Progressive damage and failure of unidirectional fiber reinforced laminates under impact loading with composite properties derived from a micro-mechanics approach

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Micromechanics theories have been used to develop macro-level constitutive relations for infinitesimal elastoplastic deformations of unidirectional fiber reinforced laminates. The matrix is assumed to be isotropic and deform elasto-plastically and the fibers transversely isotropic and linear elastically. We have analyzed damage initiation, damage progression, and failure of 16-ply unidirectional fiber reinforced laminates impacted at normal incidence by a rigid sphere. The damage is assumed to initiate when at least one of Hashin’s failure criteria is satisfied with the evolving damage modeled by an exponential relation. Transient three dimensional impact problems have solved using the finite element method (FEM) by implementing the material damage model as a user defined subroutine in the FE software ABAQUS. From strains supplied by ABAQUS the subroutine uses the free shear traction technique and values of material parameters of the constituents to find average stresses in a FE, and checks for Hashin’s failure criteria. If the damage has initiated, the subroutine evaluates the damage developed, computes resulting stresses, and provides them to ABAQUS. The irreversibility of the damage is satisfied by requiring that the damage evolved does not decrease during unloading. The delamination failure mode is simulated by using the cohesive zone model and the degradation of material properties already available in ABAQUS. The computed time histories of the axial load acting on the impactor are found to agree well with the experimental ones available in the literature. The effect of stacking sequence in the laminate upon the impact load has been ascertained.
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CHAPTER 1: Introduction and literature review

1. Introduction

Present day materials used in industry are required to meet many design specifications which are often not fulfilled by conventional monolithic materials. Composites are a very good alternative and offer a number of advantages over their conventional counterparts. Typically a composite is made of two or more materials and its constituents are tailored to meet the design specifications. For example, to optimize the aerodynamics of flight, the wings of combat aircraft have to be shaped intricately; apart from this they require varied stiffness along their wingspan. If metal parts were to be used they need to be machined to the required specifications as one single piece, and if multiple parts are used they are usually fastened. This makes it cumbersome and costs make their use prohibitive. Using composites, which generally are lighter than metals, a designer has greater flexibility because the strength and the stiffness of the structure can be tailored by stacking laminate plies in localized areas and along preferred orientations. Also, with the advances made in the manufacturing processes complex shapes can now be cured in autoclaves. Composites have significantly influenced some industries, nearly 10% by weight of current day commercial aircrafts and cars are made of composites. The airbus 350 aircraft and the Boeing 787 Dreamliner have nearly 50% composite by weight. The ever increasing growth in composites usage and their wide applicability means that they need to be studied and analyzed thoroughly.

The dissertation deals with the analysis of composites which exhibit more than one spatial scale. To understand and clarify the above statement, consider the makeup of a typical bullet proof vest. At the micro-scale the vest is made of fine polymeric fibers typically only a few micrometers in diameter. At the next higher scale are the woven fiber bundles, which consist of a collection of fibers bonded together and are typically in the millimeter range. Finally, we have the woven laminate structure. The vest thus exhibits a different structure at each spatial scale and hence may be classified as multi-scaled. In general, materials that appear to be homogeneous at the macro-scale exhibit heterogeneity at the micro-structural level. Even if one considers a defect free single crystal whose microstructure is uniform it is ultimately discrete at the atomic scale.
It is well known that the macroscopic behavior of a heterogeneous material is significantly influenced by its microstructure. For example, it is seen that the addition of rigid particles to a polymer can increase its stiffness, and improve creep resistance and fracture toughness [1]. The mechanical properties are also influenced by the size, the shape, the aspect ratio and the distribution of reinforcing particles [1]. In ceramic matrix composites the toughness is increased by the addition of soft inclusions. In a ceramic matrix composite a propagating crack when intercepted by a ductile inclusion causes it to deform plastically, blunts the crack front and thus dissipates energy [2]. In steel it is seen that the elongation of non-metallic inclusions can reduce its ductility, failure strain and fracture toughness, but through shape control of the non-metallic inclusions such deleterious effects can be greatly reduced [3]. Since the micro-structure has a profound effect on the macroscopic behavior it is important to study how the microstructure influences the macroscopic behavior. This forms the basis of the dissertation, that is, to be able to predict the macroscopic behavior using information available at the constituent level.

It is impractical and unfeasible to study deformations of each constituent in heterogeneous materials that exhibit multiple spatial scales through experiments, primarily because of the number of variables that can influence the outcome, and the difficulty in monitoring deformations at the constituent level. These problems are thus best studied through numerical investigations. Also, since the problems being analyzed span several spatial scales it is computationally expensive to carry out numerical simulations that account for the complete heterogeneous structure of the body. Approximate methods need to be devised that account for micro-structural details and bridge different spatial scales; such approximate techniques are called homogenizing schemes.

2. Objectives and scope of the work

Inelastic behavior in fiber reinforced composites is primarily due to inelastic deformations of the matrix [4, 5]. Though fibers exhibit strain rate dependent elastic and strength properties their effects could be considered negligible as compared to that of the matrix. To analyze the inelastic behavior in AS4-PEEK composites Weeks and Sun [5] developed a macroscopic viscoplastic model based on an earlier work of Sun and Chen [6] by assuming that
the composite remains elastic when loaded along the fiber direction but can undergo plastic/visco-plastic pressure-independent deformations otherwise. Zhao et al. [7] modeled AS4-PEEK composite as an elastic-plastic material with nonlinear hardening and used the yield function developed by Sun and Chen [6]. The advantage of using a macro-model is that it is computationally efficient but a major short coming is that it relies on data from experimental off-axis loading of fiber reinforced composites to establish the parameters for the macro-model.

The objective of the present work is to use a micro-mechanics based approach to first derive values of macroscopic material parameters and then use them to analyze failure and damage at the macroscopic level. The micro-mechanics approach accounts for the inelastic behavior of the matrix and hence of the composite. Failure and damage in the composite is considered using continuum damage mechanics approach. The micro-mechanics method and the failure/damage model is integrated into a finite element setting so as to be able to solve problems under general loading conditions. The validity of the approach is tested by comparing simulated results with experimental findings.

3. Historical developments and literature review of homogenization methods

Some of the earliest work in this area dates back to the early twentieth century when Voigt and Reuss found the effective properties of a heterogeneous material based on the volume fractions of its constituents. Hill [8] proved that the elastic constants obtained through the Voigt and the Reuss rules are the upper and the lower bounds of the effective properties of the heterogeneous material. The bounds are narrow when properties of the individual constituents are close to each other but as the difference between the properties increases so does the difference in the two bounds. Though these rules are still used today, they are of only limited use as they are valid only under dilute concentrations of inclusions in the composite. Hashin [9, 10, 11] in a series of papers developed variational principles for heterogeneous materials and derived bounds for the effective elastic properties of the homogenized body; these bounds are tighter than those derived by Hill and are commonly referred to as the Hashin-Shtrikman bounds. Willis [12] developed the Hashin-Shtrikman bounds in a generalized way using Green’s function approach and found the effective thermal conductivity and elastic properties of a heterogeneous material containing spheroidal inclusions, cracks and needle shape inclusions. For a more
comprehensive review of this area one should refer to Torquato [13], and for details on bounds for non-linear behavior one can refer to works by Talbot and Willis [14], Ponte Castaneda [15], Suquet [16] and references therein.

The Mori-Tanaka [17, 18] (M-T) method is often used to estimate the effective properties of composites. In spite of its limitations [19] it has gained popularity primarily because of the ease with which it can be implemented in a computer program, and the explicit relations one can derive for effective properties provided that the deformations are linear elastic and inclusions have simple geometric shapes. Other micro-mechanical models which could be classified in the same class as the M-T method include the equivalent inclusion method [20], the self consistent method introduced by Hill [21], the double inclusion method developed by Nemat-Nasser [22], and the transformation field analysis developed by Dvorak and Benveniste [23] and Dvorak[24].

The major limiting factors of the methods listed in the preceding paragraph are that they are restricted to small strains. Since they depend on the use of the Eshelby tensor to determine the effective properties, the shapes of inclusions need to be ellipsoidal. The homogenizing methods are of course not limited to the elastic regime and these have been extended to study inelastic deformations using the M-T scheme. Tandon and Weng [25] used the M-T scheme to homogenize composites in which matrix undergoes plastic deformations (based on the J_2 flow theory), and Lagoudas et al. [26] applied the M-T method to study the elasto-plastic behavior of metal matrix composites. Jiang and Batra [58] have used the energy equivalence approach and the M-T theory to deduce effective moduli of epoxies containing piezoelectric and shape memory inclusions. In recent years there has been a growing interest in extending the M-T scheme to study damage and more specifically to study de-bonding between the inclusion and the matrix. The method adopted by Baney et al. [27] assumes that de-bonding follows a Weibull probability distribution function and the failure is governed by the tri-axial tensile state at the inclusion matrix interface. Qu [28] developed relations for effective stiffness of a heterogeneous system undergoing de-bonding by explicitly considering the displacement jump across the interface between the inclusion and the matrix using the M-T scheme. Dvorak and Zhang [29] and Matous [30] have used the transformation field analysis to study the effect of de-bonding.
Gartener [59] has generalized Aboudi’s [60] method of cells to consider debonding between the matrix and the fibers.

Homogenization of composites can also be carried out using mechanics of material approach. Ha et al. [31] have derived explicit expressions for compliance/stiffness matrix for fiber reinforced composites by assuming axial strains along the fiber direction to be equal in both the matrix and fiber and along the transverse directions stresses in the fiber are in some proportion to the those in the matrix, with the proportionality factors to be determined from the experimental data. The method of cells developed by Aboudi [60] is another well established micro-mechanics approach used to model both continuous and short fiber reinforced composites. Apart from modeling elastic behavior of composites, Aboudi [32] has modified the approach to account for inelastic deformations in composites. In this approach, a representative volume element (RVE) is comprised of four parallelepiped cells; three cells are made of the matrix and one of the fiber. The volume fraction of the fiber cell equals the volume fraction of fibers in the composite. Based on continuity and equilibrium conditions satisfied on the average across interfaces between adjacent cells, explicit constitutive equations for the composite are derived. Aboudi has modeled inelastic deformations in the composite by assuming fibers can undergo only elastic deformations while the matrix can undergo inelastic deformations. Robertson and Mall [33] relaxed the continuity of tangential tractions across the interfaces between adjacent cells, and showed through numerical experiments that this assumption does not affect values of the effective material properties. Pecknold and Rahman [34, 35] have developed a three-dimensional (3D) micromechanical model similar to that developed by Aboudi [32] but the RVE considered is divided into 3 sub-cells consisting of a fiber region and two matrix regions. In Aboudi’s method of cells, the deduced effective properties of the composite are independent of the shapes of fibers since the RVE has their equivalent volume fraction rather than the actual shapes as in the M-T scheme.

The micro-mechanics approaches stated above are restrictive in the sense that they cannot be applied for large deformations and, in general, for inclusions of irregular shapes. These also do not account for the spatial distribution of the inclusions. This can, however, be overcome if a numerical approach is adopted. A finite element scheme is a preferred method to solve such
problems, though the Fast Fourier transform (FFT) method has also been used [36, 37]. To carry out the homogenization process using the finite element method (FEM) RVEs of both the inhomogeneous and the heterogeneous materials are meshed, displacement, traction and periodic boundary conditions are applied on the RVE boundaries and Hill’s energy equivalence theorem is used to obtain the homogenized elastic constants. In periodic boundary conditions, opposite surfaces of the RVE conform to the same deformed shape. From the solution of the problem associated with the RVE, the local/effective stiffness and the average/effective stress and strain can be calculated. In the method of asymptotic expansion, periodic boundary conditions are applied on the surfaces of the RVE to homogenize the heterogeneous material. For an exhaustive analysis of this subject one can refer to works by Guedes and Kikuchi [38] and Bensoussan et al. [39]. Homogenization can also be achieved by applying the boundary conditions on the RVE through Lagrange multipliers, details of such an analysis can be found in the works of Miehe [40] and Koch and Miehe [41]. Another approach proposed by Kouznetsova et al. [42] and Smit et al. [43] can be applied only to 2D problems. It involves applying tie constraints to opposite edges of the RVE so that the deformed edges on opposite sides of the RVE conform to the same shape. Though the computational methods offer greater flexibility as compared to the micro-mechanics based approaches in terms of the complexity of the problem that can be solved, they are a huge burden on the computational resources and remain impractical to study large practical problems.

4. Review of continuum damage mechanics for fiber reinforced composites

Some of the earliest work on damage in fiber reinforced laminated composites was reported by Ladeveze and Dantec [44]. Their theory assumes damage to be uniform throughout the thickness of each individual lamina that makes up the composite. It is considered to be meso-scale because it falls between the micro-scale damage analysis of its constituents (fibers and matrix) and macro-scale damage analysis of laminates [45]. In this method internal variables are used to degrade the elastic properties of the composite. Ladeveze and Dantec’s damage model [44] at the ply level accounts for tensile and compressive fiber damage and matrix cracking. Plasticity type equations are also used to account for permanent deformations due to shear loading. Johnson [46] has used this model to study damage in fabric composites under impact loads. A major short coming in utilizing this model is that it requires a large number of input
parameters that need to be determined experimentally at specific fiber orientations. Further it assumes plane stress conditions and numerical simulations using the FEM are restricted to shell elements.

Hassan and Batra [47] have used three internal variables to account for damage due to fiber breakage, matrix cracking and fiber matrix debonding with the damage applied locally to a constituent. Experimental stress strain curves are used to extract and derive the phenomenological expressions relating damage variables with their corresponding conjugate forces. This approach enables one to compute the energy dissipated in each failure mode.

Donadon et al. [48] have developed a 3D failure model to predict damage in composite structures subjected to multi-axial loading. To avoid strain localization and mesh dependence due to damage a smear crack approach has been incorporated into the problem formulation. Parameters for the damage law were obtained through experimental data, details of which can be found in [49].

Clegg et al. [50] have implemented a damage model for hyper-velocity impacts in AUTODYN. To account for the nonlinear behavior in the fiber reinforced composite it is assumed that the composite undergoes plastic deformations having a yield surface which is quadratic in stress. Softening behavior in the composite is accounted for by using damage mechanics with the damage surface defined in terms of the Cauchy stresses and damage variables. The evolutions of damage variables are expressed as a function of the critical strain, the fracture energy, the fracture stress and a local characteristic dimension.

Maa and Cheng [51] have computed the tensile strength of laminated composites containing a circular hole by using a damage model and three failure modes, namely, fiber breakage, matrix cracking and interface debonding. To account for nonlinear hardening behavior during shear loading a phenomenological model on the lines of Ramberg-Osgood plasticity relation was used to express the shear stress-the shear strain relation. The parameters for the model were obtained through curve fitting.
Hou et al. [52] have considered fiber tensile failure, tensile matrix cracking, matrix crushing and delamination to study impact analysis of fiber reinforced composites. Chang-Chang failure criteria was modified by incorporating in the failure modes additional terms depending upon the shear stresses. Once failure criteria are met stress components associated with the different failure modes are reduced to zero.

Matzenmiller et al. [53] assumed that damage initiates at a point when one of Hashin’s failure criteria is met there and proposed a damage evolution law that is an exponential function of stresses. The elastic properties are progressively decreased with the evolution of damage, and this model is often referred to as the MLT damage model. The MLT approach has also been used to study damage in woven fabric composites, e.g., Xiao et al. [54] have used it to study damage during quasi-static punch test for woven fabric composites. Williams and Vaziri [55] have evaluated the effectiveness of the MLT damage model under impact loads by studying damage in carbon fiber reinforced plastics, and Xiao [56] has used it to study damage during crushing of hollow tubes. Huang and Lee [57] have used Hashin and Yeh’s delamination failure criteria to analyze crushing of fiber reinforced composites plates. The evolution of damage variables is proportional to the reciprocal of the failure factors rather than an exponential function of damage used in the MLT approach.

In the present work the MLT damage model is used to delineate damage at a point and then degrade the material properties, and a micro-mechanics approach is adopted to find the effective material properties of a point from those of the constituents.

5. Organization of the dissertation

The rest of the dissertation is organized as follows. The second chapter focuses on using micro-mechanics theories to derive mechanical properties of a unidirectional fiber reinforced composite from those of its constituents, and the third chapter on analyzing deformations of laminated composite plates under impact loading. Major conclusions and contributions from this work are summarized in the fourth chapter.
References


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CHAPTER 2: Homogenization of elasto-plastic material parameters for unidirectional fiber-reinforced polymeric composites

Abstract

After reviewing several micromechanical theories for determining values of elastic parameters for a unidirectional fiber reinforced polymeric composite (UFPC), we find values of these parameters using four such theories for an AS4/PEEK composite. We also study elastic-plastic deformations of the UFPC in which fibers are assumed to deform elastically but the matrix undergoes both elastic and plastic deformations. The matrix plastic deformations are considered by using the associative flow rule of plasticity. Both the pressure-independent von Mises yield surface and the pressure-dependent Drucker-Prager yield surfaces are employed to model elastic-plastic deformations of the polymer and the composite. In both cases the strain hardening of the matrix is considered and values of material parameters for the matrix are obtained by computing the effective stress versus the effective plastic strain curves from experimental uniaxial stress-strain curves. Values of parameters in the yield surface for the UFPC in terms of those of the matrix and the fiber and the volume fraction of fibers are found by using a micromechanical theory. Wherever possible, the computed results are compared with the corresponding experimental results available in the literature.

Key words: Homogenization, micro-mechanics, elastic-plastic deformation, fiber reinforced composites

1. Introduction

Composite materials are being increasingly used in several applications because of their high specific strength, and they can be engineered to attain desired mechanical properties in specific directions. For example, the optimization of the aerodynamics of a flight of an aircraft may require that the wings be shaped intricately and their stiffness suitably varied along the wingspan. If the wings were made of a metal then different parts will need to be precisely machined and fastened. Most metals are isotropic materials meaning they have same properties in all directions. Thus the thickness of a part will need to be varied to get the desired stiffness in a particular direction. When using composites, one can exploit their directional properties and
still get a uniform thickness but adjust stiffness in different directions. Moreover, the mass density of a composite is generally less than that of a metal, and complex shapes can be cured in an autoclave.

Composites being inhomogeneous materials provide challenges in design since their failure mechanisms are not well understood. Furthermore, for design purposes, one needs to find properties of a homogeneous material equivalent in mechanical response to the composite. Several homogenization techniques have been proposed in the literature to find mechanical properties of the composite from those of its constituents when the deformations are linear elastic; for example, see books by Torquato [1], Suquet [2], Bensoussan et al. [3], Nemat-Nasser and Hori [4], Mura [5], Aboudi [6], Tsai and Hahn [7], Hyer [8], and Reifsnyder and Case [9], and the review paper by Charalambakis [10]. In contrast to the micromechanical theories Love and Batra [11] numerically simulated plane strain deformations of a mixture of a metal matrix composite with circular cylindrical fibers by keeping edges of the cross-section plane and deduced values of the elastic and the plastic parameters at different strain rates by assuming that under quasistatic deformations the yield stress of the composite is given by the rule of mixtures.

The unidirectional fiber reinforced polymeric composites (UFPCs) with fibers in a ply laid parallel to each other may undergo inelastic deformations due to unexpected large loads applied to them. Young’s modulus of a lamina in the direction of fibers is considerably more than that in any other direction. Most fibers undergo infinitesimal elastic deformations prior to failure but the matrix bonding the fibers can deform inelastically either because deformations of the matrix are large or its yield stress is very small.

Bridgman [12] experimentally showed that the hydrostatic pressure does not affect plastic deformations of metals. However, mechanical deformations of some polymers have been shown to be influenced by the hydrostatic pressure. Thus the yield surface for such a polymer should depend upon the hydrostatic pressure; this dependence has been considered, among others, by Caddell et al. [13], and Hu and Pae [14]. Here we consider one such yield criterion for the matrix and account for its strain hardening. The yield surface for the UFPC is deduced by making the following assumptions: (i) the UFPC is a transversely isotropic material with the
fiber axis as the axis of transverse isotropy, (ii) there is no plastic deformation in the direction of fibers, and (iii) fibers deform elastically. We use a micromechanical theory to quantify the dependence of values of material parameters appearing in the yield function as a function of the volume fraction of fibers and mechanical properties of the matrix and the fibers. The functional dependence of the effective stress of the composite upon the effective plastic strain (i.e., the strain hardening effect) is taken to be similar to that of the matrix.

The rest of the paper is organized as follows. We review in Section 2 eight micromechanical theories to determine elasticities of the composite from those of the fibers and the matrix, and the volume fraction of fibers. Section 3 describes pressure-dependent and pressure-independent yield surfaces for the matrix and the UFPC, and techniques to evaluate various material parameters appearing in the yield surface of the UFPC from those of the fiber and the matrix. For an AS4/PEEK composite, the computed numerical results are compared with the experimental findings available in the literature in Section 4, and conclusions of this study are summarized in Section 5 where values of material parameters in the yield surface for the UFPC as a function of the volume fraction of fibers are given. We have compared in the appendix results of elastic plastic deformations of the UFPC from the incremental Mori-Tanaka and the FST methods. These results show that for in-plane shear deformations the two micromechanics approaches give different values of the yield stress and the strain-hardening modulus.

2. Elastic material parameters

2.1 Preliminaries

In this section we review eight micromechanics theories that have been proposed to derive effective properties of UFPCs from those of their constituents. We assume that fibers are straight circular cylinders and the fiber material is transversely isotropic with the fiber axis as the axis of transverse isotropy. Thus we need values of five elastic moduli to characterize the fiber material. We assume that (i) the polymer can be modeled as an isotropic material that has only two elastic moduli, namely Young’s modulus $E$ and Poisson’s ratio $\nu$, to characterize its linear elastic response; (ii) fibers are aligned parallel to the $x_1$-axis, and the $x_2$- and the $x_3$-axes are in and perpendicular to the plane of the lamina, respectively, as shown in Fig. 2-1, and (iii) the
A homogenized composite material is transversely isotropic with fiber axis as the axis of transverse isotropy. Thus the coordinate axes coincide with the material principal axes of the lamina.

![Diagram of a unidirectional fiber reinforced lamina with fibers along the \( x_1 \) axis.](image)

Fig. 2-1: Schematic sketch of a unidirectional fiber reinforced lamina with fibers along the \( x_1 \) axis.

We consider only infinitesimal deformations and assume that fibers and the matrix deform elastically; subsequently, we will also consider plastic deformations of the matrix. The constitutive relation for a linear elastic material can be written as

\[
\sigma_{ij} = C_{ijkl} e_{kl} \quad i, j, k, l = 1, 2, 3
\]  

(2.1)

where \( \sigma \) is the symmetric stress tensor, \( e \) the infinitesimal strain tensor, \( C \) the elastic moduli of the material, and a repeated index implies summation over the range of the index. The infinitesimal strain \( e_{ij} \) is related to the displacement \( u_i \) by

\[
e_{ij} = \frac{1}{2} \left( u_{ij} + u_{ji} \right), \quad u_{ij} = \frac{\partial u_i}{\partial x_j}
\]  

(2.2)

We assume that

\[
C_{jkl} = C_{ijkl} = C_{klj}
\]  

(2.3)

and thus there are 21 independent elastic moduli. For an isotropic material

\[
C_{jkl} = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \delta_{ij} \delta_{kl} + \frac{E}{(1 + \nu)} I_{ijkl}
\]  

(2.4)
where the Kronecker delta, \( \delta_{ij} \), equals 1 for \( i = j \) and zero otherwise, and 
\[
I_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]
is the fourth-order identity tensor. For a transversely isotropic material with \( x_1 \)-axis as the axis of transverse isotropy,

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33}
\end{bmatrix} =
\begin{bmatrix}
\gamma_5 & \gamma_4 & \gamma_4 \\
\gamma_4 & 2\gamma_1 + \gamma_2 & \gamma_2 \\
\gamma_4 & \gamma_2 & 2\gamma_1 + \gamma_2
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33}
\end{bmatrix}
\]

\[
\sigma_{12} = \frac{\gamma_3}{2} e_{12}, \sigma_{31} = \frac{\gamma_3}{2} e_{31}, \sigma_{23} = \frac{\gamma_3}{2} e_{23},
\]

\[
(2.5)
\]

\[
\gamma_1 = \frac{2E}{(1+\nu)}, \gamma_2 = \frac{E(v\xi + \nu^2)}{\xi_\alpha - 2v^2\xi_\alpha - 2\nu\xi_\alpha}, \gamma_3 = 4G_a
\]

\[
\gamma_4 = \frac{Ev_a}{\nu\xi_a - \xi_a + 2v_a^2}, \gamma_5 = \frac{E(v - 1)}{\nu\xi_a - \xi_a + 2v_a^2}, \gamma_6 = \frac{E}{E_a}, \nu_a = \frac{E}{G_a}
\]

where \( E \) is Young’s modulus along any direction in the \( x_2 x_3 \)-plane, \( \nu \), Poisson’s ratio for deformations in the \( x_2 x_3 \)-plane, \( G_a \), the shear modulus for deformations either in the \( x_1 x_2 \)- or in the \( x_1 x_3 \)-plane, \( E_a \), Young’s modulus along the \( x_1 \)-axis, and Poisson’s ratio \( \nu_a = -\frac{e_{22}}{e_{11}} \) when the material is deformed by a uniaxial load applied along the \( x_1 \)-axis.

It is often convenient to use the Voigt notation and express \( \sigma \) and \( e \) as six-dimensional vectors \( \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}\} \) and \( \{e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{13}, 2e_{12}\} \), respectively. Then the matrix \( C \) in Eq. (2.1) becomes a 6 x 6 symmetric matrix. In order that the work done to deform a linear elastic body be positive, the matrix \( C \) must be positive definite or equivalently its six eigenvalues must be positive. We assume that the matrix \( C \) is positive-definite, denote \( C^{-1} \) by \( S \) and call it the matrix of elastic compliances.

Henceforth we use superscripts \( f \) and \( m \) to denote quantities for the fiber and the matrix, respectively; a quantity without a superscript is for the composite which is assumed to be a homogeneous material with elastic moduli equal to those of the UFPC. A micromechanics
theory gives elastic moduli of the composite in terms of those of its constituents. Several micromechanics theories have been proposed, and we briefly describe below a few.

2.2 Voigt and Reuss rules

The simplest of the micromechanics theories is the rule of mixtures (or the Voigt rule) and gives

\[ C = (1 - v_f)C_m + v_f C_f \]  \hspace{0.5cm} (2.6)

where \( v_f \) equals the volume fraction of fibers. The analogue of Eq. (2.6) in terms of the compliance matrix \( S \), i.e.,

\[ S = (1 - v_f)S_m + v_f S_f \]  \hspace{0.5cm} (2.7)

is called the Reuss rule. In terms of the stored energy (or the strain energy) density \( W \) defined by

\[ 2W = \sigma_{ij} e_{ij} = e_{ij} C_{ijkl} e_{kl} \]  \hspace{0.5cm} (2.8)

Eq. (2.6) can be derived by assuming that

\[ W = (1 - v_f)W_m + v_f W_f \]

\[ e_{ij} = e_{ij}^f = e_{ij}^m, \sigma_{ij} = \frac{\partial W}{\partial e_{ij}} \]  \hspace{0.5cm} (2.9a,b)

where \( W \) is expressed in terms of strains. Similarly, Eq. (2.7) can be derived from the assumptions

\[ W_c = (1 - v_f)W_c^m + v_f W_c^f \]

\[ \sigma_{ij} = \sigma_{ij}^f = \sigma_{ij}^m, e_{ij} = \frac{\partial W_c}{\partial \sigma_{ij}} \]  \hspace{0.5cm} (2.10a,b)

where \( W_c \) is the strain energy density expressed in terms of stresses. It is clear that estimates of \( C \) and \( S \) given, respectively, by Eqs. (2.6) and (2.7) may not be very good since Eqs. (2.9b) and (2.10b) do not hold in general. Eqs. (2.6) and (2.7) are also called, respectively, the rule of mixtures for the elasticities and the compliances.

2.3 Hill’s equivalent energy criterion

Hill [15] proposed the following energy principle to derive \( C \) from \( C_m, C_f \) and \( v_f \). Consider two overall geometrically identical representative volume elements (RVEs) – one composed of a homogeneous material with elastic moduli \( C \) and the other of the UFPC with the volume fraction \( v_f \) of fibers. We apply displacement \( u \) on the boundaries given by
to both RVEs. Hill postulated that the matrix $C$ should be such that the work done to deform the two RVEs by equal average strains is the same. Eq. (2.11) implies that the average strain produced in the two RVEs equals $e^o_{ij}$; however, strains and stresses in the RVE containing fibers and the matrix are inhomogeneous or non-uniform but those in the composite are uniform. Since in a linear elastic body deformed quasistatically the work done by external forces equals twice the strain energy of the elastic body, therefore,

$$\Omega e^o_{ij} C_{ijkl} e^o_{kl} = \int_{\Omega_f} e^f_{ij} C^f_{ijkl} e^f_{kl} \, dv + \int_{\Omega_m} e^m_{ij} C^m_{ijkl} e^m_{kl} \, dv$$

where $\Omega$ equals the volume of the RVE, and $\Omega_f$ and $\Omega_m$, respectively, are the volumes of the fiber and the matrix in the RVE. The determination of $e^f_{ij}$ and $e^m_{ij}$ requires that one solve a displacement boundary-value problem (BVP) for the RVE by assuming that the matrix and the fibers are perfectly bonded to each other. By considering several linearly independent values of $e^o_{ij}$, one can find $C_{ijkl}$.

Instead of applying displacements given by Eq. (2.11) on the boundaries, one can apply tractions

$$f_i = \sigma^o_{ij} n_j$$

and equate complementary strain energies of the two RVEs, i.e.,

$$\Omega \sigma^o_{ij} S^{ijkl}_{ijkl} \sigma^o_{kl} = \int_{\Omega_f} \sigma^f_{ij} S^{f}_{ijkl} \sigma^f_{kl} \, dv + \int_{\Omega_m} \sigma^m_{ij} S^{m}_{ijkl} \sigma^m_{kl} \, dv$$

In Eq. (2.13) $n$ is a unit outward normal to the boundary.

We note that the $C(S)$ found by using Eq. (2.12) (Eq. (2.14)) accounts for the interactions amongst fibers, and between the fiber and the matrix and may also depend upon how fibers are laid out in the RVE. Furthermore, $C$ could also depend upon the shape and the size of the fiber cross-section. Hence, several BVPs should be solved to obtain statistically meaningful average values of $C$ and $S$. 

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2.4 Strain concentration tensors

For a homogeneous strain \( \varepsilon_{ij}^{e} \) in the composite body, let strains in the fiber and the matrix be given by

\[
e^{f} = A^{f} e^{o}, \quad e^{m} = A^{m} e^{o}
\]

where the fourth-order tensors \( A^{f} \) and \( A^{m} \) are called strain concentration tensors. Since

\[
e^{o} = v^{f} e^{f} + (1 - v^{f}) e^{m}
\]

therefore

\[
I = v^{f} A^{f} + (1 - v^{f}) A^{m}
\]

and knowing \( A^{f} \) one can find \( A^{m} \) and vice versa.

Recalling that the average stress \( \sigma^{o} \) in the composite is given by

\[
\sigma^{o} = v^{f} \sigma^{f} + (1 - v^{f}) \sigma^{m}
\]

we get

\[
C = v^{f} C^{f} A^{f} + (1 - v^{f}) C^{m} A^{m} = C^{m} + v^{f} (C^{f} - C^{m}) A^{f}
\]

and knowing \( A^{f} \) one can find \( A^{m} \) and vice versa.

Substitution from Eq. (2.15) into Eq. (2.12) gives

\[
C = v^{f} (A^{f})^{T} C^{f} A^{f} + (1 - v^{f}) (A^{m})^{T} C^{m} A^{m}
\]

Note that the elasticities \( C \) derived from Eq. (2.19) are based on the requirement that the average stress in the RVE equals that in the homogenized composite but the \( C \) obtained from Eq. (2.20) follows from the condition that the strain energy of the RVE equals that of the homogenized medium. Thus the two values of the elasticity matrix may be different.

For computational work, \( C \), \( A^{f} \) and \( A^{m} \) are 6 x 6 matrices. Several methods of finding approximate values of the strain concentration tensor \( A^{f} \), and the matrix \( C \) for the composite are reviewed below.

2.5 Eshelby’s equivalent inclusion method

In Eshelby’s equivalent inclusion method one assumes that \( v^{f} \ll 1 \), inclusions do not interact with each other, and finds the strain concentration tensor \( A^{f} \) by assuming that the fiber
is replaced by the matrix but having strains different from $e'$. Eshelby [16] considered an inclusion in an infinite homogeneous and isotropic linear elastic body and replaced the composite body by a homogeneous one as follows. Let the uniform strain field $e^o(x)$ in a homogeneous matrix be disturbed by $e_d(x)$ when a fiber is placed in the matrix. Thus the strain field in the composite body is given by $e^o(x) + e_d(x)$, and

$$
\sigma^f = C^f(e^o + e_d)
$$

(2.21)
gives stresses in the fiber. Assume that the fiber region is now replaced by the matrix but having a different strain field $e^o + e_d - e_e$ such that

$$
\sigma^f = C^f(e^o + e_d) = C^m(e^o + e_d - e_e)
$$

(2.22)

Eshelby found the tensor $S^f$ so that

$$
e_d = S^f e_e
$$

(2.23)
Substitution for the eigen-strain $e_e$ from Eq. (2.23) into Eq. (2.22), solving the resulting equation for $e_d$, and substituting the result into $e^f = e^o + e_d$, we get

$$
A^f = \left[I - S^f + S^f C^{-1} C^f \right]^T
$$

(2.24)
The expression for the tensor $S^f$ depends upon the matrix properties, fiber size and shape and is given in many books, e.g., see [4, 5].

### 2.6 Mori-Tanaka method

Consider a UFPC with fibers randomly placed parallel to each other with average strains $\bar{e}_o$ and stresses $\bar{\sigma}_o$. Note that the value of $\bar{e}_o$ depends upon the distribution of fibers (inclusions) and the interaction amongst them. Assume that one of the inclusions is replaced by the matrix material. Because of the large number of inclusions randomly placed in the matrix, it is reasonable to assume that replacing one inclusion by the matrix will not affect $\bar{e}_o$ and $\bar{\sigma}_o$. Thus the only difference between Eshelby’s equivalent inclusion method and the Mori-Tanaka method is in using $\bar{e}_o$ instead of $e^o$. Hence

$$
e^f = \bar{e}_o + e_d = \bar{e}_o + S^f e_e = A^f \bar{e}_o
$$

(2.25)
where the expression for $A^f$ is the same as that for $A^f$ in Eq. (2.24). Note that $A^f$ in Eq. (2.24) (Eq. (2.25)) relates the total strain in the inclusion to the average strain over the entire composite
(the matrix surrounding the inclusion). Therefore, \( A^f \) in Eq. (2.24) represents the global strain concentration tensor but the \( A_i^f \) in Eq. (2.25) is the local strain concentration tensor. Assuming that \( A_i^f \) is the same for every inclusion in the UFPC, then the global strain concentration tensor is given by

\[
A^f = A_i^f \left[ I + v^f A_i^f \right]^T
\]  

(2.26)

where

\[
A_i^f = \left[ I - S^E + S^E C_m^{-1} C^f \right]^T
\]  

(2.27)

### 2.7 Self-Consistent method

This is similar to Eshelby’s equivalent inclusion method except that the matrix region is assigned effective properties of the composite when finding \( A^f \). Replacing \( C_m \) by \( C \) in Eq. (2.24) and substituting for \( A^f \) in Eq. (2.19) we get the following nonlinear equation for the determination of \( C \).

\[
C = C_m + v^f (C^f - C_m) \left( I - S^E + S^E C^{-1} C^f \right)^T
\]  

(2.28)

This equation is iteratively solved for \( C \).

### 2.8 Aboudi’s method of cells

Aboudi [6] proposed the method of cells in which inclusions (fibers) are assumed to be periodically distributed as shown in Fig. 2-2. For a UFPC with fibers aligned along the \( x_1 \)-axis it is assumed that

\[
e_{i1} = e_m^f = e_i^f
\]  

(2.29)

A cylindrical RVE of square cross-section is considered with the cross-section divided into two square and two rectangular cells; e.g., see Fig. 2-2. Each cell is made of a homogeneous material and is assumed to deform uniformly. The material of one rectangular cell is fiber and that of the remaining three cells the matrix. We note that a fiber of any cross-section is replaced by a square; thus values of the elastic moduli of the UFPC are independent of the fiber cross-section.

Referring to dimensions shown in Fig. 2-2,
\[ a^2 = v^f \text{ and } b = 1 - a \] (2.30a,b)

imply that the volume fraction of the fiber in the RVE equals \( v^f \). The average strain and stress, \( \bar{\varepsilon} \) and \( \bar{\sigma} \), respectively, can be expressed in terms of strains and stresses in the four cells as follows:

\[
\begin{align*}
\bar{\varepsilon} &= a^2 e^{(4)} + ab e^{(1)} + b^2 e^{(2)} + ab e^{(3)} \\
\bar{\sigma} &= a^2 \sigma^{(4)} + ab \sigma^{(1)} + b^2 \sigma^{(2)} + ab \sigma^{(3)}
\end{align*}
\] (2.31a,b)

where \( e^{(1)} \) equals \( e \) in cell 1. Eq. (2.31a,b) follows from the definition of average stress and average strain. Thus we have five equations for the determination of twenty unknowns \( e_{a2}^{(1)}, e_{a3}^{(1)}, a = 1,2,3,4; a,b = 1,2 \) in terms of \( \bar{\sigma} \). The remaining fifteen equations are provided by the continuity of displacements and surface tractions across the four interfaces between cells 1 through 4. These conditions are satisfied on the average rather than at every point which is obviously impossible since the four cells intersect at one point.

We refer the reader to Aboudi's paper for the derivation of these fifteen equations, and the assumptions made to derive them. We note that Gardner [28] has incorporated effects of debonding/separation between any two cells of the RVE.
Fig. 2-2: Left: Cross-section of a UFPC with uniform arrangement of fibers; right: representation of the cross-section of the RVE by four cells, three of which are made of the matrix and the fourth one of the fiber.

### 2.9 Free shear traction (FST) method

Robertson and Mall [17] relaxed the traction continuity requirements at the interfaces between the four cells in Aboudi’s method, and showed through numerical experiments for the Graphite/Epoxy composite that their approximation gives values of elastic moduli close to those obtained by Aboudi [18]. Robertson and Mall assumed Eq. (2.29), and satisfied the continuity of surface tractions across interfaces between adjoining cells as follows.

\[
\begin{align*}
\sigma_{22}^{(4)} &= \sigma_{22}^{(1)} \quad \sigma_{22}^{(2)} &= \sigma_{22}^{(3)} \quad \sigma_{33}^{(4)} &= \sigma_{33}^{(3)} \quad \sigma_{33}^{(1)} &= \sigma_{33}^{(2)} \\
& \quad a\sigma_{12}^{(4)} + b\sigma_{12}^{(3)} = a\sigma_{12}^{(1)} + b\sigma_{12}^{(2)} \\
& \quad a\sigma_{13}^{(4)} + b\sigma_{13}^{(3)} = a\sigma_{13}^{(1)} + b\sigma_{13}^{(2)} \\
& \quad \sigma_{23}^{(4)} = \sigma_{23}^{(1)} = \sigma_{23}^{(2)} = \sigma_{23}^{(3)} 
\end{align*}
\]  

(2.32a-i)
Eq. (2.32a-d) imply the continuity of normal tractions across interfaces perpendicular to the $x_2$ - and the $x_3$ - axes. Eqs. (2.32e) and (2.32f) follow from the requirement that the total tangential force in the $x_1$ - direction on the planes $x_2 =$ constant and $x_3 =$ constant are continuous. Eq. (2.32g-i) states that the shear stress $\sigma_{23}$ has the same constant value in the four cells. It is clear that the continuity of tangential tractions $\sigma_{12}$ is being satisfied in a way different from the continuity of tangential tractions $\sigma_{13}$ and $\sigma_{14}$. In order that the four cells do not separate from each other during any deformation Robertson and Mall [17] imposed the following requirements on the uniform strains in the four cells.

\[
\begin{align*}
(a + b)\bar{e}_{12} & = a e_{12}^{(i)} + b e_{12}^{(j)} \\
(a + b)\bar{e}_{13} & = a e_{13}^{(i)} + b e_{13}^{(j)} \\
(a + b)\bar{e}_{23} & = a e_{23}^{(i)} + b e_{23}^{(j)} \\
\end{align*}
\]

Eq. (2.33a-d) reflects the requirement that the total displacement of the RVE in the $x_2$ - and the $x_3$ - directions equals the sum of the displacements of cells in those directions. In order to motivate Eq. (2.33e-j) we assume that $u_1 = 0$. Eq. (2.33e,f,i) implies that during the shear deformation given by $e_{12}$, the average $x_2$ - displacement across the interface between cells 4 and 3, and that between cells 1 and 2 are the same. Furthermore, the sum of the $x_2$ - displacement of the interface between cells 4 and 3, and that between cells 1 and 3 equals the average displacement of the RVE in the $x_2$ - direction. Eq. (2.33k) is the same as Eq. (2.31a) for $e_{23}$. We note that Eq. (2.31a) is satisfied for other components of $e$ when Eq. (2.33b-k) holds.

We substitute for stresses in Eq. (2.32) from Eqs. (2.1) and (2.5) in terms of the elastic moduli of the fiber and the matrix and strains in the four cells, and obtain 24 linear equations for determining the twenty four unknowns $e^{(i)}, e^{(j)}, e^{(k)}$ and $e^{(l)}$ in terms of six components of $\bar{e}$. 

25
Said differently, we can find strain concentration tensors for $e^{(1)}, e^{(2)}, e^{(3)}$, and $e^{(4)}$. Thus the strain concentration tensors depend upon the elastic moduli of the matrix and the fiber, and the volume fraction of fibers but not on the shape and the distribution of fibers.

3. Elasto-plastic material parameters

3.1 Preliminaries

The polymer in a UFPC usually exhibits an elasto-plastic response. However, fibers deform elastically and are brittle. Consequently, the failure strain of a UFPC in the direction of fibers is small and one can neglect effects of geometric nonlinearities. We first briefly review elasto-plastic deformations of the polymer and the UFPC and then using the FST approach determine parameters that appear in the plastic potential as well as find the functional dependence of effective stress on effective plastic strain of a UFPC.

We note that yielding of polymers may not be independent of the hydrostatic pressure as it is for most metals. For polycarbonates and polymethylmethacrylates the yield stress is known to depend upon the hydrostatic pressure $p$. Pressure-dependent yield functions have been discussed by Caddell et al. [13], and Hu and Pae [14].

3.2 Infinitesimal elasto-plastic deformations of polymer

3.2.1 Pressure independent yielding of polymer

We employ an incremental theory of plasticity and assume that during an incremental deformation, the increment, $de$, in the strain has an additive decomposition into elastic, $de^e$, and plastic, $de^p$, strains. That is

$$de = de^e + de^p$$  (3.1)

Eq.(2.1) relates the corresponding incremental stress, $d\sigma$, with the incremental elastic strain $de^e$. For relating $de^p$ to $d\sigma$, we assume the von Mises plastic potential for the polymer, i.e.,

$$2F(\sigma_{ij}) = \frac{1}{3} (\sigma_{22} - \sigma_{33})^2 + \frac{1}{3} (\sigma_{33} - \sigma_{11})^2 + \frac{1}{3} (\sigma_{11} - \sigma_{22})^2 + 2\sigma_{23}^2 + 2\sigma_{13}^2 + 2\sigma_{12}^2$$  (3.2)
Here we have expressed the plastic potential in terms of $\sigma$ rather than in terms of the deviatoric stress $s_y = \sigma_y - \frac{1}{3} \sigma_{kk} \delta_y$. In Eq. (3.2) we have tacitly assumed that the polymeric response is isotropic.

We use the associated flow rule for relating $d e^p$ to $d \sigma$, i.e.,

$$d e^p_{ij} = d \lambda \frac{\partial F}{\partial \sigma_{ij}}$$  \hspace{1cm} (3.3)

where $d \lambda$ is the proportionality factor. Eq. (3.3) implies that $d e^p$ is normal to the yield surface

$$F(\sigma_y) = \sigma^m$$  \hspace{1cm} (3.4)

where $\sqrt{3} \sigma^m$ is the yield stress of the polymer in a simple tension or compression test.

We define the effective stress $\bar{\sigma}$ by

$$\bar{\sigma} = \sqrt{3} F(\sigma_y)$$  \hspace{1cm} (3.5)

and the effective plastic strain, $\bar{e}^p$, by requiring that in any incremental deformation,

$$d e^p_{ij} \sigma_{ij} = \bar{\sigma} d \bar{e}^p$$  \hspace{1cm} (3.6)

We note that $\bar{\sigma} = \sigma^m$ in a uniaxial tension or compression test. Substituting for $d e^p_{ij}$ in Eq. (3.6) from Eq. (3.3) and using Eq. (3.5) we get

$$d \lambda = \frac{3}{2} \frac{d \bar{\sigma}^p}{\sigma} = \frac{3}{2 H^m} \frac{d \bar{\sigma}}{\bar{\sigma}}$$  \hspace{1cm} (3.7)

where the strain-hardening modulus, $H^m = \frac{d \sigma}{d \bar{e}^p}$, has been assumed to be positive. The function $H^m(\bar{\sigma}^p)$ is found from experimental data.

Taking the differential on both sides of Eq. (3.5) we get

$$d \bar{\sigma} = \frac{3}{2 \sigma} \left( \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij} \right) d \sigma_{ij}$$  \hspace{1cm} (3.8)

Substituting for $d \lambda$ from Eq. (3.7) and for $d \bar{\sigma}$ from Eq. (3.8) into Eq. (3.3) we get
\[ de^p_{ij} = \left( \frac{9}{4H^m - \sigma^2} \right) \left( \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij} \right) \left( \sigma_{pq} - \frac{\sigma_{pp}}{3} \delta_{pq} \right) d\sigma_{pq} = S^{mp}_{ijkl} d\sigma_{kl} \] (3.9)

where \( S^{mp} \) is the plastic compliance matrix of the polymer.

Expressing strain increments on the right-hand side of Eq. (3.1) in terms of incremental stress we get

\[ de = S^{me} d\sigma + S^{mp} d\sigma \] (3.10)

where \( S^{me} = (C^m)^{-1} \) is the elastic compliance matrix for the polymer. Hence

\[ d\sigma = (S^{me} + S^{mp})^{-1} de \] (3.11)

which expresses incremental stresses in terms of incremental strains. We note that \( S^{mp} \) depends upon current values of \( \sigma \), and normal stresses may affect ensuing shear strains.

The dependence of the flow stress, \( \sigma^m \), in Eq. (3.4) upon the effective plastic strain, \( \bar{\varepsilon}^p \), is determined from the test data. It will also determine \( H^m \).

### 3.2.2 Pressure-dependent yielding of polymer

To account for the pressure sensitive yielding of polymers, the Drucker-Prager yield potential is used, and the effective stress is taken to be

\[ \bar{\sigma} = \sqrt{3F} + \alpha^m \sigma_{kk} \] (3.12a)

where the parameter \( \alpha^m \) accounts for hydrostatic effects. The Drucker-Prager yield surface is given by

\[ \sqrt{3F} + \alpha^m \sigma_{kk} = \beta^m \] (3.12b)

where \( \beta^m \) is a material parameter.

Values of parameters \( \alpha^m \) and \( \beta^m \) can be found from the magnitudes of the yield stresses \( \sigma^i_\gamma \) and \( \sigma^c_\gamma \) in simple tension and compression tests. That is

\[ \sigma^i_\gamma + \alpha^m \sigma^c_\gamma = \sigma^c_\gamma - \alpha^m \sigma^i_\gamma = \beta^m \] (3.12c)
By following the procedure used to derive Eqs. (3.8) and (3.9) one can deduce an expression for $S^{mp}$ for the Drucker-Prager yield criterion (3.12b).

3.3 Yield function for unidirectional fiber reinforced polymeric composite (UFPC)

3.3.1 Pressure independent yielding of composite

We recall that the axial Young’s modulus of reinforcing circular cylindrical fibers is nearly fifty times that of a polymeric matrix. Also, fibers exhibit brittle linear elastic response. As for the matrix discussed in subsection 3.2, we use the associative flow rule to analyze infinitesimal elastoplastic deformations of a UFPC. We assume that the response of the UFPC is transversely isotropic with the fiber axis as the axis of transverse isotropy, and there is no plastic deformation along the fiber axis. That is,

$$de^{p}_{i1} = 0 \text{ or } e^{p}_{i1} = 0$$

(3.13)

Following Spencer [20] we write the plastic potential, $F$, for a transversely isotropic UFPC as

$$F(\sigma) = \frac{1}{2} \left[ \sigma_{22} - \sigma_{33} \right]^2 + 2A_{12}\sigma_{23}^2 + 2A_{13}\sigma_{13}^2 + \sigma_{ij}\sigma_{ij}$$

(3.14)

where parameters $A_{12}$ and $A_{23}$ depend upon the constituents of the UFPC and their volume fractions. The yield surface for the UFPC is

$$F = \sigma$$

(3.15)

We follow the same procedure as that used to derive Eq. (3.9) and get

$$\{de^{p}\} = [S^{p}]\{d\sigma\}$$

(3.16)

where

$$[S^{p}] = \frac{9}{4H^{1/2}a^2} 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & (\sigma_{22} - \sigma_{33})^2 & (\sigma_{33} - \sigma_{22})(\sigma_{22} - \sigma_{33}) & 2A_{12}\sigma_{23}^2(\sigma_{22} - \sigma_{33}) & 2A_{13}\sigma_{13}^2(\sigma_{22} - \sigma_{33}) & 2A_{13}\sigma_{13}^2(\sigma_{33} - \sigma_{22}) \\
0 & 0 & (\sigma_{33} - \sigma_{22})(\sigma_{22} - \sigma_{33}) & (\sigma_{33} - \sigma_{22})^2 & 2A_{13}\sigma_{13}^2(\sigma_{33} - \sigma_{22}) & 2A_{12}\sigma_{23}^2(\sigma_{33} - \sigma_{22}) \\
0 & 0 & 0 & 0 & \sigma_{23}^2(\sigma_{23} - \sigma_{22})^2 & \sigma_{23}^2(\sigma_{23} - \sigma_{22})^2 \\
0 & 2A_{12}\sigma_{13}^2(\sigma_{22} - \sigma_{33}) & 2A_{12}\sigma_{23}^2(\sigma_{22} - \sigma_{33}) & 4A_{23}\sigma_{12}\sigma_{23}\sigma_{13} & (2A_{12}\sigma_{13}^2)^2 & 4A_{23}\sigma_{12}\sigma_{23}\sigma_{13} \\
0 & 2A_{13}\sigma_{12}^2(\sigma_{22} - \sigma_{33}) & 2A_{13}\sigma_{12}^2(\sigma_{33} - \sigma_{22}) & 4A_{23}\sigma_{12}\sigma_{23}\sigma_{13} & (2A_{12}\sigma_{13}^2)^2 & 4A_{23}\sigma_{12}\sigma_{23}\sigma_{13} \\
0 & 2A_{13}\sigma_{12}^2(\sigma_{22} - \sigma_{33}) & 2A_{13}\sigma_{12}^2(\sigma_{33} - \sigma_{22}) & 4A_{23}\sigma_{12}\sigma_{23}\sigma_{13} & (2A_{12}\sigma_{13}^2)^2 & 4A_{23}\sigma_{12}\sigma_{23}\sigma_{13} \\
\end{bmatrix}$$
\(H^p\) equals the strain hardening modulus of the UFPC, and the stress and the strain increments are written in the Voigt notation.

Knowing the elastic compliance matrix, \(S\), for the composite obtained from the micromechanics theories discussed in section 2 we can determine the elastic-plastic compliance matrix \((S + S^p)\).

### 3.3.2 Pressure dependent yielding of composite

To account for the pressure sensitive yielding of composites, the Drucker-Prager yield potential for the composite is adopted, and the effective stress is defined as

\[
\bar{\sigma} = \sqrt{3F} + \alpha(\sigma_{22} + \sigma_{33})
\]

where the parameter \(\alpha\) accounts for effects of the hydrostatic pressure on the yielding of the UFPC. The term \(\alpha(\sigma_{22} + \sigma_{33})\) in Eq. (3.21) does not contain \(\sigma_{11}\) because of the assumption \(e_{11}^p = 0\) for the UFPC. One can derive the elastic-plastic compliance matrix for this case by following the same procedure as that outlined above.

### 3.4 Determination of elastic-plastic parameters of the UFPC

#### 3.4.1 Pressure-independent yield surface

We use the FST [17] method to compute the effective stress versus the effective plastic strain curve for the UFPC from a knowledge of the material properties of the constituents of the UFPC and the volume fraction of fibers. One way to obtain this curve is to study plane stress deformations of a thin lamina with the fiber axis making an angle \(\theta\) counter clockwise with the \(x\)–axis of loading as shown in Fig 2-3. Thus with respect to the global rectangular Cartesian coordinate axes \((x, y, z)\), the only non-zero component of the stress tensor is \(\sigma_{xx}\).
Fig. 2-3: Fiber orientation with respect to the axis of loading, and the material principal axes $(x_1, x_2, x_3)$.

Using either the Mohr circle or the stress transformation equations, stresses with respect to the material principal axes $(x_1, x_2, x_3)$ are given by

$$\sigma_{11} = \sigma_{xx} \cos^2 \theta, \quad \sigma_{22} = \sigma_{xx} \sin^2 \theta \quad \text{and} \quad \sigma_{12} = -\sigma_{xx} \sin \theta \cos \theta$$

Substituting for stresses from Eq. (3.18) into Eq. (3.2), and the result into Eq. (3.5) we get

$$\sigma = \sqrt[3]{\frac{2}{3} \left[ \sin^4 \theta + 2A_{12} \sin^2 \theta \cos^2 \theta \right]} \sigma_{xx}$$

It follows from Eq. (3.6) that

$$\sigma d\bar{e}^p = d\varepsilon_{xx} \sigma_{xx}$$

and therefore

$$d\bar{e}^p = \frac{d\varepsilon_{xx}}{\sqrt[3]{\frac{2}{3} \left[ \sin^4 \theta + 2A_{12} \sin^2 \theta \cos^2 \theta \right]^{1/2}}}$$

Thus

$$\bar{e}^p = \frac{\varepsilon_{xx} - \sigma_{xx}/E_{xx}}{\sqrt[3]{\frac{2}{3} \left[ \sin^4 \theta + 2A_{12} \sin^2 \theta \cos^2 \theta \right]^{1/2}}}$$
Here $E_x$ is Young’s modulus in the $x$-direction and it is a function of the elastic constants with respect to the material principal axes $(x_1, x_2, x_3)$ and the angle $\theta$; values of elastic constants with respect to coordinates $x_1, x_2, x_3$ are determined using a micromechanics theory. From Eqs. (3.19) and (3.22) one can find $\sigma$ as a function of $\varepsilon^p$, or equivalently plot the curve $\sigma$ vs. $\varepsilon^p$ for a given angle $\theta$. The value of parameter $A_{12}$ is found by trial and error by ensuring that $\sigma$ vs. $\varepsilon^p$ curves for various values of $\theta$ are close to each other within an acceptable tolerance [26].

We follow a similar procedure to find $A_{23}$. For simple shearing deformations in the $yz$ – plane with $\theta = 0$ we have

$$\sigma = \sqrt{3}A_{23}\sigma_{yz} \quad d\varepsilon_{yz}^p = d\lambda(A_{23})\sigma_{yz}$$

Using Eqs. (3.5) and (3.6) we have

$$d\varepsilon^p = \frac{d\varepsilon_{yz}^p}{2\sqrt{3}A_{23}} \quad \text{or} \quad \sigma = \varepsilon = \frac{\varepsilon_{yz}^p}{2\sqrt{3}A_{23}}$$

where $\varepsilon_{yz}^p = \varepsilon_{yz} - \frac{\sigma_{yz}}{2G_{yz}}$ and recalling that $G_{yz} = G_{23}$.

An alternative approach to finding values of parameters $A_{12}$ and $A_{23}$ and the function $\sigma(\varepsilon^p)$ is now described. Using the FST method we numerically simulate plane stress deformations of the UFPC under a uniaxial load in the $x_2$ – direction. Thus the only non-zero component of the stress tensor is $\sigma_{22}$, and from Eqs. (3.15) and (3.14) we get

$$\sigma = \sqrt{\frac{3}{2}} \sigma_{22}$$

Theoretically the UFPC will begin to deform plastically as soon as the matrix in one of the three matrix cells of the RVE starts yielding. This definition gives unrealistically low values of the yield stress of the UFPC and depends on the type of load (e.g., tensile, compressive, simple shearing) applied to the RVE. Here we use computed values of $\sigma_{22}$ and $e_{22}$ to plot the $\sigma_{22}$ vs. $e_{22}$ curve and take the proof-stress for $e_{22} = 0.0015$ as the value of the yield stress. Recall that the proof stress equals the value of $\sigma_{22}$ where the straight line parallel to the slope of the
\( \sigma_{22} \) vs. \( e_{22} \) curve at \( e_{22} = 0 \) (Young’s modulus at zero strain) passing through the point \( (\sigma_{22} = 0, e_{22} = 0.0015) \) intersects the \( \sigma_{22} \) vs. \( e_{22} \) curve. We thus find \( \sigma_{\text{yield}} \) when the UFPC begins to yield as

\[
\sigma_{\text{yield}} = \sqrt{\frac{3}{2}} \sigma_{22}^{PS} \tag{3.26}
\]

where \( \sigma_{22}^{PS} \) equals the proof-stress \( \sigma_{22} \).

In order to find values of \( A_{12} \) and \( A_{23} \) we use the FST technique to simulate simple shearing deformations of the UFPC in the \( x_1x_2 \) and \( x_2x_3 \) planes. Using the same definition of the yield stress as that adopted in Eq. (3.26) we obtain

\[
\sigma_{\text{yield}} = \sqrt{3A_{12}} \sigma_{12}^{PS} = \sqrt{3A_{23}} \sigma_{23}^{PS} \tag{3.27a,b}
\]

We find \( \sigma_{\text{yield}} \) from Eq. (3.26), and then values of parameters \( A_{12} \) and \( A_{23} \) from Eq. (3.27a,b).

We note that in Weeks and Sun’s [26] approach values of parameters \( A_{12}, A_{23} \) and \( \sigma_{\text{yield}} \) are determined by considering \( \sigma \) vs. \( \varepsilon^p \) curves for the entire range of values of \( \varepsilon^p \) considered, and \( \sigma_{\text{yield}} = \sigma \) when \( \varepsilon^p = 0 \). Thus the value of \( \sigma_{\text{yield}} \) is determined after values of \( A_{12} \) and \( A_{23} \) have been found. The \( \sigma(\varepsilon^p) \) function is found from the computed \( \sigma \) vs. \( \varepsilon^p \) curve by the least squares method. However, in the alternative approach proposed here values of \( A_{12}, A_{23} \) and \( \sigma_{\text{yield}} \) are found from those of the proof stresses in uniaxial loading in the transverse direction, and in simple shearing deformations in the \( x_1x_2 \) and \( x_2x_3 \) planes.

### 3.4.2 Pressure-dependent yield surface

We now need to find values of parameters \( \alpha, A_{12}, A_{23} \) and \( \sigma_{\text{yield}} \). When using Weeks and Sun’s [26] approach we find values of \( \alpha \) and \( A_{12} \) by ensuring that the \( \sigma \) vs. \( \varepsilon^p \) curves for different values of the fiber-orientation \( \theta \) in Fig. 2-3 and positive and negative values of \( \sigma_{xx} \) are close to each other within the prescribed tolerance. Subsequently, the value of \( A_{23} \) is chosen so that the \( \sigma \) vs. \( \varepsilon^p \) curve for the simple shearing deformations in the \( yz \) plane is close to the \( \sigma \) vs. \( \varepsilon^p \) curve for uniaxial loading.
In the alternative approach proposed here we use the FST technique to compute \( \sigma_{22} \) vs. \( e_{22} \) curves separately for positive and negative values of \( \sigma_{22} \), and find \( \sigma_{\text{yield}}^l \) and \( \sigma_{\text{yield}}^c \) from \( \sigma_{22}^{ps} \). From Eq. (3.17) we get

\[
\alpha = \frac{3}{2} \left( \frac{\sigma_{\text{yield}}^l - \sigma_{\text{yield}}^c}{\sigma_{\text{yield}}^l + \sigma_{\text{yield}}^c} \right), \quad \sigma_{\text{yield}} = \sqrt{\frac{3}{2} \left( 2\sigma_{\text{yield}}^c \sigma_{\text{yield}}^l \right)}
\]

(3.28a,b)

Values of parameters \( A_{12} \) and \( A_{23} \) are found by following a procedure similar to that described in subsection 3.4.1.

4. Numerical results and discussion

4.1 Elastic constants for AS4 fibers

The fiber is modeled as a transversely isotropic material. With the \( x_1 \) -axis of a rectangular Cartesian coordinate system taken along the fiber axis, values of the five independent elastic moduli of AS4 fibers [21] are listed in Table 2-1.

<table>
<thead>
<tr>
<th>( E_1 )</th>
<th>( E_2 = E_3 )</th>
<th>( G_{12} )</th>
<th>( \nu_{12} )</th>
<th>( \nu_{23} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>234 GPa</td>
<td>14 GPa</td>
<td>27.6 GPa</td>
<td>0.2</td>
<td>0.25</td>
</tr>
</tbody>
</table>

4.2 Material parameters for PEEK assuming pressure independent yielding

The PEEK is assumed to be isotropic and its response in the elastic region is taken to be linear. Values of Young’s modulus, \( E^m \), and Poisson’s ratio, \( \nu^m \), obtained from Goldberg and Stouffer [22] for the PEEK tested by Bordonaro [23] equals 4 GPa and 0.35, respectively. The axial stress versus the axial strain curve for PEEK deformed in simple tension at an axial strain rate of \( 1.0 \times 10^{-6} \) /s taken from [22] is exhibited in Fig. 2-4. For a uniaxial tension test with the load applied along the \( x_1 \) -axis, the only non-zero component of the stress tensor is \( \sigma_{11} \), and from the definition (3.5) of the effective stress, \( \sigma_{eff} = \sigma \). Eq. (3.6) implies that \( de_{11}^p = d\bar{\sigma} \), and hence \( e_{11}^p = \bar{\sigma} \). From the axial stress-axial strain curve of Fig. 2-4, we find \( e_{11}^p \) by using
\[ e_{11}^p = e_{11} - \frac{\sigma_{11}}{E} \] (4.1)

and plot the \( \bar{\sigma} \) vs. \( \bar{\varepsilon}^p \) curve shown in Fig. 2-5 where it has been tacitly assumed that the yield stress of PEEK in uniaxial tension and compression equals 40 MPa. Assuming that \( \bar{\sigma} \) and \( \bar{\varepsilon}^p \) are related by

\[ \bar{\varepsilon}^p = \exp\left( \frac{\bar{\sigma} - A}{N} \right) \] (4.2)

in the range \( 94 \text{MPa} \geq \bar{\sigma} \geq 40 \text{MPa} \), values of \( A \) and \( N \) found by the least squares method and having a regression coefficient of 0.986 are \( A = 152.7 \text{MPa} \) and \( N = 13.86 \text{MPa} \).

Fig. 2-4: Computed and experimental axial stress vs. axial strain curves for PEEK polymer.
Eq. (4.2) was verified by numerically analyzing elastoplastic deformations of a bar made of PEEK and pulled in uniaxial tension. The BVP was studied using the finite element method (FEM) by implementing the material model for PEEK as a user subroutine in the software ABAQUS. The elastoplastic problem for the matrix is solved iteratively by first assuming that deformations during an incremental load are elastic. The iterative process within a load step at each integration point is stopped when

$$\frac{\|e^{k+1} - e^k\|}{\|e^k\|} \leq 1.0E - 8, \quad \|e^k\|^2 = \epsilon_{ij}^k \epsilon_{ij}^k$$

where $k$ is the iteration number.

The axial stress versus the axial strain curve so obtained is displayed in Fig. 2–4. It is clear that the computed axial stress versus the axial strain curve is very close to the experimental one.
4.3 Material parameters for PEEK assuming pressure dependent yielding

Zheng's [27] experimental uniaxial tension and compression stress strain curves for PEEK at strain rates of 1.0E-6/s and 1.0E-5/s, respectively, are shown in Fig. 2-6. From Eq. (3.12a) for uniaxial tension test, the effective stress $\bar{\sigma}$ and the effective plastic strain $\bar{\varepsilon}^p$ can be expressed in terms of the axial stress $\sigma_{ii}$ and the axial plastic strain $e_{ii}^p$, that is,

$$\bar{\sigma} = (I + \alpha^m)\sigma_{ii} \quad \text{and} \quad \bar{\varepsilon}^p = \frac{e_{ii}^p - \frac{\sigma_{ii}}{E^m}}{(I + \alpha^m)}$$  \hspace{1cm} (4.3)

Similarly, for uniaxial compression we have

$$\overline{-\sigma} = (I - \alpha^m)\sigma_{ii} \quad \text{and} \quad \overline{-\varepsilon}^p = \frac{-|e_{ii}|-\frac{\sigma_{ii}}{E^m}}{(I - \alpha^m)}$$  \hspace{1cm} (4.4)

From the experimental uniaxial stress strain curves for tension and compression tests, the effective stress and the effective plastic strains can be evaluated using Eqs. (4.3) and (4.4). The value of $\alpha^m$ is chosen so that the effective stress versus effective plastic strain curves for both tension and compression tests overlap within an acceptable tolerance. Fourth order polynomials were fit to the effective stress versus effective plastic strain curves for tension and compression data and the percentage difference was found using

$$\%\text{Difference} = \sqrt{\frac{\sum_i (\sigma_i^t - \sigma_i^c)^2}{\sum_i (\sigma_i^t)^2}} \times 100$$  \hspace{1cm} (4.5)

where $\sigma_i^t$ and $\sigma_i^c$ are the effective stress values from tension and compression tests, respectively, corresponding to the effective plastic strain $\varepsilon_i^p$. The entire range of $\bar{\varepsilon}^p$ was divided into 30 equal segments and values at the mid-point of a segment were used in Eq. (4.5).

Fig. 2-7 shows the effective stress versus the effective plastic strain curve for uniaxial compression and tension tests obtained from Zheng [27]. Also shown in Fig. 2-7 is the least squares curve fit for $\alpha^m = 0.1645$. The difference between the effective stress versus effective plastic strain curves from tension and compression data using Eq. (4.5) was found to be 1.7% for effective plastic strain up to 3.0%.
The value of $\alpha^m$ was also determined using the value of the proof stress at yield from uniaxial compression and tension tests. The expression for $\alpha^m$ in terms of the proof stress in tension, $\sigma_{t}^{PS}$, and compression, $\sigma_{c}^{PS}$, obtained from Eqs. (4.3) and (4.4) is

$$\alpha^m = \left\{ \frac{\sigma_c^{PS} - \sigma_t^{PS}}{\sigma_c^{PS} + \sigma_t^{PS}} \right\}$$

(4.6)

From the experimental curves depicted in Fig. 2-6, we found that $\sigma_t^{PS} = 57$ MPa and $\sigma_c^{PS} = 73$ MPa. Substituting for $\sigma_t^{PS}$ and $\sigma_c^{PS}$ in the expression for $\alpha^m$ in Eq. (4.6), we get $\alpha^m = 0.123$. The difference between the effective stress versus effective plastic strain curves from tension and compression data using Eq. (4.5) was found to be 16.6% for effective plastic strain of up to 3.0%. As mentioned earlier, Weeks and Sun's [26] approach uses the entire set of data for $0 \leq \tilde{\varepsilon}^p \leq 0.03$ whereas the present approach uses values of proof stresses which vary with the definition of the proof stress. For example, one could define the proof stress as the stress corresponding to the strain of 1.3% rather than 1.5%.

For the relation (4.2) values of $A$ and $N$ obtained by the least squares method with a regression coefficient of 0.954 and $\alpha^m = 0.1645$ are 144.0 MPa and 8.69 MPa, respectively. In Fig. 2-6 we have compared the experimental uniaxial stress strain curves with the curves obtained from implementing the material model for PEEK as a user subroutine in ABAQUS and employing the above listed values of $\alpha^m$, $A$ and $N$. It is evident that the computed axial stress versus the axial strain curves are close to the experimental ones.
Fig. 2-6: Comparison of computed and experimental axial stress vs. axial strain curves for PEEK polymer deformed in uniaxial tension and compression.

Fig. 2-7: Effective stress versus effective plastic strain curve for PEEK obtained from the experimental axial stress versus axial strain curve of [27].
4.4 Elastic parameters of the composite

When using Hill’s theorem, we note that it is simpler to adopt the following approach than to use Eq. (2.12). For the displacement field \( u_i = e_{ij}^0 x_j \) imposed on the boundaries of the RVE, we equate the work done by external forces on the homogenized RVE to that on the composite RVE. That is,

\[
e_{ij}^0 \sigma_{ij}^0 = \frac{1}{\Omega} \int_{\Gamma} f_i u_i ds = \frac{1}{\Omega} \int_{\Gamma} f_i e_{ij}^0 x_j ds \tag{4.7a}
\]

where \( f_i \) is the surface traction on the boundaries of the composite RVE. Since Eq. (4.7a) must hold for all values of \( e_{ij}^0 \), therefore

\[
\sigma_{ij}^0 = \frac{1}{\Omega} \int_{\Gamma} f_i x_j ds = \frac{1}{\Omega} \sum R_i x_j \tag{4.7b}
\]

where \( R_i = f_i ds \) is the reaction force and \( \Gamma \) is the boundary surface of the RVE. The RVE, depicted in Fig. 2-8, consists of nine uniformly distributed fibers in the matrix. The linear elastic BVP for the composite RVE under displacement boundary conditions \( u_i = e_{ij}^0 x_j \) is numerically solved using the commercial software ABAQUS, and reaction forces \( R_i \) at the boundary surfaces are found. The deformed configurations depicted in Fig. 2-8 of the RVE are for tensile deformations along the \( x_i \)-axis obtained by prescribing \( \{u\} = \{e_{ij}^0, x_i, 0, 0\} \) on the boundary and for pure shear deformations in the \( x_i, x_2 \)-plane obtained by assigning displacements \( \{u\} = \{e_{ij}^0, x_2, x_1, 0\} \) on the bounding surfaces of the RVE. We note that for \( \{u\} = \{e_{ij}^0, x_i, 0, 0\} \) the specimen is not deformed in uniaxial tension since the other two components of displacements are constrained to be zero. From the deformed configuration depicted in Fig. 2-8, it is evident that small matrix regions surrounding the fibers are more intensely deformed than the remainder of the matrix. For each of the six displacement boundary conditions involving \( e_{ij}^0 \), \( e_{22}^0 \), \( e_{33}^0 \), \( e_{12}^0 \), \( e_{13}^0 \) and \( e_{23}^0 \) the stress vector \( \{\sigma_{ij}^0, \sigma_{22}^0, \sigma_{33}^0, \sigma_{12}^0, \sigma_{13}^0, \sigma_{23}^0\} \) is calculated using Eq. (4.7b) from which \( C_{ijkl} \) is obtained.
Robertson and Mall [17] have shown that their FST method gives results close to those derived from Aboudi’s [6] method of cells. Consequently, we did not compute results from Aboudi’s method.

For each one of the micromechanical theories studied here values of Young’s modulus $E_1$ essentially follow the rule of mixtures and are not plotted here.

We have plotted in Figs. 2-9 through 2-13 values of $E_2$, $G_{12}$, $G_{23}$, $\nu_{12}$ and $\nu_{23}$ as a function of the volume fraction, $\nu'$, of the fibers. In general, the curves exhibited in Figs. 2-9...
through 2-13 are not straight lines. Even for $v^f = 0.3$ different micromechanics approaches do not give the same value of an elastic modulus.

For $v^f = 0.6$, the experimentally determined values of $E_2 = E_3$, $G_{12} = G_{13}$, $G_{23}$, $\nu_{12}$ and $\nu_{23}$ are exhibited in these figures. Except for possibly different curing methods, it is unclear why various experimentalists obtained different values of these elastic constants. The difference between the minimum and the maximum values of $E_2$, $G_{12}$, $\nu_{12}$ and $\nu_{23}$ equals 11, 11, 16, and 4%, respectively.

Out of the four micromechanics approaches adopted here, Hill’s and the FST methods give not only values of $E_2$ that are close to each other but these values are also close to the test data. Eshelby’s equivalent inclusion and the Mori-Tanaka techniques noticeably underestimate values of $E_2$. We did not compute results for $v^f < 0.3$ since for most composites applications $v^f > 0.3$.

For a given volume fraction of the fibers, the value of $G_{12}$ computed by Hill’s method is considerably more than that estimated by using the other three approaches, and it also far exceeds the experimentally found value. This is possibly due to the strain concentrations around nearly rigid fibers. An examination of fringe plots of stresses showed that equal and opposite stresses were developed near the North and South poles of fibers that equilibrated each other but contributed to the energy required to deform the fibers. Also for the simple shearing deformations, local values of strain components other than $e_{12}$ were non-zero even though their average values were null.

For $v^f = 0.3$, the micromechanics theories provide values of $G_{23}$ that are reasonably close to each other. Hill’s method in this case yields good values of $G_{23}$ for the composite because $G_{23}$ for the fiber is less than four times that of the matrix, and there are fewer places of strain concentrations. The Mori-Tanaka method and Hill’s equivalent energy principle provide values of $G_{23}$ that are close to each other. We note that the differences in the values of $G_{23}$ increase with an increase in the volume fractions of fibers. For this case the FST and the EIM severely underestimate values of $G_{23}$. 
For $v^f = 0.6$ the micromechanics theories that provide good values of $E_2$, $G_{12}$ and $G_{23}$ do not yield values of $\nu_{12}$ and $\nu_{23}$ that are close to the test data. For $v^f = 0.3$, the four micromechanics approaches give close values of $\nu_{12}$ but not of $\nu_{23}$.

It is clear from the plots exhibited in Fig. 2-13 that values of $\nu_{23}$ do not obey the rule of mixtures since $\nu_{23}$ for $v^f = 0.3$ is greater than the value of Poisson’s ratio for the matrix. Furthermore, the rule of mixtures implies that an elastic modulus is an affine function of $v^f$.

The foregoing observations of the computed values of the elastic moduli suggest that none of the four micromechanics approaches provides values of the five elastic moduli that are very close to the corresponding experimentally found values.

**Fig. 2-9**: Variation with $v^f$ of the elastic modulus $E_2 = E_3$ obtained using different micromechanics approaches; M-T = Mori-Tanaka, FST = free shear traction, EIM = Eshelby’s inclusion method.
Fig. 2-10: Variation with $v_f$ of the elastic modulus $G_{12} = G_{13}$ obtained using different micromechanics approaches; M-T = Mori-Tanaka, FST = free shear traction, EIM = Eshelby’s inclusion method.

Fig. 2-11: Variation with $v_f$ of the elastic modulus $G_{23}$ obtained using different micromechanics approaches; M-T = Mori-Tanaka, FST = free shear traction, EIM = Eshelby’s inclusion method.
Fig. 2-12: Variation with $v_f$ of the Poisson’s ratio $\nu_{12}$ obtained using different micromechanics approaches; M-T = Mori-Tanaka, FST = free shear traction, EIM = Eshelby’s inclusion method.

Fig. 2-13: Variation with $v_f$ of the Poisson’s ratio $\nu_{23}$ obtained using different micromechanics approaches; M-T = Mori-Tanaka, FST = free shear traction, EIM = Eshelby’s inclusion method.
4.5 Elasto-plastic properties of the composite

4.5.1 Validity of the FST approach for elasto-plastic deformations

Given values of elastic moduli of the fiber and the matrix, and the effective stress, \( \sigma \), of the matrix as a function of the effective plastic strain, \( \bar{\varepsilon}^{pm} \), for the matrix, we find values of \( A_{12} \), \( A_{23} \), and \( \bar{\sigma}\left(\bar{\varepsilon}^{p}\right) \) for the UFPC. A user defined subroutine based on the FST method of subsection 2.8 has been implemented in the FE software, ABAQUS. The three matrix cells deform elasto-plastically and the fiber cell elastically.

In order to determine the elastoplastic material response of the AS4/PEEK composite two material models for the polymer are considered; one assumes that the hydrostatic pressure contributes to plastic deformations [27] (discussed in subsection 3.2.2) and the other assumes the von Mises yield surface in which plastic deformations are independent of the hydrostatic pressure [24] (discussed in subsection 3.2.1). The elastic-plastic properties for pressure insensitive and pressure sensitive polymer are given in subsections 4.2 and 4.3, respectively. Using material properties of the constituents and the volume fraction of fibers we simulated uniaxial deformations of the composite. Since the specimen is expected to deform homogeneously only one FE is used to simulate these tests.

For \( v^f = 0.6 \), Figs. 2-14, 2-15 and 2-16 compare the computed axial stress versus the axial strain curves for off-axis loading (e.g. see Fig. 2-3) with experimental results reported by Weeks and Sun [26]. It is clear that the axial stress versus the axial strain response obtained using the pressure independent von Mises yield surface for the polymer is in better agreement with the experimental data than that when the polymer is modeled with the pressure dependent Drucker-Prager yield surface.
Fig. 2-14: For $\nu = 0.6$, comparison of the computed axial stress-axial strain curves for the $14^\circ$ off-axis loading using the FST approach with the experimental curve of Weeks and Sun [26].

Fig. 2-15: For $\nu = 0.6$, comparison of the computed axial stress-axial strain curves for the $30^\circ$ off-axis loading using the FST approach with the experimental curve of Weeks and Sun [26].
Fig. 2-16: For $\nu^f = 0.6$, comparison of the computed axial stress-axial strain curves for the $45^\circ$ off-axis loading using the FST approach with the experimental curve of Weeks and Sun [26].

4.5.2 Values of material parameters for the homogenized composite

Values of material parameters appearing in the yield surfaces of the UFPC are determined from the FST method. For $\nu^f = 0.6$, and using the techniques discussed in subsection 3.4, we get

(a) Pressure independent yield surface
(i) Weeks and Sun's approach

\[ A_{12} = 7.5 \quad A_{23} = 8.5 \quad A = 496.6 \text{ MPa} \quad N = 35.2 \text{ MPa} \]

(ii) Present approach

\[ A_{12} = 5.45 \quad A_{23} = 6.47 \quad A = 388.5 \text{ MPa} \quad N = 24.4 \text{ MPa} \]
(b) Pressure dependent yield surface \( \alpha = 0.544 \)

(i) Weeks and Sun's approach

\[ A_{12} = 2.1 \quad A_{23} = 2.2 \quad A = 250.0 \text{ MPa} \quad N = 15.4 \text{ MPa} \]

(ii) Present approach

\[ A_{12} = 3.22 \quad A_{23} = 3.47 \quad A = 303.7 \text{ MPa} \quad N = 16.7 \text{ MPa} \]

4.5.3 Comparison of results from the FST method with those from the homogenized composite

Figs. 2-17, 2-18 and 2-19 compare the axial stress versus the axial strain curves for off-axis loading obtained from the FST method with the axial stress versus the axial strain response for the pressure insensitive yielding of composites using above-listed values of material parameters determined. It is clear that the three sets of curves are close to each other.

![Graph comparing stress-strain curves](image_url)

Fig. 2-17: For \( \nu' = 0.6 \), comparison of axial stress - axial strain response for 14° off-axis loading obtained from the FST method with the response of pressure insensitive yielding of composites using values of parameters determined from the two methods.
Fig. 2-18: For $\nu = 0.6$, comparison of axial stress - axial strain response for $30^\circ$ off-axis loading obtained from the FST method with the response of pressure insensitive yielding of composites using values of parameters determined from the two methods.

Fig. 2-19: For $\nu = 0.6$, comparison of axial stress - axial strain response for $45^\circ$ off-axis loading obtained from the FST method with the response of pressure insensitive yielding of composites using values of parameters determined from the two methods.
For the pressure sensitive yield surface, Figs. 2.20, 2.21 and 2.22 compare the axial stress versus the axial strain curves for off-axis loading of a lamina obtained using the FST method with those computed by using the above-listed values of material parameters. Again, the three sets of curves are close to each other.

Fig. 2-20: For \( v' = 0.6 \), comparison of axial stress - axial strain response for 14° off-axis loading obtained from the FST method with the response of pressure sensitive yielding of composites using values of parameters determined from the two methods.
Fig. 2-21: For $\nu = 0.6$, comparison of axial stress - axial strain response for 30° off-axis loading obtained from the FST method with the response of pressure sensitive yielding of composites using values of parameters determined from the two methods.

Fig. 2-22: For $\nu = 0.6$, comparison of axial stress - axial strain response for 45° off-axis loading obtained from the FST method with the response of pressure sensitive yielding of composites using values of parameters determined from the two methods.
4.5.4 Dependence of the yield surface upon the fiber volume fraction

We only use the von Mises yield criterion for the matrix and the composite with the composite modeled as a transversely isotropic material and the fiber axis as the axis of transverse isotropy; results for the pressure dependent yield surface can be similarly obtained. Furthermore, we seek an expression for the effective stress of the composite with variables $A$ and $N$ depending upon the volume fraction of the fibers. We have exhibited in Figs. 2-23 and 2-24 the dependence of parameters $A$, $N$, $A_{12}$ and $A_{23}$ upon the volume fraction of fibers ranging from 0.3 to 0.75. The least squares cubic polynomial fits to the computed values of these variables are listed as insets in the Figures. It is clear that values of all four parameters rapidly increase with an increase in the value of the volume fraction of fibers.

![Graph](image_url)

Fig. 2-23: Variation with the fiber volume fraction of parameters $A$ and $N$ appearing in the strain hardening expression for the composite.
Fig. 2-24: Variation with the fiber volume fraction of the parameters $A_{12}$ and $A_{23}$ appearing in the yield surface.

5. Conclusions

We have used four micromechanical theories, namely, the Mori-Tanaka, Eshelby’s equivalent inclusion, the free shear traction method proposed by Robertson and Mall that is a modification of Aboudi’s method of cells, and Hill’s equivalent energy principle, to find values of material parameters for elasto-plastic deformations of AS4/PEEK composite. The elasto-plastic deformations of the matrix are modeled by two yield criteria - the von Mises that stipulates that plastic deformations are independent of the hydrostatic pressure and the Drucker-Prager that includes dependence of the yield surface upon the hydrostatic pressure. Whereas the von Mises yield surface is quadratic in deviatoric stresses the other one has a linear pressure term.

It is found that for fiber volume fraction, $v_f$, varying from 0.3 to 0.6 the four micromechanics approaches provide noticeably different values of the elastic moduli. For $v_f = 0.6$, none of the four theories give values of all five elastic parameters that are close to the

$$A_{23} = 158.0(VF)^3 - 164.4(VF)^2 + 58.44(VF) - 1.5$$

$$A_{12} = 135.8(VF)^3 - 138.8(VF)^2 + 47.77(VF) - 0.5$$
experimental results available in the literature. We note that the difference in the minimum and
the maximum values of Young’s modulus in the direction transverse to fibers found
experimentally by three research groups differ by 20%. Along the fibers, all four theories give
values of Young’s modulus that are very close to each other and can also be found by using the
rule of mixtures. When displacements are applied on the surfaces of a representative volume
element (RVE) to produce a state of simple shear in the lamina plane, the value of the in-plane
shear modulus deduced from Hill’s equivalence energy principle is significantly higher than that
provided by the other three micromechanics approaches as well as the literature experimental
value.

For uniaxial elasto-plastic deformations of the composite lamina with fibers making an
angle \( \theta \) with the axis of loading, the axial stress versus the axial strain curves computed with the
von Mises yield criterion for the PEEK agree better with the experimental results of Weeks and
Sun than those derived by assuming that the yield criterion is pressure dependent. Assuming that
the plastic deformations of the composite are also independent of the hydrostatic pressure, we
have found values as a function of the volume fraction of fibers of the two material parameters in
the yield surface and the two material parameters that characterize strain hardening of the
composite.

The pressure independent yield surface for the AS4/PEEK composite studied herein is given by

\[
\bar{\sigma} = \sqrt{3F(\sigma_y)} = A + N \ln\left( \bar{\sigma}^p \right)
\]

where

\[
F(\sigma_y) = \frac{1}{2} \left[ (\sigma_{22} - \sigma_{33})^2 + 2A_{23}\sigma_{33}^2 + 2A_{12}(\sigma_{33}^2 + \sigma_{12}^2) \right]
\]

\[
A = 2.597(VF)^3 - 1.473(VF)^2 + 0.116(VF) + 0.396
\]

\[
N = 0.107(VF)^3 + 0.008(VF)^2 - 0.050(VF) + 0.039
\]

\[
A_{23} = 158.0(VF)^3 - 164.4(VF)^2 + 58.44(VF) - 1.5
\]

\[
A_{12} = 135.8(VF)^3 - 138.8(VF)^2 + 47.77(VF) - 0.5
\]

and VF equals the volume fraction of the fibers between 0.3 and 0.75. By following a procedure
similar to that used here and simulating homogeneous deformations of the UFPC at different
strain rates one can quantify the dependence of the yield stress of the UFPC upon the strain rate characteristics of the fiber and the matrix.

6. Appendix: comparison of results for elasto-plastic deformations by the M-T and the FST techniques

We compare results for elasto-plastic deformations derived by using the incremental M-T and the FST methods. The algorithm for the M-T method has been given by Pettermann et al. [29], and we study the same composite (carbon fiber reinforced metal matrix) as that analyzed in [29]. It is assumed that only the matrix deforms plastically deformation, and it is modeled as an elasto-plastic material obeying the von-Mises yield criterion with the effective stress an affine function of the effective plastic strain. Values of material parameters for the matrix taken from [29] are listed in Table 2-2, and those for the carbon are given in Table 2-3 [29].

Table 2-2: Values of material parameters for the matrix.

<table>
<thead>
<tr>
<th>Elastic modulus</th>
<th>Poisson’s ratio</th>
<th>Yield stress</th>
<th>Hardening modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td>45 GPa</td>
<td>0.35</td>
<td>0.15 GPa</td>
<td>1.28 GPa</td>
</tr>
</tbody>
</table>

Table 2-3: Values of material parameters for the carbon fibers.

<table>
<thead>
<tr>
<th>$E_1$</th>
<th>$E_2 = E_3$</th>
<th>$G_{12} = G_{13}$</th>
<th>$v_{12} = v_{13}$</th>
<th>$v_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>224 GPa</td>
<td>14 GPa</td>
<td>14 GPa</td>
<td>0.2</td>
<td>0.25</td>
</tr>
</tbody>
</table>

In Fig. 2-25 we have plotted the axial stress versus the axial strain curves for off-axis loading of a lamina obtained by using the two approaches. In general, the results from the two method agree well with each other. However, when fibers are inclined at either 60° or 90° with the loading axis, yielding occurs at a lower value of the strain for the FST method than that with the incremental M-T approach. In Figs. 2-26 and 2-27, we have compared results for the cyclic loading along the fiber direction and the in-plane shear loading. The initial slope ($G_{23}$) of the
shear stress vs. the shear strain curve from the incremental M-T method is larger than that of the curve computed with the FST approach. The yield stress obtained from results obtained using the FST method is higher than that deduced from results of the M-T approach, and the FST method predicts a lower value of the strain hardening modulus than the input value. Thus the two micromechanics theories that give very close results for axial deformations of a composite lamina do not provide close values of the yield stress and the hardening modulus for in-plane shear deformations.

Fig. 2-25: For $v_f = 0.3$, comparison of the axial stress versus the axial strain curves for off-axis loading computed using the incremental M-T and the FST methods.
Fig. 2-26: For $v^f = 0.4$, comparison of the axial stress versus the axial strain curves for cyclic loading from the incremental M-T and the FST methods.

Fig. 2-27: For $v^f = 0.4$, comparison of the shear stress versus the shear strain curves computed using the incremental M-T and the FST approaches.
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CHAPTER 3: Damage and Failure during Impact Loading of Unidirectional Fiber-reinforced Laminates

Abstract

We analyze the damage initiation, damage progression, and failure during 3-dimensional deformations of a unidirectional fiber reinforced laminated composite impacted by a rigid sphere, and compare computed results with the experimental findings available in the literature. Damage is assumed to initiate when one of Hashin’s failure criteria is satisfied, and the damage evolution is modeled by an empirical relation proposed by Matzenmiller, Lubliner and Taylor. Instead of degrading values of the elastic parameters at a material point we account for the effect of the damage evolved by appropriately enhancing values of stresses there. The transient problem is solved by the finite element method (FEM) using 8-node brick elements with 1-point integration rule, hour-glass control and the explicit conditionally stable central-difference algorithm for integrating the coupled ordinary differential equations obtained from the Galerkin formulation of the problem. A user defined subroutine has been developed and implemented in the commercial FE software ABAQUS. From the strains supplied by ABAQUS it uses the free shear traction technique of Robertson and Mall (who relaxed the requirement of shear traction continuity in Aboudi’s method of cells) and values of material parameters of the constituents to find average stresses in a FE, and checks if Hashin’s failure criteria are satisfied. If damage has initiated in the material, the subroutine also evaluates the damage developed, computes the resulting stresses, and provides them back to ABAQUS. The irreversibility of the damage evolved is considered by requiring that the damage evolved does not decrease during unloading. The delamination failure mode is simulated by using the cohesive zone model (CZM) and the Benzegagh-Kenane criterion for the degradation of material properties in the CZM available in ABAQUS. The computed time histories of the axial load acting on the impactor are found to agree well with the experimental one available in the literature, and various damage and failure modes agree qualitatively with those observed in tests.

Keywords: micro-mechanics, damage, failure, composite laminate, impact
1. Introduction

Unidirectional fiber reinforced composites (UFPCs) are being increasingly used in automotive, aerospace, and defense industries because of their higher specific strength than those of metallic parts. Furthermore, they can be engineered to obtain optimal material properties in desired directions. One of the challenging issues in designing a UFPC is delineating various failure modes, such as fiber breakage, matrix cracking, fiber/matrix debonding, fiber kinking, and delamination between adjacent plies. The difficulty of the problem is evidenced by the fact that in the worldwide exercise conducted by Soden et al. [1] very few theories could successfully predict failure of composite coupons deformed quasi-statically. In general, the load carrying capacity of a structure does not become zero as soon as either failure or damage ensues at one of its material points and the structure can support additional load before it eventually fails. Thus it is important to quantify damage caused by the initiation of a failure mode and study the development and progression of damage and the eventual failure of a structure as the applied load is increased. For designing UFPCs to resist impact loads it is important to understand energy dissipated in each failure mode.

Failure and damage in UFPCs can be studied either by using a micro-mechanics approach that considers constituent level material properties or a continuum damage mechanics approach in which material properties of the composite have been homogenized and the lamina can be regarded as a homogeneous and anisotropic body. Generally the micro-mechanics approach requires considerable computational resources and is impractical for a real size problem. A middle ground is to use a micromechanics approach to deduce effective properties of a UFPC and analyze deformations of the structure by regarding it as a continuous body. We adopt this technique here to analyze the response of a laminated composite plate under impact loading. We note that the worldwide exercise work summarized by Soden et al. [1] lists numerous references describing failure theories and their predictive capability, and a person interested in UFPCs should study that work and references listed therein.

Togho and Weng [2] have used a statistical approach based on Weibull’s distribution of inclusions and the assumption that the inclusion carries no load after it has debonded from the matrix; they thus generalized Mori-Tanaka’s micromechanics method of deriving effective
properties of a UFPC to include the effect of fiber/matrix debonding. In Sun et al.’s [3] micromechanics-based approach the effect of progressive debonding is considered by gradually reducing the elastic constants of the inclusions. Nguyen et al. [4] modeled the debonding process by reducing strengths of the interface between the inclusion and the matrix. Meraghni et al. [5] and Desrumaux et al. [6] have studied the combined effects of micro-cracks and debonding on the effective properties of a composite.

Continuum damage theories try to capture effects of microscopic damage by using essentially Coleman and Gurtin's theory of internal variables [7]. Ladeveze and Dantec [8] have used this approach to degrade elastic properties of the composite due to fiber breakage and matrix cracking, and a plasticity theory to account for permanent deformations induced under shear loading. Hassan and Batra [9] have used three internal variables to characterize damage due to fiber breakage, matrix cracking and fiber/matrix debonding. The delamination between adjacent plies was analyzed by using a failure surface that is quadratic in the transverse normal and the transverse shear stresses. Puck and Schurmann [10] have generalized Hashin’s [11] stress-based failure criteria, and have proposed techniques to degrade elastic parameters of the UFPC subsequent to the initiation of a failure mode. Donadon et al. [12] have used a smeared crack approach to develop a failure model for predicting damage in three-dimensional (3-D) deformations of a composite structure. Clegg et al. [13] have considered plastic deformations of a composite by assuming a yield surface quadratic in stresses, and have defined a damage surface in terms of stresses to consider damage induced softening. The evolution of damage variables is expressed in terms of a critical strain, fracture energy, fracture stress and a local characteristic dimension which should help minimize the dependency of computed results upon the finite element (FE) mesh used to numerically solve a problem. Ma and Cheng [14] have employed the Ramberg-Osgood plasticity relation to account for nonlinear response of a composite under in-plane shear loads, and considered three failure modes, namely, fiber breakage, matrix cracking and interface debonding while studying the initiation and propagation of damage in a composite plate with a circular hole.

Matzenmiller et al. [15] have proposed that when one of Hashin’s failure criteria is satisfied at a point in a composite structure, damage ensues at that point and it can be
characterized by introducing five internal variables – one for each of the five failure criteria, namely, fiber breakage in tension and compression, matrix cracking in tension and compression, and crushing. These internal variables evolve depending upon values of stresses in Hashin’s failure criteria which are expressed in terms of stress invariants for a transversely isotropic body and the strength parameters for the composite. Values of the three damage variables depend upon the values of these five variables, and influence the subsequent values of material elastic constants. Alternatively, the damage variables can be used to modify the six stress components that are used to characterize future deformations of the material point. Xiao et al. [16] have used this approach to study damage during quasistatic punch test for woven fabric composites, and Williams and Vaziri [17] have evaluated the effectiveness of this technique for studying damage in carbon fiber reinforced plastics under impact loads.

Here we use Matzenmiller et al.’s [15] damage evolution criteria for studying 3-D deformations of a 16-ply UFPC laminate impacted at normal incidence by a rigid sphere, and derive effective elastoplastic properties of the composite by using the free shear traction (FST) method of Robertson and Mall [18] who relaxed the requirement of the continuity of shear tractions in Aboudi’s [19] method of cells. The matrix is assumed to deform elastoplastic and the fibers elastically. A user defined subroutine has been developed and implemented in the commercial FE software ABAQUS. This subroutine takes as input from ABAQUS values of the six strain components at an integration point of a FE, computes effective stresses there by using the constituent level properties, checks for material failure due to Hashin’s criteria, computes damage, modifies stresses due to the induced damage, and supplies them to ABAQUS. In the explicit algorithm adopted here values of effective stresses suffice to find forces due to internal stresses. Accelerations can be computed from the difference between the applied forces and forces due to internal stresses. The delamination between adjacent plies is characterized by using the cohesive zone model (CZM) available in ABAQUS. The computed time histories of the axial force experienced by the sphere are found to agree well with the experimental results of Curson et al. [20]. The initiation and the propagation of failure modes agree qualitatively with those reported by Curson et al.
The rest of the paper is organized as follows. Section 2 describes the governing equations, and a weak formulation of the problem is derived in Section 3 that also briefly outlines the time integration scheme. Hashin’s failure criteria and the damage evolution approach of Matzenmiller et al. are described in Sections 4 and 5, respectively. The element deletion criterion is listed in Section 6, and Section 7 reviews the CZM approach for characterizing delamination between adjacent plies. The yield surface for modeling plastic deformations of the matrix is given in Section 8. The micromechanics approach for finding effective stresses from strains at a point is described in Section 9. We have summarized in Section 10 the solution algorithm and how the developed subroutine interacts with the FE software ABAQUS. Computed results for quasistatic deformations of a composite laminate with different fiber orientations are compared with the corresponding experimental ones of Weeks and Sun [21] in Section 11. This Section also compares computed results for the impact loading of a composite laminate with those of Curson et al. [20]. Conclusions of the work are summarized in Section 12.

2. Problem Formulation

A schematic sketch of the problem studied is exhibited in Fig. 3-1 and resembles that employed by Curson et al. [20] in their experiments on studying the impact response of composite laminates. The problem formulation and the solution algorithm are applicable to study impact of a composite laminate except for the boundary conditions described below that are specific to the experimental set up of Curson et al. A square \( n \)-ply composite laminate of side, \( L \), and thickness, \( h \), rests on a square steel plate of the size of the laminate except that the steel plate has a circular hole of diameter, \( D \), at its center. We use global rectangular Cartesian coordinate axes \( (x, y, z) \) to describe deformations of the laminate impacted at normal incidence by a steel sphere of radius, \( R \). We note that Curson et al. [20] employed a hemi-spherical nosed cylindrical impactor in their experiments and observed that it did not deform much; here we regard the impactor to be rigid. Since the contact area is anticipated to depend upon the radius of the hemi-spherical nose of the projectile, and the damage induced in the laminate upon the kinetic energy of the impactor, replacing the cylindrical projectile by a spherical one should not significantly affect the computed damage in the laminate.
Recall that fibers exhibit brittle failure at an axial strain of approximately 1% and the matrix may deform elasto-plastically. Accordingly we assume that elasto-plastic deformations of the laminate prior to deleting an element are infinitesimal, and the material properties in each lamina have been homogenized. Furthermore, the lamina material can be modeled as transversely isotropic with the fiber axis as the axis of transverse isotropy. Thus deformations of the composite laminate are governed by

$$\rho \ddot{u}_i = \sigma_{ij} + \rho b_i, \ i=1,2,3 \tag{2.1}$$

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \tag{2.2}$$

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{2.3}$$

$$d\varepsilon_{ij} = d\varepsilon_{ij}^c + \dot{\varepsilon}_{ij}^p \tag{2.4}$$

Here $\rho$ is the mass density, $u_i$ the displacement of a material point along the $x_i$-axis, $u_{ij} = \frac{\partial u_i}{\partial x_j}, \varepsilon_{ij}$ the infinitesimal strain tensor, $\sigma_{ij}$ the stress tensor, $C_{ijkl}$ the tensor of material elasticities, $b_i$ the body force per unit mass, a superimposed dot indicates the material time derivative, and $d\varepsilon_{ij}^c$ and
are the elastic and the plastic parts of the strain increment \( \varepsilon_{ij} \). We note that neither \( p_{ij} \) nor \( d_{ij} \) can be found from the gradients of the displacement increment \( du_i \). However, \( d_{ij} \) is determined from \( du_i \) by using Eq. (2.3).

We use the micromechanics approach based on the FST technique of Robertson and Mall to study elasto-plastic deformations of the composite; the method is a slight modification of Aboudi’s [19] method of cells and is briefly reviewed in Section 8.

In order to complete the problem formulation, we need to specify the yield function, the flow rule, the failure criterion, the damage parameters, evolution laws for damage parameters, contact conditions at interfaces between different materials, and initial and boundary conditions. We assume that initially the laminate particles are at rest and initial displacements are null. That is,

\[
\begin{align*}
    u_i(x,0) &= 0, \dot{u}_i(x,0) = 0 \\
    &= (2.5)
\end{align*}
\]

For the boundary conditions we take

\[
    f_i = \sigma_{ij} n_j = 0, x_1 = 0, L, x_2 = 0, L
\]

That is, the bounding edges are traction free. On the top surface \( x_3 = h \) of the laminate, boundary conditions are either

\[
    \sigma_{3i}(x_1, x_2, h) = 0
\]

or

\[
    u_3(x_1, x_2, h) = g(x_1, x_2), f_1 = -\mu_1 \sqrt{\frac{u_1}{|u_1|}}, f_2 = -\mu_2 \sqrt{\frac{u_2}{|u_2|}}
\]

\[
    (2.7a,b,c,d)
\]

The boundary condition (2.7a) holds at a point that is not contacting the impactor, and boundary conditions (2.7b-d) apply to laminate points contacting the impactor. The function \( g(x_1, x_2) \) is such that the deformed shape of the laminate surface conforms to the shape of the impactor, and is to be determined as a part of the solution of the problem. Since material properties in the \( x_1 \)- and the \( x_2 \)-directions may be different, therefore \( g(x_1, x_2) \) need not be a symmetric function of \( x_1 \) and \( x_2 \); i.e., the contact surface in the \( x_1 \) \( x_2 \) – plane is an ellipse. Similarly, the coefficients of
friction $\mu_1$ and $\mu_2$ in the $x_1$- and the $x_2$-directions may be different. The normal surface traction, $f_i$, is also to be determined as a part of the solution of the problem. Because of infinitesimal deformations considered, we have taken the normal to the deformed surface under the indenter to be along the $x_3$-axis as is done in the Hertz contact theory. For a given depth of penetration, the function $g(x_1, x_2)$ can be found by requiring that the deformed shape of the laminate prior to failure match with that of the impactor. Subsequent to the failure initiation, we require that there is no interpenetration between the laminate and the impactor. The boundary condition on the bottom surface $x_3 = 0$ of the laminate is

$$
either u_i(x_1, x_2, 0, t) = 0 \text{ or } \sigma_{i3}(x_1, x_2, 0, t) = 0$$

(2.8a,b)

We regard the bottom steel plate to be rigid. The boundary condition (2.8a) holds when the particle is contacting the steel plate and we impose the boundary condition (2.8b) when the particle has lifted off from the plate.

At an interface between two adjoining layers, prior to delamination, we assume that

$$\lbrack u \rbrack = 0, \lbrack f \rbrack = 0$$

(2.9)

where $\lbrack u \rbrack$ equals the difference in the values of $u$ at the common point of the two layers. These ensure that the displacements and the surface tractions are continuous across the two layers, or equivalently, the two layers are perfectly bonded to each other. Subsequent to delamination, the relation between $\lbrack f \rbrack$ and $\lbrack u \rbrack$ of the type given in Section 7 needs to be specified.

At the risk of repetition, we emphasize that except for the initial and the boundary conditions specified for the problem experimentally studied by Curson et al. [20], the material presented in Sections 2 through 10 is applicable to any transient problem involving impact and failure of UFPC laminates.

3. Weak Formulation of Governing Equations and the Time Integration Scheme

Let $\varphi_i (i = 1, 2, 3)$ be a smooth real-valued function defined on the domain, $\Omega$, occupied by the body in the reference configuration such that $\varphi_i = 0$ on the part $\Gamma_u$ of the boundary where
displacements are prescribed. Taking the inner product of both sides of Eq. (2.1) with \( \phi_i \),
integrating the resulting equation on \( \Omega \), and using the divergence theorem, we get
\[
\int_{\Omega} \rho \ddot{u}_i \phi_i d\Omega = \int_{\Gamma_f} f_i \phi_i dA - \int_{\Omega} \sigma_{ij} u_{ij} d\Omega + \int_{\Omega} \rho b_i \phi_i d\Omega
\]
where surface traction, \( f \), is prescribed on the part \( \Gamma_f = \partial \Omega - \Gamma_0 \) of the boundary \( \partial \Omega \) of the domain \( \Omega \). If we interpret \( \phi_i \) as a virtual displacement then the left-hand side of Eq. (3.1) equals the work done by inertia forces. The first, the second and the third terms on the right-hand side of Eq. (3.1) equal, respectively, the work done by surface tractions applied on \( \Gamma_f \), the internal stresses, and the body force. Thus one can regard Eq. (3.1) as a statement of the principle of virtual work. Henceforth we take \( b = 0 \). We note that the displacement \( u_i \) in Eq. (3.1) depends upon the position \( x \) of a material particle of \( \Omega \) and the time \( t \), but the test function \( \phi \) depends only upon \( x \).

In terms of the basis functions \( \phi_a \) \( (a = 1,2,...,N) \) defined on \( \Omega \), we write
\[
\begin{align*}
    u_i(\mathbf{x},t) &= d_{ai} \phi_a(\mathbf{x}) \\
    \phi_i(\mathbf{x}) &= c_{ai} \phi_a(\mathbf{x})
\end{align*}
\]
where summation on the repeated index \( a \) is implied. Because of the retention of only \( N \) terms in Eq. (3.2), the right-hand sides of Eqs. (3.2a) and (3.2b) are approximations of functions \( u_i(\mathbf{x},t) \) and \( \phi_i(\mathbf{x}) \), respectively. Substitution from Eq. (3.2) into Eq. (3.1) and recalling that Eq. (3.1) holds for all choices of \( \phi \) and hence \( c_{ai} \), we obtain the following set of coupled ordinary differential equations for \( d_{ai} \).
\[
\begin{align*}
    M_{ab} \ddot{d}_b &= F_{ai}^{ext} - F_{ai}^{int} \\
    M_{ab} = \int_{\Omega} \rho \phi_a \phi_b d\Omega
\end{align*}
\]
\[
\begin{align*}
    F_{ai}^{ext} &= \int_{\Gamma_f} f_i \phi_a dA \\
    F_{ai}^{int} &= \int_{\Omega} \sigma_{ij} \phi_{ai} d\Omega
\end{align*}
\]
In Eq. (3.3), \( M \) is the mass matrix, and \( F^{ext} \) and \( F^{int} \) are, respectively, due to the applied external forces and internal stresses. Note that \( F^{int} \) depends upon \( d \) since \( \sigma \) in Eq. (3.3d) is a function of \( u \) and hence of \( d \).
Here we use the finite element (FE) basis functions, and hence the finite element method (FEM) to solve the problem. That is, we divide the domain $\Omega$ into a collection $\{\Omega_e\}$ of non-overlapping sub-domains called FEs. On each element, $\Omega_e$, we select special points, called nodes. The collection of elements and nodes is called the FE mesh. For the problem studied here, $\Omega_e$ is an 8-node rectangular parallelipiped, and $\varphi_a$ is a polynomial defined piecewise on $\Omega$ such that $\varphi_a$ equals 1 at node $a$ and zero at the remaining nodes. Furthermore, $\varphi_a$ vanishes on those faces of elements that do not pass through node $a$. Because $\varphi_a(x^{(b)}) = \delta_{ab}$, $u_i(x^{(b)}, t) = d_{bi}(t)$. Here $x^{(b)}$ represents the position vector for node $b$. Thus $d_{bi}$ equals the displacement of node $b$ in the $x_i$-direction.

The integrals appearing in Eq. (3.3b-d) can be written as

$$
M_{ab} = \sum_e \int_{\Omega_e} \rho \varphi_a \varphi_b d\Omega = \sum_e M^{(e)}_{ab}, \quad F^{a\text{ext}}_{ai} = \sum_e \int_{\Omega_e \cap \Gamma_j} f_i \varphi_a dA = \sum_e F^{a\text{ext}}_{ai}
$$

$$
F^{a\text{int}}_{ai} = \sum_e \int_{\Omega_e} \sigma_{ij} \varphi_{a,j} d\Omega = \sum_e F^{a\text{int}}_{ai}
$$

(3.4)

Here $M^{(e)}_{ab}$ is the mass matrix for element $e$, and the summation is over all elements in the FE mesh. Generally integrals in Eq. (3.4) are numerically evaluated using a Gauss quadrature rule, and the accuracy of the results depends upon the number of Gauss integration points used.

Thus the problem of finding an approximate solution of the initial-boundary-value problem (IBVP) defined by Eqs. (2.1) through (2.9) has been reduced to that of integrating the system of coupled nonlinear ordinary differential Eqs. (3.3) subject to the following initial conditions:

$$
M_{ab} d_{bi}(0) = F_{ai}(0), \quad M_{ab} \dot{d}_{bi}(0) = F^{d}_{ai}(0),
$$

(3.5a,b)

where

$$
F_{ai}(0) = \int_{\Omega} \rho \varphi_{a, i}(x, 0) d\Omega, \quad \bar{F}_{ai}(0) = \int_{\Omega} \rho \varphi_{a, i}(x, 0) d\Omega
$$

(3.5c,d)

Eqs. (3.5a) and (3.5b) are derived, respectively, from Eq. (3.2) by multiplying both sides with $\rho \varphi_a$, substituting for $u_i(x, t)$ from Eq. (3.2) and integrating the resulting equations over $\Omega$. 

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The central difference method is used to integrate with respect to time $t$ the coupled nonlinear ordinary differential Eqs. (3.3) subject to the initial conditions (3.4). The time interval $[0,T]$ of interest is divided into several times steps $[0,t_1],[t_1,t_2],...$, and for the time step $[t_n,t_{n+1}]$ the solution $\mathbf{d}_{(n+1)}$ at time $t_{n+1}$ is expressed in terms of the solution $\mathbf{d}_{(n)}$ at time $t_n$ as follows:

$$
\begin{align*}
\mathbf{d}_{(n+1)} &= \mathbf{d}_{(n)} + \Delta t \mathbf{d}_{(n)} + \frac{\Delta t^2}{2} \mathbf{\ddot{d}}_{(n)}, \\
\mathbf{\ddot{d}}_{(n+1)} &= \mathbf{\ddot{d}}_{(n)} + \frac{\Delta t}{2} (\mathbf{\ddot{d}}_{(n)} + \mathbf{\ddot{d}}_{(n+1)}),
\end{align*}
$$

(3.6a,b,c)

Here $\mathbf{d}_{(n+1)}$ denotes values of nodal displacements at time $t_{n+1}$. For a 3-D problem and the domain $\Omega$ having $N$ nodes, $\mathbf{d}$ is a $3N$-D vector. In Eq. (3.6c) $\mathbf{F}^\text{int}_{(n+1)}$ is unknown since it depends upon stresses evaluated at time $t_{n+1}$ which in turn are functions of $\mathbf{d}_{(n+1)}$. In the time step $[t_n,t_{n+1}]$, Eq. (3.6) is solved iteratively till the computed solution has converged within the prescribed tolerance $\delta_{tol}$. That is, in the $k^{th}$ iteration in the time step $[t_n,t_{n+1}]$, Eq. (3.3a) is written as

$$
\mathbf{M} \mathbf{\ddot{d}}_{(n+1)}^{(k)} = \mathbf{F}_{(n+1)}^{\text{ext}(k)} - \mathbf{F}_{(n+1)}^{\text{int}(k)}
$$

(3.7)

where $\mathbf{F}_{(n+1)}^{\text{int}(k)}$ is evaluated from stresses computed with displacements $\mathbf{d}_{(n+1)}^{(k)}$. The iterative process is stopped when

$$
\frac{\|\mathbf{F}_{(n+1)}^{\text{int}(k)} - \mathbf{F}_{(n+1)}^{\text{int}(k-1)}\|}{\|\mathbf{F}_{(n+1)}^{\text{int}(k)}\|} \leq \delta_{tol} \quad \|\mathbf{F}\| = F_{ai} F_{ai}
$$

(3.8a,b)

The mass matrix $\mathbf{M}$ is diagonalized by the row sum technique. That is,

$$
M_{aa} = \sum_{b=1}^{N} M_{ab}, \text{ no sum on } a
$$

(3.9a,b)

$$
M_{ab} = 0 \text{ if } a \neq b
$$

For diagonal $\mathbf{M}$ Eq. (3.6c) can be easily solved and the explicit algorithm so obtained is conditionally stable. The time step, $\Delta t = t_{n+1} - t_n$, must be less than the time taken for the fastest wave to travel through the smallest length of any element in the FE mesh. Thus as an element, $\Omega_e$, gets distorted during deformations of the body, its minimum dimension controls the time step size which can tremendously increase the CPU time. This can be avoided by either remeshing the deformed region or by deleting severely deformed elements. The former approach requires mapping values of solution variables from nodes of the old mesh to those of the new mesh.
mesh, and the latter introduces a void of the size of $\Omega_c$ in the region. Here we adopt the second approach and discuss below the damage/failure initiation and the element deletion criteria.

The essential (displacement) boundary conditions prescribed on $\Gamma_u$ are satisfied by modifying the right-hand of Eq. (3.7) for nodes on $\Gamma_u$. For example, for node $b$ on $\Gamma_u$ with $u(x^{(b)}, t) = u^b(t)$, we replace the resultant force vector $(F_{ext}^{(a+1)} - F_{int}^{(a+1)})$ by $M_{ba}u^b$ (no sum on $b$).

4. Damage and Failure Initiation Criteria

Hashin [11] assumed that a UFPC can be modeled as a homogeneous linear elastic transversely isotropic body with the fiber axis as the axis of transverse isotropy. He proposed failure criteria in terms of the following five stress invariants:

$$I_1 = \sigma_{11}, I_2 = \sigma_{22} + \sigma_{33}, I_3 = (\sigma_{22} - \sigma_{33})^2 + 4\sigma_{23}^2$$

$$I_4 = \sigma_{12}^2 + \sigma_{23}^2, I_5 = 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2$$

Here a bar superimposed on a stress component implies that its value is with respect to the material principal axes with the $\bar{x}_1$-axis aligned along the fiber axis, $\bar{x}_2$-axis perpendicular to the $\bar{x}_1$-axis but in the plane of the lamina and the $\bar{x}_3$-axis perpendicular to the plane of the lamina. The failure criteria are assumed to be at most quadratic in stresses; hence they do not depend upon $I_5$. We note that

$$4\sigma_{23}^2 + (\sigma_{22} - \sigma_{33})^2 - (\sigma_{22} + \sigma_{33})^2 = 4(\sigma_{23}^2 - \sigma_{22}\sigma_{33})$$

Thus $(\sigma_{23}^2 - \sigma_{22}\sigma_{33})$ is also a stress invariant. We list below Hashin’s five failure criteria in terms of the failure indices $f_i (i = 1, 2, 3, 4, 5)$ with the understanding that the failure initiates when the corresponding failure index equals 1.0.

(i) Fiber tensile failure index

$$f_i = \left( \frac{\sigma_{11}}{X_T} \right)^2 + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2}, \sigma_{11} > 0$$

$$f_i > 1$$
(ii) Fiber compressive failure index

\[ f_z \equiv \frac{\left| \sigma_{12} \right|}{X_c}, \sigma_{11} < 0 \]  

(4.4)

(iii) Lamina crush index

\[ f_z \equiv \frac{\left| \sigma_{33} \right|}{Z_c}, \sigma_{33} < 0 \]  

(4.5)

(iv) Matrix tensile failure index

\[ (f_d)^2 \equiv \left( \frac{\sigma_{22} + \sigma_{33}}{Y_T} \right)^2 + \frac{\sigma_{23}^2 - \sigma_{22} \sigma_{33}}{S_T^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2}, (\sigma_{22} + \sigma_{33}) > 0 \]  

(4.6)

(v) Matrix compressive failure index

\[ (f_d)^2 \equiv \left( \frac{Y_C}{2S_T} - 1 \right)^2 \left( \frac{\sigma_{22} + \sigma_{33}}{Y_C} \right)^2 + \left( \frac{\sigma_{22} + \sigma_{33}}{4S_T} \right)^2 + \frac{\sigma_{23}^2 - \sigma_{22} \sigma_{33}}{S_T^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2}, (\sigma_{22} + \sigma_{33}) < 0 \]  

(4.7)

In these equations $X_T(Y_C)$ is the axial tensile (compressive) stress along the $\bar{x}_T$ axis. Furthermore, $Z_c$ is the lamina crush strength, $Y_T(Y_c)$ and $S_T$ are, respectively, the transverse tensile (compressive) strengths along the $\bar{x}_T$ axis and the shear strengths in the $\bar{x}_2\bar{x}_3$ -plane. The shear strength in either the $\bar{x}_j\bar{x}_3$ - plane or the $\bar{x}_j\bar{x}_2$ - plane is denoted by $S$. We note that $f_2 = 1$ does not imply fiber failure due to kinking and buckling – these failure modes are not considered here. $\sigma_{33} > 0$ generally leads to delamination and is considered separately in Section 7.

5. Damage Evolution

Matzenmiller et al. [15] postulated that when one of the five failure indices in Eqs. (4.3)-(4.7) equals at least 1, the failure initiates at that point but the material surrounding that point has not necessarily lost all of its strength. They assumed that the initiation of failure as indicated by
Hashin’s failure criteria is synonymous with the initiation of damage at that point. The material loses its load carrying capacity when the damage accumulated at a material point has reached a critical value. The damage evolution at a material point is defined in terms of an internal variable \( Q_\alpha \) \((\alpha = 1, 2, 3, 4, 5)\) associated with the failure index \( f_\alpha \) by the following empirical relation.

\[
Q_\alpha = 1 - \exp \left[ \frac{1}{m} (1 - f_\alpha) \right], \quad (\alpha = 1, 2, 3, 4, 5); \quad f_\alpha \geq 1
\]  

(5.1)

The value of the parameter \( m \) controls the damage evolution rate; a large (small) value of \( m \) implies that the damage evolves quickly (slowly). In the absence of the availability of the pertinent test data, the value of \( m \) is chosen by trial and error.

One way to consider the effect of the damage at a point is to degrade (or reduce) values of the elastic parameters at that point. Alternatively, one can enhance values of stresses there because the effective area supporting surface tractions is less than the geometric area because of the damage induced. The two approaches will have similar consequences when one is using a stress based failure criteria like Eqs. (4.3) - (4.7). Here we adopt the second approach and enhance stresses as follows:

\[
\hat{\sigma} = D\sigma, \quad \hat{\sigma} = D\tilde{\sigma}
\]  

(5.2a,b)

\[
\langle D \rangle = \left\langle (1 - \omega_1)^{-l}, (1 - \omega_2)^{-l}, (1 - \omega_3)^{-l}, (1 - \omega_4)^{-l}, (1 - \omega_5)^{-l}, (1 - \omega_6)^{-l} \right\rangle
\]  

(5.3)

\[
\omega_1 = \omega_2 = \text{Max}(Q_1, Q_2, Q_3),
\]

\[
\omega_2 = \omega_4 = \text{Max}(Q_3, Q_4, Q_5),
\]

\[
\omega_5 = Q_5,
\]

\[
\omega_6 = \text{Max}(Q_1, Q_2, Q_3, Q_4, Q_5)
\]  

(5.4a-f)

Here \( D \) is a diagonal matrix, and the right-hand side of Eq. (5.3) gives the diagonal elements of \( D \). Furthermore, \( \sigma \) is written as a 6-dimensional vector \( \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\} \). The motivation for definitions (5.4a-f) of the six damage variables \( \omega_1, \omega_2, ..., \omega_6 \) is given in [22].

The irreversibility of the internal variable is accounted for by requiring that \( dQ_\alpha = 0 \) whenever \( df_\alpha \leq 0 \) where \( df_\alpha \) represents an increment in the value of the failure index \( f_\alpha \) when
the time is increased from \( t_n \) to \( t_{n+1} \). Eq. (5.4a-f) implies that more than one failure index influences the value of a damage parameter; for example, the value of \( Q_j \) affects all six components of the damage vector \( \omega \). Thus even if an internal variable does not increase, the value of a damage parameter may increase.

The effect of the damage evolved on the subsequent deformations of a material point and on the value of the failure index \( f_a \) is considered by replacing \( \sigma \) and \( \sigma \) in Eqs. (4.3)-(4.7) and (3.6c) by \( \hat{\sigma} \) and \( \hat{\sigma} \), respectively. That is \( F_{(n+1)}^{\text{int}} \) in Eq. (3.6c) is evaluated by using \( \hat{\sigma}_{(n+1)} \) rather than \( \sigma_{(n+1)} \).

6. Element Deletion Criterion

In the numerical evaluation of integrals appearing in Eq. (3.5) one needs values of stress, \( \hat{\sigma}_{ij} \), at the integration points which depend upon values of the damage variables there. An element is assumed to have failed when at least one of the five damage variables \( Q_1, Q_2, \ldots, Q_5 \) exceeds 0.95 and either the ratio of the final volume of the element to its initial volume is less than 0.1 or more than 4.0 or the axial strain along the fiber direction equals at least 0.05. We have followed the work of Xiao et al. [16] in adopting this element deletion criterion. Thus for an element to be deleted not only one of the damage variables must exceed 0.95 but one of the three auxiliary conditions outlined above must also be satisfied.

7. Cohesive Zone Model for Delamination

The delamination between adjacent layers usually results from the mismatch between the material properties of the adjacent layers, and strongly depends upon the strength of the interface between the two layers. Here we use the delamination criterion included in Abaqus. Accordingly, the description of the criterion follows that given in Abaqus theory manual. Similar to Hashin’s failure criteria described in Section 4, we assume that the delamination ensues at a material point on the interface between two contacting layers when the failure index, \( f_{\text{del}} \), defined by
\[(f_{rel})^2 \equiv \begin{cases} \left(\frac{\hat{\sigma}_{33}}{X_N}\right)^2 + \left(\frac{\hat{\sigma}_{31}}{X_{SL}}\right)^2 + \left(\frac{\hat{\sigma}_{32}}{X_{S2}}\right)^2, & \hat{\sigma}_{33} > 0 \\ \left(\frac{\hat{\sigma}_{31}}{X_{SL}}\right)^2 + \left(\frac{\hat{\sigma}_{32}}{X_{S2}}\right)^2, & \hat{\sigma}_{33} < 0 \end{cases} \] (7.1a,b)

equals 1. Here \(X_N\), \(X_{SL}\) and \(X_{S2}\) are, respectively, strengths of the interface along the \(\bar{x}_3\)-, the \(\bar{x}_1\)-, and the \(\bar{x}_2\)-axes. Subsequent to the initiation of delamination, the relative normal displacement, \(d_{3}^{rel}\), and the relative tangential displacements, \(d_{1}^{rel}\) and \(d_{2}^{rel}\), between the adjoining points of the two layers, are governed by
\[K_{j}d_{j}^{rel} = \hat{\sigma}_{33}, \quad K_{2}d_{2}^{rel} = \hat{\sigma}_{32}, \quad K_{1}d_{1}^{rel} = \hat{\sigma}_{31}\] (7.2)

where \(K_{j}, K_{2}\) and \(K_{3}\) can be viewed as stiffnesses of the interface along the \(\bar{x}_1\)-, the \(\bar{x}_2\)- and the \(\bar{x}_3\)-axes, respectively. Thus \(d_{3}^{rel}\) measures the relative separation between the surfaces, and \(d_{1}^{rel}\) and \(d_{2}^{rel}\) the relative sliding between them. Eq. (7.2) relates jumps in tractions and displacements when Eq. (2.9) does not hold, and implies that subsequent to the onset of delamination the jump in surface tractions gradually diminish to zero. The values of \(K_{j}, K_{2}\) and \(K_{3}\) are degraded by the same damage parameter \(\bar{D}\) defined by
\[\bar{D} = \frac{\tilde{\delta}_{m}}{\tilde{\delta}_{m}^{\max}} \left(\frac{\tilde{\delta}_{m}^{\max}}{\tilde{\delta}_{m}^{\max}}\right)\] (7.3)

where
\[\tilde{\delta}_{m}^{2} = (d_{1}^{rel})^2 + (d_{2}^{rel})^2 + (d_{3}^{rel})^2\]
\[\tilde{\delta}_{m}^{\max} = \max_{0 \leq t_{eff}} \tilde{\delta}_{m}\]
\[t_{eff}^{\infty} \tilde{\delta}_{m}^{\infty} = 2G_{IC} + 2(G_{II} - G_{IC}) \left(\frac{G_{II} + G_{III}}{G_{TOTAL}}\right)^{\eta}\] (7.4a-d)
\[t_{eff}^{2} = \begin{cases} \left(\hat{\sigma}_{31}\right)^2 + \left(\hat{\sigma}_{32}\right)^2 + \left(\hat{\sigma}_{33}\right)^2, & \hat{\sigma}_{33} > 0 \\ \left(\hat{\sigma}_{31}\right)^2 + \left(\hat{\sigma}_{32}\right)^2, & \hat{\sigma}_{33} < 0 \end{cases} \]
The damage evolution law given by Eq. (7.4) is due to Benzeggah and Kenane [23]. Furthermore, \( \delta_m^o \) and \( t_{\text{eff}}^o \) are equal, respectively, to the values of \( \delta_m \) and \( t_{\text{eff}} \) at the instant of the initiation of the delamination, and \( \eta \) is a material parameter. The new value of \( K_i \), for example, is given by

\[
K_i^{\text{dam}} = K_i (1 - D) \tag{7.5}
\]

### 8. Elastoplastic Deformations of the Matrix

We assume that fibers deform elastically and the matrix elasto-plastically. As reported by Raghava and Caddell [24], unlike most metals, plastic deformations of polymers generally depend upon the hydrostatic pressure. Here we presume that plastic deformations of the polymer (PEEK) obey the von Mises yield criterion. The von Mises potential function [28] is given by

\[
2F(\sigma_{ij}) = \frac{1}{3} (\sigma_{22} - \sigma_{33})^2 + \frac{1}{3} (\sigma_{33} - \sigma_{11})^2 + \frac{1}{3} (\sigma_{11} - \sigma_{22})^2 + 2 \sigma_{23}^2 + 2 \sigma_{13}^2 + 2 \sigma_{12}^2 \tag{8.1}
\]

Thus we have neglected the dependence of the yield surface upon the hydrostatic pressure.

We use the associative flow rule of plasticity to find \( d\varepsilon_{ij}^p \). Thus

\[
d\varepsilon_{ij}^p = d\lambda \frac{\partial F}{\partial \sigma_{ij}} \tag{8.2}
\]

where \( d\lambda \) is the proportionality (also sometimes called the consistency) parameter. The effective stress associated with the yield function given in Eq. (8.1) is defined as

\[
\bar{\sigma} = \sqrt[3]{3F(\sigma_{ij})} \tag{8.3}
\]

and the effective plastic strain, \( \bar{\varepsilon}^p \), by requiring that in any incremental deformation,

\[
d\epsilon_{ij}^p \sigma_{ij} = \bar{\sigma} d\varepsilon^p \tag{8.4}
\]

Substituting for \( d\varepsilon_{ij}^p \) from Eq. (8.2) into Eq. (8.4) and using Eq. (8.3) we get

\[
d\lambda = \frac{3}{2} \frac{d\varepsilon^p}{\bar{\sigma}} = \frac{3}{2H^m} \frac{d\bar{\sigma}}{\bar{\sigma}} \tag{8.5}
\]

where the strain-hardening modulus, \( H^m = \frac{d\bar{\sigma}}{d\varepsilon^p} \), has been assumed to be positive. The function \( H^m(\bar{\sigma}^p) \) is found from experimental data.
Taking the differential of both sides of Eq. (8.3) we get

\[ d\bar{\sigma} = \frac{3}{2\bar{\sigma}} \left( \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij} \right) d\sigma_{ij} \]  

(8.6)

Substituting for \( d\lambda \) from Eq. (8.5) and \( d\bar{\sigma} \) from Eq. (8.6) into Eq. (8.2) we get

\[ d\varepsilon_{ij}^p = \left( \frac{9}{4H^m \bar{\sigma}^2} \right) \left( \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij} \right) \left( \sigma_{pq} - \frac{\sigma_{ll}}{3} \delta_{pq} \right) d\sigma_{pq} = S_{ijpq}^p d\sigma_{pq} \]  

(8.7)

\( S^p \) may be called the plastic compliance matrix of the polymer.

Expressing incremental strains in terms of incremental stresses we have

\[ d\varepsilon = S^E d\sigma + S^p d\sigma \]  

(8.8)

where \( S^E = C^{-1} \) is the elastic compliance matrix for the polymer. Hence

\[ d\sigma = \left( S^E + S^p \right)^{-1} d\varepsilon \]  

(8.9)

which relates incremental stresses to incremental strains and the current stresses.

**9. Micromechanics Approach for Finding Stresses in a Finite Element**

The iterative solution of the initial-boundary-value problem formulated in Section 3 for the time interval \( [t_n, t_{n+1}] \) provides values of incremental displacements and hence incremental strains \( \Delta \varepsilon_{ij} \) in a FE. We use the free shear traction (FST) scheme of Robertson and Mall [18] to find the stress increment corresponding to the strain increment \( \Delta \varepsilon_{ij} \) and known values of material parameters for the matrix and the fiber as well as of the fiber volume fraction. Details of this technique are described in Chapter 2, and are omitted here. The stress increment so computed is used to update the force vector \( F_{(n+1)} \) in Eq. (3.7).

Robertson and Mall [18] modified Aboudi’s [19] method of cells by relaxing the continuity of tractions across the interfaces between four cells of which one is made of the fiber and the other three of the matrix. The ratio of the volume of the fiber cell to that of the four cells combined equals the volume fraction, \( \nu_f \), of fibers in the composite. Since we find stresses in the fiber and the matrix before finding the average stress in the FE which is divided into the four cells, we could potentially use failure and damage criteria at the constituent level. This approach could simulate well the failure of fibers and of the matrix but will not be very good for predicting
debonding of fibers from the matrix. Gardner [25] has incorporated fiber/matrix debonding in Aboudi’s [18] method of cells by inserting springs between faces of adjoining cells. The approach is similar to the cohesive zone modeling (CZM) technique discussed above for simulating delamination between adjoining layers.

10. Summary of the Solution Algorithm

The numerical solution of a given initial-boundary-value problem involves the following steps.

(i) Solve Eq. (3.7) for \( d^{(k+1)}_{(n+1)} \) and from incremental displacements \( \Delta d = d^{(k+1)}_{(n+1)} - d^{(k)}_{(n+1)} \), find incremental strains by using Eqs. (3.2a) and (2.3). Check for convergence of the solution by using Eq. (3.8). When the solution has converged, go to step (v).

(ii) Use the micromechanics approach of Section 9 to find \( \Delta \sigma_{ij} \) and hence \( \sigma_{(n+1)} \).

(iii) Find stress \( \sigma_{(n+1)} \) in material principal coordinates by using tensor transformation rules and check if any one of the five failure indices, \( f_a \), either equals or exceeds 1.

(iv) If \( f_a \geq 1 \), use Eqs. (5.1) - (5.4) to find \( \hat{\sigma}_{(n+1)} \). Return to step (i).

(v) Check for delamination between adjoining layers by finding the value of \( f_{del} \), defined by Eq. (7.1).

(vi) If the solution time \( t_{n+1} \) is less than the final time \( T \) for the computation of the solution, increment \( t_{n+1} \) by \( \Delta t \) and go to step (i). Otherwise, post-process stresses, strains and displacements to find quantities of interest.

11.1 Numerical Results and Discussion

The composite laminates studied in this work are AS4/PEEK with material properties of the fiber and the matrix listed in Tables 3-1 and 3-2, respectively. Values of strength parameters appearing in Hashin’s failure initiation criteria (4.3) – (4.7) are given in Table 3-3.
Table 3-1: Values of material parameters for the AS4 fiber [26].

<table>
<thead>
<tr>
<th>$E_1$</th>
<th>$E_2 = E_3$</th>
<th>$G_{12} = G_{13}$</th>
<th>$G_{23}$</th>
<th>$v_{12} = v_{13}$</th>
<th>$v_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>234 GPa</td>
<td>14 GPa</td>
<td>27.6 GPa</td>
<td>5.6 GPa</td>
<td>0.2</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 3-2: Values of material parameters for PEEK polymer [27].

<table>
<thead>
<tr>
<th>$E$</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 GPa</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Table 3-3: Values in GPa of strength parameters for the AS4/PEEK composite with 60% volume fraction of fibers [28].

<table>
<thead>
<tr>
<th>$X_T$</th>
<th>$X_C$</th>
<th>$Y_T$</th>
<th>$Y_C$</th>
<th>$Z_C$</th>
<th>$S_T$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>1.1</td>
<td>0.08</td>
<td>0.21</td>
<td>3.0</td>
<td>0.07</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Values of strength parameters $X_T(Y_T), X_C(Y_C)$ were obtained as the last data points reported in the experimental uniaxial tension and uniaxial compression tests with the loading along (perpendicular to) the fiber axis [28]. It is clear that the composite strength for uniaxial tensile load along the fiber axis is nearly 21% more than that for the compressive load. However, for uniaxial transverse load, the strength in compression is about 2.6 times that in tension. The tensile strength in the transverse direction is nearly $1/17^{th}$ of that in the axial direction. The value of the crush strength, $Z_C$, has not been reported by several researchers and values as high as 10 GPa have been used [29]. Here we have estimated $Z_C = 3$ GPa. A flat laminate impacted by a sphere normal to the face sheet usually does not fail due to crushing; thus the precise value of $Z_C$ is not critical for our work. The value of the shear strength parameter, $S$, is taken equal to the last data point reported in the shear stress-shear strain curve of Kyriakides et al. [28], and we have set $S_T = S$. 

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Goldberg and Stouffer’s [27] reported experimental uniaxial stress-strain curve for PEEK tested at the strain rate of $10^6$/s by Bordonaro [30]; it is reproduced in Fig. 3-2. For a given value of the axial stress, $\sigma_{11}$, the value of the axial plastic strain, $\varepsilon_{11}^p$, is computed by using the relation

$$\varepsilon_{11}^p = \varepsilon_{11} - \frac{\sigma_{11}}{E}$$

(11.1)

where $E = 4$ GPa as listed in Table 3-2 and provided by Goldberg and Stouffer [27]. Points $(\varepsilon_{11}^p, \sigma_{11})$ are plotted in Fig. 3-3. A least squares fit to these data points with a regression coefficient of 0.98 is

$$\bar{e}^p = \exp\left(\frac{\bar{\sigma} - A}{N}\right) \text{ for } 94 \text{ MPa} \geq \bar{\sigma} \geq 40 \text{ MPa}$$

(11.2)

where $A = 153$ MPa and $N = 13.86$ MPa. In Eq. (11.2) we have set $\bar{e}^p = \varepsilon_{11}^p$ and $\bar{\sigma} = \sigma_{11}$, which follow from Eqs. (8.4) and (8.5).

---

Fig. 3-2: Computed and experimental axial stress vs. axial strain curves for PEEK polymer.
Fig. 3-3: Effective stress versus effective plastic strain curve for PEEK obtained from the experimental axial stress versus axial strain curve.

11.2 Comparison of Computed Results with Experimental Findings

11.2.1 Simple quasistatic deformations

In these simulations $m$ in Eq. (5.1) is assigned the value 100. Thus the damage evolves very rapidly at a material point. There is no experimental data to determine the rate of evolution for the damage. The value of $m = 100$ is probably appropriate for the fiber breakage but may be too high for the matrix failure mode.

We have compared in Fig. 3-4a-e computed results with the experimental ones for uniaxial tensile and compressive loading along and perpendicular to the fiber axis as well as simple shear deformations in the $x_1x_2$-plane. The specimen is assumed to deform homogeneously, thus one FE is used to simulate these tests. Also the stress-strain relations, the damage criteria and the damage evolution laws have been assumed to be independent of the strain rate. Hence the loading rate does not affect the computed stresses for strains prescribed
via assigning appropriate displacements at the bounding surfaces. The developed subroutine, VUMAT, in conjunction with the commercial FE software ABAQUS is used to compute results using an 8-node brick element with one-point integration rule. The possible emergence of spurious energy modes with one-point integration rule is suppressed by using the default values of parameters in the hourglass shape functions.

Fig. 3-4a: Comparison of the experimental and the numerical axial stress versus the axial strain curves for AS4/PEEK composite for tension test along the fiber direction.
Fig. 3-4b: Comparison of the experimental and the numerical axial stress versus the axial strain curves for AS4/PEEK composite for compression test along the fiber direction.

It is clear from results depicted in Fig. 3-4a,b that the present approach mimics well the experimentally observed axial stress versus the axial strain response in tension but not so well in compression. The magnitude of the experimental axial strain at failure in tension and compression equals about 1.03%. The computed value of the axial strain at failure in tension is close to 1.03% but that in compression is about 0.8% which can be attributed to the fact that the value of $X_c$ is less than that of $X_T$. We note that there is no distinction between the uniaxial compressive and tensile loading in the method of cells and our formulation of the elasto-plastic deformations of the matrix. Young’s modulus from the experimental curve for uniaxial compressive loading is less than that for uniaxial tensile loading. The vertical drop at failure in the experimental curves is our concoction since that portion of the curve was not captured in experiments.
For uniaxial tensile and compressive loading in the transverse ($x_2$-) direction, see Fig. 3-4c,d, the computed values of the transverse axial strain at failure in compression is nearly four times that in tension primarily because $Y_c/Y_T = 2.6$ and the composite deforms elasto-plastically in compression but mostly elastically in tension. The curves from the computed results are close to those from experimental findings but the corresponding values of the strain at failure differ by about 16%. Both for the tensile and the compressive loading, the experimental value of Young’s modulus is higher than that computed from the micromechanics approach.

Fig. 3-4c: Comparison of the experimental and the numerical axial stress versus the axial strain curves for AS4/PEEK composite for tension test normal to the fiber direction.
For simple shear deformations in the $x_1x_2$-plane, there are significant plastic deformations induced in the composite. The shear stress versus shear strain curve from the computed results agrees well with that from the test data till a shear strain of 4% when the specimen failed in the tests. The computed shear strain at failure of 5.2% exceeds the experimental one by 30%.
We now simulate experiments conducted by Weeks and Sun [21] involving uniaxial loads on a lamina with unidirectional fibers inclined at an angle $\theta$ to the loading axis. These are simulated by first finding strains in the material principal axes, supplying these to the user defined subroutine, VUMAT, computing stresses by the FST method in the material principal coordinates, checking for Hashin’s failure criteria, modifying them due to the damage induced, transforming the modified stresses to the global coordinate axes, and supplying these to ABAQUS for the next iteration or the load step (time increment). Results presented in Figs. 3-5a,b,c for $\theta = 14^\circ$, $30^\circ$ and $45^\circ$ clearly show that, prior to the onset of failure, the computed axial stress versus axial strain curves agree well with the corresponding experimental ones. The maximum difference between the computed and the experimental values of the axial strain at failure is less than 20%.
Fig. 3-5a: Comparison of the experimental and the computed axial stress versus axial strain curves for AS4/PEEK composite with fibers making an angle of 14° with the loading axis.

Fig. 3-5b: Comparison of the experimental and the computed axial stress versus axial strain curves for AS4/PEEK composite with fibers making an angle of 30° with the loading axis.
11.2.2 Impact Loading of a Composite Laminate

We now simulate transient deformations of a 75 mm x 75 mm x 2 mm 16-layer $[-45/0/45/90]_{2S}$ AS4/PEEK composite plate impacted at normal incidence by a 500 g 12.7 mm diameter rigid sphere. Fig. 3-6 shows the stacking sequence of the top half of the symmetric composite plate, the bottom half is its mirror image.
Fig. 3-6: Stacking sequence of the top half of the symmetric 16 layer composite laminate.

As displayed in Fig. 3-1 the composite laminate is supported on a rigid steel plate having a 50 mm diameter circular opening. The assumption of modeling the sphere as rigid is reasonable since the 500 g 12.7 mm hemispherical nosed cylindrical impactor used in experiments underwent very little deformations. All three displacement components of the base plate are set equal to zero, and the coefficients of friction $\mu_1$ and $\mu_2$ (cf. Eq. (2.7)) between the spherical impactor and the composite laminate are set equal to 0.25. Values of material parameters in Eqs. (7.1) and (7.4) are listed below in Table 3-4; values of $G_{IC}$, $G_{IIIC}$, $G_{IIIc}$ and $\eta$ used herein are typical for carbon fiber reinforced polymer composites, see e.g. [31]. There is, in general, considerable scatter in values of strength parameters $X_T$ and $Y_T$; for example, Curson
et al. [20] reported $X_T = 2.1 \text{GPa}$ and $Y_T = 0.135 \text{GPa}$ which are higher than those listed in Table 3-3.

Table 3-4: Values of interfacial strength parameters for the composite laminate [31].

<table>
<thead>
<tr>
<th>$X_N$</th>
<th>$X_{SI} = X_{S2}$</th>
<th>$G_{IC}$</th>
<th>$G_{IIC} = G_{IIIC}$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80 MPa</td>
<td>150 MPa</td>
<td>150 J/m$^2$</td>
<td>500 J/m$^2$</td>
<td>1.7</td>
</tr>
</tbody>
</table>

Because of infinitesimal deformations of the laminate the contact force, $F_C$, acting on the rigid sphere can be defined as

$$F_c = \int_A \hat{\sigma}_{33} d\gamma_1 d\gamma_2$$

where $A_c$ is the time-dependent contact surface.

The time step size is adaptively controlled in ABAQUS by finding the maximum frequency of the structure. We further scaled it down to 25% of that computed by ABAQUS using the stability limit.

We used the default values 0.06 and 1.2, respectively, for the linear and the quadratic artificial bulk viscosities. The linear bulk viscosity damps out “ringing” in the highest element frequency and the quadratic bulk viscosity is introduced only in those elements that have compressive volume strain to prevent them from collapsing. These viscosities are introduced by adding an artificial hydrodynamic pressure to the right-hand side of Eq. (2.2).

Numerical simulations involving contact problems using ABAQUS EXPLICIT were carried out using the penalty based algorithm “GENERAL CONTACT” in which one does not have to define the "MASTER" and the "SLAVE" surfaces. The algorithm introduces an additional stiffness when it detects the penetrating body. The value of the penalty stiffness influences the time step size, and high values of the penalty stiffness not only drastically reduce
the time step size but can also lead to numerical instabilities. In all simulations the default value of the penalty stiffness has been used.

In order to ensure that we are using the code correctly and a contact problem can be analyzed with ABAQUS, we first studied the indentation by a 50 mm radius steel sphere into a 500 mm x 500 mm x 125 mm aluminum plate with both materials taken to be homogeneous and isotropic, and the contact surface between them to be smooth; value of material parameters are listed in Table 3-5 and a schematic sketch of the problem is shown in Fig. 3-7. The graded FE mesh for the plate had elements of side length 5 mm in the anticipated contact area and larger away from it. The bottom surface is rigidly clamped while other surfaces, except that contacting the sphere, are traction free.

![Fig. 3-7: Schematic sketch of the Hertzian contact problem.](image)

<table>
<thead>
<tr>
<th>$E_{\text{sphere}}$</th>
<th>$\nu_{\text{sphere}}$</th>
<th>$E_{\text{plate}}$</th>
<th>$\nu_{\text{plate}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200 GPa</td>
<td>0.25</td>
<td>10 GPa</td>
<td>0.25</td>
</tr>
</tbody>
</table>

The analytical expression [35] for the axial load $P$ in terms of the indentation depth $\delta$ is

$$\delta^3 = \left(\frac{3}{4E^*}\right)^2 P^2 R \quad (11.4)$$

where $R$ is the radius of the spherical indenter, and the effective modulus $E^*$ is given by

$$\frac{1}{E^*} = \frac{1-\nu_{\text{sphere}}}{E_{\text{sphere}}} + \frac{1-\nu_{\text{plate}}}{E_{\text{plate}}} \quad (11.5)$$
The computed contact force acting on the sphere versus the indentation depth is compared with that obtained from the analytical solution in Fig. 3-7. The two curves agree well with each other for the indentation depth up to 1.5 mm after which they begin to deviate. There are oscillations in the numerical solution because we solved a transient problem whereas the analytical solution is for a static problem. This exercise insures that we are correctly using the software, and one can get good results for a contact problem with the software.

![Graph](image)

Fig. 3-8: Comparison of the computed contact force versus the indentation depth with that obtained from the analytical solution.

For the initial velocity = 4 m/s of the sphere (initial kinetic energy = 4 J), we have compared in Fig. 3-9 the computed time history of the contact force, $F_c$, with the experimental one. The time is reckoned from the instant of contact between the sphere and the laminate, and results have been computed for two sets of values of $X_T$ and $Y_T$ listed in subsection 11.2.2. During the initial 0.25 ms of contact, the total force between the sphere and the laminate is more than that measured experimentally, and different values of $X_T$ and $Y_T$ have negligible effect on the magnitude of $F_c$. For $t > 0.25$ ms, the lower values of $X_T$ and $Y_T$ result in a smaller value of
The time, $t_p$, when the two computed values of $F_c$ peak agrees well with that when the experimentally measured $F_c$ has the maximum value. In the neighborhood of $t = t_p$, the amplitude of oscillations in the experimental values of $F_c$ is considerably more than that in the computed values of $F_c$. The time, $t_f$, when the sphere loses contact with the laminate is the least for $X_T = 2.1\, \text{GPa}$ and $Y_T = 0.135\, \text{GPa}$, and the most for $X_T = 1.4\, \text{GPa}$ and $Y_T = 0.08\, \text{GPa}$. The sphere remains in contact with the laminate for about 2.5 ms.

![Graph showing comparison of computed and experimental contact force histories.](image)

**Fig. 3-9**: Comparison of the computed and the experimental time histories of the contact force.

We have displayed in Fig. 3-10a-f fringe plots of stresses at $t = 0.55\, \text{ms}$, this is the time when a failure first occurs in the composite. In order to clearly show stresses within the laminate, the system has been cut by the plane $x_2 = \text{constant}$ that passes through the sphere.
Peak values of $\hat{\sigma}_{11}$ close to 1.5 GPa occur in the four elements directly underneath the spherical impactor that are on the bottom surface of the laminate. In the third and the fourth ply from the bottom surface, one can see large values, around 1.28 GPa, of $\hat{\sigma}_{11}$. The average values of Young's moduli $E_1$ and $E_2$ are equal. However, the 3-D analysis seems to capture effects of the location of plies. The magnitudes of stresses are essentially either symmetric or anti-symmetric about the plane $x_2 = \text{constant}$ passing through the sphere center. The maximum value of $\hat{\sigma}_{12}$ is about 45% of the maximum value of $\hat{\sigma}_{11}$. The magnitudes of $\hat{\sigma}_{13}$ and $\hat{\sigma}_{23}$ are approximately equal and are about 4.5% of the maximum value of $\hat{\sigma}_{11}$. $\hat{\sigma}_{33}$ varies from 82.5 MPa in tension along the periphery of the impactor to -0.28 GPa in compression below the impactor. Since $X_v \equiv 0.5X_{xj} \equiv 0.11X_T$, the plies in the bottom half of the laminate have delaminated in the region directly underneath the sphere.

Fig. 3-10a: Fringe plot of stress $\hat{\sigma}_{11}$ at $t = 0.55$ ms after impact.
Fig. 3-10b: Fringe plot of the normal stress $\dot{\sigma}_{22}$ at $t = 0.55$ ms after impact.

Fig. 3-10c: Fringe plot of the normal stress $\dot{\sigma}_{33}$ at $t = 0.55$ ms after impact.
Fig. 3-10d: Fringe plot of the shear stress $\hat{\sigma}_{12}$ at $t = 0.55$ ms after impact.

Fig. 3-10e: Fringe plot of the shear stress $\hat{\sigma}_{13}$ at $t = 0.55$ ms after impact.
At $t = 1.1 \text{ms}$ we have exhibited in Fig. 3-11a-f fringe plots of the internal variables $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ and $\overline{D}$ that characterize damage due to fiber tensile failure, fiber compressive failure, laminate crushing, the matrix tensile failure, the matrix compressive failure, and the delamination between adjacent plies, respectively. In the magnified view of Fig. 3-11b one can see several elements that failed due to the matrix failing in tension and were deleted during the analysis. There are several elements directly underneath the spherical impactor in Fig. 3-11c in which the fibers failed due to the excessive axial compressive stress. Fringe plots of the internal variable, $\varphi_3$, associated with the crush damage shown in Fig. 3-11e reveal that there is no damage induced due to the laminate crushing. However, fringe plots of the damage variable, $\overline{D}$, displayed in Fig. 3-11f lead to the conclusion that the delamination between adjacent plies occurs over a very large region.
Fig. 3-11a, b: Fringe plots of internal variables $Q_1$ and $Q_4$ associated with the fiber and the matrix tensile damage at $t = 1.1$ ms; and a magnified view of the severely damaged region.

Fig. 3-11c,d: Fringe plots of internal variables $Q_2$ and $Q_5$ associated with the fiber and the matrix compressive damage at $t = 1.1$ ms.
The bending of the laminate is clearly visible in these plots. Even though one usually tries to enhance the laminate strength by using high strength fibers this analysis brings out the importance of the matrix strength at least for a low speed impact problem.

11.2.3. **Evolution of damage during impact loading**

The evolution of internal variables $Q_3$ (associated with the five damage modes) is described here and is correlated with the corresponding points on the contact force time history plot; this should help understand how damage influences the contact force during impact. In Fig. 3-12 arrows pointing at different locations on the contact force time history plot allude to points of significance with respect to damage initiation and damage evolution.
Fig. 3-12: Contact force time history plot with points of significance for the damage initiation and propagation.

Our analysis predicts that the fiber compressive damage associated with the internal variable $Q_2$ initiates first at $\sim 0.25$ ms, and occurs at a point contacting the impactor; this is indicated as '1' in Fig. 3-12. The corresponding fringe plot of $Q_2$ shown in Fig 3-13a suggest that the damage due to fiber compression is localized in a narrow region below the impactor and it does not affect the contact force.
Fig. 3-13a: Fringe plot of internal variable $Q_2$ (fiber compressive damage) at $t \approx 0.25$ ms.

The matrix tensile damage, quantified by values of $Q_4$, initiates next at the bottom-most layer of the composite plate at $\sim 0.3$ ms after impact. It is indicated as point '2' in Fig. 3-12 and fringe plots of the fiber compressive and the matrix tensile damage are shown in Fig. 3-13b. The value of the fiber compressive damage variable has increased to nearly 1 but the damage remains localized at points below the impactor. The initiation of the two damage modes does not seem to influence the force between the impactor and the laminate. Even though the value of the internal variable has exceeded 0.95, the other two auxiliary criteria given in Section 6 for element deletion have not been satisfied.
Fig. 3-13b: Fringe plots of internal variables $Q_2$ (fiber compressive damage) and $Q_4$ (matrix tensile damage) at $t \approx 0.3$ ms.

The fiber tensile damage, signified by values of $Q_1$, initiates along the top-most layers of the composite plate at ~0.4 ms after impact and is indicated as point '3' in Fig. 3-12. The corresponding fringe plots for the three damage modes shown in Fig. 3-13c suggest that the matrix tensile damage has increased in the bottom layers of the plate while the fiber compressive damage has spread out in the top layers of the plate. Also noticeable in the contact force time history plot is a sharp drop in the contact force at this instant. This drop is attributed to both the fiber tensile damage and the growth of the fiber compressive and the matrix tensile damage modes. The drop in the contact force is followed by a further increase in the contact force but
now at a rate less than that prior to the drop. Schoepner and Abrate [32] called this point as the "damage threshold load" (DTL). DTL is the impact load at which there is sufficient accumulation of damage in the composite plate causing a reduction in its stiffness and hence a change in the slope in the contact force time history plot.

Fig. 3-13c: Fringe plots of internal variables $Q_1$ (matrix tensile damage), $Q_2$ (fiber compressive damage) and $Q_4$ (matrix tensile damage) at $t = 0.4$ ms.

The compressive matrix damage, decipherable from values of $Q_5$, initiates at $t = 0.45$ ms but is limited to points near the top surface of the composite plate. At $t = 0.52$ ms the second drop in the contact force is observed (indicated as ‘4’ in Fig. 3-12). The fringe plot of the compressive damage mode shown in Fig. 3-13d indicates that the damage is insignificant to warrant a drop in the contact force. The fringe plot of the fiber tensile damage mode reveals that significant fiber tensile damage occurs in the bottom layers of the composite plate but the region of the fiber tensile damage has not grown in the top layers of the plate. The magnitudes of other damage modes have also not increased much. Hence it can be concluded that the second drop in the
contact force at point '4' is due to the accumulation of the fiber tensile damage along the bottom layers of the composite laminate.

Fig. 3-13d: Fringe plots of internal variables $Q_1$ (matrix tensile damage), $Q_2$ (fiber compressive damage), $Q_4$ (matrix tensile damage) and $Q_5$ (matrix compressive damage) at $t \approx 0.52$ ms.

At $t \approx 0.55$ ms, indicated as point '5' in Fig. 3-12, we see another sharp drop in the contact force whose magnitude is much larger than that of the previous two drops in the contact force. Fringe plots of the internal variables $Q_1$ (fiber tensile damage mode) and $Q_4$ (matrix tensile damage mode) shown in Fig 3-13e indicate that elements have been deleted from the FE mesh which may have reduced the laminate stiffness and caused the contact force to drop. The other damage modes are primarily restricted to the top layers of the composite plate, and no element has been deleted in that region.

At $t \approx 1.25$ ms the impactor begins to rebound (indicated as point '6' in Fig. 3-12). A significant number of elements have failed by this time.
11.2.4. Effect of laminate stacking sequence on the impact response of the laminate

Experimental results of Hitchen and Kemp [33] indicate that the damage area and the contact force on carbon fiber reinforced composite laminates are influenced by the stacking sequence. Hitchen and Kemp [33] and Dorey [34] have shown that the presence of a $45^\circ$ lamina on the impact surface of a composite plate offers better impact resistance than having a $0^\circ$ lamina there. Here the influence of stacking sequence is examined by studying the impact problem for the $[0]_{16}, [-45/0/45/90]_{25}$ and $[0/90]_{45}$ laminates.

We briefly discuss the evolution of damage in the $[0]_{16}$ and the $[0/90]_{45}$ laminates and compare it with that in the $[-45/0/45/90]_{25}$ laminate.

For the $[0]_{16}$ composite plate, the matrix tensile damage initiates first in the top-most layer at $t \approx 0.032 \text{ ms}$. It then develops along the bottom-most layer of the composite plate at
At $t = 0.125\, ms$ the matrix compressive damage initiates in the top layers of the composite plate, and the matrix tensile damage in the bottom layers of the composite plate has grown to a significant area. Multiple matrix tensile damage paths develop and run parallel to each other (e.g., see Fig. 3-14a). The fiber compressive damage initiates at $t = 0.25\, ms$ in the top-most layer of the composite plate. The DTL corresponding to the first drop in load occurs at $t = 0.3\, ms$, and is indicated as point '1' in Fig. 3-16. The fringe plot of the internal variable $Q_4$ in Fig. 3-14a clearly evinces the extensive damage due to matrix tensile failure along the bottom layers of the composite plate. The fiber tensile damage has not yet occurred when the DTL is reached for the $[0]_{10}$ composite plate since it initiates at $t = 0.325\, ms$. The fiber tensile, the fiber compressive, and the matrix compressive damage modes at $t = 0.325\, ms$ are located only in the top layers of the composite plate.

The first element is deleted from the FE mesh at $t = 0.45\, ms$ due to the fiber and the matrix tensile damage occurring in the bottom layers of the composite plate and this coincides with the second drop in the contact force (point '2' in Fig. 3-16). Fringe plots of the internal
variables plotted in Fig. 3-14b show that the matrix tensile damage is very extensive and contributes more to the failure of the composite than any other mode.

In successive time steps many more elements get deleted in the bottom layers of the composite plate which results in severe loss of stiffness, and prevents the peak load from rising much above the DTL (cf. Fig. 3-16).

Fig. 3-14b: Fringe plots of internal variables $Q_1$ (matrix tensile damage), $Q_2$ (fiber compressive damage), $Q_4$ (matrix tensile damage) and $Q_5$ (matrix compressive damage) at $s \approx 0.45 \, ms$.

In Fig. 3-14c we have plotted fringes of internal variables $Q_1$ (fiber tensile damage) and $Q_4$ (matrix tensile damage) at $t = 2 \, ms$ (point at which the sphere starts to rebound). It is clear that severe damage has occurred along the bottom layers of the plate; the bottom twelve layers of the $[0]_{16}$ composite laminate have been damaged in comparison with only five bottom layers of the $[-45/0/45/90]_{25}$ laminate (see Fig. 3-11b). One can conclude from the plot of the time history of the contact force included in Fig. 3-16 that the contact time between the sphere and the
plate has noticeably increased for the \( [0]_{16} \) laminate in comparison with that for the \( [-45/0/45/90]_{2S} \) laminate.

![Fringe plots of internal variables \( Q_1 \) (fiber tensile damage) and \( Q_4 \) (matrix tensile damage) at \( t \approx 2 \text{ ms} \).](image)

For the \([0/90]_{4S}\) laminate the matrix tensile damage initiates first at \( t = 0.2 \text{ ms} \) along the bottom layers of the composite plate, and the fiber compressive damage initiates at \( t = 0.3 \text{ ms} \) along the top-most layer of the composite just underneath the spherical impactor. The fiber tensile and the matrix compressive damage initiate at \( t = 0.32 \text{ ms} \) along the top layers of the composite. The DTL for the \([0/90]_{4S}\) laminate (point '3' in Fig. 3-16) occurs at \( t \approx 0.35 \text{ ms} \). As seen from the fringe plots of the internal variables in Fig. 3-15a, all of the damage modes, except the crush damage, have initiated by this time. Unlike the two other laminate stacking sequences
discussed, for the {0/90}_4S laminate elements get deleted from the FE mesh soon after the DTL is reached. The damaged area in the {0/90}_4S laminate is not as wide spread as that in the {0}_{16} composite laminate but is larger than that for the {-45/0/45/90}_2S laminate.

![Fringe plots of internal variables Q₁ (fiber tensile damage), Q₂ (fiber compressive damage), Q₄ (matrix tensile damage), and Q₅ (matrix compressive damage) at t ≈ 0.35 ms.](image)

The spherical impactor begins to rebound at t = 1.9 ms and fringe plots of the internal variables included in Fig. 3.15b imply that some portions of the bottom eight layers are damaged due to the impact load.
Fig. 3-15b: Fringe plots of internal variables $Q_1$ (fiber tensile damage), $Q_2$ (fiber compressive damage), $Q_4$ (matrix tensile damage), and $Q_5$ (matrix compressive damage) at $t = 1.9\ ms$.

The time histories of the contact force for the three stacking sequences plotted in Fig. 3-16 reveal that the initial rate of increase of the contact force (before the DTL) for the $[0]_{16}$ stacking sequence is much less than that for the other two stacking sequences. In terms of either the damage threshold or the peak contact load the stacking sequence $[-45/0/45/90]_{25}$ offers a better impact resistance than that offered by the other two stacking sequences. This qualitatively agrees with the experimental findings of Hitchen and Kemp [33] and Dorey [34].
Fig. 3-16: Comparison of contact force time histories for three laminate stacking sequences.

In Table 3-6 we have summarized results computed for the impact loading of the 3 laminate sequences.

Table 3-6: Summary of results for the three laminate sequences under impact loading.

<table>
<thead>
<tr>
<th></th>
<th>[-45/0/45/90]_{2S}</th>
<th>[0]_{16}</th>
<th>[0/90]_{4S}</th>
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</thead>
<tbody>
<tr>
<td>DTL (kN)</td>
<td>1.567</td>
<td>0.85</td>
<td>1.3</td>
</tr>
<tr>
<td>Slope after DTL (kN/ms)</td>
<td>47.602</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Peak load (kN)</td>
<td>1.806</td>
<td>1.2</td>
<td>1.33</td>
</tr>
<tr>
<td>Time when impactor begins to rebound (ms)</td>
<td>1.4</td>
<td>2</td>
<td>1.9</td>
</tr>
<tr>
<td>Maximum plate deflection (mm)</td>
<td>3.24</td>
<td>4.5</td>
<td>3.9</td>
</tr>
</tbody>
</table>
11.2.5. Mesh sensitivity analysis

To see how sensitive the results are to the FE mesh, we consider 3 element sizes namely 0.65 mm (mesh 1), 0.775 mm (mesh 2) and 0.96 mm (mesh 3); these dimensions are for elements that are in close vicinity of the impactor. The maximum size of an element along the outer edges of the lamina varies from 1 mm for mesh 1 to 1.25 mm for mesh 3, see Fig. 3-17. This results in approximately 120,000 elements for mesh 1, 85,000 elements for mesh 2, and 70,000 elements for mesh 3. The regions having the coarse and the fine mesh are shown in Fig. 3-17. The contact force time history plots for the \([-45/0/45/90]_2\) laminate for the three FE meshes, shown in Fig. 3-18, suggest that the mesh size has only a marginal effect on the contact force time history. Each of these FE meshes has only one element through the thickness of a ply. Also even for the finest FE mesh the element aspect ratio is more than the ideal one. However, results provide a qualitative information about various aspects of the problem, and are not necessarily converged values of various parameters.

Fig. 3-17: A graded FE mesh used in the impact analysis of the composite laminate.
Fig. 3-18: Contact force time histories plot for three different FE meshes.

We have summarized in Table 3-7 results obtained with the three different FE meshes.

Table 3-7: Summary of results for the $[-45/0/45/90]^{2s}$ laminate obtained with three different meshes.

<table>
<thead>
<tr>
<th></th>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
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<tr>
<td>DTL (kN)</td>
<td>1.3</td>
<td>1.4</td>
<td>1.56</td>
</tr>
<tr>
<td>Slope after DTL (kN/ms)</td>
<td>47.66</td>
<td>47.63</td>
<td>47.6</td>
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<tr>
<td>Peak load (kN)</td>
<td>1.74</td>
<td>1.87</td>
<td>1.81</td>
</tr>
<tr>
<td>Time when the sphere begins to rebound (ms)</td>
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<td>1.395</td>
<td>1.4</td>
</tr>
<tr>
<td>Plate deflection (mm)</td>
<td>3.36</td>
<td>3.32</td>
<td>3.24</td>
</tr>
<tr>
<td>Final K. E. of impactor (J)</td>
<td>2.06</td>
<td>2.1</td>
<td>2.17</td>
</tr>
</tbody>
</table>
Here the slope after DTL has been found by fitting a straight line by the least squares method to the appropriate points of the contact force time history curve (see Fig. 3-12). Because of oscillations in the curve, the least squares fit line had regression coefficient of 0.85. For plates having the peak load much greater than the DTL one could potentially take corrective measures once the DTL is reached but before the structure fails.

11.2.6. Effect of the rate of evolution of the damage

The value of the parameter $m$ in Eq. (5.1) determines how quickly the damage evolves. A large (small) positive value of $m$ causes the damage to evolve quickly (slowly). For negative values of $m$ the slope of the stress-strain curve is positive after the initiation of damage which is unrealistic for real systems. In Fig. 3-19 we have exhibited the effect of the value of $m$ on the axial stress versus the axial strain curve for a composite deformed in uniaxial tension.

![Graph showing effect of m on stress-strain curve](image)

**Fig. 3-19:** The axial stress vs. the axial strain curves for different values of $m$ for a composite laminate deformed in uniaxial tension.
Xiao et al. [16] have used $m = 2$ for S2 glass/SC-15 epoxy composite and have suggested tailoring the value of the parameter $m$ based on experimental observations. Here we have used the same value of $m$ for all modes of damage. However, one intuitively expects that the fiber damage would result in the immediate failure of the composite whereas the matrix damage is more gradual and one should use a small value of $m$ for it. In Fig. 3.20 we have displayed time histories of the contact force for the 16 ply AS4/PEEK composite plate studied in subsection 11.2.2.

![Fig. 3-20: Comparison of the contact force time histories for $m=4$ and 100.](image)

We have listed in Table 3-8 values of various quantities computed with $m = 4$ and 100.
Table 3-8: Summary of results for the $[-45/0/45/90]_{2S}$ laminate for $m = 4$ and 100.

<table>
<thead>
<tr>
<th></th>
<th>$m = 4$</th>
<th>$m = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTL (kN)</td>
<td>1.567</td>
<td>1.4</td>
</tr>
<tr>
<td>Slope after DTL (kN/ms)</td>
<td>47.602</td>
<td>-</td>
</tr>
<tr>
<td>Peak load (kN)</td>
<td>1.806</td>
<td>1.48</td>
</tr>
<tr>
<td>Time when the sphere begins to rebound (ms)</td>
<td>1.4</td>
<td>1.55</td>
</tr>
<tr>
<td>Maximum plate deflection (mm)</td>
<td>3.24</td>
<td>3.49</td>
</tr>
</tbody>
</table>

11.3. Effect of presence of the base plate on the response of the composite laminate

Since the composite laminate simply rests on the base plate, the laminate edges are lifted from the plate during impact as should be clear from the fringe plots of the normal displacement included in Fig. 3-21. The laminate points in yellow, brown and red regions have positive values of $u_3$ signifying that the back surface of the laminate in these regions is not contacting the supporting steel plate.
Fig. 3-21: Fringe plot of the normal displacement ($u_3$) on the back surface of the composite laminate.

12. Conclusions

We have analyzed by the finite element method transient elasto-plastic deformations of a unidirectional fiber reinforced AS4/PEEK laminate impacted at normal incidence by a rigid sphere moving at a speed not large enough to punch a hole in the laminate. The matrix is assumed to deform elasto-plastically and fibers elastically. For each finite element the material properties of the composite are found by using a micromechanics approach, namely, the free shear traction method. Based on the stresses induced in the element we find whether or not a failure mode has initiated by using Hashin’s failure criteria. An internal variable is associated with each failure mode whose evolution is used to account for the irreversible damage induced in
the element. Thus values of material parameters of the constituents rather than those of the composite are needed. However, values of strength parameters for the composite are required.

The computed time history of the total axial force acting on the impactor agrees well with the experimental one available in the literature. For the problem studied the delamination failure occurs over an extensive region beneath the spherical impactor. Fibers below the impactor fail in compression, and the matrix in the bottom-most plies fails in tension. A wide crack develops in the third ply from the bottom surface for the $\{-45/0/45/90\}_{2S}$ laminate. Results computed for the three stacking sequences for the laminate suggest that the locations and the extent of the failed regions strongly depend upon the stacking sequence. Also, the matrix may fail in tension before fibers fail.

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CHAPTER 4: Conclusions and Contributions

1. Conclusions

We have used four micromechanical theories, namely, the Mori-Tanaka, Eshelby’s equivalent inclusion, the free shear traction method, and Hill’s equivalent energy principle, to find values of material parameters for elasto-plastic deformations of AS4/PEEK composite. The elasto-plastic deformations of the matrix are modeled by two yield criteria - the pressure-independent von Mises and the pressure-dependent Drucker-Prager. Whereas the von Mises yield surface is quadratic in deviatoric stresses the Drucker-Prager also has a linear term in the hydrostatic pressure.

It is found that for fiber volume fraction, $v_f$, varying from 0.3 to 0.6 the four micromechanics approaches provide noticeably different values of the elastic moduli. For $v_f = 0.6$, none of the four theories give values of all five elastic parameters that are close to the experimental results available in the literature. We note that the difference in the minimum and the maximum values of Young’s modulus in the direction transverse to fibers found experimentally by three research groups equals 20%. Along the fibers, all four theories give values of Young’s modulus that are very close to each other, and it can also be found by using the rule of mixtures. When displacements are applied on all surfaces of a representative volume element to produce a state of simple shear in the lamina plane, the value of the in-plane shear modulus deduced from Hill’s equivalence energy principle is significantly higher than that provided by the other three micromechanics approaches as well as the literature experimental value.

For uniaxial elasto-plastic deformations of the composite lamina with fibers making an angle $\theta$ with the axis of loading, the effective stress versus the effective plastic strain curves computed with the von Mises yield criterion for the PEEK agree better with the experimental results of Weeks and Sun than those derived by assuming that the yield criterion is pressure dependent. Assuming that plastic deformations of the composite are also independent of the hydrostatic pressure, we have found values, as a function of the volume fraction of fibers, of the two material parameters in the yield surface and of the two material parameters that characterize the strain dependent yield stress of the composite.
The pressure independent yield surface for the AS4/PEEK composite studied herein is given by

$$\bar{\sigma} = \sqrt{3 F (\sigma_y)} = A + N \ln(\bar{\epsilon}^{\text{p}})$$

where

$$F(\sigma_y) = \frac{L}{2} \left[ (\sigma_{22} - \sigma_{33})^2 + 2A_{23} \sigma_{33}^2 + 2A_{12} (\sigma_{13}^2 + \sigma_{23}^2) \right]$$

$$A = 2.597(VF)^3 - 1.473(VF)^2 + 0.116(VF) + 0.396$$
$$N = 0.107(VF)^3 + 0.008(VF)^2 - 0.050(VF) + 0.039$$
$$A_{23} = 158.0(VF)^3 - 164.4(VF)^2 + 58.44(VF) - 1.5$$
$$A_{12} = 135.8(VF)^3 - 138.8(VF)^2 + 47.77(VF) - 0.5$$

and VF equals the volume fraction of the fibers between 0.3 and 0.75, and values of A and N are in MPa. By following a procedure similar to that used here and simulating homogeneous deformations of the unidirectional fiber reinforced polymeric composite (UFPC) at different strain rates, one can quantify the dependence of the yield stress of the UFPC upon the strain rate characteristics of the fiber and the matrix.

We have also analyzed by the finite element method transient elasto-plastic deformations of a unidirectional fiber reinforced AS4/PEEK laminate impacted at normal incidence by a rigid sphere moving at a speed not large enough to punch a hole in the laminate. The matrix is assumed to deform elasto-plastically and fibers elastically. For each finite element the material properties of the composite are found by using a micromechanics approach, namely, the free shear traction method. Based on the stresses induced in the element we find whether or not a failure mode has initiated by using Hashin’s failure criteria. An internal variable is associated with each failure mode whose evolution is used to account for the irreversible damage induced in the element. Thus values of material parameters of the constituents rather than those of the composite are needed. However, values of strength parameters for the composite are required.

The computed time history of the total axial force acting on the impactor agrees well with the experimental one available in the literature. For the problem studied the delamination failure occurs over an extensive region beneath the spherical impactor. Fibers below the impactor fail in
compression, and the matrix in the bottom-most plies fails in tension. A wide crack develops in the third ply from the bottom surface for the $\{ -45/0/45/90 \}_2$ laminate.

2. Contributions

A micromechanical analysis which uses constituent level properties to establish the overall response of a heterogeneous material is computationally intensive, implying that it is ideally suited to analyze simple problems under ideal loading conditions. One of the main accomplishments of this work is to show the use of a micromechanics method for solving realistic problems where there is non-uniform loading, e.g., impact problems, while considering failure and damage during the loading process. Contributions of this work are summarized below.

1. The free shear traction method for finding material properties of the lamina from those of its constituents has been integrated with the finite element scheme to solve realistic problems involving non-uniformly distributed loads. Thus only constituent level properties are needed for the analysis. However, values of strength parameters for the lamina and those of parameters appearing in the delamination criterion are also required for studying damage and failure.

2. A user defined subroutine has been developed to analyze damage and failure for 3-dimensional problems, and has been integrated with the commercial FE software, ABAQUS.

3. The software has been used to analyze impact at normal incidence of a composite laminate by a rigid sphere while considering various damage modes.
Appendix A
ABAQUS input file for the impact analysis of UFPC laminate

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** Generated by: Abaqus/CAE Version 6.8-2
*Preprint, echo=NO, model=NO, history=NO, contact=NO
** ------------------------------------------------ ----------------
** II PART INSTANCE: LAY1-ANG45N
**
*Node
1, 0.0500000007, 0., 0.000125000006
2, 0.0500000007, 0., 0.

8843, 0.0269230772, 0.0237499997, 0.000125000006
8844, 0.0259615388, 0.0237499997, 0.000125000006

*Element, type=C3D8R
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2, 1033, 1394, 1395, 1034, 70, 33, 34, 69

4290, 858, 785, 20, 19, 8369, 8844, 646, 590

*Section: AS4-PEEK-SECTION
*Solid Section, elset=LAY1-ANG45N__PickedSet19, material=AS4-PEEK

*Surface, type=ELEMENT, name=CONTACT-FORCE _CONTACT-FORCE_SPOS, SPOS
*Rigid Body, ref node=SPHERE-1-RefPt_, elset=SPHERE-1

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100.
*User Material, constants=14
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0.25, 4.0e+09, 0.35, 1.527e8, 1.386e7, 0.60
*Material, name=STEEL
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8000.,
*Elastic
2.1e+11, 0.25

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VELOCITY, 2, 2
** Name: FIX-BASE Type: Symmetry/Antisymmetry/Encastre
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*Contact property assignment
/ / SPHERE-PLATE
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LAY12_ANG45N-TOP, LAY11_ANG0-BOT, ADJUST-CLEARANCE
LAY13_ANG90-TOP, LAY12_ANG45N-BOT, ADJUST-CLEARANCE
LAY14_ANG45-TOP, LAY13_ANG90-BOT, ADJUST-CLEARANCE
LAY15_ANG0-TOP, LAY14_ANG45-BOT, ADJUST-CLEARANCE
LAY16_ANG45N-TOP, LAY15_ANG0-BOT, ADJUST-CLEARANCE
*****************************************************************************
** ------------------------------------------------ ----------------
** STEP: IMPACT
**
*Step, name=IMPACT, ngeom = YES/NO
*Dynamic, Explicit, scale factor=0.45
, 0.004
*Bulk Viscosity
0.06, 1.2
**
** INTERACTIONS
**
** Interaction: PLATE-SPHERE
*****************************************************************************
*Contact, op=MOD
*CONTACT INCLUSIONS
LAY2_ANG0-TOP, LAY1_ANG45N-BOT
LAY3_ANG45-TOP, LAY2_ANG0-BOT
LAY4_ANG90-TOP, LAY3_ANG45-BOT
LAY5_ANG45N-TOP, LAY4_ANG90-BOT
LAY6_ANG0-TOP, LAY5_ANG45N-BOT
LAY7_ANG45-TOP, LAY6_ANG0-BOT
LAY8_ANG90-TOP, LAY7_ANG45-BOT
LAY9_ANG90-TOP, LAY8_ANG90-BOT
LAY10_ANG45-TOP, LAY9_ANG90-BOT
LAY11_ANG0-TOP, LAY10_ANG45-BOT
LAY12_ANG45N-TOP, LAY11_ANG0-BOT
LAY13_ANG90-TOP, LAY12_ANG45N-BOT
LAY14_ANG45-TOP, LAY13_ANG90-BOT
LAY15_ANG0-TOP, LAY14_ANG45-BOT
LAY16_ANG45N-TOP, LAY15_ANG0-BOT
**CONTACT PROPERTY ASSIGNMENT**
LAY2_ANG0-TOP, LAY1_ANG45N-BOT, COHESIVE-INTERACTION
LAY3_ANG45-TOP, LAY2_ANG0-BOT, COHESIVE-INTERACTION
LAY4_ANG90-TOP, LAY3_ANG45-BOT, COHESIVE-INTERACTION
LAY5_ANG45N-TOP, LAY4_ANG90-BOT, COHESIVE-INTERACTION
LAY6_ANG0-TOP, LAY5_ANG45N-BOT, COHESIVE-INTERACTION
LAY7_ANG45-TOP, LAY6_ANG0-BOT, COHESIVE-INTERACTION
LAY8_ANG90-TOP, LAY7_ANG45-BOT, COHESIVE-INTERACTION
LAY9_ANG90-TOP, LAY8_ANG90-BOT, COHESIVE-INTERACTION
LAY10_ANG45-TOP, LAY9_ANG90-BOT, COHESIVE-INTERACTION
LAY11_ANG0-TOP, LAY10_ANG45-BOT, COHESIVE-INTERACTION
LAY12_ANG45N-TOP, LAY11_ANG0-BOT, COHESIVE-INTERACTION
LAY13_ANG90-TOP, LAY12_ANG45N-BOT, COHESIVE-INTERACTION
LAY14_ANG45-TOP, LAY13_ANG90-BOT, COHESIVE-INTERACTION
LAY15_ANG0-TOP, LAY14_ANG45-BOT, COHESIVE-INTERACTION
LAY16_ANG45N-TOP, LAY15_ANG0-BOT, COHESIVE-INTERACTION

******************************************************
** OUTPUT REQUESTS
**
** Restart, write, number interval=1, time marks=NO
**
** FIELD OUTPUT: F-Output-1
**
*Output, field, number interval=250
*Node Output
RF, U, V
*Element Output, directions=YES
E, S, SDV, STATUS
*Contact Output
CSDMG, CSQUADSCRT, CSTRESS
**
** HISTORY OUTPUT: H-Output-2
**
*Output, history, time interval=5e-06
*Contact Output, surface=CONTACT-FORCE
CFN3, CFNM
**
** HISTORY OUTPUT: H-Output-1
**
*Node Output, nset=VELOCITY
U3, V3
*End Step
Appendix B

VUMAT code to analyze damage and failure in UFPC

```
subroutine vumat
  ! Read only -
  1 jblock, ndir, nshr, nstatev, nfieldv, nprops, lanneal,
  2 stepTime, totalTime, dt, cmname, coordMp, charLength,
  3 props, density, strainInc, relSpinInc,
  4 tempOld, stretchOld, defgradOld, fieldOld,
  5 stressOld, stateOld, enerInternOld, enerInelasOld,
  6 tempNew, stretchNew, defgradNew, fieldNew,
  ! Write only -
  7 stressNew, stateNew, enerInternNew, enerInelasNew
  !
  include 'vaba_param.inc'

  dimension jblock(*), props(nprops), density(*), coordMp(*),
  1 charLength(*), strainInc(*),
  2 relSpinInc(*), tempOld(*),
  3 stretchOld(*),
  4 defgradOld(*),
  5 fieldOld(*), stressOld(*),
  6 stateOld(*), enerInternOld(*),
  7 enerInelasOld(*), tempNew(*),
  8 stretchNew(*),
  9 defgradNew(*),
  10 fieldNew(*),
  2 stressNew(*), stateNew(*),
  3 enerInternNew(*), enerInelasNew(*)

  character*80 cmname

  parameter (  
    1 i_umt_nblock = 1,
    2 i_umt_npt = 2,
    3 i_umt_layer = 3,
    4 i_umt_kspt = 4,
    5 i_umt_noel = 5  
  )

  call vumatXtrArg ( jblock(i_umt_nblock),
  1 ndir, nshr, nstatev, nfieldv, nprops, lanneal,
  2 stepTime, totalTime, dt, cmname, coordMp, charLength,
  3 props, density, strainInc, relSpinInc,
  4 tempOld, stretchOld, defgradOld, fieldOld,
  5 stressOld, stateOld, enerInternOld, enerInelasOld,
  6 tempNew, stretchNew, defgradNew, fieldNew,
  7 stressNew, stateNew, enerInternNew, enerInelasNew,
  8 jblock(i_umt_noel), jblock(i_umt_npt),
  9 jblock(i_umt_layer), jblock(i_umt_kspt))

return
end
```

```
subroutine vumatXtrArg
  ! read only -
  1 nblock, ndir, nshr, nstatev, nfieldv, nprops, lanneal,
  2 stepTime, totaltime, dt, cmname, coordMp, charLength,
  3 props, density, strainInc, relSpinInc,
  4 tempOld, stretchOld, defgradOld, fieldOld,
  5 stressOld, stateOld, enerInternOld, enerInelasOld,
  6 tempNew, stretchNew, defgradNew, fieldNew,
  ! write only -
  5 stressNew, stateNew, enerInternNew, enerInelasNew,
  ! read only extra arguments -
  6 nElement, nMatPoint, nLayer, nSecPoint
  !
  include 'vaba_param.inc'

  all arrays dimensioned by (*) are not used in this algorithm
  dimension props(nprops), density(nblock),
```

130
1 strainInc(nblock,ndir+nshr), relSpinInc(nblock,nshr), defgradOld(nblock,9),
2 stressOld(nblock,ndir+nshr), stateOld(nblock,nstatev), enerInternOld(nblock),
3 enerInelasOld(nblock), stretchNew(nblock,ndir+nshr), defgradNew(nblock,9),
4 stressNew(nblock,ndir+nshr)
5
dimension enerInelasNew(nblock),stateNew(nblock,nstatev),
6 enerInternNew(nblock),
7 dimension nElement(nblock),nMatPoint(nblock),nLayer(nblock),
8 nSecPoint(nblock)
9 character*80 cmname
10
DOUBLE PRECISION, PARAMETER :: PI = 3.141592653589793D0
11 DOUBLE PRECISION, PARAMETER :: TOL_SOLUTION = 1.0E-8
12 INTEGER, PARAMETER :: ITERATE_LOOP = 20
13
DOUBLE PRECISION, PARAMETER :: DAMAGE_TOL = 0.95
15 DOUBLE PRECISION, PARAMETER :: STRAIN_TOL = 0.15
17 DOUBLE PRECISION, PARAMETER :: TENSILE_STRAIN_LMT = 0.05
19 DOUBLE PRECISION, PARAMETER :: COMPRRESS_VOL = 0.1
21 DOUBLE PRECISION, PARAMETER :: EXPAND_VOL = 4.0
23
C_____ELASTIC PROPERTIES OF THE FIBER________
25 DOUBLE PRECISION :: E11_FIB, E22_FIB, E33_FIB
27 DOUBLE PRECISION :: G12_FIB, G13_FIB, G23_FIB
29 DOUBLE PRECISION :: NU12_FIB, NU13_FIB, NU23_FIB,
31           NU21_FIB, NU31_FIB, NU32_FIB
33
C_____MATRIX PROPERTIES_____
35 DOUBLE PRECISION :: E_MAT, NU_MAT, A_plast, N_pow
37
C_____VOLUME FRACTION___
39 DOUBLE PRECISION :: Vm, Vf
41 DOUBLE PRECISION :: RVE_a, RVE_b, RVE_c
43
C_____TRANSFORMATION MATRIX___
45 INTEGER, DIMENSION(500000)::ALONG_0, ALONG_90, ALONG_45, ALONG_45N
47 INTEGER                                        :: LNTH_0,  LNTH_90,  LNTH_45,  LNTH_45N
49 DOUBLE PRECISION :: TETA
51 DOUBLE PRECISION, DIMENSION(6,6) :: ROT_1   , ROT_2 ,
53       INV_ROT_1,  INV_ROT_2
55
C_____COMPLIANCE AND STIFFNESS  MATRIX OF FIBER AND MATRIX___
57 DOUBLE PRECISION, DIMENSION(6,6) :: C_INC, C_MAT, S_INC, S_MAT
59
C_____COEFFICIENT MATRIX FOR NORMAL STRAIN___
61 DOUBLE PRECISION, DIMENSION(8,8) :: COEFF_NORMAL, INV_COEFF_NORMAL
63
C_____VECTOR FOR NORMAL STRAIN AND RHS FOR SOLUTION_____
65 DOUBLE PRECISION, DIMENSION(8,1) :: SOLUTION_NORMAL,
67
C_____COEFFICIENT MATRIX FOR SHEAR STRAIN___
69 DOUBLE PRECISION, DIMENSION(12,12) :: COEFF_SHEAR, INV_COEFF_SHEAR
71
C_____VECTOR FOR SHEAR STRAIN AND RHS FOR SOLUTION_ _____
73 DOUBLE PRECISION, DIMENSION(12,1) :: SOLUTION_SHEAR,
75
C_____TOTAL SOLUTION____
77 DOUBLE PRECISION, DIMENSION(20) :: OLD_TOTALsol, NEW_TOTALsol,
79       DIFF_sol
81
C_____NORM TO CHECK FOR CONVERGENCE____
83 DOUBLE PRECISION :: NORM_DIFF, NORM_SOL
85
C_____STRAIN VECTORS IN EACH PHASE___
87 DOUBLE PRECISION, DIMENSION(6,1) ::OLDstrain_F,
89     OLDstrain_M1, OLDstrain_M2, OLDstrain_M3
91 DOUBLE PRECISION, DIMENSION(6,1) ::NEWstrain_F,
93     NEWstrain_M1, NEWstrain_M2, NEWstrain_M3
95
C_____STRESS VECTORS IN EACH PHASE____
97 DOUBLE PRECISION, DIMENSION(6,1) ::OLDstress_F,
99 OLDstress_M1, OLDstress_M2, OLDstress_M3
101 DOUBLE PRECISION, DIMENSION(6,1) ::NEWstress_F,
103     NEWstress_M1, NEWstress_M2, NEWstress_M3
105
C_____PLASTIC STRAIN IN EACH PHASE____
107 DOUBLE PRECISION, DIMENSION(6,1) ::OLDplastic_M1, OLDplastic_M2,
109     OLDplastic_M3
111
131
1  NEWplastic_M3

C___ EFFECTIVE PLASTIC STRAIN IN EACH PHASE
DOUBLE PRECISION :: YIELDfn_MAX_M1, YIELDfn_MAX_M2, YIELDfn_MAX_M3

C___ INCREMENTAL STRAINS IN EACH PHASE
DOUBLE PRECISION, DIMENSION(6,1) :: Dstrain_F, Dstrain_M1, Dstrain_M2, Dstrain_M3, Dstrain_RVE

C___ INCREMENTAL PLASTIC STRAIN IN EACH PHASE
DOUBLE PRECISION, DIMENSION(6,1) :: Dplastic_M1, Dplastic_M2, Dplastic_M3

C___ STRESS AND STRAIN IN RVE
DOUBLE PRECISION, DIMENSION(6,1) :: STRESS_RVE, STRAIN_RVE

C___ ENERGY DENSITY
DOUBLE PRECISION :: ENERGY_DENSITY

C___ ELEMENT NUMBER
INTEGER :: ELEMENT_NO, LOGIC

C___ DAMAGE RELATED PARAMETERS
DOUBLE PRECISION, DIMENSION(6,1) :: DAMAGEold_VECT, DAMAGEnew_VECT
DOUBLE PRECISION, DIMENSION(6,1) :: DAMAGE_STRESS
DOUBLE PRECISION :: Rv1, Rv2, Rv3, Rv4, Rv5
DOUBLE PRECISION :: PHI_1, PHI_2, PHI_3, PHI_4, PHI_5

C___ VOLUME EXPANSION VARIABLES
DOUBLE PRECISION, DIMENSION(3,3) :: F_GRAD
DOUBLE PRECISION :: DET_F1, DET_F2, DET_F3, DET_F
DOUBLE PRECISION :: VOL_RATIO

C___ VARIABLES TO DEFINE LOOP
INTEGER :: I, J, K, L, IFLAG, LOOP

C___ STORAGE OF HISTORY VARIABLES ALL IN LOCAL C/S
C__ stateOld(1:6) = STRAIN_RVE
C
C__ stateOld(7:12) = OLDstrain_F
C__ stateOld(13:18) = OLDstrain_M1
C__ stateOld(19:24) = OLDstrain_M2
C__ stateOld(25:30) = OLDstrain_M3
C
C__ stateOld(31:36) = OLDstress_F
C__ stateOld(37:42) = OLDstress_M1
C__ stateOld(43:48) = OLDstress_M2
C__ stateOld(49:54) = OLDstress_M3
C
C__ stateOld(55:60) = OLDplastic_M1
C__ stateOld(61:66) = OLDplastic_M2
C__ stateOld(67:72) = OLDplastic_M3
C
C__ stateOld(73) = YIELDfn_MAX_M1
C__ stateOld(74) = YIELDfn_MAX_M2
C__ stateOld(75) = YIELDfn_MAX_M3

C****************************************
C__ stateOld(80:85) = DAMAGE VARIABLES
C__ stateOld(86:90) = PHI's
C__ stateOld(91:95) = RV's

C___ ELASTIC PROPERTIES OF FIBER
E11_FIB = PROPS(1)
E22_FIB = PROPS(2)
E33_FIB = PROPS(3)
G12_FIB = PROPS(4)
G13_FIB = PROPS(5)
G23_FIB = PROPS(6)
NU12_FIB = PROPS(7)
NU13_FIB = PROPS(8)
NU23_FIB = PROPS(9)

C___ CALCULATED POISSON'S RATIO OF FIBER
NU21_FIB = NU12_FIB*(E22_FIB/E11_FIB)
NU31_FIB = NU13_FIB*(E33_FIB/E11_FIB)
NU32_FIB = NU23_FIB*(E33_FIB/E22_FIB)

C___ PROPERTIES OF MATRIX
E_MAT = PROPS(10)
NU_MAT = PROPS(11)
A_plast = CONSTANT
N_pow = SLOPE
Vf = PROPS(14)
C____VOLUME FRACTION OF MATRIX____
   Vm = 1-Vf
C____RVE DIMENSIONS_____
   RVE_a = SQRT(Vf)
   RVE_b = 1.0D0 - RVE_a
   RVE_c = RVE_b
C__ELEMENT SETS ASSOCIATED WITH ROTATION MATRIX___
   CALL ELEMENT_SET(ALONG_0, ALONG_90, ALONG_45, ALONG_45N,  
                    LNTH_0, LNTH_90, LNTH_45, LNTH_45N)
C
C____COMPLIANCE MATRIX OF FIBER_____
   CALL COMPLIANCE_TRANS(E11_FIB, E22_FIB, E33_FIB,  
                          G12_FIB, G13_FIB, G23_FIB,  
                          NU12_FIB, NU13_FIB, NU23_FIB, S_INC)
   C_INC = S_INC
   CALL MATINV(C_INC,6,6,IFLAG)
C
C____COMPLIANCE MATRIX OF MATRIX___
   CALL COMPLIANCE_ISO(E_MAT, NU_MAT, S_MAT)
   C_MAT = S_MAT
   CALL MATINV(C_MAT,6,6,IFLAG)
C
C____COEFFICIENT MATRIX FOR NORMAL STRAINS_____
   COEFF_NORMAL = 0.0D0
   C____ROW 1____
   COEFF_NORMAL(1,1) = RVE_a
   COEFF_NORMAL(1,3) = RVE_b
   C____ROW 2____
   COEFF_NORMAL(2,5) = RVE_b
   COEFF_NORMAL(2,7) = RVE_a
   C____ROW 3____
   COEFF_NORMAL(3,2) = RVE_a
   COEFF_NORMAL(3,8) = RVE_c
   C____ROW 4____
   COEFF_NORMAL(4,4) = RVE_a
   COEFF_NORMAL(4,6) = RVE_c
   C____ROW 5____
   COEFF_NORMAL(5,1) = C_INC(2,2)
   COEFF_NORMAL(5,2) = C_INC(2,3)
   COEFF_NORMAL(5,3) = -C_MAT(2,2)
   COEFF_NORMAL(5,4) = -C_MAT(2,3)
   C____ROW 6____
   COEFF_NORMAL(6,5) = C_MAT(2,2)
   COEFF_NORMAL(6,6) = C_MAT(2,3)
   COEFF_NORMAL(6,7) = -C_MAT(2,2)
   COEFF_NORMAL(6,8) = -C_MAT(2,3)
   C____ROW 7____
   COEFF_NORMAL(7,1) = C_INC(3,2)
   COEFF_NORMAL(7,2) = C_INC(3,3)
   COEFF_NORMAL(7,7) = -C_MAT(3,2)
   COEFF_NORMAL(7,8) = -C_MAT(3,3)
   C____ROW 8____
   COEFF_NORMAL(8,3) = C_MAT(3,2)
   COEFF_NORMAL(8,4) = C_MAT(3,3)
   COEFF_NORMAL(8,5) = -C_MAT(3,2)
   COEFF_NORMAL(8,6) = -C_MAT(3,3)

   INV_COEFF_NORMAL = COEFF_NORMAL
   CALL MATINV(INV_COEFF_NORMAL,8,8,IFLAG)
C
C____COEFFICIENT MATRIX FOR SHEAR STRAINS______
   COEFF_SHEAR = 0.0D0
   C____ROW 1____
   COEFF_SHEAR(1,3) = 1.0D0
   COEFF_SHEAR(1,12) = -1.0D0
   C____ROW 2____
   COEFF_SHEAR(2,6) = 1.0D0
   COEFF_SHEAR(2,9) = -1.0D0
   C____ROW 3____
   COEFF_SHEAR(3,3) = RVE_a
   COEFF_SHEAR(3,6) = RVE_b
C_____ROW 4_____
COEFF_SHEAR(4,2)  =  1.0D0
COEFF_SHEAR(4,5)  = -1.0D0
C_____ROW 5_____
COEFF_SHEAR(5,8)  =  1.0D0
COEFF_SHEAR(5,11) = -1.0D0
C_____ROW 6_____
COEFF_SHEAR(6,2)  =  RVE_a
COEFF_SHEAR(6,11) =  RVE_c
C_____ROW 7_____
COEFF_SHEAR(7,1)  =  RVE_a*RVE_a
COEFF_SHEAR(7,4)  =  RVE_a*RVE_b
COEFF_SHEAR(7,7)  =  RVE_b*RVE_c
COEFF_SHEAR(7,10) =  RVE_a*RVE_c
C_____ROW 8_____
COEFF_SHEAR(8,3)  =   RVE_a*C_INC(6,6)
COEFF_SHEAR(8,6)  =  -RVE_a*C_MAT(6,6)
COEFF_SHEAR(8,9)  =  -RVE_c*C_MAT(6,6)
COEFF_SHEAR(8,12) =   RVE_c*C_MAT(6,6)
C_____ROW 9_____
COEFF_SHEAR(9,2)  =   RVE_a*C_INC(5,5)
COEFF_SHEAR(9,5)  =   RVE_b*C_MAT(5,5)
COEFF_SHEAR(9,8)  =  -RVE_b*C_MAT(5,5)
COEFF_SHEAR(9,11) =  -RVE_a*C_MAT(5,5)
C_____ROW 10____
COEFF_SHEAR(10,1) =  C_INC(4,4)
COEFF_SHEAR(10,10)= -C_MAT(4,4)
C_____ROW 11____
COEFF_SHEAR(11,1) =  C_INC(4,4)
COEFF_SHEAR(11,4) = -C_MAT(4,4)
C_____ROW 12____
COEFF_SHEAR(12,1) =  C_INC(4,4)
COEFF_SHEAR(12,7) = -C_MAT(4,4)
C_________________
INV_COEFF_SHEAR = COEFF_SHEAR
CALL MATINV(INV_COEFF_SHEAR,12,12,IFLAG)
C
C*****MAKE COMPUTATIONS AT EVERY MATERIAL POINT********
DO KM = 1,nblock
   LOGIC=0
   ELEMENT_NO = nElement(KM)
   C_________ASSIGN ANGLE TO EACH ELEMENT___
   IF(LOGIC .EQ. 0)THEN
      DO I = 1, LNTH_0
         IF(ELEMENT_NO .EQ. ALONG_0(I))THEN
            TETA =0.0
            LOGIC=1
            EXIT
         END IF
      END DO
      END IF
   IF(LOGIC .EQ. 0)THEN
      DO I = 1, LNTH_90
         IF(ELEMENT_NO .EQ. ALONG_90(I))THEN
            TETA = (1.0D0/2.0D0)*PI
            LOGIC=1
            EXIT
         END IF
      END DO
      END IF
   IF(LOGIC .EQ. 0)THEN
      DO I = 1, LNTH_45
         IF(ELEMENT_NO .EQ. ALONG_45(I))THEN
            TETA = (1.0D0/4.0D0)*PI
            LOGIC=1
            EXIT
         END IF
      END DO
      END IF
   END IF
END IF
C

IF(ELEMENT_NO .EQ. ALONG_45N(I)) THEN
  TETA = -(1.0D0/4.0D0)*PI
  LOGIC=1
  EXIT
END IF
END DO
END IF
CALL ROTATION(TETA, ROT_1, ROT_2, INV_ROT_1, INV_ROT_2)

C_________INCREMENTAL STRAIN IN GLOBAL COORDINATE SYSTEM______
Dstrain_RVE(1,1) = strainInc(KM,1)
Dstrain_RVE(2,1) = strainInc(KM,2)
Dstrain_RVE(3,1) = strainInc(KM,3)
Dstrain_RVE(4,1) = 2.0D0*strainInc(KM,5)
Dstrain_RVE(5,1) = 2.0D0*strainInc(KM,6)
Dstrain_RVE(6,1) = 2.0D0*strainInc(KM,4)

C_________ROTATE INCREMENTAL STRAIN TO LOCAL COORDINATE SYSTEM______
Dstrain_RVE = MATMUL(ROT_2, Dstrain_RVE)

C_________stateOld(1:6) = STRAIN_RVE
C_________STRAIN IN RVE (LOCAL C/S): PRESENT TIME STEP_____
DO I=1,6
  STRAIN_RVE(I,1) = stateOld(KM,I) + Dstrain_RVE(I,1)
END DO

C_________stateOld(7:12) = OLDstrain_F
C_________stateOld(13:18) = OLDstrain_M1
C_________stateOld(19:24) = OLDstrain_M2
C_________stateOld(25:30) = OLDstrain_M3
DO I=1,6
  OLDstrain_F(I,1) = stateOld(KM,I+6)
  OLDstrain_M1(I,1) = stateOld(KM,I+12)
  OLDstrain_M2(I,1) = stateOld(KM,I+18)
  OLDstrain_M3(I,1) = stateOld(KM,I+24)
END DO

C_________stateOld(31:36) = OLDstress_F
C_________stateOld(37:42) = OLDstress_M1
C_________stateOld(43:48) = OLDstress_M2
C_________stateOld(49:54) = OLDstress_M3
DO I=1,6
  OLDstress_F(I,1) = stateOld(KM,I+30)
  OLDstress_M1(I,1) = stateOld(KM,I+36)
  OLDstress_M2(I,1) = stateOld(KM,I+42)
  OLDstress_M3(I,1) = stateOld(KM,I+48)
END DO

C_________stateOld(55:60) = OLDplastic_M1
C_________stateOld(61:66) = OLDplastic_M2
C_________stateOld(67:72) = OLDplastic_M3
DO I=1,6
  OLDplastic_M1(I,1) = stateOld(KM,I+54)
  OLDplastic_M2(I,1) = stateOld(KM,I+60)
  OLDplastic_M3(I,1) = stateOld(KM,I+66)
END DO

C_________stateOld(73) = YIELDfn_MAX_M1
C_________stateOld(74) = YIELDfn_MAX_M2
C_________stateOld(75) = YIELDfn_MAX_M3
YIELDfn_MAX_M1 = stateOld(KM, 73)
YIELDfn_MAX_M2 = stateOld(KM, 74)
YIELDfn_MAX_M3 = stateOld(KM, 75)

C_________START ITERATIVE LOOP FOR CONVERGENCE OF STRAINS_____
C_________ASSUME INCRMENTAL PLASTIC STRAIN TO BE ZERO_____
Dplastic_M1 = 0.0D0
Dplastic_M2 = 0.0D0
Dplastic_M3 = 0.0D0
C_________INITIALIZE SOLUTION VECTOR TO BE ZERO_____
OLD_TOTALsol = 0.0D0
C_________________________________________________________
DO LOOP = 1, ITERATE_LOOP
C_________RHS VECTOR FOR NORMAL STRAINS_____
VECTOR_NORMAL_RHS = 0.0D0
  VECTOR_NORMAL_RHS(1,1) = (RVE_a + RVE_b)*STRAIN_RVE(2,1)
VECTOR_NORMAL_RHS(2,1) = (RVE_a + RVE_b)*STRAIN_RVE(2,1)
VECTOR_NORMAL_RHS(3,1) = (RVE_a + RVE_c)*STRAIN_RVE(3,1)
VECTOR_NORMAL_RHS(4,1) = (RVE_a + RVE_c)*STRAIN_RVE(3,1)
VECTOR_NORMAL_RHS(5,1) = (C_MAT(2,1) - C_INC(2,1))*
1                  STRAIN_RVE(1,1)
2                  - C_MAT(2,1)*(OLDplastic_M1(1,1) + Dplastic_M1(1,1))
3                  - C_MAT(2,2)*(OLDplastic_M2(2,1) + Dplastic_M2(2,1))
4                  - C_MAT(2,3)*(OLDplastic_M3(3,1) + Dplastic_M3(3,1))
VECTOR_NORMAL_RHS(6,1) =
1                  C_MAT(2,1)*(OLDplastic_M2(1,1) + Dplastic_M2(1,1))
2                  + C_MAT(2,2)*(OLDplastic_M3(2,1) + Dplastic_M3(2,1))
3                  + C_MAT(2,3)*(OLDplastic_M3(3,1) + Dplastic_M3(3,1))
VECTOR_NORMAL_RHS(7,1) = (C_MAT(3,1) - C_INC(3,1))*
1                  STRAIN_RVE(1,1)
2                  - C_MAT(3,1)*(OLDplastic_M3(1,1) + Dplastic_M3(1,1))
3                  - C_MAT(3,2)*(OLDplastic_M3(2,1) + Dplastic_M3(2,1))
4                  - C_MAT(3,3)*(OLDplastic_M3(3,1) + Dplastic_M3(3,1))
VECTOR_NORMAL_RHS(8,1) =
1                  C_MAT(3,1)*(OLDplastic_M1(1,1) + Dplastic_M1(1,1))
2                  + C_MAT(3,2)*(OLDplastic_M2(2,1) + Dplastic_M2(2,1))
3                  + C_MAT(3,3)*(OLDplastic_M2(2,1) + Dplastic_M2(2,1))
VECTOR_SHEAR_RHS(1,1) = 0.0D0
VECTOR_SHEAR_RHS(2,1) = 0.0D0
VECTOR_SHEAR_RHS(3,1) = (RVE_a + RVE_b)*STRAIN_RVE(6,1)
VECTOR_SHEAR_RHS(4,1) = 0.0D0
VECTOR_SHEAR_RHS(5,1) = 0.0D0
VECTOR_SHEAR_RHS(6,1) = (RVE_a + RVE_c)*STRAIN_RVE(5,1)
VECTOR_SHEAR_RHS(7,1) = (RVE_a + RVE_b)*
1       (RVE_a + RVE_c)*STRAIN_RVE(4,1)
VECTOR_SHEAR_RHS(8,1) =
1       C_MAT(6,6)*RVE_c*(OLDplastic_M3(6,1) + Dplastic_M3(6,1))
2       - C_MAT(6,6)*RVE_a*(OLDplastic_M1(6,1) + Dplastic_M1(6,1))
3       - C_MAT(6,6)*RVE_c*(OLDplastic_M2(6,1) + Dplastic_M2(6,1))
VECTOR_SHEAR_RHS(9,1) =
1       C_MAT(5,5)*RVE_b*(OLDplastic_M1(5,1) + Dplastic_M1(5,1))
2       - C_MAT(5,5)*RVE_a*(OLDplastic_M3(5,1) + Dplastic_M3(5,1))
3       - C_MAT(5,5)*RVE_b*(OLDplastic_M2(5,1) + Dplastic_M2(5,1))
VECTOR_SHEAR_RHS(10,1) =
1       - C_MAT(4,4)*(OLDplastic_M3(4,1) + Dplastic_M3(4,1))
VECTOR_SHEAR_RHS(11,1) =
1       - C_MAT(4,4)*(OLDplastic_M1(4,1) + Dplastic_M1(4,1))
VECTOR_SHEAR_RHS(12,1) =
1       - C_MAT(4,4)*(OLDplastic_M2(4,1) + Dplastic_M2(4,1))
C________SOLUTION FOR NORMAL STRAINS___
SOLUTION_NORMAL = MATMUL(INV_COEFF_NORMAL,VECTOR_NORMAL_RHS)
C
C________SOLUTION FOR SHEAR STRAINS___
SOLUTION_SHEAR = MATMUL(INV_COEFF_SHEAR,VECTOR_SHEAR_RHS)
C
C________STRAIN IN EACH PHASE___
C_______STRAIN IN FIBER____
NEWstrain_F(1,1)  =  STRAIN_RVE(1,1)
NEWstrain_F(2,1)  =  SOLUTION_NORMAL(1,1)
NEWstrain_F(3,1)  =  SOLUTION_NORMAL(2,1)
NEWstrain_F(4,1)  =  SOLUTION_SHEAR(1,1)
NEWstrain_F(5,1)  =  SOLUTION_SHEAR(2,1)
NEWstrain_F(6,1)  =  SOLUTION_SHEAR(3,1)
C
C_______STRAIN IN MATRIX: M1____
NEWstrain_M1(1,1) =  STRAIN_RVE(1,1)
NEWstrain_M1(2,1) =  SOLUTION_NORMAL(3,1)
NEWstrain_M1(3,1) =  SOLUTION_NORMAL(4,1)
NEWstrain_M1(4,1) =  SOLUTION_SHEAR(4,1)
NEWstrain_M1(5,1) =  SOLUTION_SHEAR(5,1)
NEWstrain_M1(6,1) =  SOLUTION_SHEAR(6,1)
C
C_______STRAIN IN MATRIX: M2____
NEWstrain_M2(1,1) =  STRAIN_RVE(1,1)
NEWstrain_M2(2,1) =  SOLUTION_NORMAL(5,1)
NEWstrain_M2(3,1) =  SOLUTION_NORMAL(6,1)
NEWstrain_M2(4,1) =  SOLUTION_SHEAR(7,1)
NEWstrain_M2(5,1) =  SOLUTION_SHEAR(8,1)
NEWstrain_M2(6,1) =  SOLUTION_SHEAR(9,1)
C
C_______STRAIN IN MATRIX: M3____
NEWstrain_M3(1,1) =  STRAIN_RVE(1,1)
NEWstrain_M3(2,1) =  SOLUTION_NORMAL(7,1)
NEWstrain_M3(3,1) =  SOLUTION_NORMAL(8,1)
NEWstrain_M3(4,1) =  SOLUTION_SHEAR(10,1)
NEWstrain_M3(5,1) =  SOLUTION_SHEAR(11,1)
NEWstrain_M3(6,1) =  SOLUTION_SHEAR(12,1)
C
C________INCREMENTAL STRAINS IN EACH PHASE______
Dstrain_F  = NEWstrain_F  - OLDstrain_F
Dstrain_M1 = NEWstrain_M1 - OLDstrain_M1
Dstrain_M2 = NEWstrain_M2 - OLDstrain_M2
Dstrain_M3 = NEWstrain_M3 - OLDstrain_M3
C
C________TOTAL SOLUTION__
DO I = 1, 8
NEW_TOTALsol(I) = SOLUTION_NORMAL(I,1)
END DO
DO I = 1, 12
NEW_TOTALsol(I+8) = SOLUTION_SHEAR(I,1)
END DO
C
C________NORM OF SOLUTION VECTORS TO CHECK FOR CONVERGENCE______
NORM_SOL  = DOT_PRODUCT(NEW_TOTALsol, NEW_TOTALsol)
NORM_SOL  = SQRT(NORM_SOL)
NORM_DIFF = DOT_PRODUCT(DIFF_SOL, DIFF_SOL)
NORM_DIFF = SQRT(NORM_DIFF)
IF(NORM_SOL  EQ. 0.0D0)NORM_SOL = 1.0D0
IF(ABS(NORM_DIFF/NORM_SOL) .LE. TOL_SOLUTION)EXIT
OLD_TOTALsol = NEW_TOTALsol
C
C________CALL PLASTIC ROUTINE TO CHECK FOR INELASTIC DEFORMATION______
C________PLASTIC DEFORMATION: M1______
CALL PLASTIC (C_MAT, A_plast, N_pow, YIELDfn_MAX_M1,
1   Dstrain_M1, OLDstress_M1, NEWstress_M1,
2   OLDstrain_M1, NEWstrain_M1, OLDplastic_M1,  NEWplastic_M1,
3   Dplastic_M1, totalTime, timeinc)
C________PLASTIC DEFORMATION: M2
CALL PLASTIC (C_MAT, A_plast, N_pow, YIELDfn_MAX_M2,
  1                     Dstrain_M2, OLDstress_M2, NEWstress_M2,
  2   OLDstrain_M2, NEWstrain_M2, OLDplastic_M2, NEWplastic_M2,
  3                             Dplastic_M2, totalTime, timeinc)
C________PLASTIC DEFORMATION: M3
CALL PLASTIC (C_MAT, A_plast, N_pow, YIELDfn_MAX_M3,
  1                     Dstrain_M3, OLDstress_M3, NEWstress_M3,
  2   OLDstrain_M3, NEWstrain_M3, OLDplastic_M3, NEWplastic_M3,
  3                             Dplastic_M3, totalTime, timeinc)
C
END DO
C#####################################################################
C________STRESS IN EACH PHASE___
C________STRESS IN FIBER___
NEWstress_F   = MATMUL(C_INC, NEWstrain_F)
C_______STRESS IN M1, M2 & M3 CALCULATED FROM PLASTICITY ROUTINE________
C
C________STRESS IN RVE___
STRESS_RVE(1,1) =   RVE_a*RVE_a*NEWstress_F(1,1)
  1                     + RVE_a*RVE_b*NEWstress_M1(1,1)
  2                     + RVE_b*RVE_c*NEWstress_M2(1,1)
  3                     + RVE_c*RVE_c*NEWstress_M3(1,1)
STRESS_RVE(1,1) =  STRESS_RVE(1,1)/(RVE_a  + RVE_b)
STRESS_RVE(1,1) =  STRESS_RVE(1,1)/(RVE_a  + RVE_c)
C
STRESS_RVE(2,1) =  RVE_a*NEWstress_M1(2,1)
  1                     + RVE_c*NEWstress_M2(2,1)
STRESS_RVE(2,1) =  STRESS_RVE(2,1)/(RVE_a  + RVE_c)
C
STRESS_RVE(3,1) =  RVE_a*NEWstress_M3(3,1)
  1                     + RVE_b*NEWstress_M2(3,1)
STRESS_RVE(3,1) =  STRESS_RVE(3,1)/(RVE_a  + RVE_b)
C
STRESS_RVE(4,1) =  NEWstress_F(4,1)
C
STRESS_RVE(5,1) =  RVE_a*NEWstress_F(5,1)
  1                     + RVE_b*NEWstress_M1(5,1)
STRESS_RVE(5,1) =  STRESS_RVE(5,1)/(RVE_a  + RVE_b)
C
STRESS_RVE(6,1) =  RVE_a*NEWstress_F(6,1)
  1                     + RVE_c*NEWstress_M3(6,1)
STRESS_RVE(6,1) =  STRESS_RVE(6,1)/(RVE_a  + RVE_c)
C
C_________STORAGE OF DAMAGE HISTORY VARIABLES______
C_________stateOld(100)   =  DELETION
C_________stateOld(80:85) =  DAMAGE VARIABLES
C_________stateOld(86:90) =  PHI's
C_________stateOld(91:95) =  RV's
DO I=1,6
  DAMAGEold_VECT(I,1)= stateOld(KM,79+I)
END DO
C_________INITIALIZE VISUALIZING VARIABLES_______
PHI_1 =  stateOld(KM,86)
PHI_2 =  stateOld(KM,87)
PHI_3 =  stateOld(KM,88)
PHI_4 =  stateOld(KM,89)
PHI_5 =  stateOld(KM,90)
Rv1   =  stateOld(KM,91)
Rv2   =  stateOld(KM,92)
Rv3   =  stateOld(KM,93)
RV4   =  stateOld(KM,94)
RV5   =  stateOld(KM,95)
C_________CALL DAMAGE SUBROUTINE____
CALL HASHIN_DAMAGE(STRESS_RVE, DAMAGEold_VECT, DAMAGEnew_VECT,
  1                     Rv1, Rv2, Rv3, Rv4, Rv5, PHI_1, PHI_2, PHI_3, PHI_4, PHI_5)
C_________DAMAGED STRESS_____
DAMAGE_STRESS(1,1)=(1.0-DAMAGEnew_VECT(1,1))*STRESS_RVE(1,1)
DAMAGE_STRESS(2,1)=(1.0-DAMAGEnew_VECT(2,1))*STRESS_RVE(2,1)
DAMAGE_STRESS(3,1)=(1.0-DAMAGEnew_VECT(3,1))*STRESS_RVE(3,1)
DAMAGE_STRESS(4,1)=(1.0-DAMAGEnew_VECT(4,1))*STRESS_RVE(4,1)
DAMAGE_STRESS(5, 1) = (1.0 - DAMAGEnew_VECT(5, 1)) * STRESS_RVE(5, 1)
DAMAGE_STRESS(6, 1) = (1.0 - DAMAGEnew_VECT(6, 1)) * STRESS_RVE(6, 1)

C__ UPDATE HISTORY VARIABLES IN L/C SYSTEM ______________
C__ stateNew(1:6) = STRAIN_RVE
DO I = 1, 6
    stateNew(KM, I) = STRAIN_RVE(I, 1)
END DO
C__ stateNew(7:12) = NEWstrain_F
C__ stateNew(13:18) = NEWstrain_M1
C__ stateNew(19:24) = NEWstrain_M2
C__ stateNew(25:30) = NEWstrain_M3
DO I = 1, 6
    stateNEW(KM, I+6) = NEWstrain_F(I, 1)
    stateNEW(KM, I+12) = NEWstrain_M1(I, 1)
    stateNEW(KM, I+18) = NEWstrain_M2(I, 1)
    stateNEW(KM, I+24) = NEWstrain_M3(I, 1)
END DO
C__ stateNew(31:36) = NEWstress_F
C__ stateNew(37:42) = NEWstress_M1
C__ stateNew(43:48) = NEWstress_M2
C__ stateNew(49:54) = NEWstress_M3
DO I = 1, 6
    stateNEW(KM, I+30) = NEWstress_F(I, 1)
    stateNEW(KM, I+36) = NEWstress_M1(I, 1)
    stateNEW(KM, I+42) = NEWstress_M2(I, 1)
    stateNEW(KM, I+48) = NEWstress_M3(I, 1)
END DO
C__ stateNew(55:60) = NEWplastic_M1
C__ stateNew(61:66) = NEWplastic_M2
C__ stateNew(67:72) = NEWplastic_M3
DO I = 1, 6
    stateNEW(KM, I+54) = NEWplastic_M1(I, 1)
    stateNEW(KM, I+60) = NEWplastic_M2(I, 1)
    stateNEW(KM, I+66) = NEWplastic_M3(I, 1)
END DO
C__ stateNew(73) = YIELDfn_MAX_M1
C__ stateNew(74) = YIELDfn_MAX_M2
C__ stateNew(75) = YIELDfn_MAX_M3

C__ STORAGE OF DAMAGE HISTORY VARIABLES ______________
C__ stateOld(80:85) = DAMAGE VARIABLES
C__ stateOld(86:90) = PHI's
C__ UPDATE DAMAGE VARIABLES ______________
DO I = 1, 6
    stateNew(KM, 79+I) = DAMAGEnew_VECT(I, 1)
END DO
C__ UPDATE DAMAGE VISUALIZATION VARIABLES __________
stateNew(KM, 86) = PHI_1
stateNew(KM, 87) = PHI_2
stateNew(KM, 88) = PHI_3
stateNew(KM, 89) = PHI_4
stateNew(KM, 90) = PHI_5
stateNew(KM, 91) = RV1
stateNew(KM, 92) = RV2
stateNew(KM, 93) = RV3
stateNew(KM, 94) = RV4
stateNew(KM, 95) = RV5

C__ ROTATE STRESS BACK TO GLOBAL COORDINATE SYSTEM ______
DAMAGE_STRESS = MATMUL(INV_ROT_1, DAMAGE_STRESS)

C__ stressNew(KM, 1) = DAMAGE_STRESS(1, 1)
stressNew(KM, 2) = DAMAGE_STRESS(2, 1)
stressNew(KM, 3) = DAMAGE_STRESS(3, 1)
stressNew(KM, 4) = DAMAGE_STRESS(4, 1)
stressNew(KM, 5) = DAMAGE_STRESS(5, 1)
stressNew(KM, 6) = DAMAGE_STRESS(6, 1)

C__ ENERGY UPDATE ____________
ENERGY_DENSITY = 0.5D0 * (1
    + (stressOld(KM, 1) + stressNew(KM, 1)) * strainInc(KM, 1)

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\[ \text{enerInternNew(KM)} = \text{enerInternOld(KM)} + \]
\[ \text{ENERGY_DENSITY/density(KM)} \]

\[ \text{VOLUMETRIC STRAIN} \]

\[ F_{\text{GRAD}}(1,1) = \text{defgradNew(KM,1)} \]
\[ F_{\text{GRAD}}(1,2) = \text{defgradNew(KM,4)} \]
\[ F_{\text{GRAD}}(1,3) = \text{defgradNew(KM,9)} \]

\[ F_{\text{GRAD}}(2,1) = \text{defgradNew(KM,7)} \]
\[ F_{\text{GRAD}}(2,2) = \text{defgradNew(KM,2)} \]
\[ F_{\text{GRAD}}(2,3) = \text{defgradNew(KM,5)} \]

\[ F_{\text{GRAD}}(3,1) = \text{defgradNew(KM,6)} \]
\[ F_{\text{GRAD}}(3,2) = \text{defgradNew(KM,8)} \]
\[ F_{\text{GRAD}}(3,3) = \text{defgradNew(KM,3)} \]

\[ \text{DET}_F_1 = F_{\text{GRAD}}(2,2)*F_{\text{GRAD}}(3,3) - F_{\text{GRAD}}(2,3)*F_{\text{GRAD}}(3,2) \]
\[ \text{DET}_F_1 = F_{\text{GRAD}}(1,1)*\text{DET}_F_1 \]
\[ \text{DET}_F_2 = F_{\text{GRAD}}(2,1)*F_{\text{GRAD}}(3,3) - F_{\text{GRAD}}(2,3)*F_{\text{GRAD}}(3,1) \]
\[ \text{DET}_F_2 = F_{\text{GRAD}}(1,2)*\text{DET}_F_2 \]
\[ \text{DET}_F_3 = F_{\text{GRAD}}(2,1)*F_{\text{GRAD}}(3,2) - F_{\text{GRAD}}(2,2)*F_{\text{GRAD}}(3,1) \]
\[ \text{DET}_F_3 = F_{\text{GRAD}}(1,3)*\text{DET}_F_3 \]

\[ \text{VOL}_\text{RATIO} = \text{DET}_F_1 - \text{DET}_F_2 + \text{DET}_F_3 \]

\[ \text{ELEMT DELETE OPTION} \]

\[ \text{IF(DAMAGE}_{\text{new VECT}}(1,1) .GE. DAMAGE_TOL) \text{ AND.} \]
\[ \text{STATE}_{\text{NEW}}(KM, 100) = 0 \]
\[ \text{IF(VOL}_\text{RATIO} \text{ LT. CMPRSS_VOL OR.} \text{ VOL}_\text{RATIO} \text{ GT. EXPAND_VOL) THEN} \]
\[ \text{IF(DAMAGE}_{\text{new VECT}}(1,1) .GE. DAMAGE_TOL) \text{ STATE}_{\text{NEW}}(KM, 100) = 0 \]
\[ \text{IF(DAMAGE}_{\text{new VECT}}(2,1) .GE. DAMAGE_TOL) \text{ STATE}_{\text{NEW}}(KM, 100) = 0 \]
\[ \text{IF(DAMAGE}_{\text{new VECT}}(3,1) .GE. DAMAGE_TOL) \text{ STATE}_{\text{NEW}}(KM, 100) = 0 \]
\[ \text{IF(DAMAGE}_{\text{new VECT}}(4,1) .GE. DAMAGE_TOL) \text{ STATE}_{\text{NEW}}(KM, 100) = 0 \]
\[ \text{IF(DAMAGE}_{\text{new VECT}}(5,1) .GE. DAMAGE_TOL) \text{ STATE}_{\text{NEW}}(KM, 100) = 0 \]
\[ \text{IF(DAMAGE}_{\text{new VECT}}(6,1) .GE. DAMAGE_TOL) \text{ STATE}_{\text{NEW}}(KM, 100) = 0 \]
\[ \text{END IF} \]
\[ \text{END DO} \]
\[ \text{RETURN} \]
\[ \text{END} \]

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**SUBROUTINE TO DETERMINE ORIENTATION**

**SUBROUTINE ELEMENT_SET**

**IMPLICIT NONE**

**INTEGER, DIMENSION(500000) :: ALONG_0, ALONG_90, ALONG_45, ALONG_45N**

**INTEGER :: I, J, K**

**INTEGER :: LOOP_LNTH**

**LNTH_0 = 17160**

**LNTH_90 = 17160**

**LNTH_45 = 17160**

**LNTH_45N = 17160**

**LOOP_LNTH = 17160**

**ELEMENT SET ALONG -45 DIRECTION**

**DO I=1,LOOP_LNTH**

**ALONG_45N(I) = I**

**END DO**

**ELEMENT SET ALONG 0 DIRECTION**

**DO I=1,LOOP_LNTH**

**ALONG_0(I) = I + 17160**

**END DO**

**ELEMENT SET ALONG 45 DIRECTION**

**DO I=1,LOOP_LNTH**

**ALONG_45(I) = I + 34320**

**END DO**

**ELEMENT SET ALONG 90 DIRECTION**

**RETURN**

**END**

---

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DO I=1,LOOP_LENGTH
   ALONG_90(I) = I + 51480
END DO
END SUBROUTINE ELEMENT_SET

SUBROUTINE COMPLIANCE_TRANS(E11, E22, E33, G12, G13, G23, NU12, NU13, NU23, COMPLIANCE)
IMPLICIT NONE
DOUBLE PRECISION :: E11, E22, E33, G12, G13, G23
DOUBLE PRECISION :: NU12, NU13, NU23
DOUBLE PRECISION, DIMENSION(6,6) :: COMPLIANCE
NU21 = NU12*(E22/E11)
NU31 = NU13*(E33/E11)
NU32 = NU23*(E33/E22)
C COMPLIANCE MATRIX
COMPLIANCE = 0.0D0
COMPLIANCE(1,1)= 1/E11
COMPLIANCE(1,2)= -NU12/E11
COMPLIANCE(1,3)= -NU13/E11
C
COMPLIANCE(2,1)= -NU21/E22
COMPLIANCE(2,2)= 1/E22
COMPLIANCE(2,3)= -NU23/E22
C
COMPLIANCE(3,1)= -NU31/E33
COMPLIANCE(3,2)= -NU32/E33
COMPLIANCE(3,3)= 1/E33
C
COMPLIANCE(4,4)= 1/G23
COMPLIANCE(5,5)= 1/G13
COMPLIANCE(6,6)= 1/G12
C
END SUBROUTINE COMPLIANCE_TRANS

SUBROUTINE COMPLIANCE_ISO(E_MAT, NU_MAT, COMPLIANCE)
IMPLICIT NONE
DOUBLE PRECISION :: E_MAT, NU_MAT
DOUBLE PRECISION, DIMENSION(6,6) :: COMPLIANCE
C COMPLIANCE MATRIX
COMPLIANCE = 0.0D0
COMPLIANCE(1,1)= 1.0D0
COMPLIANCE(1,2)= -NU_MAT
COMPLIANCE(1,3)= -NU_MAT
C
COMPLIANCE(2,1)= -NU_MAT
COMPLIANCE(2,2)= 1.0D0
COMPLIANCE(2,3)= -NU_MAT
C
COMPLIANCE(3,1)= -NU_MAT
COMPLIANCE(3,2)= -NU_MAT
COMPLIANCE(3,3)= 1.0D0
C
COMPLIANCE(4,4)= 2.0D0*(1.0D0 + NU_MAT)
COMPLIANCE(5,5)= 2.0D0*(1.0D0 + NU_MAT)
COMPLIANCE(6,6)= 2.0D0*(1.0D0 + NU_MAT)
C
COMPLIANCE = (1/E_MAT)*COMPLIANCE
C
END SUBROUTINE COMPLIANCE_ISO

SUBROUTINE ROTATION(TETA, ROT_1, ROT_2, INV_ROT_1, INV_ROT_2)
IMPLICIT NONE
DOUBLE PRECISION :: TETA
DOUBLE PRECISION, DIMENSION(6,6) :: ROT_1, ROT_2, INV_ROT_1, INV_ROT_2

END SUBROUTINE ROTATION

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INTEGER :: IFLAG

C______ROTATION MATRIX 1______
ROT_1 = 0.0D0

C____ROW_1______
ROT_1(1,1) = DCOS(TETA)*DCOS(TETA)
ROT_1(1,2) = DSIN(TETA)*DSIN(TETA)
ROT_1(1,6) = 2.0D0*DCOS(TETA)*DSIN(TETA)

C____ROW_2______
ROT_1(2,1) = DSIN(TETA)*DCOS(TETA)
ROT_1(2,2) = DCOS(TETA)*DCOS(TETA)
ROT_1(2,6) = -2.0D0*DCOS(TETA)*DSIN(TETA)

C____ROW_3______
ROT_1(3,3) = 1.0D0

C____ROW_4______
ROT_1(4,4) = DCOS(TETA)
ROT_1(4,5) = -DSIN(TETA)

C____ROW_5______
ROT_1(5,4) = DSIN(TETA)
ROT_1(5,5) = DCOS(TETA)

C____ROW_6______
ROT_1(6,1) = -DSIN(TETA)*DCOS(TETA)
ROT_1(6,2) = DSIN(TETA)*DCOS(TETA)
ROT_1(6,6) = DCOS(TETA)*DCOS(TETA) - DSIN(TETA)*DSIN(TETA)

C______ROTATION MATRIX 2______
ROT_2 = 0.0D0

C____ROW_1______
ROT_2(1,1) = DCOS(TETA)*DCOS(TETA)
ROT_2(1,2) = DSIN(TETA)*DSIN(TETA)
ROT_2(1,6) = DCOS(TETA)*DSIN(TETA)

C____ROW_2______
ROT_2(2,1) = DSIN(TETA)*DSIN(TETA)
ROT_2(2,2) = DCOS(TETA)*DCOS(TETA)
ROT_2(2,6) = -DCOS(TETA)*DSIN(TETA)

C____ROW_3______
ROT_2(3,3) = 1.0D0

C____ROW_4______
ROT_2(4,4) = DCOS(TETA)
ROT_2(4,5) = -DSIN(TETA)

C____ROW_5______
ROT_2(5,4) = DSIN(TETA)
ROT_2(5,5) = DCOS(TETA)

C____ROW_6______
ROT_2(6,1) = -2.0D0*DSIN(TETA)*DSIN(TETA)
ROT_2(6,2) = 2.0D0*DSIN(TETA)*DCOS(TETA)
ROT_2(6,6) = DCOS(TETA)*DCOS(TETA) - DSIN(TETA)*DSIN(TETA)

C______INVERSE ROTATION MATRIX______
INV_ROT_1 = ROT_1
CALL MATINV(INV_ROT_1,6,6,IFLAG)

INV_ROT_2 = ROT_2
CALL MATINV(INV_ROT_2,6,6,IFLAG)
END SUBROUTINE ROTATION

C******************************************************************************
C***********SUBROUTINE FOR PLASTIC DEFORMATION***** ********************
C******************************************************************************

SUBROUTINE PLASTIC(STIFFNESS, A_plast, N_pow, YIELDfn_MAX, INC_STRAIN, OLD_STRESS, NEW_STRESS, OLD_STRAIN, NEW_STRAIN, PLASTIC_STRAINold, PLASTIC_STRAINnew, INC_PLASTIC_STRAIN, PRESENT_TIME, INC_TIME)
IMPLICIT NONE
DOUBLE PRECISION :: PRESENT_TIME, INC_TIME
DOUBLE PRECISION :: A_plast, N_pow, YIELDfn_MAX, INC_STRAIN, OLD_STRESS, NEW_STRESS
DOUBLE PRECISION, DIMENSION(6,1) :: INC_STRAIN, INC_STRESS
DOUBLE PRECISION, DIMENSION(6,1) :: OLD_STRAIN, NEW_STRAIN
DOUBLE PRECISION, DIMENSION(6,1) :: Dplastic_M1, totalTime, timeinc

C________NONLINEAR HARDENING______
CALL PLASTIC (C_MAT, A_plast, N_pow, YIELDfn_MAX_M1, Dstrain_M1, OLDstress_M1, NEWstress_M1, OLDstrain_M1, NEWstrain_M1, OLDplastic_M1, NEWplastic_M1, Dplastic_M1, totalTime, timeinc)
DOUBLE PRECISION, DIMENSION(6,1) :: PLASTIC_STRAINold,
       PLASTIC_STRAINnew,
       INC_PLASTIC_STRAIN
DOUBLE PRECISION, DIMENSION(6,6) :: COMPLIANCE, STIFFNESS
DOUBLE PRECISION, DIMENSION(6,6) :: COMPLIANCE_P, STIFFNESS_P
DOUBLE PRECISION, DIMENSION(6,6) :: COMPLIANCE_EP, STIFFNESS_EP
DOUBLE PRECISION :: SUM_1, SUM_2, SUM_3
DOUBLE PRECISION :: DIFF_1, DIFF_2, DIFF_3
DOUBLE PRECISION :: Hp, TRM, SIG_BAR
INTEGER :: I, J, K, L, IFLAG, LOOP

C  
C__COMPLIANCE MATRIX__
C
COMPLIANCE = STIFFNESS
CALL MATINV(COMPLIANCE, 6, 6, IFLAG)

C  
C___TERMS REQUIRED FOR ELASTO-PLASTIC MATRIX___
C
SUM_1 = 2.0D0*OLD_STRESS(1,1) - OLD_STRESS(2,1) - OLD_STRESS(3,1)
SUM_2 = 2.0D0*OLD_STRESS(2,1) - OLD_STRESS(1,1) - OLD_STRESS(3,1)
SUM_3 = 2.0D0*OLD_STRESS(3,1) - OLD_STRESS(2,1) - OLD_STRESS(1,1)

C
DIFF_1 = OLD_STRESS(1,1) - OLD_STRESS(2,1)
DIFF_2 = OLD_STRESS(1,1) - OLD_STRESS(3,1)
DIFF_3 = OLD_STRESS(3,1) - OLD_STRESS(2,1)

C___YIELD FUNCTION___
YIELDfn = (1.0/3.0)*(DIFF_1*DIFF_1 + DIFF_2*DIFF_2 + DIFF_3*DIFF_3)
YIELDfn = YIELDfn + 2.0D0*OLD_STRESS(4,1)*OLD_STRESS(4,1)
YIELDfn = YIELDfn + 2.0D0*OLD_STRESS(5,1)*OLD_STRESS(5,1)
YIELDfn = YIELDfn + 2.0D0*OLD_STRESS(6,1)*OLD_STRESS(6,1)
YIELDfn = YIELDfn/2.0D0

C___SIGMA_BAR___
SIG_BAR = SQRT(3.0D0*YIELDfn)
IF(SIG_BAR .LE. 40.0E6 .OR. YIELDfn_MAX .GT. YIELDfn)THEN
  COMPLIANCE_EP = COMPLIANCE
ELSE
  YIELDfn_MAX = YIELDfn
  Hp = (SIG_BAR - A_plast)/ N_pow
  Hp = EXP(Hp)
  Hp = N_pow/Hp
  TRM = 1.0D0/(4.0D0*Hp*SIG_BAR*SIG_BAR)
C
COMPLIANCE_EP(1,1) = COMPLIANCE(1,1) + TRM*SUM_1*SUM_1
COMPLIANCE_EP(1,2) = COMPLIANCE(1,2) + TRM*SUM_1*SUM_2
COMPLIANCE_EP(1,3) = COMPLIANCE(1,3) + TRM*SUM_1*SUM_3
COMPLIANCE_EP(1,4) = COMPLIANCE(1,4) + 6.0D0*TRM*SUM_1*OLD_STRESS(4,1)
COMPLIANCE_EP(1,5) = COMPLIANCE(1,5) + 6.0D0*TRM*SUM_1*OLD_STRESS(5,1)
COMPLIANCE_EP(1,6) = COMPLIANCE(1,6) + 6.0D0*TRM*SUM_1*OLD_STRESS(6,1)

C
COMPLIANCE_EP(2,1) = COMPLIANCE(2,1) + TRM*SUM_2*SUM_1
COMPLIANCE_EP(2,2) = COMPLIANCE(2,2) + TRM*SUM_2*SUM_2
COMPLIANCE_EP(2,3) = COMPLIANCE(2,3) + TRM*SUM_2*SUM_3
COMPLIANCE_EP(2,4) = COMPLIANCE(2,4) + 6.0D0*TRM*SUM_2*OLD_STRESS(4,1)
COMPLIANCE_EP(2,5) = COMPLIANCE(2,5) + 6.0D0*TRM*SUM_2*OLD_STRESS(5,1)
COMPLIANCE_EP(2,6) = COMPLIANCE(2,6) + 6.0D0*TRM*SUM_2*OLD_STRESS(6,1)

C
COMPLIANCE_EP(3,1) = COMPLIANCE(3,1) + TRM*SUM_3*SUM_1
COMPLIANCE_EP(3,2) = COMPLIANCE(3,2) + TRM*SUM_3*SUM_2
COMPLIANCE_EP(3,3) = COMPLIANCE(3,3) + TRM*SUM_3*SUM_3
COMPLIANCE_EP(3,4) = COMPLIANCE(3,4) + 6.0D0*TRM*SUM_3*OLD_STRESS(4,1)
COMPLIANCE_EP(3,5) = COMPLIANCE(3,5) + 6.0D0*TRM*SUM_3*OLD_STRESS(5,1)
COMPLIANCE_EP(3,6) = COMPLIANCE(3,6) + 6.0D0*TRM*SUM_3*OLD_STRESS(6,1)

C
COMPLIANCE_EP(4,1) = COMPLIANCE(4,1) +
COMPLIANCE_EP(4,2) = COMPLIANCE(4,2) + 
6.0D0*TRM*SUM_1*OLD_STRESS(4,1)

COMPLIANCE_EP(4,3) = COMPLIANCE(4,3) + 
6.0D0*TRM*SUM_2*OLD_STRESS(4,1)

COMPLIANCE_EP(4,4) = COMPLIANCE(4,4) + 
36.0D0*TRM*OLD_STRESS(4,1)*OLD_STRESS(4,1)

COMPLIANCE_EP(4,5) = COMPLIANCE(4,5) + 
36.0D0*TRM*OLD_STRESS(4,1)*OLD_STRESS(5,1)

COMPLIANCE_EP(4,6) = COMPLIANCE(4,6) + 
36.0D0*TRM*OLD_STRESS(4,1)*OLD_STRESS(6,1)

COMPLIANCE_EP(5,1) = COMPLIANCE(5,1) + 
6.0D0*TRM*SUM_1*OLD_STRESS(5,1)

COMPLIANCE_EP(5,2) = COMPLIANCE(5,2) + 
6.0D0*TRM*SUM_2*OLD_STRESS(5,1)

COMPLIANCE_EP(5,3) = COMPLIANCE(5,3) + 
6.0D0*TRM*SUM_3*OLD_STRESS(5,1)

COMPLIANCE_EP(5,4) = COMPLIANCE(5,4) + 
36.0D0*TRM*OLD_STRESS(5,1)*OLD_STRESS(4,1)

COMPLIANCE_EP(5,5) = COMPLIANCE(5,5) + 
36.0D0*TRM*OLD_STRESS(5,1)*OLD_STRESS(5,1)

COMPLIANCE_EP(5,6) = COMPLIANCE(5,6) + 
36.0D0*TRM*OLD_STRESS(5,1)*OLD_STRESS(6,1)

COMPLIANCE_EP(6,1) = COMPLIANCE(6,1) + 
6.0D0*TRM*SUM_1*OLD_STRESS(6,1)

COMPLIANCE_EP(6,2) = COMPLIANCE(6,2) + 
6.0D0*TRM*SUM_2*OLD_STRESS(6,1)

COMPLIANCE_EP(6,3) = COMPLIANCE(6,3) + 
6.0D0*TRM*SUM_3*OLD_STRESS(6,1)

COMPLIANCE_EP(6,4) = COMPLIANCE(6,4) + 
36.0D0*TRM*OLD_STRESS(6,1)*OLD_STRESS(4,1)

COMPLIANCE_EP(6,5) = COMPLIANCE(6,5) + 
36.0D0*TRM*OLD_STRESS(6,1)*OLD_STRESS(5,1)

COMPLIANCE_EP(6,6) = COMPLIANCE(6,6) + 
36.0D0*TRM*OLD_STRESS(6,1)*OLD_STRESS(6,1)

END IF
STIFFNESS_EP = COMPLIANCE_EP
CALL MATINV(STIFFNESS_EP, 6, 6, IFLAG)

STRESS INCREMENT
INC_STRESS = MATMUL(STIFFNESS_EP, INC_STRAIN)

NEW STRESS
NEW_STRESS = OLD_STRESS + INC_STRESS

INCREMENTAL PLASTIC STRAIN
INC_PLASTIC_STRAIN = INC_STRAIN - MATMUL(COMPLIANCE,INC_STRESS)
PLASTIC_STRAINnew = PLASTIC_STRAINold + INC_PLASTIC_STRAIN

SUBROUTINE HASHIN_DAMAGE(EFF_STRESS,DAMAGEold_VECT,DAMAGEnew_VECT,
Rv1, Rv2, Rv3, Rv4, Rv5, PHI_1, PHI_2, PHI_3, PHI_4, PHI_5)
IMPLICIT NONE
DOUBLE PRECISION, PARAMETER :: DAMAGE_TOL = 0.95
DOUBLE PRECISION, DIMENSION(6,1) :: EFF_STRESS, DMG_STRESS
DOUBLE PRECISION, DIMENSION(6,6) :: EFF_TENSOR
DOUBLE PRECISION :: OMEGA_1, OMEGA_2, OMEGA_3, OMEGA_4, OMEGA_5
DOUBLE PRECISION, DIMENSION(6) :: ACTIVE
DOUBLE PRECISION :: R1, R2, R3, R4, R5
DOUBLE PRECISION :: POW
DOUBLE PRECISION :: X_T, X_C, Y_T, Y_C, Z_C, S_T, S
DOUBLE PRECISION :: PHI_1, PHI_2, PHI_3, PHI_4, PHI_5
DOUBLE PRECISION :: Rv1, Rv2, Rv3, Rv4, Rv5
DOUBLE PRECISION :: TMP1, TMP2, TMP3, TMP4, TMP5

STRAIN TERMS
DOUBLE PRECISION :: STRESS11, STRESS22, STRESS33,
STRESS12, STRESS23, STRESS13

END SUBROUTINE HASHIN_DAMAGE
DOUBLE PRECISION :: MAC_S11, MAC_nS11
DOUBLE PRECISION :: ABS_SUM, SUM
DOUBLE PRECISION :: MAC_S33, MAC_nS33
INTEGER :: I, J, K, L, M

C______DAMAGE PARAMETERS____
X_T = 2.1E9
X_C = 1.1E9
Y_T = 0.135E9
Y_C = 0.21E9
Z_C = 3.0E9
S = 0.07E9
S_T = 0.07E9
POW = 4.0D0

C____STRESS VALUES____
STRESS11 = EFF_STRESS(1,1)
STRESS22 = EFF_STRESS(2,1)
STRESS33 = EFF_STRESS(3,1)
STRESS12 = EFF_STRESS(6,1)
STRESS23 = EFF_STRESS(4,1)
STRESS13 = EFF_STRESS(5,1)

C____MACAULY STRESS VARIABLES____
IF(STRESS11 .GE. 0.0D0)THEN
    MAC_S11  = STRESS11
    MAC_nS11 = 0.0D0
ELSE
    MAC_S11  = 0.0D0
    MAC_nS11 = -STRESS11
END IF
IF(STRESS33 .GT. 0.0D0)THEN
    MAC_S33  = STRESS33
    MAC_nS33 = 0.0D0
ELSE
    MAC_S33  = 0.0D0
    MAC_nS33 = -STRESS33
END IF
SUM = STRESS22 + STRESS33
ABS_SUM = ABS(STRESS22 + STRESS33)

C____DAMAGE-FIBER TENSILE FAILURE: MODE1___
R1 = (MAC_S11/X_T)**2 + 1
    (STRESS12*STRESS12 + STRESS13*STRESS13)/(S*S)
IF(R1 .GE. Rv1)Rv1 = R1
IF(R1 .LT. 1.0D0)THEN
    R1 = 1.0D0
ELSE
    R1 = SQRT(R1)
END IF

C____DAMAGE-FIBER COMPRESSIVE FAILURE: MODE2____
R2 = (MAC_nS11/X_C)**2
IF(R2 .GE. Rv2)Rv2 = R2
IF(R2 .LT. 1.0D0)THEN
    R2 = 1.0D0
ELSE
    R2 = SQRT(R2)
END IF

C____DAMAGE-FIBER CRUSH FAILURE: MODE3___
R3 = (MAC_nS33/Z_C)**2
IF(R3 .GE. Rv3)Rv3 = R3
IF(R3 .LT. 1.0D0)THEN
    R3 = 1.0D0
ELSE
    R3 = SQRT(R3)
END IF

C____DAMAGE-TENSILE TRANSVERSE FAILURE: MODE 4___
IF(SUM .GE. 0.0D0)THEN
    R4 = (SUM/Y_T)**2 + 1
        (STRESS23*STRESS23 - STRESS22*STRESS33)/(S_T*S_T)
        + (STRESS12*STRESS12 + STRESS13*STRESS13)/(S*S)
ELSE
    R4 = 0.0D0
END IF
IF(R4 .GE. Rv4)Rv4 = R4
IF(R4 .LT. 1.0D0) THEN
  R4 = 1.0D0
ELSE
  R4 = SQRT(R4)
END IF

C____DAMAGE- COMPRESSIVE TRANSVERSE FAILURE: MODE 5____

IF(SUM .LT. 0.0D0) THEN
  R5 = (1.0D0/Y_C)*(((Y_C/(2.0D0*S_T))**2)-1.0D0)*SUM
  + (ABS_SUM/(2.0D0*S_T))**2
  + (STRESS23*STRESS23 - STRESS22*STRESS33)/(S_T*S_T)
  + (STRESS12*STRESS12 + STRESS13*STRESS13)/(S*S)
ELSE
  R5 = 0.0D0
END IF

IF(R5 .GE. Rv5) Rv5 = R5
IF(R5 .LT. 1.0D0) THEN
  R5 = 1.0D0
ELSE
  R5 = SQRT(R5)
END IF

C____DAMAGE VARIABLES____

OMEGA_1 = 1.0D0 - EXP((1.0D0/POW)*(1.0D0-R1**POW))
OMEGA_2 = 1.0D0 - EXP((1.0D0/POW)*(1.0D0-R2**POW))
OMEGA_3 = 1.0D0 - EXP((1.0D0/POW)*(1.0D0-R3**POW))
OMEGA_4 = 1.0D0 - EXP((1.0D0/POW)*(1.0D0-R4**POW))
OMEGA_5 = 1.0D0 - EXP((1.0D0/POW)*(1.0D0-R5**POW))

C____DAMAGE ASSOCIATED WITH E11_COMP: DIFFERENT MODES____

ACTIVE(1) = OMEGA_1
ACTIVE(2) = OMEGA_2
ACTIVE(3) = OMEGA_3
ACTIVE(4) = 0.0D0
ACTIVE(5) = 0.0D0
ACTIVE(6) = 0.0D0
DAMAGEnew_VECT(1,1) = MAXVAL(ACTIVE)
IF(DAMAGEnew_VECT(1,1) .GT. DAMAGE_TOL)
  1  DAMAGEnew_VECT(1,1) = DAMAGE_TOL
IF(DAMAGEnew_VECT(1,1) .LT. DAMAGEold_VECT(1,1))
  1  DAMAGEnew_VECT(1,1) = DAMAGEold_VECT(1,1)

C____DAMAGE ASSOCIATED WITH E22_COMP: DIFFERENT MODES____

ACTIVE(1) = 0.0D0
ACTIVE(2) = 0.0D0
ACTIVE(3) = OMEGA_3
ACTIVE(4) = OMEGA_4
ACTIVE(5) = OMEGA_5
ACTIVE(6) = 0.0D0
DAMAGEnew_VECT(2,1) = MAXVAL(ACTIVE)
IF(DAMAGEnew_VECT(2,1) .GT. DAMAGE_TOL)
  1  DAMAGEnew_VECT(2,1) = DAMAGE_TOL
IF(DAMAGEnew_VECT(2,1) .LT. DAMAGEold_VECT(2,1))
  1  DAMAGEnew_VECT(2,1) = DAMAGEold_VECT(2,1)

C____DAMAGE ASSOCIATED WITH E33_COMP: DIFFERENT MODES____

ACTIVE(1) = 0.0D0
ACTIVE(2) = 0.0D0
ACTIVE(3) = OMEGA_3
ACTIVE(4) = 0.0D0
ACTIVE(5) = 0.0D0
ACTIVE(6) = 0.0D0
DAMAGEnew_VECT(3,1) = MAXVAL(ACTIVE)
IF(DAMAGEnew_VECT(3,1) .GT. DAMAGE_TOL)
  1  DAMAGEnew_VECT(3,1) = DAMAGE_TOL
IF(DAMAGEnew_VECT(3,1) .LT. DAMAGEold_VECT(3,1))
  1  DAMAGEnew_VECT(3,1) = DAMAGEold_VECT(3,1)

C____DAMAGE ASSOCIATED WITH G12_COMP: DIFFERENT MODES____

ACTIVE(1) = OMEGA_1
ACTIVE(2) = OMEGA_2
ACTIVE(3) = OMEGA_3
ACTIVE(4) = OMEGA_4
ACTIVE(5) = OMEGA_5
ACTIVE(6) = 0.0D0
DAMAGEnew_VECT(6,1) = MAXVAL(ACTIVE)
IF(DAMAGEnew_VECT(6,1) .GT. DAMAGE_TOL)
```fortran
1 DAMAGEnew_VECT(6,1) = DAMAGE_TOL
  IF(DAMAGEnew_VECT(6,1).LT. DAMAGEold_VECT(6,1))
1 DAMAGEnew_VECT(6,1) = DAMAGEold_VECT(6,1)
C____DAMAGE ASSOCIATED WITH G23_COMP: DIFFERENT MODES___
ACTIVE(1) = 0.0D0
ACTIVE(2) = 0.0D0
ACTIVE(3) = OMEGA_3
ACTIVE(4) = OMEGA_4
ACTIVE(5) = OMEGA_5
ACTIVE(6) = 0.0D0
DAMAGEnew_VECT(4,1) = MAXVAL(ACTIVE)
  IF(DAMAGEnew_VECT(4,1) .GT. DAMAGE_TOL)
1 DAMAGEnew_VECT(4,1) = DAMAGE_TOL
  IF(DAMAGEnew_VECT(4,1) .LT. DAMAGEold_VECT(4,1))
1 DAMAGEnew_VECT(4,1) = DAMAGEold_VECT(4,1)
C____DAMAGE ASSOCIATED WITH G13_COMP: DIFFERENT MODES___
ACTIVE(1) = OMEGA_1
ACTIVE(2) = OMEGA_2
ACTIVE(3) = OMEGA_3
ACTIVE(4) = 0.0D0
ACTIVE(5) = 0.0D0
ACTIVE(6) = 0.0D0
DAMAGEnew_VECT(5,1) = MAXVAL(ACTIVE)
  IF(DAMAGEnew_VECT(5,1) .GT. DAMAGE_TOL)
1 DAMAGEnew_VECT(5,1) = DAMAGE_TOL
  IF(DAMAGEnew_VECT(5,1) .LT. DAMAGEold_VECT(5,1))
1 DAMAGEnew_VECT(5,1) = DAMAGEold_VECT(5,1)
DO I=1,6
  IF(DAMAGEnew_VECT(I,1) .GT. DAMAGE_TOL)
1          DAMAGEnew_VECT(I,1) = DAMAGE_TOL
END DO
C____DAMAGE TENSOR___
DMG_TENSOR = 0.0D0
DMG_TENSOR(1,1) = (1.0D0-DAMAGEnew_VECT(1,1))
DMG_TENSOR(2,2) = (1.0D0-DAMAGEnew_VECT(2,1))
DMG_TENSOR(3,3) = (1.0D0-DAMAGEnew_VECT(3,1))
DMG_TENSOR(4,4) = (1.0D0-DAMAGEnew_VECT(4,1))
DMG_TENSOR(5,5) = (1.0D0-DAMAGEnew_VECT(5,1))
DMG_TENSOR(6,6) = (1.0D0-DAMAGEnew_VECT(6,1))
C____DAMAGE STRESS___
DMG_STRESS = MATMUL(DMG_TENSOR, EFF_STRESS)
C_____INTERNAL VARIABLES TO IDENTIFY DAMAGE MODE INITIATION___
IF(OMEGA_1 .GT. PHI_1) PHI_1 = OMEGA_1
IF(OMEGA_2 .GT. PHI_2) PHI_2 = OMEGA_2
IF(OMEGA_3 .GT. PHI_3) PHI_3 = OMEGA_3
IF(OMEGA_4 .GT. PHI_4) PHI_4 = OMEGA_4
IF(OMEGA_5 .GT. PHI_5) PHI_5 = OMEGA_5
END SUBROUTINE HASHIN_DAMAGE
C************************************************************************************
C***************SUBROUTINE TO OBTAIN INVERSE OF A MATRIX*************************
C************************************************** **********************************
SUBROUTINE MATINV (A,LDA,N,IFLAG)
C-------------------------------------------------- ---------------------
C   DEFINITION OF PASSED PARAMETERS
C     * A = MATRIX (SIZE NXN) TO BE INVERTED (DOUBLE PRECISION)
C   * LDA = LEADING DIMENSION OF MATRIX A [LDA>=N] (INTEGER)
C     * N = NUMBER OF ROWS AND COLUMNS OF MATRIX A (INTEGER)
C   IFLAG = ERROR INDICATOR ON OUTPUT (INTEGER)   I NTERPRETATION:
C           -2 -> TOO MANY ROW INTERCHANGES NEEDED - INCREASE MX
C           -1 -> N>LDA
C            0 -> NO ERRORS DETECTED
C            K -> MATRIX A FOUND TO BE SINGULAR AT THE KTH STEP OF
C                 THE CROUT REDUCTION (1<=K<=N)
C
C   * INDICATES PARAMETERS REQUIRING INPUT VALUES
C-----------------------------------------------------------------------
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
PARAMETER (MX=100)
DIMENSION A(LDA,*),IEX(MX,2)
IFLAG = 0
C---- CHECK CONSISTENCY OF PASSED PARAMETERS
```
IF (N.GT.LDA) THEN
  IFLAG = -1
  RETURN
ENDIF

C---- COMPUTE A = LU BY THE CROUT REDUCTION WHERE L IS LOWER TRIANGULAR
C---- AND U IS UNIT UPPER TRIANGULAR (ALGORITHM 3.4, P. 138 OF THE REFERENCE)
NEX = 0
DO K = 1, N
  DO I = K, N
    S = A(I,K)
    DO L = 1, K-1
      S = S - A(I,L)*A(L,K)
    END DO
    A(I,K) = S
  END DO
  C---- INTERCHANGE ROWS IF NECESSARY
  Q = 0.0
  L = 0
  DO I = K, N
    R = ABS(A(I,K))
    IF (R.GT.Q) THEN
      Q = R
      L = I
    ENDIF
  END DO
  IF (L.EQ.0) THEN
    IFLAG = K
    RETURN
  ENDIF
  IF (L.NE.K) THEN
    NEX = NEX+1
    IF (NEX.GT.MX) THEN
      IFLAG = -2
      RETURN
    ENDIF
    IEX(NEX,1) = K
    IEX(NEX,2) = L
    DO J = 1, N
      Q = A(K,J)
      A(K,J) = A(L,J)
      A(L,J) = Q
    END DO
  END IF
END DO

C---- END ROW INTERCHANGE SECTION
DO J = K+1, N
  S = A(K,J)
  DO L = 1, K-1
    S = S - A(K,L)*A(L,J)
  END DO
  A(K,J) = S/A(K,K)
END DO

C---- INVERT THE LOWER TRIANGLE L IN PLACE (SIMILAR TO ALGORITHM 1.5, P. 110 OF THE REFERENCE)
DO K = N, 1, -1
  A(K,K) = 1.0/A(K,K)
  DO I = K-1, 1, -1
    S = 0.0
    DO J = I+1, K
      S = S + A(J,I)*A(K,J)
    END DO
    A(K,I) = -S/A(I,I)
  END DO
END DO

C---- INVERT THE UPPER TRIANGLE U IN PLACE (ALGORITHM 1.5, P. 110 OF THE REFERENCE)
DO K = N, 1, -1
  DO I = K-1, 1, -1
    S = A(I,K)
    DO J = I+1, K-1
      S = S + A(I,J)*A(J,K)
    END DO
    A(I,K) = -S
C--- COMPUTE \text{INV}(A) = \text{INV}(U) \ast \text{INV}(L)
DO I = 1, N
  DO J = 1, N
    IF (J.GT.I) THEN
      S = 0.0
      L = J
    ELSE
      S = A(I,J)
      L = I+1
    ENDIF
    DO K = L, N
      S = S+A(I,K) \ast A(K,J)
    END DO
    A(I,J) = S
  END DO
END DO

C--- INTERCHANGE COLUMNS OF INV(A) TO REVERSE EFFECT OF ROW INTERCHANGES OF A
DO I = NEX, 1, -1
  K = IEX(I,1)
  L = IEX(I,2)
  DO J = 1, N
    Q = A(J,K)
    A(J,K) = A(J,L)
    A(J,L) = Q
  END DO
END DO
RETURN
END SUBROUTINE MATINV
Appendix C
VUMAT code to analyze inelastic deformation in homogenized UFPC

```
subroutine vumat ( 
  Read only - 
  1 jblock, ndir, nshr, nstatev, nfieldv, nprops, lanneal, 
  2 stepTime, totalTime, dt, cmname, coordMp, charLength, 
  3 props, density, strainInc, relSpinInc, 
  4 tempOld, stretchOld, defgradOld, fieldOld, 
  5 stressOld, stateOld, enerInternOld, enerInelasOld, 
  6 tempNew, stretchNew, defgradNew, fieldNew, 
  Write only - 
  7 stressNew, stateNew, enerInternNew, enerInelasNew )

  include 'vaba_param.inc'

dimension jblock(*), props(nprops), density(*), coordMp(*), 
  1 charLength(*), strainInc(*), 
  2 relSpinInc(*), tempOld(*), 
  3 stretchOld(*), 
  4 defgradOld(*), 
  5 fieldOld(*), stressOld(*), 
  6 stateOld(*), enerInternOld(*), 
  7 enerInelasOld(*), tempNew(*), 
  8 stretchNew(*), 
  9 defgradNew(*), 
  1 fieldNew(*), 
  2 stressNew(*), stateNew(*), 
  3 enerInternNew(*), enerInelasNew(*)

c     character*80 cmname

c     parameter ( 
  1 i_umt_nblock = 1, 
  2 i_umt_npt = 2, 
  3 i_umt_layer = 3, 
  4 i_umt_ksp = 4, 
  5 i_umt_noel = 5 )

  call vumatXtrArg ( jblock(i_umt_nblock), 
  1 ndir, nshr, nstatev, nfieldv, nprops, lanneal, 
  2 stepTime, totalTime, dt, cmname, coordMp, charLength, 
  3 props, density, strainInc, relSpinInc, 
  4 tempOld, stretchOld, defgradOld, fieldOld, 
  5 stressOld, stateOld, enerInternOld, enerInelasOld, 
  6 tempNew, stretchNew, defgradNew, fieldNew, 
  7 stressNew, stateNew, enerInternNew, enerInelasNew, 
  8 jblock(i_umt_noel), jblock(i_umt_npt), 
  9 jblock(i_umt_layer), jblock(i_umt_ksp))

  return
end

subroutine vumatXtrArg ( 
  read only - 
  1 nblock, ndir, nshr, nstatev, nfieldv, nprops, lanneal, 
  2 stepTime, totalTime, timeinc, cmname, coordMp, charLength, 
  3 props, density, strainInc, relSpinInc, 
  4 tempOld, stretchOld, defgradOld, fieldOld, 
  5 stressOld, stateOld, enerInternOld, enerInelasOld, 
  6 tempNew, stretchNew, defgradNew, fieldNew, 
  write only - 
  5 stressNew, stateNew, enerInternNew, enerInelasNew, 
  read only extra arguments - 
  6 nElement, nMatPoint, nLayer, nSecPoint)

  include 'vaba_param.inc'

c     all arrays dimensioned by (*) are not used in this algorithm
```

150
dimension props(nprops), density(nblock),
1 strainInc(nblock,ndir+nshr),
2 relSpinInc(nblock,nshr), defgradOld(nblock,9),
4 stressOld(nblock,ndir+nshr),
5 stateOld(nblock,nstatev), enerInternOld(nblock),
6 enerInelasOld(nblock),
7 stretchNew(nblock,ndir+nshr), defgradNew(nblock,9),
8 stressNew(nblock,ndir+nshr)

dimension enerInelasNew(nblock), stateNew(nblock,nstatev),
1 enerInternNew(nblock)
1 nElement(nblock), nMatPoint(nblock), nLayer(nblock),
1 nSecPoint(nblock)
character*80 cmname

DOUBLE PRECISION, PARAMETER :: PI = 3.141592653589793D0
DOUBLE PRECISION, PARAMETER :: TOL_STRAIN = 1.0E-8
DOUBLE PRECISION, PARAMETER :: A22 = 1.0D0, A33 = 1.0D0,
1 A23 = -1.0D0
DOUBLE PRECISION, PARAMETER :: BILINEAR = 2.20E8

C____PLASTIC PROPERTIES OF THE COMPOSITE
DOUBLE PRECISION :: A_plast, N_pow, A44, A66, A55, TRM
DOUBLE PRECISION :: SIGold_BAR, SIGnew_BAR
DOUBLE PRECISION :: YIELDfn, YIELDfn_MAX

C____ELASTIC PROPERTIES OF THE COMPOSITE________
DOUBLE PRECISION :: E11_COMP, E22_COMP, E33_COMP
DOUBLE PRECISION :: G12_COMP, G13_COMP, G23_COMP
DOUBLE PRECISION :: NU12_COMP, NU13_COMP, NU23_COMP,
1 NU21_COMP, NU31_COMP, NU32_COMP

C_____TRANSFORMATION MATRIX___
INTEGER, DIMENSION(500000) :: ALONG_0, ALONG_90
INTEGER :: LNTH_0, LNTH_90
DOUBLE PRECISION :: TETA
DOUBLE PRECISION, DIMENSION(6,6) :: ROT_1, ROT_2,
1 INV_ROT_1, INV_ROT_2

C_____COMPLIANCE AND STIFFNESS MATRIX OF THE COMPOSITE___
DOUBLE PRECISION, DIMENSION(6,6) :: C_COMP, S_COMP,
1 C_COMPep, S_COMPep

C_____STRESS AND STRAIN IN COMPOSITE_____
DOUBLE PRECISION, DIMENSION(6,1) :: STRESS_OLD, STRESS_NEW,
1 Dstrain_COMP, STRAIN_COMP

C____ENERGY DENSITY_______
DOUBLE PRECISION :: ENERGY_DENSITY

C_____ELEMENT NUMBER____
INTEGER :: ELEMENT_NO, LOGIC

C____VARIABLES TO DEFINE LOOP_____
INTEGER :: I, J, K, L, IFLAG, LOOP_OUT, LOOP_IN

C____storage of history variables all in local c/s_____
C stateOld(1:6) = STRAIN_COMP
C stateOld(7:12) = STRESS_COMP
C stateOld(13) = YIELDfn_MAX

C____ELASTIC PROPERTIES OF FIBER_______
E11_COMP = PROPS(1)
E22_COMP = PROPS(2)
E33_COMP = PROPS(3)
G12_COMP = PROPS(4)
G13_COMP = PROPS(5)
G23_COMP = PROPS(6)
NU12_COMP = PROPS(7)
NU13_COMP = PROPS(8)
NU23_COMP = PROPS(9)

C____CALCULATED POISSONS RATIO OF FIBER____
NU21_COMP = NU12_COMP*(E22_COMP/E11_COMP)
NU31_COMP = NU13_COMP*(E33_COMP/E11_COMP)
NU32_COMP = NU23_COMP*(E33_COMP/E22_COMP)

C____PLASTICITY PARAMETERS FOR COMPOSITE____
C A_plast = CONSTANT N_pow = SLOPE
A_plast = PROPS(10)
N_pow = PROPS(11)
A66 = PROPS(12)
A44 = PROPS(13)
A55 = A66

C___ELEMENT SETS ASSOCIATED WITH ROTATION MATRIX___
CALL ELEMENT_SET(ALONG_0, ALONG_90, LNTH_0, LNTH_90)

C___COMPLIANCE MATRIX OF COMPOSITE___
CALL COMPLIANCE_TRANS(E11_COMP, E22_COMP, E33_COMP,
1                    G12_COMP, G13_COMP, G23_COMP,
2                    NU12_COMP, NU13_COMP, NU23_COMP, S_COMP)
C_COMP = S_COMP
CALL MATINV(C_COMP,6,6,IFLAG)

C*****MAKE COMPUTATIONS AT EVERY MATERIAL POINT********
DO KM = 1,nblock
   LOGIC=0
   ELEMENT_NO = nElement(KM)
   IF(LOGIC .EQ. 0) THEN
      DO I = 1, LNTH_0
         IF(ELEMENT_NO .EQ. ALONG_0(I)) THEN
            TETA =45.0D0*PI/180.0D0
            LOGIC=1
            EXIT
         END IF
      END DO
   END IF
   IF(LOGIC .EQ. 0) THEN
      DO I = 1, LNTH_90
         IF(ELEMENT_NO .EQ. ALONG_90(I)) THEN
            TETA = 0.0*(1.0D0/3.0D0)*PI
            LOGIC=1
            EXIT
         END IF
      END DO
   END IF
   CALL ROTATION(TETA, ROT_1, ROT_2, INV_ROT_1, INV_ROT_2)
   C_________INCREMENTAL STRAIN IN GLOBAL COORDINATE SYSTEM______
   Dstrain_COMP(1,1) = strainInc(KM,1)
   Dstrain_COMP(2,1) = strainInc(KM,2)
   Dstrain_COMP(3,1) = strainInc(KM,3)
   Dstrain_COMP(4,1) = 2.0D0*strainInc(KM,5)
   Dstrain_COMP(5,1) = 2.0D0*strainInc(KM,6)
   Dstrain_COMP(6,1) = 2.0D0*strainInc(KM,4)
   C_________ROTATE INCREMENTAL STRAIN TO LOCAL COORDINATE SYSTEM______
   Dstrain_COMP = MATMUL(ROT_2, Dstrain_COMP )
   C_________STRAIN IN COMPOSITE (LOCAL C/S): PRESENT TIME STEP_____
   DO I=1,6
      STRAIN_COMP(I,1) = stateOld(KM,I) + Dstrain_COMP(I,1)
   END DO
   C_________STRESS IN COMPOSITE (LOCAL C/S): PREVIOUS TIME STEP_____
   DO I=1,6
      STRESS_OLD(I,1) = stateOld(KM,I+6)
   END DO
   C________MAXIMUM YIELD FUNCTION VALUE____
   YIELDfn_MAX = stateOld(KM,13)
   C________CALCULATE ELASTO-PLASTIC COMPLIANCE MATRIX___
   SIGold_BAR = (STRESS_OLD(2,1)- STRESS_OLD(3,1))
   SIGold_BAR = SIGold_BAR* (STRESS_OLD(2,1)- STRESS_OLD(3,1))
   SIGold_BAR = SIGold_BAR + 2*A44*STRESS_OLD(4,1)*STRESS_OLD(4,1)
   SIGold_BAR = SIGold_BAR + 1.0D0*A66*(STRESS_OLD(5,1)*STRESS_OLD(5,1)
   + STRESS_OLD(6,1)*STRESS_OLD(6,1))
   SIGold_BAR = SQRT(1.5D0*SIGold_BAR)
   C_________YIELD FUNCTION_______
   YIELDfn = SIGold_BAR*SIGold_BAR/3.0D0
   C S_COMPep = 0.0D0
   WRITE(*,*)"SIGold_BAR", SIGold_BAR
   WRITE(*,*)"YIELDfn", YIELDfn, "YIELDfn_MAX", YIELDfn_MAX
   IF(SIGold_BAR .EQ. 0.0D0 .OR. YIELDfn_MAX .GT. YIELDfn)THEN
S_COMPep = S_COMP
ELSE
YIELDfn_MAX = YIELDfn
Hp = (SIGold_BAR - A_plast)/ N_pow
Hp = EXP(Hp)
Hp = N_pow/Hp
TRM = 9.0D0/(Hp*SIGold_BAR*SIGold_BAR)
ROW 1
S_COMPep(1,1) = S_COMP(1,1)
S_COMPep(1,2) = S_COMP(1,2)
S_COMPep(1,3) = S_COMP(1,3)
S_COMPep(1,4) = 0.0D0
S_COMPep(1,5) = 0.0D0
S_COMPep(1,6) = 0.0D0
ROW 2
S_COMPep(2,1) = S_COMP(2,1)
S_COMPep(2,2) = A22*STRESS_OLD(2,1)+ A23*STRESS_OLD(3,1)
S_COMPep(2,2) = S_COMPep(2,2)*S_COMPep(2,2) + (TRM/4.0D0)* S_COMPep(2,2)
S_COMPep(2,3) = A23*STRESS_OLD(2,1)+ A33*STRESS_OLD(3,1)
S_COMPep(2,3) = S_COMPep(2,3) + (TRM/4.0D0) * S_COMPep(2,3)
S_COMPep(2,4) = A22*A44*STRESS_OLD(2,1)*STRESS_OLD(4,1)
S_COMPep(2,4) = S_COMPep(2,4) + (TRM/2.0D0) *S_COMPep(2,4)
S_COMPep(2,5) = A22*A66*STRESS_OLD(2,1)*STRESS_OLD(5,1)
S_COMPep(2,5) = S_COMPep(2,5) + (TRM/2.0D0) *S_COMPep(2,5)
S_COMPep(2,6) = A22*A66*STRESS_OLD(2,1)*STRESS_OLD(6,1)
S_COMPep(2,6) = S_COMPep(2,6) + (TRM/2.0D0) *S_COMPep(2,6)
ROW 3
S_COMPep(3,1) = S_COMP(1,3)
S_COMPep(3,2) = S_COMPep(2,3)
S_COMPep(3,3) = A23*STRESS_OLD(2,1)+ A33*STRESS_OLD(3,1)
S_COMPep(3,4) = A23*STRESS_OLD(3,1)*STRESS_OLD(4,1)
S_COMPep(3,4) = S_COMPep(3,4) + (TRM/2.0D0) *S_COMPep(3,4)
S_COMPep(3,5) = A23*STRESS_OLD(3,1)*STRESS_OLD(5,1)
S_COMPep(3,5) = S_COMPep(3,5) + (TRM/2.0D0) *S_COMPep(3,5)
S_COMPep(3,6) = A23*STRESS_OLD(3,1)*STRESS_OLD(6,1)
S_COMPep(3,6) = S_COMPep(3,6) + (TRM/2.0D0) *S_COMPep(3,6)
ROW 4
S_COMPep(4,1) = S_COMPep(1,4)
S_COMPep(4,2) = S_COMPep(2,4)
S_COMPep(4,3) = S_COMPep(3,4)
S_COMPep(4,4) = S_COMP(4,4) + TRM*A44*A44*STRESS_OLD(4,1)*STRESS_OLD(4,1)
S_COMPep(4,5) = S_COMP(4,5) +
S_COMPep(5, 1) = S_COMPep(1, 5)
S_COMPep(5, 2) = S_COMPep(2, 5)
S_COMPep(5, 3) = S_COMPep(3, 5)
S_COMPep(5, 4) = S_COMPep(4, 5)
S_COMPep(5, 5) = S_COMP(5, 5) + 
TRM*A66*A66*STRESS_OLD(5,1)*STRESS_OLD(5,1)
S_COMPep(5, 6) = S_COMP(5, 6) + 
TRM*A66*A66*STRESS_OLD(5,1)*STRESS_OLD(6,1)

S_COMPep(6, 1) = S_COMPep(1, 6)
S_COMPep(6, 2) = S_COMPep(2, 6)
S_COMPep(6, 3) = S_COMPep(3, 6)
S_COMPep(6, 4) = S_COMPep(4, 6)
S_COMPep(6, 5) = S_COMPep(5, 6)
S_COMPep(6, 6) = S_COMP(6, 6) + 
TRM*A66*A66*STRESS_OLD(6,1)*STRESS_OLD(6,1)

END IF
C_COMPep = S_COMPep
CALL MATINV(C_COMPep,6,6,IFLAG)

NEW STRESS IN COMPOSITE
STRESS_NEW = STRESS_OLD + MATMUL(C_COMPep,Dstrain_COMP)

UPDATE HISTORY VARIABLES IN L/C SYSTEM
DO I=1,6
stateNew(KM, I) = STRAIN_COMP(I,1)
stateNew(KM, I+6) = STRESS_NEW(I,1)
END DO
stateNew(KM,13) = YIELDfn_MAX

ROTATE STRESS BACK TO GLOBAL COORDINATE SYSTEM
STRESS_NEW = MATMUL(INV_ROT_1, STRESS_NEW)

stressNew(KM, 1) = STRESS_NEW(1,1)
stressNew(KM, 2) = STRESS_NEW(2,1)
stressNew(KM, 3) = STRESS_NEW(3,1)
stressNew(KM, 4) = STRESS_NEW(4,1)
stressNew(KM, 5) = STRESS_NEW(5,1)
stressNew(KM, 6) = STRESS_NEW(6,1)

ENERGY UPDATE
ENERGY_DENSITY = 0.5D0*(
1 (stressOld(KM,1) + stressNew(KM,1))*strainInc(KM,1)
2 + (stressOld(KM,2) + stressNew(KM,2))*strainInc(KM,2)
3 + (stressOld(KM,3) + stressNew(KM,3))*strainInc(KM,3)
4 + 2.0D0*(stressOld(KM,4) + stressNew(KM,4))*strainInc(KM,4)
5 + 2.0D0*(stressOld(KM,5) + stressNew(KM,5))*strainInc(KM,5)
6 + 2.0D0*(stressOld(KM,6) + stressNew(KM,6))*strainInc(KM,6))
enerInternNew(KM) = enerInternOld(KM) +
ENERGY_DENSITY/density(KM)

END DO
RETURN
END

C********************************************************
C******SUBROUTINE TO DETERMINE ORIENTATION*************
C******************************************************
SUBROUTINE ELEMENT_SET(ALONG_0, ALONG_90, LNTH_0, LNTH_90)
IMPLICIT NONE
INTEGER, DIMENSION(500000) :: ALONG_0
INTEGER, DIMENSION(500000) :: ALONG_90
INTEGER :: LNTH_0, LNTH_90
INTEGER :: I
LNTH_0 = 1
LNTH_90 = 1
C___ ELEMENT SET ALONG FIBER DIRECTION___
DO I=1,LNTH_0
   ALONG_0(I) = I
END DO
C___READING ELEMENT SET NORMAL TO FIBER DIRECTION___
DO I=1,LNTH_90
   ALONG_90(I) = I + 1200
END DO
END SUBROUTINE ELEMENT_SET
C**********************************************************************************
C***SUBROUTINE TO CALCULATE COMPLIANCE MATRIX OF TRANSVERSELY ISOTROPIC MATERIAL***
C**********************************************************************************
SUBROUTINE COMPLIANCE_TRANS(E11, E22, E33, G12, G13, G23,
                           NU12, NU13, NU23, COMPLIANCE)
IMPLICIT NONE
DOUBLE PRECISION :: E11, E22, E33, G12, G13, G23
DOUBLE PRECISION :: NU12, NU13, NU23, NU21, NU31, NU32
DOUBLE PRECISION, DIMENSION(6,6) :: COMPLIANCE
NU21 = NU12*(E22/E11)
NU31 = NU13*(E33/E11)
NU32 = NU23*(E33/E22)
C_____COMPLIANCE MATRIX___
COMPLIANCE(1,1) = 1/E11
COMPLIANCE(1,2) = -NU12/E11
COMPLIANCE(1,3) = -NU13/E11
C
COMPLIANCE(2,1) = -NU21/E22
COMPLIANCE(2,2) = 1/E22
COMPLIANCE(2,3) = -NU23/E22
C
COMPLIANCE(3,1) = -NU31/E33
COMPLIANCE(3,2) = -NU32/E33
COMPLIANCE(3,3) = 1/E33
C
COMPLIANCE(4,4) = 1/G23
COMPLIANCE(5,5) = 1/G13
COMPLIANCE(6,6) = 1/G12
C
END SUBROUTINE COMPLIANCE_TRANS
C**********************************************************************************
C***SUBROUTINE TO CALCULATE COMPLIANCE MATRIX OF ISOTROPIC MATERIAL***
C**********************************************************************************
SUBROUTINE COMPLIANCE_ISO(E_MAT, NU_MAT, COMPLIANCE)
IMPLICIT NONE
DOUBLE PRECISION :: E_MAT, NU_MAT
DOUBLE PRECISION, DIMENSION(6,6) :: COMPLIANCE
C_____COMPLIANCE MATRIX___
COMPLIANCE = 0.0D0
COMPLIANCE(1,1) = 1/E_MAT
COMPLIANCE(1,2) = -NU_MAT
COMPLIANCE(1,3) = -NU_MAT
C
COMPLIANCE(2,1) = -NU_MAT
COMPLIANCE(2,2) = 1/E_MAT
COMPLIANCE(2,3) = -NU_MAT
C
COMPLIANCE(3,1) = -NU_MAT
COMPLIANCE(3,2) = -NU_MAT
COMPLIANCE(3,3) = 1/E_MAT
C
COMPLIANCE(4,4) = 2.0D0*(1.0D0 + NU_MAT)
COMPLIANCE(5,5) = 2.0D0*(1.0D0 + NU_MAT)
COMPLIANCE(6,6) = 2.0D0*(1.0D0 + NU_MAT)
C
END SUBROUTINE COMPLIANCE_ISO
COMPLIANCE(6,6) = 2.0D0*(1.0D0 + NU_MAT)
C
COMPLIANCE = (1/E_MAT)*COMPLIANCE
C
END SUBROUTINE COMPLIANCE_ISO
C************************************************************************************
C***********SUBROUTINE TO OBTAIN ROTATION MATRICES******************************
C************************************************************************************
SUBROUTINE ROTATION(TETA, ROT_1, ROT_2, INV_ROT_1, INV_ROT_2)
IMPLICIT NONE
DOUBLE PRECISION :: TETA
DOUBLE PRECISION, DIMENSION(6,6) :: ROT_1, ROT_2,
1                                   INV_ROT_1, INV_ROT_2
INTEGER :: IFLAG
C______ROTATION MATRIX 1______
ROT_1 = 0.0D0
C_____ROW_1______
ROT_1(1,1) = DCOS(TETA)*DCOS(TETA)
ROT_1(1,2) = DSIN(TETA)*DSIN(TETA)
ROT_1(1,6) = 2.0D0*DCOS(TETA)*DSIN(TETA)
C_____ROW_2______
ROT_1(2,1) = DSIN(TETA)*DSIN(TETA)
ROT_1(2,2) = DCOS(TETA)*DCOS(TETA)
ROT_1(2,6) = -2.0D0*DCOS(TETA)*DSIN(TETA)
C_____ROW_3______
ROT_1(3,3) = 1.0D0
C_____ROW_4______
ROT_1(4,4) = DCOS(TETA)
ROT_1(4,5) = -DSIN(TETA)
C_____ROW_5______
ROT_1(5,4) = DSIN(TETA)
ROT_1(5,5) = DCOS(TETA)
C_____ROW_6______
ROT_1(6,1) = -DSIN(TETA)*DCOS(TETA)
ROT_1(6,2) =  DSIN(TETA)*DCOS(TETA)
ROT_1(6,6) =  DCOS(TETA)*DCOS(TETA)-DSIN(TETA)*DSIN(TETA)
C______ROTATION MATRIX 2______
ROT_2 = 0.0D0
C_____ROW_1______
ROT_2(1,1) = DCOS(TETA)*DCOS(TETA)
ROT_2(1,2) = DSIN(TETA)*DSIN(TETA)
ROT_2(1,6) = DCOS(TETA)*DSIN(TETA)
C_____ROW_2______
ROT_2(2,1) = DSIN(TETA)*DSIN(TETA)
ROT_2(2,2) = DCOS(TETA)*DCOS(TETA)
ROT_2(2,6) = -DCOS(TETA)*DSIN(TETA)
C_____ROW_3______
ROT_2(3,3) = 1.0D0
C_____ROW_4______
ROT_2(4,4) = DCOS(TETA)
ROT_2(4,5) = -DSIN(TETA)
C_____ROW_5______
ROT_2(5,4) = DSIN(TETA)
ROT_2(5,5) = DCOS(TETA)
C_____ROW_6______
ROT_2(6,1) = -2.0D0*DSIN(TETA)*DCOS(TETA)
ROT_2(6,2) =  2.0D0*DSIN(TETA)*DCOS(TETA)
ROT_2(6,6) =  DCOS(TETA)*DCOS(TETA)-DSIN(TETA)*DSIN(TETA)
C_____INVERSE ROTATION MATRIX______
INV_ROT_1 = ROT_1
CALL MATINV(INV_ROT_1,6,6,IFLAG)
INV_ROT_2 = ROT_2
CALL MATINV(INV_ROT_2,6,6,IFLAG)
END SUBROUTINE ROTATION
C************************************************************************************
C***************SUBROUTINE TO OBTAIN INVERSE OF A MATRIX**************************
C************************************************************************************
SUBROUTINE MATINV (A,LDA,N,IFLAG)
C---------------------------------------------------------------------
C   DEFINITION OF PASSED PARAMETERS
C     * A = MATRIX (SIZE NXN) TO BE INVERTED (DOUBLE PRECISION)
C * LDA = LEADING DIMENSION OF MATRIX A [LDA>=N] (INTEGER)
C * N = NUMBER OF ROWS AND COLUMNS OF MATRIX A (INTEGER)
C IFLAG = ERROR INDICATOR ON OUTPUT (INTEGER) INTERPRETATION:
C -2 -> TOO MANY ROW INTERCHANGES NEEDED - INCREASE MX
C -1 -> N>LDA
C 0 -> NO ERRORS DETECTED
C K -> MATRIX A FOUND TO BE SINGULAR AT THE KTH STEP OF
C THE CROUT REDUCTION (1<=K<=N)
C
C * INDICATES PARAMETERS REQUIRING INPUT VALUES
C-----------------------------------------------------------------------
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
PARAMETER (MX=100)
DIMENSION A(LDA,*),IEX(MX,2)
IFLAG = 0
C--- CHECK CONSISTENCY OF PASSED PARAMETERS
IF (N.GT.LDA) THEN
  IFLAG = -1
  RETURN
ENDIF
C--- COMPUTE A = LU BY THE CROUT REDUCTION WHERE L IS LOWER TRIANGULAR
C--- AND U IS UNIT UPPER TRIANGULAR (ALGORITHM 3.4, P. 138 OF THE--- REFERENCE)
NEX = 0
DO K = 1, N
  DO I = K, N
    S = A(I,K)
    DO L = 1, K-1
      S = S-A(I,L)*A(L,K)
    END DO
    A(I,K) = S
  END DO
  C--- INTERCHANGE ROWS IF NECESSARY
  Q = 0.0
  L = 0
  DO I = K, N
    R = ABS(A(I,K))
    IF (R.GT.Q) THEN
      Q = R
      L = I
    END IF
  END DO
  IF (L.EQ.0) THEN
    IFLAG = K
    RETURN
  END IF
  IF (L.NE.K) THEN
    NEX = NEX+1
    IF (NEX.GT.MX) THEN
      IFLAG = -2
      RETURN
    END IF
    IEX(NEX,1) = K
    IEX(NEX,2) = L
    DO J = 1, N
      Q = A(K,J)
      A(K,J) = A(L,J)
      A(L,J) = Q
    END DO
  END IF
END DO
C--- INVERT THE LOWER TRIANGLE L IN PLACE (SIMILAR TO ALGORITHM 1.5,--- P. 110 OF THE REFERENCE)
DO K = 1, N
  DO J = K+1, N
    S = A(K,J)
    DO L = 1, K-1
      S = S-A(K,L)*A(L,J)
    END DO
    A(K,J) = S/A(K,K)
  END DO
END DO
DO I = K-1, 1, -1
    S = 0.0
    DO J = I+1, K
        S = S+A(J,I)*A(K,J)
    END DO
    A(K,I) = -S/A(I,I)
END DO

C--- INVERT THE UPPER TRIANGLE U IN PLACE (ALGORITHM 1.5, P. 110 OF--- THE REFERENCE)
DO K = N, 1, -1
    DO I = K-1, 1, -1
        S = A(I,K)
        DO J = I+1, K-1
            S = S+A(I,J)*A(J,K)
        END DO
        A(I,K) = -S
    END DO
END DO

C--- COMPUTE INV(A) = INV(U)*INV(L)
DO I = 1, N
    DO J = 1, N
        IF (J.GT.I) THEN
            S = 0.0
            L = J
        ELSE
            S = A(I,J)
            L = I+1
        ENDIF
        DO K = L, N
            S = S+A(I,K)*A(K,J)
        END DO
        A(I,J) = S
    END DO
END DO

C--- INTERCHANGE COLUMNS OF INV(A) TO REVERSE EFFECT OF ROW INTERCHANGES OF A
DO I = NEX, 1, -1
    K = IEX(I,1)
    L = IEX(I,2)
    DO J = 1, N
        Q = A(J,K)
        A(J,K) = A(J,L)
        A(J,L) = Q
    END DO
END DO
RETURN
END SUBROUTINE MATINV
Appendix D
MATLAB code for Mori-Tanaka method

clear all
cclc

% REUSS RULE ______
C_MAT(1:6,1:6)=0;
D_MAT(1:6,1:6)=0;
C_INC(1:6,1:6)=0;
D_INC(1:6,1:6)=0;
fi=.64;

%--6*6 matrix [Cijkl]=[C]
D_MAT=inv(W)*inv(C_MAT)*inv(W);

%---6*6 matrix [Cijkl]=[C]
[P_MAT]=eshelby(nu_matx);

E_mat=4.0E9;
u_mat=0.35;
u_matx=nu_mat;
TRM_MAT=E_mat/((1+nu_mat)*(1-2*nu_mat));

%____MATRIX: ELASTIC AND ISOTROPIC____ [Cijkl]=[C]=STIFFNESS
% [Dijkl]=[D]=COMPLIANCE
E1=234.5E9;
E2=14E9;
E3=E2;
G12=27.6E9;
G13=G12;
u23=0.25;
u12=0.2;
u13=nu12;
u21=nu12*E2/E1;
u31=nu13*E3/E1;

%------ matrix Dijkl=[D]
D_INC=inv(D_INC);

%----6*6 matrix [Cijkl]=[C]
[P_MAT]=eshelby(nu_matx);

%---CALL ESHELBY TENSOR
ID_4=inv(W);

%-------------------------------------- Eqs.(7.3.16-17)p.168
% FUNDAMENTALS OF MACRO MECHANICS OF SOLIDS (J. QU & M. CHERKAOUI)
% EQUATIONS ARE IN 4TH-ORDER TENSOR FORMULATION
% [T]=6*6 matrix
%------------------ T:H = [T][W][H] -----> T and H 4th-order double contraction -->
% change all to 6*6 matrix
% T:G = [T][W][G] -----> 4TH-ORDER 2ND ORDER double contraction --> change all to 6*6 matrix
%------------------
% Micromechanics: Overall Properties of Heterogeneous Materials (S. NASSER & M. HORI)

T_temp=ID_4+P_MAT*W*(D_MAT*W*C_INC-ID_4);
T_INC=inv(W)*inv(T_temp)*inv(W);
L1=(1-fi)*C_MAT+fi*C_INC*T_INC;
L2_TEMP=(1-fi)*ID_4 + fi*T_INC;
L2=inv(W)*inv(L2_TEMP)*inv(W);
L = C_COMP=L1*W*L2;

nu12_COMP = -E11_COMP*D_COMP(1,2)
nu21_COMP = -E22_COMP*D_COMP(2,1)
nu13_COMP = -E11_COMP*D_COMP(1,3)
nu31_COMP = -E33_COMP*D_COMP(3,1)
nu23_COMP = -E22_COMP*D_COMP(2,3)
nu32_COMP = -E33_COMP*D_COMP(3,2)

t=1/D_COMP(1,1)
E11_COMP = 1/D_COMP(1,1)
E22_COMP = 1/D_COMP(2,2)
E33_COMP = 1/D_COMP(3,3)
nu12_COMP = -E11_COMP*D_COMP(1,2)
nu21_COMP = -E22_COMP*D_COMP(2,1)
nu13_COMP = -E11_COMP*D_COMP(1,3)
nu31_COMP = -E33_COMP*D_COMP(3,1)
nu23_COMP = -E22_COMP*D_COMP(2,3)
nu32_COMP = -E33_COMP*D_COMP(3,2)

e11_comp = 1/D_COMP(1,1)
ed22_comp = 1/D_COMP(2,2)
ed33_comp = 1/D_COMP(3,3)
u12_comp = -ed11_comp*ed33_comp(1,2)
u21_comp = -ed22_comp*ed33_comp(2,1)
u13_comp = -ed11_comp*ed33_comp(1,3)
u31_comp = -ed33_comp*ed33_comp(3,1)
u23_comp = -ed22_comp*ed33_comp(2,3)
u32_comp = -ed33_comp*ed33_comp(3,2)

e11_comp = 1/D_COMP(1,1)
ed22_comp = 1/D_COMP(2,2)
ed33_comp = 1/D_COMP(3,3)
u12_comp = -ed11_comp*ed33_comp(1,2)
u21_comp = -ed22_comp*ed33_comp(2,1)
u13_comp = -ed11_comp*ed33_comp(1,3)
u31_comp = -ed33_comp*ed33_comp(3,1)
u23_comp = -ed22_comp*ed33_comp(2,3)
u32_comp = -ed33_comp*ed33_comp(3,2)

function [ESHB]=eshelby(nu_mat)
%------------------ ESHELBY TENSOR FOR SPHERICAL INCLUSION
% ESHB(1:6,1:6)=0;
% nu=nu_mat;
% ESHB(1,1)=(7-5*nu)/15/(1-nu);
% ESHB(2,2)=ESHB(1,1);
% ESHB(3,3)=ESHB(1,1);
% ESHB(1,2)=(5*nu-1)/15/(1-nu);
% ESHB(2,3)=ESHB(1,2);
% ESHB(3,1)=ESHB(1,2);
% ESHB(1,3)=ESHB(1,2);
% ESHB(2,1)=ESHB(1,2);
% ESHB(3,2)=ESHB(1,2);
% ESHB(6,6)=(4-5*nu)/15/(1-nu);
% ESHB(4,4)=ESHB(6,6);
% ESHB(5,5)=ESHB(6,6);

%------------------ ESHELBY TENSOR FOR CYLINDRICAL INCLUSION

ESHB(1:6,1:6)=0;
ESHB(1,1)=(7-5*nu)/15/(1-nu);
ESHB(2,2)=ESHB(1,1);
ESHB(3,3)=ESHB(1,1);
ESHB(1,2)=(5*nu-1)/15/(1-nu);
ESHB(2,3)=ESHB(1,2);
ESHB(3,1)=ESHB(1,2);
ESHB(1,3)=ESHB(1,2);
ESHB(2,1)=ESHB(1,2);
ESHB(3,2)=ESHB(1,2);
ESHB(6,6)=(4-5*nu)/15/(1-nu);
ESHB(4,4)=ESHB(6,6);
ESHB(5,5)=ESHB(6,6);

function [ESHB]=voigt(ESHB_4);

function [CHK_MATRIX]=voigt(TENSOR)

CHK_MATRIX(1:6,1:6)=0;

% VOIGT OPERATION

for p=1:3
    for q=1:3
        for r=1:3
            for s=1:3
                if (p==1 & q==1)
                    i=1;
                elseif (p==1 & q==2)
                    i=6;
                elseif (p==1 & q==3)
                    i=5;
                elseif (p==2 & q==1)
                    i=6;
                elseif (p==2 & q==2)
                    i=2;
                elseif (p==2 & q==3)
                    i=4;
                elseif (p==3 & q==1)
                    i=5;
                elseif (p==3 & q==2)
                    i=4;
                else
                    i=3;
                end

                if (r==1 & s==1)
                    j=1;
                elseif (r==1 & s==2)
                    j=6;
                elseif (r==1 & s==3)
                    j=5;
                elseif (r==2 & s==1)
                    j=6;
                elseif (r==2 & s==2)
                    j=2;
                elseif (r==2 & s==3)
                    j=4;
                elseif (r==3 & s==1)
                    j=5;
                elseif (r==3 & s==2)
                    j=4;
                else
                    j=3;
                end

                CHK_MATRIX(i,j)=TENSOR(p,q,r,s);
            end
        end
    end
end
Appendix E
MATLAB code for Eshelby’s equivalent Inclusion method

```matlab
% (11->1, 22->2, 33->3, 23->4, 31->5, 12->6)
clear all
c=eye(6);

% REUSS RULE
C_MAT(1:6,1:6)=0;
D_MAT(1:6,1:6)=0;
C_INC(1:6,1:6)=0;
D_INC(1:6,1:6)=0;
fi=.7;

% MATRIX: ELASTIC AND ISOTROPIC
E_mat=5.35E9;
nu_mat=0.354;
nu_matx=nu_mat;
TRM_MAT=E_mat/((1+nu_mat)*(1-2*nu_mat));
G=E_mat/2/(1+nu_mat);
C_MAT(1,1)=1-nu_mat;
C_MAT(1,2)=nu_mat;
C_MAT(1,3)=nu_mat;
C_MAT(2,1)=C_MAT(1,2);
C_MAT(2,2)=C_MAT(1,1);
C_MAT(2,3)=nu_mat;
C_MAT(3,1)=C_MAT(1,3);
C_MAT(3,2)=C_MAT(2,3);
C_MAT(3,3)=C_MAT(1,1);
C_MAT=TRM_MAT*C_MAT;
C_MAT(4,4)=G;
C_MAT(5,5)=G;
C_MAT(6,6)=G;

% FIBER: ELASTIC AND TRANSVERSELY ISOTROPIC
E1=232.0E9;
E2=15.0E9;
E3=E2;
G12=24.0E9;
G13=G12;
nu12=0.279;
nu13=nu12;
nu21=nu12*(E2/E1);
nu31=nu13*(E3/E1);
G23=E2/2/(1+nu23);
G23=5.0E9;
nu23=0.49;
nu12=0.279;
nu13=nu12;
nu21=nu12*(E2/E1);
nu31=nu13*(E3/E1);
G23=E2/2/(1+nu23);
G23=5.0E9;
nu23=0.49;
nu12=0.279;
nu13=nu12;
nu21=nu12*(E2/E1);
D_INC(1,1)=1/E1;
D_INC(1,2)=nu12/E1;
D_INC(1,3)=nu13/E1;
D_INC(2,1)=nu21/E2;
D_INC(2,2)=1/E2;
D_INC(2,3)=nu23/E2;
D_INC(3,1)=nu31/E3;
D_INC(3,2)=nu32/E3;
D_INC(3,3)=1/E3;
D_INC(4,4)=1/G23;
D_INC(5,5)=1/G13;
D_INC(6,6)=1/G12;
C_INC=inv(D_INC);

%-- 6*6 matrix [Cijkl]=[C]

[P_MAT]=eshelby(nu_matx);  %-- CALL ESHELBY TENSOR
ID_4=inv(W);  %-- Eqs. (7.3.16-17)p.168
```

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% FUNDAMENTALS OF MACRO MECHANICS OF SOLIDS (J. QU & M. CHERKAOUI)
% EQUATIONS ARE IN 4TH-ORDER TENSOR FORMULATION
% [[]=6*6 matrix

%------------------ T:H = [T][W][H]------ T and H 4th-order double contraction -->
% change all to 6*6 matrix
%------------------ T:G = [T][W][G]------ 4th-order : 2nd order double contraction --> change all to 6*6 matrix

% Micromechanics: Overall Properties of Heterogeneous Materials (S. NASSER & M. HORT)

T_temp = ID_4 + P_MAT*W*(D_MAT*W*C_INC-ID_4); % [I+S1(D_MAT*C_INC-I)] : 4TH ORDER TENSOR --> WRITTEN 6*6 MATRIX NOTATION

TINC = inv(W)*inv(T_temp)*inv(W); % [I+S1(D_MAT*C_INC-I)]^-1 : 4TH ORDER TENSOR --> WRITTEN 6*6 MATRIX NOTATION

C_COMP = C_MAT + fi*(C_INC-C_MAT)*W*T_INC;

D_COMP = inv(C_COMP);

E11_COMP = 1/D_COMP(1,1)
E22_COMP = 1/D_COMP(2,2)
E33_COMP = 1/D_COMP(3,3)
nu12_COMP = -E11_COMP*D_COMP(1,2)
nu21_COMP = -E22_COMP*D_COMP(2,1)
nu13_COMP = -E11_COMP*D_COMP(1,3)
nu31_COMP = -E33_COMP*D_COMP(3,1)
nu23_COMP = -E22_COMP*D_COMP(2,3)
nu32_COMP = -E33_COMP*D_COMP(3,2)
G13_COMP = 1/D_COMP(5,5)
G12_COMP = 1/D_COMP(6,6)
G23_COMP = 1/D_COMP(4,4)

function [ESHB]=eshelby(nu_mat)
%----------------ESHELBY TENSOR FOR SPHERICAL INCLUSION
% ESHB(1:6,1:6)=0;

nu=nu_mat;
% ESHB(1,1)=(7-5*nu)/15/(1-nu);
% ESHB(2,2)=ESHB(1,1);
% ESHB(3,3)=ESHB(1,1);
% ESHB(1,2)=(5*nu-1)/15/(1-nu);
% ESHB(2,3)=ESHB(1,2);
% ESHB(3,1)=ESHB(1,2);
% ESHB(1,3)=ESHB(1,2);
% ESHB(2,1)=ESHB(1,2);
% ESHB(3,2)=ESHB(1,2);
% ESHB(6,6)=(4-5*nu)/15/(1-nu);
% ESHB(4,4)=ESHB(6,6);
% ESHB(5,5)=ESHB(6,6);

%---------------- ESHELBY TENSOR FOR CYLINDRICAL INCLUSION

ESHB(1:6,1:6)=0;

nu=nu_mat;
% ESHB(1,1)=(7-5*nu)/15/(1-nu);
% ESHB(2,2)=ESHB(1,1);
% ESHB(3,3)=ESHB(1,1);
% ESHB(1,2)=(5*nu-1)/15/(1-nu);
% ESHB(2,3)=ESHB(1,2);
% ESHB(3,1)=ESHB(1,2);
% ESHB(1,3)=ESHB(1,2);
% ESHB(2,1)=ESHB(1,2);
% ESHB(3,2)=ESHB(1,2);
% ESHB(6,6)=(4-5*nu)/15/(1-nu);
% ESHB(4,4)=ESHB(6,6);
% ESHB(5,5)=ESHB(6,6);

function [CHK_MATRIX]=voigt(TENSOR)
%_____VOIGT OPERATION_____
for p=1:3
    for q=1:3
        for r=1:3
            for s=1:3
                if (p==1 && q==1)
                    i=1;
                elseif (p==1 && q==2)
                    i=6;
                elseif (p==1 && q==3)
                    i=5;
                elseif (p==2 && q==1)
                    i=6;
                elseif (p==2 && q==2)
                    i=2;
                elseif (p==2 && q==3)
                    i=4;
                elseif (p==3 && q==1)
                    i=5;
                elseif (p==3 && q==2)
                    i=4;
                else
                    i=3;
                end
                if (r==1 && s==1)
                    j=1;
                elseif (r==1 && s==2)
                    j=6;
                elseif (r==1 && s==3)
                    j=5;
                elseif (r==2 && s==1)
                    j=6;
                elseif (r==2 && s==2)
                    j=2;
                elseif (r==2 && s==3)
                    j=4;
                elseif (r==3 && s==1)
                    j=5;
                elseif (r==3 && s==2)
                    j=4;
                else
                    j=3;
                end
                CHK_MATRIX(i,j)=TENSOR(p,q,r,s);
            end
        end
    end
end