^2 \) and \( I \in \text{obj}(\mathbb{T}) \) be given. We have that \( (a,b) + I \in \text{obj}(\mathbb{T}) \).

Because \( \mathcal{M} \) is a closed, shift-invariant, and linear subspace of \( \mathbb{L}^q \), we have \( \sigma_1^{n_1} \sigma_2^{n_2} \mathcal{M} = \mathcal{M} \) for all \( n_1, n_2 \in \mathbb{Z} \). This translates to the following result.

**Lemma 5.2.1.** Let \( (a,b) \in \mathbb{Z}^2 \) and \( I \in \text{obj}(\mathbb{T}) \) be given. We have

\[
\mathcal{B}(I) \cong \mathcal{B}((a,b) + I).
\]
5.2.3 Chains and Boundary

In [6], chains and boundary maps are defined; these form what is called a behavioral complex. Because of the low-dimensionality, they are easily defined.

\[ C_2 \mathcal{B}(k,l;m,n) = \mathcal{B}(\mathbb{Z}^2) \]
\[ C_1 \mathcal{B}(k,l;m,n) = \mathcal{B}(E(k)) \times \mathcal{B}(N(m)) \times \mathcal{B}(W(l)) \times \mathcal{B}(S(n)) \]
\[ C_0 \mathcal{B}(k,l;m,n) = \mathcal{B}(E(k) \cap N(m)) \times \mathcal{B}(N(m) \cap W(l)) \times \mathcal{B}(W(l) \cap S(n)) \times \mathcal{B}(S(n) \cap E(k)) \]

From here two boundary maps are specified.

\[ d_1(w) = (w|_{E(k)}, w|_{N(m)}, w|_{W(l)}, w|_{S(n)}) \]
\[ d_0(w_E, w_N, w_W, w_S) = (w_E|_{E(k) \cap N(m)} - w_N|_{E(k) \cap N(m)}, w_N|_{N(m) \cap W(l)} - w_W|_{N(m) \cap W(l)}, w_W|_{W(l) \cap S(n)} - w_S|_{W(l) \cap S(n)}, w_S|_{S(n) \cap E(k)} - w_E|_{S(n) \cap E(k)}) \]

These boundary maps are defined so that they yields the complex

\[ 0 \longrightarrow C_2 \mathcal{B}(k,l;m,n) \overset{d_1}{\longrightarrow} C_1 \mathcal{B}(k,l;m,n) \overset{d_0}{\longrightarrow} C_0 \mathcal{B}(k,l;m,n) \longrightarrow 0. \]

By construction, \( d_0 \circ d_1 = 0 \) and thus we have a bonafide chain complex. The homology is then defined as \( H_i(\mathcal{B}(k,l;m,n)) = \ker d_{i-1}/\text{im} d_i \).

By the shift-invariance of \( \mathcal{B} \), we have that the homology is also shift invariant. This leads us to the observation

**Observation 5.2.2.**

\[ H_i(\mathcal{B}(k,l;m,n)) = H_i(\mathcal{B}(k+a,l+a;m+b,n+b)) \quad a, b \in \mathbb{Z}. \]

Furthermore, the only important values in the homology are \( l - k \) and \( n - m \).

Before moving on, we wish to introduce a pictorial view of what is occurring. This will be one of the motivating factors in the \( n \)-dimensional extension of this work. In Figure 5.2, the 1-chains are considered points and the 2-chains are considered lines. Although this is counter-intuitive, we can see that the orientation of the lines specifies the canonical boundary map of the simplex.

If we consider the directed and labeled graph in Figure 5.2, we almost have the correct view. However, notice that if we try to label the vertices, we have an inconsistent “induced simplicial boundary map.” The labeling in [6] is \( W = 1, N = 2, E = 3, \) and \( S = 4 \). This labeling yields that motion towards the greater (or lesser) vertex will not work. It is for this reason, that when we extend into higher dimensions, we fix this problem so that we have boundary maps occur naturally from the gluing of two \( n \)-simplexes. Thankfully, this does not interfere with the results in any way.
5.2.4 Results on Homology Groups

We present some of the results found in [6]. Keep in mind that the homology groups were rigged so that these results fall out.

**Theorem 5.2.2.** A system is autonomous if and only if \( H_2(B(l+1, k-1; n+1, m-1)) = 0 \) for all \( l - k > 0 \) and \( n - m > 0 \).

**Proof.** This falls out of the construction of the chains and boundary maps\(^1\). First assume that \( H_2(B(k, l; m, n)) = 0 \), i.e. \( \ker d_1 = 0 \).

This implies that the only maps on \([k, l] \times [m, n]\) which are zero are ones which are zero everywhere. This is the definition of autonomy. Provided \( l - k \) and \( n - m \) are large enough, we have that the system is autonomous. The converse direction is true by definition. \( \square \)

**Theorem 5.2.3.** A system is controllable if and only if \( H_1(B(l+1, k-1; n+1, m-1)) = 0 \) for all \( l - k > 0 \) and \( n - m > 0 \).

**Proof.** By the definition of \( d_0 \), this implies that the signals agree on their respective quadrants. Exactness in the complex implies that there is a global signal that can gives us the desired concatenation. That is, there exists \( w \in B(\mathbb{Z}^2) \) such we can specify the values on the quadrants and fill in the rectangle \([k, l] \times [m, n]\). \( \square \)

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\(^1\)The idea on motivating our boundary maps by simplexes does not change this.
5.3 Behavioral Complexes

When one wishes to discuss homology it is necessary to construct a complex by defining chains and a boundary map. This should be done in a systematic manner so that it can be used as a computational tool. In this section we define chains of signal maps and the boundary maps between these chains. We begin by “fixing” the 2-dimensional scenario presented in [6] and then extend to the 3-dimensional scenario and then try to understand the underlying concepts required to make the extension to \( n \)-dimensions. Note, however, we avoid “brute forcing” the construction by finding underlying structure that makes the extension systematic.

5.3.1 The \( i \)-orthant Set \( O_i(n) \)

Our goal in this section is to lay out a framework for working with domain decomposition so that we may define the homology of a behavior in the later sections. Consider the 2-D scenario (i.e. operators of two variables in the shift.)

**Definition 5.3.1.** Let \( a \in \mathbb{Z} \) be given. We define the half-planes

\[
\begin{align*}
S_0(a) &= E(a) = [a, \infty) \times \mathbb{Z} \\
S_1(a) &= N(a) = \mathbb{Z} \times [a, \infty) \\
S_2(a) &= W(a) = (-\infty, a] \times \mathbb{Z} \\
S_3(a) &= S(a) = \mathbb{Z} \times (-\infty, a].
\end{align*}
\]

When we wish to consider quarter-planes they are the intersection of half-planes. Namely, the north-east quarter plane with edge \((a, a)\) is \(N(a) \cap E(a) = S_1(a) \cap S_0(a)\). It is here that we notice the ordering. In particular, the four quarter planes at \((a, a)\) are

\[
\begin{align*}
S_0(a) \cap S_1(a) & \quad S_0(a) \cap S_3(a) \\
S_1(a) \cap S_2(a) & \quad S_2(a) \cap S_3(a).
\end{align*}
\]

See Figure 5.3. We have the property that quarter-planes are made from intersecting half-planes with sub-indices in different residue classes of \(\mathbb{Z}_2\). This motivates a more “systematic” approach to decomposing spaces into orthants. We begin with the following definition.

**Definition 5.3.2.** Let \( k \in \mathbb{N} \) be given and define \( S = \{0, 1, 2, \ldots, 2k - 1\} \). For \( i \in \mathbb{Z}, 0 \leq i \leq k \), we define the family of sets

\[
O_i(k) = \{\{a_1, \ldots, a_i\} \subset S : a_1 < a_2 < \cdots < a_i, \ [a_m]_{z_k} \neq [a_n]_{z_k} \text{ for } m \neq n\}.
\]

We shall call \( O_i(k) \) the \( i \)-orthant set of dimension \( k \).
Some examples for \( k = 2 \) and \( k = 3 \) are

**Example 5.3.1.** For \( k = 2 \), we have \( S = \{0, 1, 2, 3\} \) and the sets
\[
\mathcal{O}_2 = \{\{0, 1\}, \{0, 3\}, \{1, 2\}, \{2, 3\}\}
\]
\[
\mathcal{O}_1 = \{\{0\}, \{1\}, \{2\}, \{3\}\}.
\]

**Example 5.3.2.** For \( k = 3 \), we have \( S = \{0, 1, 2, 3, 4, 5\} \) and the sets
\[
\mathcal{O}_3 = \{\{0, 1, 2\}, \{0, 1, 5\}, \{0, 2, 4\}, \{0, 4, 5\}, \{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \{3, 4, 5\}\}
\]
\[
\mathcal{O}_2 = \{\{0, 1\}, \{0, 2\}, \{0, 4\}, \{0, 5\}, \{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}
\]
\[
\mathcal{O}_1 = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}.
\]

One interesting observation is that we can identify elements of \( \mathcal{O}_i(k) \) with a set of indexing maps. Namely, for \( L \in \mathcal{O}_i(k) \) we have a map
\[
\sigma : \{1, 2, \ldots, i\} \longrightarrow L.
\]

It is for this reason that we may abuse this notation and consider \( \mathcal{O}_i(n) \) as the set of these maps. We shall also be concerned with the sets from \( \mathcal{O}_i(k) \) that are contained in an element from \( \mathcal{O}_{i+1}(k) \). This will play an important role when we define our boundary maps.

**Definition 5.3.3.** Let \( L \in \mathcal{O}_i(k) \) be given. We define the set valued map
\[
\mathcal{H}_{i,k}(L) = \{\hat{L} \in \mathcal{O}_{i-1}(k) : \hat{L} \subset L\}.
\]

Knowledge of the size of this set will be useful when we begin using it. We therefore present the following lemma.

**Lemma 5.3.1.** For \( i, k \in \mathbb{N} \), \( 0 \leq i \leq k \),
\[
|\mathcal{H}_{i,k}(L)| = i
\]

**Proof.** \( \mathcal{H}_{i,k}(L) \) is formed as choosing \( i - 1 \) indices from \( i \) possible indices. It follows that
\[
\frac{i!}{(i - (i - 1))!(i - 1)!} = i.
\]

**Example 5.3.3.** Let \( L \in \mathcal{O}_3(3) \) be equal to \( \{0, 1, 2\} \). When we look at \( \mathcal{H}_{3,3}(L) \), we see
\[
\mathcal{H}_{3,3}(L) = \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}.
\]

As Lemma 5.3.1 states, \( |\mathcal{H}_{3,3}(L)| = 3. \)
We now impose an ordering of the elements of $\mathcal{O}_i(n)$. Let $(k_1, \ldots, k_i), (\hat{k}_1, \ldots, \hat{k}_i) \in \mathcal{O}_i(n)$ be two distinct elements. Let $j$ be such that $k_\ell = \hat{k}_\ell \quad \ell < j$.

If $k_j < \hat{k}_j$ then we write $(k_1, \ldots, k_i) < (\hat{k}_1, \ldots, \hat{k}_i)$. Notice that under this ordering

1. $\mathcal{O}_i(n)$ is a totally ordered set.
2. A consequence of the above is that the elements of $\mathcal{O}_i(n)$ can be indexed.

It is this second observation that we shall be concerned with when defining the boundary map of chains. Considering that $|\mathcal{H}_{i,k}(L)| = i$, we write

$$\text{index}(\cdot) : \mathcal{H}_{i,k}(L) \longrightarrow \{0, 1, 2, \ldots, i - 1\}$$

as the indexing map based on order. The following is an example of this map.

**Example 5.3.4.** Let $L \in \mathcal{O}_3(3)$ be equal to $\{0, 1, 2\}$. When we look at $\mathcal{H}_{3,3}(L)$, we see

$$\mathcal{H}_{3,3}(L) = \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}.$$

We have the ordering

$$\{0, 1\} < \{0, 2\} < \{1, 2\}.$$

This gives us the indexing map,

$$\text{index}(\{0, 1\}) = 0 \quad \text{index}(\{0, 2\}) = 1 \quad \text{index}(\{1, 2\}) = 2.$$

Now comes some substantial results that we shall use when demonstrating that the boundary maps we define allow us to form a complex. We begin with a simple result that allows us to understand how orthants are forming.

**Lemma 5.3.2.** Let $\mathcal{H}_{i,k}(L) = \{w_1, \ldots, w_i\}$ for $L \in \mathcal{O}_i(k)$. Then

$$\left| \bigcup_{j=1}^{i} \mathcal{H}_{i-1,k}(w_j) \right| = \frac{i(i - 1)}{2}.$$

**Proof.** We know that the elements of $\bigcup_{j=1}^{i} \mathcal{H}_{i-1,k}(w_j)$ are determined by picking subsets of size $i - 2$ from $L$, which contains $i$ elements. It follows that

$$\left| \bigcup_{j=1}^{i} \mathcal{H}_{i-1,k}(w_j) \right| = \frac{i!}{(i - (i - 2))!(i - 2)!} = \frac{i(i - 1)}{2}.$$
Example 5.3.5. Let $L \in \mathcal{O}_3(3)$ be equal to \{0, 1, 2\}. When we look at $\mathcal{H}_{3,3}(L)$, we see

$$\mathcal{H}_{3,3}(L) = \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}.$$  

It follows that

$$\bigcup_{j=1}^{i} \mathcal{H}_{i-1,k}(w_j) = \{\{0\}, \{1\}, \{2\}\}.$$  

This agrees with Lemma 5.3.2 because

$$|\{\{0\}, \{1\}, \{2\}\}| = \frac{3(2)}{2} = 3.$$  

Now we must observe how the spaces are decomposing when we intersect. In particular, when we intersect $\mathcal{H}_{i-1,k}(w_j)$ and $\mathcal{H}_{i-1,k}(w_\ell)$, what do they have in common? We must keep track of this when we form our boundary map.

Proposition 5.3.3. Let $\mathcal{H}_{i,k}(L) = \{w_1, \ldots, w_i\}$ for $L \in \mathcal{O}_i(k)$. Then for $1 \leq j < \ell \leq i$,

$$|\mathcal{H}_{i-1,k}(w_j) \cap \mathcal{H}_{i-1,k}(w_\ell)| = 1$$

Proof. First we show that

$$|\mathcal{H}_{i-1,k}(w_j) \cap \mathcal{H}_{i-1,k}(w_\ell)| \geq 1.$$  

Because $w_j \neq w_\ell$ and each contains $i - 1$ of $i$ elements, we know that they contain $i - 2$ of the same elements from $L$. In particular, $w_j \cap w_\ell \in \mathcal{H}_{i-1,k}(w_j) \cap \mathcal{H}_{i-1,k}(w_\ell)$. Now we have to demonstrate that they cannot contain more than one element. This follows from a rather slick counting argument. Notice that

$$\left|\bigcup_{j=1}^{i} \mathcal{H}_{i-1,k}(w_j)\right| = \sum_{j=1}^{i} |\mathcal{H}_{i-1,k}(w_j)| - \sum_{1 \leq j < \ell \leq i} |\mathcal{H}_{i-1,k}(w_j) \cap \mathcal{H}_{i-1,k}(w_\ell)|.$$  

In particular, we have that each $\mathcal{H}_{i-1,k}(w_j)$ contains $(i - 1)$ elements. By Lemma 5.3.2 we have that

$$\frac{i(i - 1)}{2} = i(i - 1) - \sum_{1 \leq j < \ell \leq i} |\mathcal{H}_{i-1,k}(w_j) \cap \mathcal{H}_{i-1,k}(w_\ell)|.$$  

It follows that

$$\sum_{1 \leq j < \ell \leq i} |\mathcal{H}_{i-1,k}(w_j) \cap \mathcal{H}_{i-1,k}(w_\ell)| = \frac{i(i - 1)}{2}. \quad (5.1)$$
Now we have to observe what (5.2) really means. Notice that for \( j = 1 \) there are \( (i - 1) \) different intersections possible. For \( j = 2 \) there are \( (i - 2) \) different intersections and in general there are \( (i - j) \) intersections. If for any one of these the cardinality of the intersection is greater than or equal to two, then we have that there is some \( c \geq 1 \) and \( \alpha \) such that
\[
\sum_{j=1}^{i-1} j + c \delta_{j,\alpha} = \frac{i(i - 1)}{2}
\]
which is not true. We conclude with the desired result.

Continuing with our tried and true example, we verify the above proposition.

**Example 5.3.6.** Let \( L \in \mathcal{O}_3(3) \) be equal to \( \{0, 1, 2\} \). When we look at \( \mathcal{H}_{3,3}(L) \), we see
\[
\mathcal{H}_{3,3}(L) = \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}.
\]

It follows that
\[
\mathcal{H}_{i-1,k}(\{0, 1\}) = \{\{0\}, \{1\}\} \quad \mathcal{H}_{i-1,k}(\{0, 2\}) = \{\{0\}, \{2\}\} \quad \mathcal{H}_{i-1,k}(\{1, 2\}) = \{\{1\}, \{2\}\}.
\]

From here intersect any of the two sets to see
\[
\mathcal{H}_{i-1,k}(\{0, 1\}) \cap \mathcal{H}_{i-1,k}(\{0, 2\}) = \{0\} \\
\mathcal{H}_{i-1,k}(\{0, 1\}) \cap \mathcal{H}_{i-1,k}(\{1, 2\}) = \{1\} \\
\mathcal{H}_{i-1,k}(\{0, 2\}) \cap \mathcal{H}_{i-1,k}(\{1, 2\}) = \{2\}.
\]

A interesting observation here is the ordering of the intersections based on the ordering of \( \mathcal{H}_{3,3}(L) \). Our work ahead is built on understanding this observation.

We now have to look at the structure of \( \mathcal{H}_{i,k}(L) \). Notice
\[
L = \{a_1, \ldots, a_i\} \quad a_1 < a_2 < \cdots < a_i.
\]

This, however, lends itself quite nicely to a convenient structure of the ordering of \( \mathcal{H}_{i,k}(L) \).
\[
\mathcal{H}_{i,k}(L) = \{w_1, \ldots, w_{i}\} \quad w_1 < \cdots < w_{i}.
\]

If we consider \( \gamma = \{a_1, \ldots, a_{i-1}\} \) then \( \gamma \leq w_j \) for \( 1 \leq j \leq i \). In particular, it must be \( w_1 \).

This idea extends! We can easily see that \( w_j \) is missing \( a_{i-j} \) because of the order we impose. That is, if \( 1 \leq j < \ell \leq n \),
\[
\{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_i\} > \{a_1, \ldots, a_{\ell-1}, a_{\ell+1}, \ldots, a_i\}
\]
since \( a_j < a_{j+1} \) and \( a_j \in \{a_1, \ldots, a_{\ell-1}, a_{\ell+1}, \ldots, a_i\} \). We have just proven the following lemma.
Lemma 5.3.4. Let \(1 \leq j \leq i\) be given. Then for \(\gamma = \{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_i\}\),
\[
\text{index}(\gamma) = (i + 1) - j.
\]

This result gives us a hashing function for the elements of \(H_{i,k}(L)\) for \(L \in O_i(k)\). This will

We just need to understand the index of
\[
H_{i-1,k}(w_j) \cap H_{i-1,k}(w_\ell)
\]
to finish our observations.

Theorem 5.3.5. Let \(1 \leq j \leq \ell \leq i\). Then for
\[
\gamma = H_{i-1,k}(w_j) \cap H_{i-1,k}(w_\ell)
\]
we have that
\[
\text{index}_{H_{i-1,k}(w_\ell)}(\gamma) - \text{index}_{H_{i-1,k}(w_j)}(\gamma) = 1 + j - \ell.
\]

Proof. By Lemma 5.3.4 we have that \(w_\ell\) is missing \(a_{(i+1)-\ell}\) must contain \(a_{(i+1)-j}\).
\[
w_\ell: \ a_1 \ \cdots \ \ a_{i-\ell+2} \ \cdots \ a_{i-j+1} \ \cdots \ a_i
\]
\[
w_j: \ a_1 \ \cdots \ a_{i-\ell+1} \ a_{i-\ell+2} \ \cdots \ \cdots \ a_i
\]
and so as far as ordering in their respective sets goes, we have
\[
w_\ell: \ \hat{a}_1 \ \cdots \ \hat{a}_{i-\ell} \ \cdots \ \hat{a}_{i-j} \ \cdots \ \hat{a}_{i-1}
\]
\[
w_j: \ \tilde{a}_1 \ \cdots \ \tilde{a}_{i-\ell} \ \tilde{a}_{i-\ell+1} \ \cdots \ \cdots \ \tilde{a}_{i-1}
\]
As far as \(H_{i-1,k}(w_\ell)\) we have that \(\text{index}_{H_{i-1,k}(w_\ell)}(\gamma) = i - (i - j) = j\) and for \(H_{i-1,k}(w_j)\) we have that \(\text{index}_{H_{i-1,k}(w_j)}(\gamma) = i - (i - \ell + 1) = \ell - 1\). We thus have that
\[
\text{index}_{H_{i-1,k}(w_\ell)}(\gamma) - \text{index}_{H_{i-1,k}(w_j)}(\gamma) = j - (\ell - 1) = 1 + j - \ell.
\]

We thus have the following corollary.

Corollary 5.3.6. Let \(1 \leq j < \ell \leq i\). Then for
\[
\gamma = H_{i-1,k}(w_j) \cap H_{i-1,k}(w_\ell)
\]
we have that
\[
\left(\text{index}_{H_{i,k}(w_\ell)} - \text{index}_{H_{i,k}(w_j)}\right) + \left(\text{index}_{H_{i-1,k}(w_\ell)}(\gamma) - \text{index}_{H_{i-1,k}(w_j)}(\gamma)\right) = 1.
\]
This tells us that the index of the common elements always differs by one; this is pivotal when defining a boundary map. We conclude this section with an example of Corollary 5.3.6.

**Example 5.3.7.** Let \( L \in \mathcal{O}_3(3) \) be equal to \( \{0, 1, 2\} \). When we look at \( \mathcal{H}_{3,3}(L) \), we see

\[
\mathcal{H}_{3,3}(L) = \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}.
\]

We have the ordering

\[
\{0, 1\} < \{0, 2\} < \{1, 2\}
\]

which gives us the indexing map,

\[
\text{index}(\{0, 1\}) = 0 \quad \text{index}(\{0, 2\}) = 1 \quad \text{index}(\{1, 2\}) = 2.
\]

It follows that

\[
\mathcal{H}_{i-1,k}(\{0, 1\}) = \{\{0\}, \{1\}\} \quad \mathcal{H}_{i-1,k}(\{0, 2\}) = \{\{0\}, \{2\}\} \quad \mathcal{H}_{i-1,k}(\{1, 2\}) = \{\{1\}, \{2\}\}.
\]

From here intersect any of the two sets to see

\[
\begin{align*}
\mathcal{H}_{i-1,k}(\{0, 1\}) \cap \mathcal{H}_{i-1,k}(\{0, 2\}) &= \{0\} \\
\mathcal{H}_{i-1,k}(\{0, 1\}) \cap \mathcal{H}_{i-1,k}(\{1, 2\}) &= \{1\} \\
\mathcal{H}_{i-1,k}(\{0, 2\}) \cap \mathcal{H}_{i-1,k}(\{1, 2\}) &= \{2\}.
\end{align*}
\]

If we observe the equality in Corollary 5.3.6, we see for \( \mathcal{H}_{i-1,k}(\{0, 1\}) \cap \mathcal{H}_{i-1,k}(\{0, 2\}) \),

\[
(1 - 0) + (0 - 0) = 1.
\]

For \( \mathcal{H}_{i-1,k}(\{0, 2\}) \cap \mathcal{H}_{i-1,k}(\{1, 2\}) \) we see,

\[
(2 - 1) + (1 - 1) = 1.
\]

This should give us some intuition on what is stated in Corollary 5.3.6.

### 5.3.2 Breakups

Now that we have the \( i \)-orthant sets, we must consider how they play in defining behaviors. We present the following formalism that systematically forms the possible domains of signals.

**Definition 5.3.4.** Let \( i \in \mathbb{N} \) and \( (S, \wedge) \) be a non-empty set with cardinality \( 2k \) for \( k \in \mathbb{N} \) and operation between elements \( \wedge \). Denote by \( \Lambda^i_{\wedge}S \) the set of all

\[
s_{\sigma(1)} \wedge s_{\sigma(2)} \wedge \cdots \wedge s_{\sigma(i)}
\]

where \( \sigma \in \mathcal{O}_i(k) \). We shall call \( \Lambda^i_{\wedge}S \) the *breakup* of order \( i \) for \( S \).
From here we have the straightforward result.

**Lemma 5.3.7.** For \( a \in \mathbb{Z} \),

\[
(S = \{S_0(a), S_1(a), S_2(a), S_3(a)\}, \cap)
\]

we have that \( \Lambda_2^2 S \) is equal to the set of quarter-planes with origin \((a, a)\).

When one transitions to 3-D, we have a similar look; however, we gain two more sets and thus look for a similar relation to that of the 2-D scenario. See Figure 5.3.

**Definition 5.3.5.** Let \( a \in \mathbb{Z} \) be given. We define

\[
\begin{align*}
S_0(a) &= \mathbb{Z} \times [a, \infty) \times \mathbb{Z} \\
S_1(a) &= \mathbb{Z}^2 \times [a, \infty) \\
S_2(a) &= [a, \infty) \times \mathbb{Z}^2 \\
S_3(a) &= \mathbb{Z} \times (-\infty, a] \times \mathbb{Z} \\
S_4(a) &= \mathbb{Z}^2 \times (-\infty, a] \\
S_5(a) &= (-\infty, a] \times \mathbb{Z}^2.
\end{align*}
\]

Notice that 3-orthants are formed by intersecting with sets that have sub-index in a different residue class of \( \mathbb{Z}_3 \).

**Example 5.3.8.** Let \( a = 0 \). Then

\[
S_0(0) \cap S_1(0) \cap S_2(0) = [0, \infty)^3 = \mathbb{R}_+^3.
\]

**Lemma 5.3.8.** For \( a \in \mathbb{Z} \),

\[
(S = \{S_0(a), S_1(a), S_2(a), S_3(a), S_4(a), S_5(a)\}, \cap)
\]

we have that \( \Lambda_2^3 S \) is equal to the set of 2-orthants.

**Lemma 5.3.9.** For \( a \in \mathbb{Z} \),

\[
(S = \{S_0(a), S_1(a), S_2(a), S_3(a), S_4(a), S_5(a)\}, \cap)
\]

we have that \( \Lambda_3^3 S \) is equal to the set of 3-orthants.
We now make the final abstraction to $n$-dimensions with the following definition.

**Definition 5.3.6.** Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ be given. Let $\{e_i\}$ be the standard basis for the free abelian group $\mathbb{Z}^n$. We define for $i = 0, 2, \ldots, n - 1$

\[
S_i(a) = \mathbb{Z}^n \setminus [a, \infty)e_i
\]

\[
S_{i+n}(a) = \mathbb{Z}^n \setminus (-\infty, a]e_i.
\]

For $n \in \mathbb{N}$ we define $\mathbb{T}^n$ to be the set that contains $\mathbb{Z}^n$ and

\[
S_{k_1}(a) \cap S_{k_2}(a), \ldots, \bigcap_{i=1}^{n} S_{k_i}(a)
\]

for all $a \in \mathbb{Z}$ and $[k_j]_{\mathbb{N}_n} \neq [k_i]_{\mathbb{Z}_n}$ for $i \neq j$. We shall refer to $\mathbb{T}^n$ as the *set of all infinite orthants*.

We conclude this section with the trivial result

**Lemma 5.3.10.** For $n \in \mathbb{N}$ we have that

\[
\mathbb{T}^n = \bigcup_{a \in \mathbb{Z}} \bigcup_{i=1}^{n} \Lambda_i^a S(a)
\]

where $S(a) = \{S_1(a), \ldots, S_{2n}(a)\}$ as in Definition 5.3.6

### 5.3.3 n-D Behaviors

We shall extend the definitions of the category $\mathbb{T}$ and the functor $\mathbb{B}$ to multiple dimensions. First, as in the two dimensional case, we must construct the necessary categories.

**The Category $\mathbb{T}^n$**

We may construct the category $\mathbb{T}^n$ where the objects are the sets we have just defined in the above section. Let $A, B$ be two objects; we define $\text{hom}(A, B)$ to consist of only the restriction map, i.e.

\[
\text{hom}(A, B) = \left\{ \begin{array}{ll}
\{\pi_{A \cap B}\} & A \cap B \in \text{obj}(\mathbb{T}^n) \\
\emptyset & \text{otherwise}
\end{array} \right.
\]
Functor

The extension of the functor $B$ is a straightforward extension as well. Denote by $F\mathcal{K}(\mathbb{F}^q)$ the category of Frechét Coordinate Spaces over the vector space $\mathbb{F}^q$. The category is comprised of

- (Objects). $(\mathbb{F}^q)^I$, $I \in \text{obj}(\mathbb{T}^n)$.

- (Morphisms). For $(\mathbb{F}^q)^I$, $(\mathbb{F}^q)^J$, the morphisms between these two spaces is induced from $\mathbb{T}^n$, i.e.

$$ \text{hom}((\mathbb{F}^q)^I, (\mathbb{F}^q)^J) = \begin{cases} \{\pi((\mathbb{F}^q)^I \cap J)\} & I \cap J \in \text{obj}(\mathbb{T}^n) \\ \emptyset & \text{otherwise} \end{cases} $$

Recall that for an autoregressive system over $\mathbb{Z}^n$, that the behavior is defined as the kernel of a Laurent polynomial matrix. Such behaviors were closed, shift-invariant, and linear subspaces of $L_q = (\mathbb{R}^q)^{\mathbb{Z}^n}$. Let $\mathcal{M}$ be such a subspace; we construct the category of restrictions of a given closed, shift-invariant, and linear subspace of $L_q$, denote by $\text{AR}(\mathcal{M})$, as

- (Objects). $\mathcal{M}|_I$ for $I \in \text{obj}(\mathbb{T}^n)$

- (Morphisms). The induced restriction morphism from $\mathbb{T}^n$.

This entire setup is rigged so that we can define a behavior functor, $\mathcal{B}$, that maps from $\mathbb{T}$ to $\text{AR}(\mathcal{M})$. Let $I \in \text{obj}(\mathbb{T}^n)$ be given. We define

$$ \mathcal{B}(I) = \mathcal{M}_I. $$

We define $\mathcal{B}$ on the morphisms in the same manner. As in the two dimensional case, we have that $\mathcal{B}$ is shift-invariant.

5.3.4 Homology

In order to discuss the homology of a linear and time-invariant $n$-dimensional behavior, we must introduce a complex. We begin with the two dimensional scenario to motivate our actions. Let the four-tuple $(a, b, c, d) \in \mathbb{Z}^4$ be given. Define

$$ C_2 \mathcal{B}(a, b, c, d) = \mathcal{B}(\mathbb{Z}^2) $$

$$ C_1 \mathcal{B}(a, b, c, d) = \mathcal{B}(S_0(a)) \times \mathcal{B}(S_1(b)) \times \mathcal{B}(S_2(c)) \mathcal{B}(S_3(d)) $$

$$ C_0 \mathcal{B}(a, b, c, d) = \mathcal{B}(S_0(a) \cap S_1(b)) \times \mathcal{B}(S_0(a) \cap S_3(d)) \times \mathcal{B}(S_1(b) \cap S_2(c)) \times \mathcal{B}(S_2(c) \cap S_3(d)). $$
We can define boundary maps between the chains as

\[
\begin{align*}
d_1(w) &= (w|_{S_0(a)}, w|_{S_1(b)}, w|_{S_2(c)}, w|_{S_3(d)}) \\
d_0(w_0, w_1, w_2, w_3) &= (w_0|_{S_0(a) \cap S_1(b)} - w_1|_{S_0(a) \cap S_1(b)}, \\
&\quad w_0|_{S_0(a) \cap S_3(d)} - w_3|_{S_0(a) \cap S_3(d)}, \\
&\quad w_1|_{S_1(b) \cap S_2(c)} - w_2|_{S_1(b) \cap S_2(c)}, \\
&\quad w_2|_{S_2(c) \cap S_3(d)} - w_3|_{S_2(c) \cap S_3(d)}).
\end{align*}
\]

See Figure 5.1.

**Lemma 5.3.11.** [6]

\[
0 \rightarrow C_2\mathcal{B}(a, b, c, d) \xrightarrow{d_1} C_1\mathcal{B}(a, b, c, d) \xrightarrow{d_0} C_0\mathcal{B}(a, b, c, d) \rightarrow 0
\]

is a chain-complex.

**Proof.** \(d_0 \circ d_1 = 0\) is a straightforward computation. \(\Box\)

In Figure 5.4, we can see that the vertices of the simplex are naturally defining the boundary map. For a 1-simplex, the negative signs in the boundary are determined by the index.

We now introduce the \(n\)-dimensional generalization. Let the \(2n\)-tuple \(\vec{k} = (k_0, \ldots, k_{2n-1})\) be given where \(k_i \in \mathbb{Z}, i = 1, 2, \ldots, 2n\). Define

\[
C_n\mathcal{B}(\vec{k}) = \mathcal{B}(\mathbb{Z}^n).
\]
For lagniappe we also define

\[ C_{n-1}B(\vec{k}) = B(S_0(k_0)) \times B(S_1(k_1)) \cdots \times B(S_{n-1}(k_{n-1})) \]

\[ C_n B(\vec{k}) = \bigoplus_{\sigma \in O_i(n)} B(S_{\sigma(1)}(k_{\sigma(1)}) \cap \cdots \cap S_{\sigma(i)}(k_{\sigma(i)})) \]

Initially, this might seem like a big complicated mess. The following example computes the chains over \( \mathbb{Z}^3 \).

**Example 5.3.9.** Let \( n = 3 \). Then we have

\[ C_1 B(\vec{k}) = B(S_0(k_0) \cap S_1(k_1)) \times B(S_0(k_0) \cap S_2(k_2)) \]

\[ \times B(S_0(k_0) \cap S_4(k_4)) \times B(S_0(k_0) \cap S_5(k_5)) \]

\[ \times B(S_1(k_1) \cap S_2(k_2)) \times B(S_1(k_1) \cap S_3(k_3)) \]

\[ \times B(S_1(k_1) \cap S_5(k_5)) \times B(S_4(k_4) \cap S_5(k_5)) \]

\[ \times B(S_2(k_2) \cap S_3(k_3)) \times B(S_2(k_2) \cap S_4(k_4)) \]

\[ \times B(S_3(k_3) \cap S_4(k_4)) \times B(S_3(k_3) \cap S_5(k_5)) \]

\[ C_0 B(\vec{k}) = B(S_0(k_0) \cap S_1(k_1) \cap S_2(k_2)) \]

\[ \times B(S_0(k_0) \cap S_2(k_2) \cap S_4(k_4)) \]

\[ \times B(S_0(k_0) \cap S_4(k_4) \cap S_5(k_5)) \]

\[ \times B(S_1(k_1) \cap S_2(k_2) \cap S_3(k_3)) \]

\[ \times B(S_1(k_1) \cap S_3(k_3) \cap S_5(k_5)) \]

\[ \times B(S_2(k_2) \cap S_3(k_3) \cap S_4(k_4)) \]

\[ \times B(S_0(k_0) \cap S_1(k_1) \cap S_5(k_5)) \]

Notice that this definition is quite complicated for values of \( n > 2 \) and the computations become rather tedious. Thankfully, this is not an issue when defining boundary maps. The only important issue is that there is an ordering on \( O_i(n) \).

**Definition 5.3.7.** Let for \( i = 0, 1, \ldots, n-1 \) we define for \( v \in C_{n-i+1}B(\vec{k}) \),

\[ \partial_{n-i}(v) = \bigoplus_{\sigma \in O_i(n)} \sum_{\omega \in H_{i,k}(\sigma)} (-1)^{\text{index}(\omega)+1} v_\omega|_{S_{\omega(1)}(k_{\omega(1)}) \cap \cdots \cap S_{\omega(i)}(k_{\omega(i)})}, \]

where \( v_\omega \) is a signal defined on \( S_{\omega(1)}(k_{\omega(1)}) \cap \cdots \cap S_{\omega(i-1)}(k_{\omega(i-1)}) \).

**Theorem 5.3.12.** For \( n \in \mathbb{N} \) we have that

\[ 0 \rightarrow C_n B(\vec{k}) \xrightarrow{\partial_{n-1}} C_{n-1} B(\vec{k}) \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_0} C_0 B(\vec{k}) \rightarrow 0 \]

is a chain complex.
Proof. We need only show that \( \partial_{n-i-1} \partial_{n-i} = 0 \) for \( i \in \mathbb{N}, 1 \leq i \leq n - 1 \). Let \( v \in C_i \mathcal{B}(\vec{k}) \) be given. Then we have that

\[
\dot{v} = \partial_{i-1}(v) = \bigoplus_{\sigma \in \mathcal{O}_i(n)} \sum_{\omega \in \mathcal{H}_{i,k}(\sigma)} (-1)^{\text{index}(\omega)+1} v_\omega \big|_{S_{\sigma(1)}(k_{\sigma(1)}) \cap \cdots \cap S_{\sigma(i)}(k_{\sigma(i)})}.
\]

Moreover,

\[
\partial_{n-i-1}(\dot{v}) = \bigoplus_{\sigma \in \mathcal{O}_{i+1}(n)} \sum_{\omega \in \mathcal{H}_{i+1,n}(\sigma)} \sum_{\gamma \in \mathcal{H}_{i,n}(\omega)} (-1)^{\text{index}_{i-1,n}(\sigma)(\omega) + \text{index}_{i,n}(\omega)(\gamma)} v_\gamma \big|_{S_{\sigma(1)}(k_{\sigma(1)}) \cap \cdots \cap S_{\sigma(i+1)}(k_{\sigma(i+1)})}.
\]

From Corollary 5.3.6 it follows that for all \( \sigma \in \mathcal{O}_{i+1}(n) \),

\[
\sum_{\omega \in \mathcal{H}_{i+1,n}(\sigma)} \sum_{\gamma \in \mathcal{H}_{i,n}(\omega)} (-1)^{\text{index}_{i-1,n}(\sigma)(\omega) + \text{index}_{i,n}(\omega)(\gamma)} v_\gamma \big|_{S_{\sigma(1)}(k_{\sigma(1)}) \cap \cdots \cap S_{\sigma(i+1)}(k_{\sigma(i+1)})} = 0.
\]

\[\square\]

5.4 Autonomy

5.4.1 n-D Autonomy

Autonomous systems are best understood in time-domain systems; it is in the situation that they are considered the signals where the past defines the future. In [15] Willems demonstrates the ties between autonomous systems and their memory. In many ways, autonomy is simply “not controllability.” It is in this point of view that autonomy and controllability clearly separate signals into two groups. However, when we lift to two or more dimension then there are different versions of controllability; it follows that no longer can we say “not controllability” is autonomy. The question becomes: what controllability is orthogonal to autonomy? Another interesting question is: what type of autonomy is orthogonal to a given definition of controllability?

In [20], a definition of controllability(4) is given that results in a sufficient, but not necessary, condition for the decomposition\(^2\)

\[\mathcal{B} = \mathcal{B}^c \oplus \mathcal{B}^a.\]

This is in contrast with the fact that for the many definitions of controllability, there is the decomposition

\[\mathcal{B} = \mathcal{B}^c + \mathcal{B}^a\]

with the possibility \( \mathcal{B}^c \cap \mathcal{B}^a \neq 0 \).

5.4.2 The Homology Group $H_n$

Based on the definitions of chains and boundary, we have

$$C_n\mathcal{B}(\vec{k}) = \mathcal{B}(\mathbb{Z}^n)$$
$$C_{n-1}\mathcal{B}(\vec{k}) = \mathcal{B}(S_1(k_1)) \times \mathcal{B}(S_2(k_2)) \cdots \times \mathcal{B}(S_n(k_n))$$

The because there are no higher chains, we have for $\vec{k} = (k_1, \ldots, k_{2n})$ where $k_i \in \mathbb{Z}$ for $i = 1, \ldots, 2n$, that

$$H_n\mathcal{B}(\vec{k}) = \ker \partial_{n-1}$$

$$= \ker \bigoplus_{\sigma \in \mathcal{O}_n} \sum_{\omega \in \mathcal{H}_{i,k}(\sigma)} (-1)^{\text{index}(\omega)+1} v_{\omega}|_{S_{\sigma(1)}(k_{\sigma(1)}) \cap \cdots \cap S_{\sigma(i)}(k_{\sigma(i)})}$$

After breaking down the indexing sets,

$$\partial_{n-1}(v) = \bigoplus v|_{S_i(k_i)}.$$  

This implies that

$$\ker (\partial_{n-1}) = \{ v \in \mathcal{B}(\mathbb{Z}^n) : v|_{S_i(k_i)} = 0 \ \forall \ i \}.$$  \hspace{1cm} (5.2)

If

$$\mathbb{Z}^n \setminus \left( \bigcap_i S_i(k_i) \right) = 0$$

the kernel is not very interesting. However, in the case that it is some orthotope ($n$-rectangle), it has meaning; provided $\mathcal{B}$ linear and shift-invariant, the dimensions of the orthotope tell us about the dimensions of kernel representation’s polynomials.

**Example 5.4.1.** Let $R(\sigma_1, \sigma_2, \sigma_3) = \sigma_1 - 1$. If $\mathcal{B} = \ker R$, then $\mathcal{B}$ is linear and shift-invariant. As expected, $H_3\mathcal{B}$ is non-zero since the values in the $y$ and $z$ dimension can be anything on the rectangle. If, however, we have

$$R(\sigma_1, \sigma_2, \sigma_3) = [\sigma_1 - 1, \sigma_2 - 1, \sigma_3 - 1]$$

and $\mathcal{B} = \ker R$, then we see that $H_3\mathcal{B}(\vec{k}) = 0$ on any orthotope. This follows because the only signals that are admissible are constant signals.

**Remark 5.4.1.** In (5.2) we see that off of some orthotope if the values are zero, then the only possible signal that could have generated such a boundary is the zero map. In particular, this says that the kernel representation has quite a bit of structure. We might be able to pull out that the kernel representation is full column rank. (See page 17 of [20].)
We have agreement with the two dimensional results when we consider classifying autonomous systems.

**Theorem 5.4.1.** There exists $\vec{k} = (k_0, \ldots, k_{2n-1})$ where $k_i \in \mathbb{Z}$ for $i = 0, \ldots, 2n - 1$ where

$k_i > k_{i+n} \quad i = 0, \ldots, 2n - 1$

we have that

$$H_n \mathcal{B}(\vec{k}) = 0$$

if and only if $\mathcal{B}$ is autonomous.

**Proof.** This is straightforward definition matching. Assume that $H_n \mathcal{B}(\vec{k}) = 0$, this implies that the only global signal that has zero restriction on the respective sets is the zero map. This implies that $\mathcal{B}$ is autonomous.

Assume that $\mathcal{B}$ is autonomous. It follows that there exists an $n$-rectangle $T$ such that for a global signal $w$, $w|_T = 0$ implies that $w = 0$. That is, the ker $\partial_{n-1} = 0$. It follows that $H_n \mathcal{B}(\vec{k}) = 0$ where $\vec{k}$ is derived from $T$. 

$\square$
Bibliography


