A Distributed Parameter Approach to Optimal Filtering and Estimation with Mobile Sensor Networks

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(ABSTRACT)

In this thesis we develop a rigorous mathematical framework for analyzing and approximating optimal sensor placement problems for distributed parameter systems and apply these results to PDE problems defined by the convection-diffusion equations. The mathematical problem is formulated as a distributed parameter optimal control problem with integral Riccati equations as constraints. In order to prove existence of the optimal sensor network and to construct a framework in which to develop rigorous numerical integration of the Riccati equations, we develop a theory based on Bochner integrable solutions of the Riccati equations. In particular, we focus on $\mathcal{S}_p$-valued continuous solutions of the Bochner integral Riccati equation. We give new results concerning the smoothing effect achieved by multiplying a general strongly continuous mapping by operators in $\mathcal{S}_p$. These smoothing results are essential to the proofs of the existence of Bochner integrable solutions of the Riccati integral equations. We also establish that multiplication of continuous $\mathcal{S}_p$-valued functions improves convergence properties of strongly continuous approximating mappings and specifically approximating $C_0$-semigroups. We develop a Galerkin type numerical scheme for approximating the solutions of the integral Riccati equation and prove convergence of the approximating solutions in the $\mathcal{S}_p$-norm. Numerical examples are given to illustrate the theory.

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Dedication

Quiero dedicar mi tesis a mi familia y a todos los que me han enseñado alguna vez.

A mis padres Carlos G. Rautenberg y Ana L. Miranda, a mi hermano Francisco E. Rautenberg, quienes se toman el tiempo a diario para hacerme sentir en casa.

A Carlos E. D’Attellis, Ricardo H. Pichel y Daniel R. Bes. A los que les debo innumerables cosas y me siguen honrando con su amistad. No me alcanzaría ninguna dedicatoria para expresar la deuda que tengo con ellos.

A Flavio Sánchez, por su dedicación a la enseñanza.
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Chapter 1

Introduction and Problem Statement

State estimation problems in a distributed parameter setting are a major source of engineering and applied science problems. In practice, almost all sources of information of a process (the temperature distribution inside a room, the concentration of a certain substance in the sea, etc.) are produced by devices capable of measuring a magnitude (temperature, concentration of a toxin, etc.) whose location is either a design variable or a known fact. We call sensors to these type of devices.

This thesis focuses on the construction of a rigorous mathematical framework and the corresponding computational science tools that can be used to address problems of state estimation for spatially dependent systems. We develop a framework to address filtering problems for distributed parameter systems when stationary and mobile (dynamic) sensor networks are used to provide system measurements. We use this framework to construct corresponding practical computational tools. The framework can be used for systems governed by parabolic partial differential equations and hence has application to a diverse set of problems ranging from estimating sources and boundaries of biological or chemical plumes to optimal sensor location for the design and control of energy-efficient buildings. We will formulate the problems as hybrid systems on infinite dimensional spaces (coupled systems of partial and ordinary differential equations) and use infinite dimensional theory to develop practical and rigorous computational algorithms for each of the problems.

This research is motivated by applications to two distinct but related
problem areas: (1) Determining optimal sensor/actuator locations for complex hybrid spatial systems to enhance tracking, estimation, information and effectiveness while limiting energy consumption. This type of problem can be posed in various ways (e.g. a differential game with PDE constraints, or an infinite dimensional stochastic control problem) and it is not clear which approach provides the best framework to develop the necessary computational tools for such complex systems. (2) The second area is focused on optimal design and control of dynamic sensor fleets that include the dynamics of the mobile sensors. In this setting one must consider sensor dynamics as a constraint in the problems of optimal estimation and allow for information delays. The computational challenges require the development of both high-fidelity “off-line” computational simulation tools and real-time numerical algorithms to optimize and evaluate potential scenarios and to test the performance of real-time algorithms.

1.1 Mobile Sensor Networks for Defense

Consider a scenario as depicted in Figure 1.1 where a chemical agent is released and there are mobile sensors (Micro Air Vehicles, MAVs) to be used to determine the nature of the chemical, the density of the cloud and perhaps estimate the track of the chemical agent “cloud boundary”. The primary zone of interest might be limited to the green zone shown. The problem is to control the MAV sensors in such a way that the sensed information coming from the mobile sensors provides the “best information” about the cloud, its direction and the amount of interaction with the green zone. The problem clearly has many practical constraints such as the dynamics of the sensors, limits on the control authority and the time one has to react to the information being gathered. Of course, terms like “best information” need to be defined precisely and, when formulating this type of problem as a mathematical question, the choice of performance measure greatly impacts the nature of the solution and the corresponding computational requirements.

Before addressing the dynamic sensor problem, it is reasonable to ask if a mobile sensor network is needed to deal with a scenario of this type. In particular, one should ask, “How well can one do with ‘optimally placed’ static sensors and how much ‘additional information’ is gained from a mobile sensor network such as the one suggested in the above scenario?” There are simple examples (see below) where the answer is “not much”, and there
are cases where mobile sensor networks offer considerable improvement over fixed sensor networks.

From an intuitive point of view, if one knows that the zone of interest is the green zone and that the prevailing winds are towards the east-northeast, then strategically placing fixed biochemical sensors around and in the green zone might be as effective as a mobile sensor network. Moreover, although the computational challenges associated with this type of problem can be huge, there is a basic theoretical foundation in place to begin the development of computational algorithms for such problems. On the other hand, if one is given real-time weather information (say wind direction data), then the ability to move the sensor network could offer better information. In such cases however, one is faced with a data driven computational problem and the computational science needed to deal with this issue is not well understood and requires considerable advances in the construction of a proper theoretical framework and the development of practical numerical algorithms.
1.2 Sensor Placement for Energy Efficient Buildings

It is important to note that whole buildings are very complex multi-scale (in time and space) systems as Fig. 1.2 below illustrates. Optimal design and control of these systems are very challenging problems and are often done by first developing a reduced-order model and then basing the design on the simplified model. The basic problem is best described as a distributed parameter control system and this framework can provide useful information about how to develop computational tools for building design and control.

Figure 1.2: A Whole Building is a Complex System

In order to illustrate the idea, consider the problem illustrated by a single room shown in Fig. 1.3 below. Here, the goal is to design the room (locate vents, place sensors, etc.) in order to control the room temperature near the workspace and minimize energy. The problems of design and control should be considered simultaneously because the type and effectiveness of the controller depends on the type and quality of the sensed information and conversely. In this problem the system is governed by the Navier-Stokes...
equations in the room denoted by $\Omega$ and given by

$$\frac{\partial \mathbf{v}(t, \mathbf{x})}{\partial t} + \mathbf{v}(t, \mathbf{x}) \cdot \nabla \mathbf{v}(t, \mathbf{x}) = -\nabla p(t, \mathbf{x}) + \frac{1}{\text{Re}}\Delta \mathbf{v}(t, \mathbf{x})$$

(1.1)

$$\nabla \cdot \mathbf{v}(t, \mathbf{x}) = 0,$$

(1.2)

$$\frac{\partial T(t, \mathbf{x})}{\partial t} + \mathbf{v}(t, \mathbf{x}) \cdot \nabla T(t, \mathbf{x}) = \frac{1}{\text{RePr}}\Delta T(t, \mathbf{x}) + B(t, \mathbf{x}),$$

(1.3)

where $\mathbf{x} \in \Omega \subset \mathbb{R}^3$, $\mathbf{v}(t, \mathbf{x})$ is the velocity vector, $p(t, \mathbf{x})$ is the pressure and $T(t, \mathbf{x})$ is the temperature. Nondimensionalization has been carried out such that $\text{Re}$ is the Reynolds number and $\text{Pr}$ is the Prandtl number. Note that for this study, the energy equation (1.3) does not influence the momentum equation (1.1) and is a convection-diffusion problem once the flow field is known. Consequently, we begin with a study of these questions for the linear convection-diffusion equations.

### 1.3 History of the Optimal Estimation Problem

The background for this problem goes back to the early 1970’s when people first started to think about optimal sensor/actuator location problems for distributed parameter systems. Much of the initial research on mobile
sensors and actuators focused on achieving more practical observability and controllability conditions.

In the case of finite dimensional systems, necessary conditions similar to those obtained later for the optimal filtering problem in the infinite dimensional setting were first developed by M. Athans ([2]) and these optimality conditions were obtained through Pontryagin’s Maximum Principle.

The first mathematically rigorous attempt to solve the optimal filtering problem for a wide class of linear distributed parameter systems was given by Bensoussan (see for example [8]). Also this was the first attempt to provide a rigorous derivation of the Kalman-Bucy filter for linear distributed parameter systems (see Curtain’s paper [17] for a detailed historical development of the optimal filtering problem).

A comprehensive treatment of the optimal sensor location problem was presented by Bensoussan (see [7]) where necessary conditions similar to the ones given previously by Athans (in [2]) were obtained for the infinite dimensional setting. Curtain obtained a general form of the Kalman-Bucy filter in infinite dimensions for general bounded generators of evolution operators (see [18]). Balakrishnan developed the Kalman-Bucy filter equations in another approach using integral equation theory.

The problem of optimally placing fixed sensors to achieve “maximal observability” of a distributed parameter system is fundamental to estimation and control of such systems. However, the term “maximal observability” is not always precisely defined even for finite dimensional systems. During the mid 1990’s, Khapalov produced a series of papers on the design of optimal mobile sensors for a robust filtering problem and applied his results to parabolic and hyperbolic systems (see [34, 35, 36, 37, 38]). This work focused on questions of observability. We shall focus on optimal estimation criterion.

### 1.4 A PDE Filtering Problem with Mobile Sensors

Consider the convection-diffusion process in the $n$–dimensional unit cube $\Omega = (0, 1)^n \subset \mathbb{R}^n$ defined by

$$\frac{\partial}{\partial t} T = (c^2 \Delta + a(x) \cdot \nabla) T + b(t, x) \eta(t),$$

(1.4)
where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the $n$-dimensional Laplacian, $a(x) \cdot \nabla = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$ is the convection operator and the functions $x \mapsto a_i(x)$ are smooth on $x \in \Omega$. Here $\eta$ is a real-valued Wiener process (a zero mean Gaussian process) and for each $t \in [0, t_f]$ the function $b(t, \cdot)$ belongs to $L^2(\Omega)$. We also assume boundary and initial conditions are of the form

$$T(t, x) \bigg|_{\partial\Omega} = 0, \quad T(0, x) = T_0(x) + \xi,$$

where $T_0(\cdot) \in L^2(\Omega)$ and $\xi$ is a $L^2(\Omega)$-valued gaussian random variable. Since the boundary $\partial\Omega$ is of Lipschitz class, the natural state space for the problem is $L^2(\Omega)$ and the domain of the differential operator $A = (c^2 \Delta + a(x) \cdot \nabla)$ is $\mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega)$ when $c > 0$.

Assume that one has $p$ sensor-platforms (vehicles) moving in $\Omega$, each with a sensor capable of measuring an average value of $T(t, x)$ within a fixed range of the location of the platform. Let $\bar{x}_i(t) \in \Omega$, $i = 1, 2, \ldots, p$ denote the position of the $i^{th}$ sensor and time $t \in [0, t_f]$ and let

$$h_i(t) = \int_{B_\delta(\bar{x}_i(t)) \cap \Omega} k(x) T(t, x) \, dx + \nu_i(t). \quad (1.5)$$

denote the measured output which is the weighted average of the field $T(t, x)$ with weight $k(x)$ and sensor range

$$B_\delta(\bar{x}_i(t)) \equiv \{ x \in \mathbb{R}^n : \| x - \bar{x}_i(t) \| < \delta \}. \quad (1.6)$$

Here, each $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ is a zero-mean white noise process and is uncorrelated with the process disturbance $\eta$. Observe that one could also define a dynamic local sensor by

$$h_i(t) = \int_{\Omega} \chi(x, \bar{x}_i(t)) T(t, x) \, dx + \nu_i(t), \quad (1.7)$$

where $\chi(x, \bar{x}(t))$ is a (normalized) characteristic function defined by

$$\chi(x, \bar{x}(t)) = \begin{cases} 1/(\int_{B_\delta(0)} 1 \, dx), & x \in B_\delta(\bar{x}(t)) \cap \Omega \\ 0, & x \notin B_\delta(\bar{x}(t)) \cap \Omega \end{cases}. \quad (1.8)$$

This is the definition used by Khapalov (see [34], [35], [36], [37] and [38]) and offers a certain structure that allows for rigorous analysis when the dynamics of the vehicle network are included.
For a network of vehicle trajectories \( \bar{x}_i(t) \in \Omega, i = 1, 2, \ldots, p \), we define the output map \( C(t) : L^2(\Omega) \rightarrow \mathbb{R}^p \) by

\[
C(t)\varphi(\cdot) = \begin{bmatrix}
C_1(t)\varphi(\cdot) \\
C_2(t)\varphi(\cdot) \\
\vdots \\
C_p(t)\varphi(\cdot)
\end{bmatrix} \in \mathbb{R}^p, \tag{1.9}
\]

where

\[
C_i(t)\varphi(\cdot) = \int_{\Omega} \chi(x, \bar{x}_i(t))\varphi(x) \, dx. \tag{1.10}
\]

The previous definitions can be used to formulate an abstract (infinite dimensional) model of the form

\[
\dot{z}(t) = Az(t) + B(t)\eta(t) \in L^2(\Omega), \tag{1.11}
\]

with \( z(0) = z_0 \in L^2(\Omega) \) and measured output

\[
h(t) = C(t)z(t) + \nu(t). \tag{1.12}
\]

Here, the state of the distributed parameter system is \( z(t)(\cdot) = T(t, \cdot) \in L^2(\Omega) \). This is the standard abstract formulation of the convection-diffusion equations as a distributed parameter control system. This abstract model can be extended to include the case where \( \eta \) is a Wiener process with values in some separable Hilbert space \( X \) and \( B(t) \in \mathcal{L}(X, L^2(\Omega)) \) for each \( t \in [0, t_f] \). This extension will be important for the work presented here.

One approach to optimal estimation is to observe that the variance equation for the optimal estimator is the (weak) solution to an infinite dimensional Riccati (partial) differential equation of the form

\[
\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A^* + BR_1R_2^*B^*(t) - \Sigma(t)(C^*R_1^{-1}C)(t)\Sigma(t), \tag{1.13}
\]

with initial condition \( \Sigma(0) = \Sigma_0 \). The operators \( R_1(\cdot) \) and \( R_2(\cdot) \) are the incremental covariances of the uncorrelated Wiener processes \( \eta \) and \( \nu \), respectively, and \( \Sigma_0 \) is the covariance operator of the \( L^2(\Omega) \)-valued Gaussian random variable \( \xi \) (see [17] and [7]). The expected value of \( \|z(t) - \hat{z}(t)\|^2 \) is the trace of the solution to the infinite dimensional Riccati equation at time \( t \), i.e.,

\[
\mathbb{E}\{\|z(t) - \hat{z}(t)\|^2\} = \text{Tr} \, \Sigma(t),
\]
where $\hat{z}(\cdot)$ is the stochastic $L^2(\Omega)$-valued process solution to the generalized Kalman-Bucy filter (see [7] and [8]). Therefore, for a sensor network defined by $t \mapsto C(t)$, the trace of the solution to the Riccati equation is a measure of error between the state and the state estimator. We use this measure to define the optimal sensor location problem.

In particular, we proceed as in [42] and consider the distributed parameter optimal control problem of finding $C_{\text{opt}}(t)$ to minimize

$$J(C(\cdot)) = \int_0^{t_f} \text{Tr} \ Q(t) \ \Sigma(t) \ dt$$

(1.14)

where $\Sigma(\cdot)$ is the mild solution of (1.13), $C(\cdot)$ is defined by (1.9)-(1.10), and for each $t \in [0,t_f]$, the operator $Q(t) : L_2(\Omega) \mapsto L_2(\Omega)$ is a bounded linear operator. The (time-varying) map $Q(\cdot)$ allows one to weigh significant parts of the state estimate. For example, assume one has a control defined by a feedback operator $G(t) : Z \mapsto \mathbb{R}^m$. If a re-constructed state (observer) is to be used in a feedback controller, then one might choose $Q(t) = G^*G(t)$, in effect minimizing the error in the control produced by variance in the state estimate.

To complete the problem formulation for mobile sensors we include the sensor dynamics. Consider the case where $n = 3$ corresponding to a MAV type flying aircraft. In particular, let $\bar{x}_i(t)$ denote the location of the $i^{th}$ sensor platform at time $t$. The state of the sensor platform is defined by position and velocity so that

$$\bar{\theta}_i(t) = [\bar{x}_i(t), \dot{\bar{x}}_i(t)]^T$$

where $\dot{\bar{x}}_i(t)$ represents the velocity of the sensor. The dynamics of the sensor are assumed to be governed by some controlled ordinary differential equations in $\mathbb{R}^6$ given by

$$\dot{\bar{\theta}}_i(t) = f_i(t, \bar{\theta}_i(t), u_i(t)),$$

$$\bar{\theta}_i(0) = \bar{\theta}_i^0 = [\bar{x}_i^0, \dot{\bar{x}}_i^0]^T$$

(1.15) (1.16)

where $u_i(\cdot)$ belongs to some admissible control set $\mathcal{U}$ and the initial conditions $\bar{\theta}_i^0$ belong to some compact set $\Theta_0 \subset \mathbb{R}^6$. For this thesis we focus only on the position of the mobile sensor and observe that

$$\bar{x}_i(t) = [I \ 0] \dot{\bar{\theta}}_i(t) = [I \ 0] \begin{bmatrix} \bar{x}_i(t) \\ \dot{\bar{x}}_i(t) \end{bmatrix} \triangleq M \dot{\bar{\theta}}_i(t)$$

(1.17)
is the output to the controlled system (1.15) - (1.16). However, the theoretical results in Section 2 apply to the more general case. In particular, the position of the sensor is assumed to be given by \( \bar{x}_i(t) = M\bar{\theta}_i(t) \) where \( M \in \mathbb{R}^{3 \times 6} \) is a constant matrix and \( \bar{\theta}_i(t) \in \mathbb{R}^6 \) for each \( t \in [0, t_f] \). Observe that initial condition \( \bar{x}_i^0 \) for \( \bar{x}_i(t) \) is given by \( \bar{x}_i^0 = M\bar{\theta}_i^0 \) so that \( X_0 = M\Theta_0 \subset \Omega \) is the set of possible initial positions for the sensor location \( \bar{x}_i(t) \). Finally, the moving sensor problem becomes

\[
\textbf{Problem}(\mathcal{P}): \text{Find } u_{opt}(\cdot) \in \mathcal{U} \text{ and } \bar{\theta}_{opt}^0 \in \Theta_0 \text{ that minimizes } \\
J(\bar{\theta}_0, u(\cdot)) = \int_0^{t_f} \text{Tr} \left( Q(t) \Sigma(t, \bar{x}(t)) \right) dt,
\]

on the set \((\Theta_0, \mathcal{U})\), where \( t \mapsto \Sigma(t, \bar{x}(t)) \) is the solution to the constraint (1.13) and the output map \( t \mapsto C(t) \) is of the form (1.10) and it is determined by the trajectory \( t \mapsto \bar{x}(t) = M\bar{\theta}(t, \theta_0, u) \).

There are several technical and computational challenges that must be addressed in order to solve \textbf{Problem}(\mathcal{P}) above. We cite the following issues:

1. Since the variance equation is infinite dimensional, one must prove that the operator \( \Sigma(t) \) is of trace class and integrable so that the cost functional (1.18) is well defined over the interval \([0, t_f]\). This can be a nontrivial problem, but the results in \([19], [21], [28], [29], [41], \) and \([46]\) provide a background to develop the necessary structure. We will provide proofs of these results in the following sections. In addition, the same framework developed to prove that the problem is well-posed will be used to establish convergence for a Galerkin approximation scheme.

2. The solution to the problem requires the introduction of approximations. Also, proving the convergence of the corresponding numerical algorithms is essential in order to verify the method. The basic theory and approximation schemes developed in \([13], [15], [21], [28], [29], [33], [41], \) and \([46]\) will be used as a starting points. We will consider the Galerkin approximation as the main tool for this problem.
1.5 Goals of this Thesis

In this thesis, we develop a rigorous mathematical and computational framework for addressing a general class of optimal sensor location problems. This framework is based on strong solutions (in the space of nuclear or trace class operators) to the Riccati integral equations that define optimal estimators for distributed parameter systems. In particular, we assume the distributed parameter system has the general form (1.11) - (1.12) where $A$ generates a $C_0$-semigroup $S(t)$ and the input mapping $t \mapsto B(t)$ and output mapping $t \mapsto C(t)$ have values in spaces of bounded operators. We formulate the Riccati differential equation (1.13) in a space of trace class operators and prove well-posedness in the strong topology. Existence of optimal solutions to Problem ($\mathcal{P}$) is established for a wide class of output mappings.

Finally, we develop a computational scheme based on Galerkin type approximations and prove convergence of this algorithm. Numerical examples are given to illustrate the theoretical results.
Chapter 2

Notation and Preliminaries

In this section we provide the basic background material. In particular, we review definitions, present known results and introduce the notation to be used throughout the thesis.

2.1 The Trace Ideals $I_p$

Let $\mathcal{H}$ be a separable complex Hilbert space. The space of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ is denoted by $L(\mathcal{H})$. If $A \in L(\mathcal{H})$, then $\|A\|$ denotes the usual operator norm. The subspace of compact bounded linear operators acting on $\mathcal{H}$ is denoted by $I_\infty(\mathcal{H})$ and, when $\mathcal{H}$ is clear, we simply use $I_\infty$ for $I_\infty(\mathcal{H})$. It is well known (see for example [44]) that $I_\infty$ is a two-sided $*$-ideal in the ring $L(\mathcal{H})$, i.e., $I_\infty$ is a vector space and;

1) If $A \in I_\infty$ and $B \in L(\mathcal{H})$, then $AB \in I_\infty$ and $BA \in I_\infty$.

2) If $A \in I_\infty$ then $A^* \in I_\infty$.

Also, finite rank operators are dense (in the operator norm) in $I_\infty$ and if $A_n \in I_\infty$ for each $n \in \mathbb{N}$ and $\|A_n - A\| \to 0$ as $n \to \infty$, then $A \in I_\infty$.

**Definition 1 (Non-negative, Positive and Strictly Positive Operators).** An operator $A \in L(\mathcal{H})$, is said to be

(a) non-negative if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$,

(b) positive if $\langle Ax, x \rangle > 0$ for all nonzero $x \in \mathcal{H}$,
(c) **strictly positive** if there is a \( c > 0 \) such \( \langle Ax, x \rangle \geq c\|x\|^2 \) for all \( x \in \mathcal{H} \).

The notation \( A \geq 0, A > 0 \) and \( A \gg 0 \) (or \( 0 \leq A, 0 < A \) and \( 0 \ll A \)) is standard for non-negative, positive and strictly positive operators, respectively.

Suppose that \( A \geq 0 \), and that both \( \{\phi_n\} \) and \( \{\psi_n\} \) are orthonormal bases of \( \mathcal{H} \), then it follows that \( \sum_n \langle \phi_n, A\phi_n \rangle = \sum_n \langle \psi_n, A\psi_n \rangle \) (we allow the case where both quantities are infinite). This observation allow one to define the trace of \( A \) by:

**Definition 2 (Trace of a positive operator).** If \( A \geq 0 \), then the **trace** of \( A \) is defined by

\[
\text{Tr} (A) = \sum_{n=1}^{\infty} \langle \phi_n, A\phi_n \rangle,
\]

where \( \{\phi_n\}_{n=1}^{\infty} \) is any orthonormal basis of \( \mathcal{H} \). Note that \( 0 \leq \text{Tr} (A) \leq \infty \).

Each operator \( A \in \mathcal{L}(\mathcal{H}) \) admits a **polar decomposition** (see for example [44]) analogous to the decomposition \( z = e^{i\text{Arg}(z)}|z| \) when \( z \in \mathbb{C} \). In particular, let \( |A| \) be defined to be the unique non-negative operator such that \( A = U|A| \), where \( U \) is the unique partial isometry such \( \text{Ker} \ U = \text{Ker} \ |A| \). Since \( |A| \geq 0 \), then \( |A|^p \geq 0 \) for any \( p \in \mathbb{N} \) and applying standard continuous functional calculus we can prove that \( |A|^p \geq 0 \) for any \( 1 \leq p < \infty \). Hence the quantity \( \text{Tr} (|A|^p) \) is well defined and lead to the following definition.

**Definition 3 (The \( I_p \) Class).** Let \( I_p(\mathcal{H}) \) for \( 1 \leq p < \infty \) (or simply \( I_p \) when the space \( \mathcal{H} \) is understood) denote the set of all bounded operators over \( \mathcal{H} \) such that \( \text{Tr} (|A|^p) < \infty \). If \( A \in I_p(\mathcal{H}) \), then the \( I_p \)-norm (or just the \( p \)-norm) of \( A \) is defined as \( \|A\|_p = \left( \text{Tr} (|A|^p) \right)^{1/p} < \infty \).

If \( \mathcal{H} \) is a complex separable Hilbert space, then the linear space \( I_p \) is a Banach space with norm \( \|A\|_p = \left( \text{Tr} (|A|^p) \right)^{1/p} < \infty \) (see [45]). We focus on the spaces \( I_1 \) and \( I_2 \) in order to develop a proper framework in which to study the solutions of the Riccati equation. The classes \( I_1 \) and \( I_2 \) are called

\[\footnote{This result is actually true for \( 0 < p < 1 \) also, but in this case \( \left( \text{Tr} (|A|^p) \right)^{1/p} \) does not define a norm.} \]
the space of Trace Class (or Nuclear) operators and the space of Hilbert-Schmidt operators, respectively. Actually, the space \( \mathcal{I}_2 \) is a Hilbert space under the inner product
\[
\langle A, B \rangle_{\mathcal{I}_2} = \sum_{n=1}^{\infty} \langle A\phi_n, B\phi_n \rangle_{\mathcal{H}},
\]
where \( A, B \in \mathcal{I}_2 \) and \( \{\phi_n\}_{n=1}^{\infty} \) is any orthonormal basis of \( \mathcal{H} \). Note that \( \langle A, A \rangle_{\mathcal{I}_2} = \sum_{n=1}^{\infty} \langle \phi_n, A^*A\phi_n \rangle_{\mathcal{H}} \). The operator \( |A| \) is given by \( |A| = \sqrt{A^*A} \), and the continuous functional calculus implies that \( |A|^2 = (\sqrt{A^*A})^2 = A^*A \).

Consequently \( \langle A, A \rangle_{\mathcal{I}_2} = \text{Tr} (|A|^2) = \|A\|_2^2 \).

As a result of this embedding, it follows by setting \( p_2 = \infty \), that every operator in \( \mathcal{I}_p \) is compact (See [23], [30] or [45]) and that \( \|A\|_p \leq \|A\|_p \) for all \( 1 \leq p \leq \infty \). We shall also need the following results (see [23], [30] and/or [45] for proof).

**Lemma 1.** If \( A \in \mathcal{I}_p \) with \( 1 \leq p \leq \infty \) and \( B \in \mathcal{I}_q \) where \( 1/p + 1/q = 1 \), then \( AB, BA \in \mathcal{I}_1 \) and
\[
\|AB\|_1 \leq \|A\|_p \|B\|_q, \quad \|BA\|_1 \leq \|A\|_p \|B\|_q.
\]

Moreover, \( \|A\|_p = \|A^*\|_p \) and for any positive integer \( r \) we have \( A^r \in \mathcal{I}_{p/r} \) and \( \|A^r\|_{p/r} \leq (\|A\|_p)^r \).

The trace can be shown to be a continuous linear functional over \( \mathcal{I}_1 \) (see [23]). This result, combined with the previous Lemma, give a simple characterization to the dual spaces of \( \mathcal{I}_p \) (see [30]) given by the following Proposition.

\[\text{If } X \text{ and } Y \text{ are normed spaces with norms } \|\cdot\|_X \text{ and } \|\cdot\|_Y \text{ we say that } X \text{ is continuously embedded in } Y \text{ and denote } "X \hookrightarrow Y" \text{ if for all } x \in X, \text{ then } x \in Y \text{ and } \|x\|_Y \leq C\|x\|_X \text{ for some fixed } C > 0.\]
Proposition 1. Let $\varphi$ be a continuous linear functional over $I_p$ with $1 < p \leq \infty$, then there is an operator $A \in I_q$ with $1/p + 1/q = 1$ such that
\[
\varphi(X) = \text{Tr} (AX), \quad \text{for all } X \in I_p,
\] (2.2)
and $\|\varphi\|_{L(X,C)} = \|A\|_q$.

If $\varphi$ is a bounded linear functional on $I_1$, then there is a bounded linear operator $A \in L(H)$ such that $\varphi(X) = \text{Tr} (AX)$ for all $X \in I_1$ and $\|\varphi\|_{L(X,C)} = \|A\|$.

The previous proposition implies that $(I_p)^* \simeq I_q$ when when $1 < p \leq \infty$ and then $I_p$ is reflexive when $1 < p < \infty$. Moreover $(I_1)^* \simeq L(H)$. If $A \in I_\infty$, then it is well known (see [30]) that it has a norm convergent expansion given by
\[
A(\cdot) = \sum_{n=1}^\omega s_n(A)\langle \phi_n, \cdot \rangle \psi_n,
\]
with $\omega$ possibly infinite and $\{\phi_n\}_{n=1}^\omega$ and $\{\psi_n\}_{n=1}^\omega$ orthonormal sequences in $H$. The elements of the sequence $\{s_n(A)\}_{n=1}^\omega$ are uniquely determined and called the singular values of $A$. In addition the singular values satisfy $s_n(A) \geq 0$ and $s_1(A) \geq s_2(A) \geq \cdots \geq 0$.

There are several equivalent ways to define the norm $\|A\|_p$ for an $A \in I_p$. The following result uses the singular values of $A$ and the results of the dual space of $I_p$ to characterize $\|A\|_p$ (see [23] and [15]).

Proposition 2. Let $A \in I_p$ and $\{s_j(A)\}_{j=1}^\omega$ be its singular values.

i. If $\frac{1}{p} + \frac{1}{q} = 1$, then
\[
\|A\|_p = \sup_{B \text{ finite rank}} \frac{|\text{Tr} (BA)|}{\|B\|_q}. \quad (2.3)
\]

ii. Moreover,
\[
\|A\|_p = \left( \sum_{j=1}^\omega s_j^p(A) \right)^{1/p}. \quad (2.4)
\]

Suppose that $I$ is a real interval (bounded or unbounded) and that $X$ is a Banach space. We define the space $C(I; X)$ by
\[
C(I; X) = \left\{ F : I \mapsto X : t \mapsto F(t) \text{ is continuous in } \| \cdot \|_X \right\}.
\]
If \( I \) is closed, then \( C(I;X) \) is a Banach space under the usual sup norm; 
\[
\|F(\cdot)\|_{C(I;X)} = \sup_{t \in I} \|F(t)\|_X.
\]
If \( 1 \leq p_1 < p_2 \leq \infty \), then the continuous embedding \( \mathcal{I}_{p_1} \hookrightarrow \mathcal{I}_{p_2} \), implies that
\[
C(I; \mathcal{I}_{p_1}) \hookrightarrow C(I; \mathcal{I}_{p_2}).
\]
Also, since \( \mathcal{I}_p \hookrightarrow \mathcal{L}(\mathcal{H}) \) for any \( 1 \leq p \leq \infty \), it follows that \( C(I; \mathcal{I}_p) \hookrightarrow C(I; \mathcal{L}(\mathcal{H})) \).

We turn to a short review of the Bochner integral. We shall use the properties of this integral to obtain the well-posedness of problem \((P)\) and to obtain convergence of the numerical scheme.

### 2.2 Bochner Measurability and \( \mathcal{I}_p \)-valued Mappings

Throughout this section we assume that \( X \) is a complex Banach space and \( I \) is an interval (bounded or unbounded) in \( \mathbb{R} \). A function \( f : I \mapsto X \) is called simple if it is of the form \( f(t) = \sum_{r=1}^{n} x_r \chi_{\Delta_r}(t) \) for some \( n \in \mathbb{N} \), \( x_r \in X \) and Lebesgue measurable sets \( \Delta_r \). Here, the measure of \( \Delta_r \) is denoted by \( m(\Delta_r) \) and \( \chi_{\Delta_r} \) is the characteristic function of the set \( \Delta_r \). The function \( f \) is called a step function if each \( \Delta_r \) can be chosen to be an interval.

**Definition 4 (Bochner Measurability).** A function \( f : I \mapsto X \) is called Bochner measurable (or simply measurable) if there is a sequence of simple functions \( f_n : I \mapsto X \) such that
\[
\lim_{n \to \infty} f_n(t) = f(t),
\]
a.e. for \( t \in I \).

It is not difficult to prove that if the function \( f \) is measurable, then the functions \( f_n \) in the previous definition can be chosen to be step functions (see \[2\]). We now will state several well known results that will be used throughout this thesis (for a proof see \[2\]).

**Proposition 3.** Assume that \( f : I \mapsto X \) and \( h : I \mapsto \mathbb{C} \) where \( I \) is a real interval and \( X \) is a Banach space. The following statements hold:

i. If \( f \) and \( h \) are measurable, then \( t \mapsto f(t)h(t) \) is measurable.
ii. If $f$ is measurable, then $\|f(\cdot)\| : I \mapsto \mathbb{R}$ is measurable.

iii. If $f_n : I \mapsto X$ for $n \in \mathbb{N}$ are measurable, and $f_n(t) \to f(t)$ as $n \to \infty$ a.e. for $t \in I$, then $f$ is measurable.

iv. If $f(\cdot) \in \mathcal{C}(I; X)$, then $f$ is measurable.

We now define the Bochner integral for a simple function. If $f : I \mapsto X$ is the simple function $f(t) = \sum_{r=1}^{n} x_r \chi_{\Delta_r}(t)$, then the Bochner integral of $f$ is defined by

$$\int_I f(t) \, dt = \sum_{r=1}^{n} x_r m(\Delta_r).$$

The natural extension to this definition to general measurable functions leads to the following definition (see [1]).

**Definition 5 (Bochner Integrability).** A function $f : I \mapsto X$ on a real interval $I$ with range in the Banach space $X$, is called Bochner integrable (or simply integrable) if there is a sequence of simple functions $f_n : I \mapsto X$ such that $f(t) = \lim_{n \to \infty} f_n(t)$, a.e. for $t \in I$ and that

$$\lim_{n \to \infty} \int_I \|f(t) - f_n(t)\|_X \, dt = 0.$$

By definition the Bochner integral of $f$ on $I$ is defined as

$$\int_I f(t) \, dt = \lim_{n \to \infty} \int_I f_n(t) \, dt.$$

The space of Bochner integrable functions admits the following elegant characterization that we cite without proof.

**Theorem 1 (Bochner).** A function $f : I \mapsto X$ is Bochner integrable if and only if $f$ is measurable and $\|f\|$ is integrable. If $f : I \mapsto X$ is Bochner integrable, then

$$\|\int_I f(t) \, dt\|_X \leq \int_I \|f(t)\|_X \, dt.$$

One of the important features of the Bochner integral is that it is “well behaved” under bounded transformations (see [1] and [32]) and satisfies the Dominated Convergence Theorem. In particular, we have the following results.
Proposition 4. Let \( T : X \mapsto Y \) be a bounded linear operator from the Banach spaces \( X \) to the Banach space \( Y \). If \( f : I \mapsto X \) is a Bochner integral function, then \( t \mapsto T(f(t)) \) is Bochner integrable and
\[
T \int_I f(t) \, dt = \int_I T(f(t)) \, dt.
\]

Theorem 2 (Dominated Convergence). Let \( f_n : I \mapsto X \) be a sequence of Bochner integrable functions. If \( f(t) = \lim_{n \to \infty} f_n(t) \) exists a.e. and if there exists an integrable function \( g : I \mapsto \mathbb{R} \) such that \( \| f_n(t) \|_X \leq g(t) \) a.e. for \( t \in I \) and all \( n \in \mathbb{N} \), then \( f \) is Bochner integrable and
\[
\int_I f(t) \, dt = \lim_{n \to \infty} \int_I f_n(t) \, dt.
\]
In addition, \( \int_I \| f(t) - f_n(t) \|_X \, dt \to 0 \) as \( n \to \infty \).

We recall the definitions of the standard Banach spaces \( L^p(I;X) \). For \( 1 \leq p < \infty \) the space \( L^p(I;X) \) is defined to be the space of (equivalence classes) of measurable functions \( f : I \mapsto X \) such that
\[
\| f \|_{L^p(I;X)} = \left( \int_I \| f(t) \|_X^p \, dt \right)^{1/p} < \infty,
\]
and \( L^\infty(I;X) \) is defined to be the space of (equivalence classes) of measurable functions such that
\[
\| f \|_{L^\infty(I;X)} = \text{ess sup}_{t \in I} \| f(t) \|_X < \infty.
\]

When \( I \) is unbounded, then we can define the spaces \( L^p_{loc}(I;X) \) as all (equivalence classes) of measurable functions such that their restriction to any compact interval \([a,b] \subset I\) belongs to \( L^p(I;X)\), i.e., \( f(\cdot) \in L^p_{loc}(I;X) \) if \( (f \chi_{[a,b]})(\cdot) \in L^p(I;X) \) for any \([a,b] \subset I\) where \( \chi_{[a,b]} \) is the characteristic function of the set \([a,b]\).

Remark 2.1. It is important to note that if \( X \) is infinite dimensional, then the step functions are dense in \( L^p(I;X) \) (with \( 1 \leq p < \infty \)). However, step functions are not dense in \( L^\infty(I;X) \) as we will show later.
Remark 2.2. If \( X \) is an operator space, for example \( X = \mathcal{L}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \), then we can define a weaker form of measurability. We say that a map \( T : I \mapsto \mathcal{L}(\mathcal{H}) \) is strongly measurable, if for any \( x \in \mathcal{H} \), the map \( t \mapsto T(t)x \) is Bochner measurable as an \( \mathcal{H} \)-valued function. The differences between these two definitions of measurability play an important role in defining solutions to Riccati equations and the way we approximate these equations. For example, let \( S(t) \) be a \( C_0 \)-semigroup over \( \mathcal{H} \), then \( t \mapsto S(t)x \) is norm continuous for every \( x \in \mathcal{H} \), which implies that the map \( S : I \mapsto \mathcal{L}(\mathcal{H}) \) is strongly measurable. However, \( t \mapsto S(t) \) is Bochner measurable if and only if \( t \mapsto S(t) \) is norm continuous for \( t > 0 \) (see Hille and Phillips book [32]). Therefore, if \( t \mapsto S(t) \) is not norm continuous for \( t > 0 \), then the mapping \( t \mapsto S(t) \) is not Bochner measurable and hence the Bochner integral \( \int_0^1 S(t) \, dt \) is not well-defined. However, one can define a bounded linear operator by

\[
Vx = \int_0^1 S(t)x \, dt
\]

for each \( x \in \mathcal{H} \), since \( \|S(t)\| \) is uniformly bounded on \( t \in [0,1] \). This is often called the strong Bochner integral.

Let \( I \) be a compact interval. Since \( \mathcal{L}_{p_1} \hookrightarrow \mathcal{L}_{p_2} \) for \( 1 \leq p_1 \leq p_2 \leq \infty \), it follows that \( \mathcal{C}(I; \mathcal{L}_{p_1}) \hookrightarrow \mathcal{C}(I; \mathcal{L}_{p_2}) \). Then, \( \mathcal{C}(I; \mathcal{L}_{p_1}) \hookrightarrow \mathcal{L}(I; \mathcal{L}_{p_2}) \) for all \( 1 \leq p \leq \infty \), and

\[
\left( \int_I \|f(t)\|_{p_2}^p \, dt \right)^{1/p} \leq (m(I))^{1/p} \sup_{t \in I} \|f(t)\|_{p_1}.
\]

Also, if \( f(\cdot) \in \mathcal{L}^p(I; \mathcal{L}_r) \) and \( g(\cdot) \in \mathcal{L}^q(I; \mathcal{L}_s) \), where \( 1/p + 1/q = 1 \) and \( 1/r + 1/s = 1 \), then \( (fg)(\cdot) \) and \( (gf)(\cdot) \) map \( I \) to \( \mathcal{L}_1 \) and they are Bochner measurable. This follows immediately by considering step functions \( f_n \) and \( g_n \) that converge point-wise a.e. to \( f \) and \( g \) in their respective norms. Since \( f_ng_n \) and \( g_nf_n \) are simple \( \mathcal{L}_1 \)-valued and converge point-wise a.e. to \( fg \) and \( gf \) respectively. Finally, we note that

\[
\int_I \|(fg)(t)\|_1 \, dt \leq \int_I \|f(t)\|_r \|g(t)\|_s \, dt \leq \left( \int_I \|f(t)\|_r^p \, dt \right)^{1/p} \left( \int_I \|g(t)\|_s^q \, dt \right)^{1/q}
\]
and the same bound holds for $t \mapsto (gf)(t)$. Therefore,
\[
\| (fg)(\cdot) \|_{L^1(I;X_1)} \leq \| f(\cdot) \|_{L^p(I;X_r)} \| g(\cdot) \|_{L^q(I;X_s)},
\]
and
\[
\| (gf)(\cdot) \|_{L^1(I;X_1)} \leq \| f(\cdot) \|_{L^p(I;X_r)} \| g(\cdot) \|_{L^q(I;X_s)}.
\]

### 2.3 Smoothing Results

It is well known that if $K \in \mathcal{L}(\mathcal{H})$ is a compact operator and $S : \mathbb{R}^+ \mapsto \mathcal{L}(\mathcal{H})$ is a $C_0$-semigroup of linear operators over $\mathcal{H}$, then the mappings $t \mapsto S(t)K$ and $t \mapsto KS(t)$ are continuous in the operator norm (for a proof see for example [25]; the proof is disguised as the eventual norm continuity of eventually compact semigroups). This smoothing property of compact operators is stronger when $K$ is not only in $\mathcal{I}_\infty$, but also belongs to $\mathcal{I}_p$, with $1 \leq p < \infty$. We extend these smoothing results to the case where the semigroup $S(t)$ is replaced by a general be a strongly continuous mapping $T : \mathbb{R}^+ \mapsto \mathcal{L}(\mathcal{H})$. Throughout this section we assume that $\mathbb{R}^+ = [0, \infty)$, $\mathcal{H}$ is a complex separable Hilbert space and that mapping $T : \mathbb{R}^+ \mapsto \mathcal{L}(\mathcal{H})$ is strongly continuous (but not necessarily a $C_0$-semigroup).

**Proposition 5.** Let $T : \mathbb{R}^+ \mapsto \mathcal{L}(\mathcal{H})$ be a strongly continuous mapping (i.e. for each $x \in \mathcal{H}$ the mapping $t \mapsto T(t)x$ is continuous in $\mathbb{R}^+$) and let $K \in \mathcal{I}_p$, for some $1 \leq p \leq \infty$. Then, $t \mapsto T(t)K$ and $t \mapsto KT^*(t)$ belong to $\mathcal{C}(\mathbb{R}^+, \mathcal{I}_p)$.

**Proof.** Consider the case where $t \in [0, \tau]$ for a fixed finite $\tau > 0$. Since $K \in \mathcal{I}_p$ and $T(t) \in \mathcal{L}(\mathcal{H})$, it follows that $T(t)K \in \mathcal{I}_p$ for each $t \in [0, \tau]$. For an operator $K \in \mathcal{I}_p$,
\[
\| K \|_p = \sup_{B \neq 0 \text{ of finite rank}} \frac{\left| \text{Tr}(BK) \right|}{\| B \|_q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1,
\]
(see Proposition [2] and [23] for a proof). We first prove that $t \mapsto T(t)K$ is $\mathcal{I}_p$-norm continuous when $K$ is a rank one operator.

If $K$ be defined as $Kx = \langle \psi, x \rangle \varphi$ for some fixed $\psi, \varphi \in \mathcal{H}$, and all $x \in \mathcal{H}$, then $T(t)Kx = \langle \psi, x \rangle T(t)\varphi$. Let $t$ and $t_0$ be in $[0, \tau]$ and $\{\phi_n\}_{n=1}^\infty$
an orthonormal basis of $\mathcal{H}$. If $B \neq 0$ is of finite rank, then the Cauchy-Schwartz inequality and Lemma 1 imply

$$
|\text{Tr} \left( B(T(t)K - T(t_0)K) \right) | \leq \sum_{n=1}^{\infty} | \langle \phi_n, B(T(t) - T(t_0))K\phi_n \rangle |
$$

$$
= \sum_{n=1}^{\infty} | \langle \psi, \phi_n \rangle | | \langle \phi_n, B(T(t) - T(t_0))\phi \rangle |
$$

$$
\leq \left( \sum_{n=1}^{\infty} | \langle \psi, \phi_n \rangle |^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} | \langle \phi_n, B(T(t) - T(t_0))\phi \rangle |^2 \right)^{1/2}
$$

$$
= \| \psi \| \| B(T(t)\phi - T(t_0)\phi) \| 
$$

$$
\leq \| \psi \| \| B \|_q \| T(t)\phi - T(t_0)\phi \|
$$

for any $1 \leq q \leq \infty$. Therefore, we have

$$
\| T(t)K - T(t_0)K \|_p = \sup_{B \neq 0 \text{ of finite rank}} \frac{|\text{Tr} \left( B(T(t)K - T(t_0)K) \right) |}{\| B \|_q}
$$

(2.5)

\[
\leq \| \psi \| \| T(t)\phi - T(t_0)\phi \|
\]

and hence $\| T(t)K - T(t_0)K \|_p \to 0$ as $t \to t_0$ because $t \mapsto T(t)$ is strongly continuous on $\mathbb{R}^+$. Therefore, $t \mapsto T(t)K$ is $\mathcal{I}_p$-norm continuous on $[0, \tau]$ when $K$ is of rank one. We now use induction to establish the same result when $K$ has finite rank.

Let $K = K_0 + K_1$ where the mapping $t \mapsto T(t)K_0$ continuous in the $\mathcal{I}_p$-norm and $K_1$ is of rank one. Since

$$
\| T(t)K - T(t_0)K \|_p \leq \| T(t)K_0 - T(t_0)K_0 \|_p + \| T(t)K_1 - T(t_0)K_1 \|_p
$$

it follows that $t \mapsto T(t)K$ is continuous in the $\mathcal{I}_p$-norm. Therefore, for any finite rank operator $K$, the map $t \mapsto T(t)K$ is $\mathcal{I}_p$-norm continuous on $[0, \tau]$.

Now we extend this result for any $K \in \mathcal{I}_p$. Let $K \in \mathcal{I}_p$ and $\{K_n\}_{n=1}^{\infty}$ be a sequence of finite rank operators such that $K_n \to K$ in the $\mathcal{I}_p$-norm (note that finite rank operators are dense in $\mathcal{I}_p$, in the corresponding norm). The triangle inequality implies

$$
\| T(t)K - T(t_0)K \|_p \leq \| T(t)(K - K_n) \|_p + \| T(t)(K_n) - T(t_0)K_n \|_p
$$

$$
\| T(t_0)(K_n - K) \|_p + \| T(t_0)K_n - T(t_0)K_n \|_p.
$$
and the Uniform Boundedness Theorem yields \(\|T(t)\| \leq M_\tau\) for some \(M_\tau > 0\) and for any \(t \in [0, \tau]\). It now follows from Lemma after that

\[
\|T(t)K - T(t_0)K\|_p \leq 2M_\tau \|K - K_n\|_p + \|T(t)K_n - T(t_0)K_n\|_p.
\]

For any \(\epsilon > 0\), there is an \(N(\epsilon)\) such that \(\|K - K_n\|_p < \epsilon/4M_\tau\) for \(n \geq N(\epsilon)\).

Since each \(K_n\) is of finite rank, for a fixed \(n\), there is a \(\delta > 0\) such that for \(t \in (t_0 - \delta, t_0 + \delta) \cap [0, \tau]\) we observe \(\|T(t)K_n - T(t_0)K_n\|_p < \epsilon/2\). Hence, it follows that for each \(\epsilon > 0\)

\[
\|T(t)K - T(t_0)K\|_p < \epsilon,
\]

for \(t \in (t_0 - \delta, t_0 + \delta) \cap [0, \tau]\) for some \(\delta = \delta(\epsilon)\). Therefore, if \(K \in \mathcal{S}_p\), the map \(t \mapsto T(t)K\) is \(\mathcal{S}_p\)-norm continuous on \([0, \tau]\). Moreover, since \(\tau > 0\) was arbitrary, \(t \mapsto T(t)K\) is \(\mathcal{S}_p\)-norm continuous on \(\mathbb{R}^+\).

Finally, if \(K \in \mathcal{S}_p\), then \(K^* \in \mathcal{S}_p\) and \(T(\cdot)K^* \in \mathcal{C}([0, \tau]; \mathcal{S}_p)\). Hence \(KT^*(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{S}_p)\) since \(\|T(t)K^* - T(t_0)K^*\|_p = \|KT^*(t) - KT^*(t_0)\|_p\) and this completes the proof.

If the mapping \(t \mapsto T(t)\) does not satisfy some additional property (for example, the semigroup property \(T(t+s) = T(t)T(s)\) for \(t, s > 0\) and \(T(0) = I\) “strong continuity” can not be replaced by “weak continuity”. If the semigroup property and \(T(0) = I\) are satisfied, then the strong continuity is implied by the weak continuity (for a proof, see [32] and [43]).

**Proposition 6.** Let \(T : [0, \tau] \mapsto \mathcal{L}(\mathcal{H})\) be weakly continuous but not strongly continuous. Then, there is a \(K \in \mathcal{S}_p\) such that \(t \mapsto T(t)K\) is not strongly continuous and \(t \mapsto KT^*(t)\) is not continuous in the operator norm in \([0, \tau]\).

**Proof.** By the initial hypotheses there is a nonzero \(\phi \in \mathcal{H}\) and a \(s \in [0, \tau]\) such that for any sequence \(\{s_n\}_{n=1}^\infty\) in \([0, \tau]\) satisfying \(s_n \to s\), we observe \(\lim_{n \to \infty} \langle \eta, T(s_n)\phi \rangle = \langle \eta, T(s)\phi \rangle\) for all \(\eta \in \mathcal{H}\), but \(\lim_{n \to \infty} T(s_n)\phi \neq T(s)\phi\). Let \(K\) be the rank one operator defined by \(K\psi = \langle \phi, \psi \rangle \phi\) for all \(\psi \in \mathcal{H}\). Then \(t \mapsto T(t)K\) fails to be strongly continuous at \(t = s\) and \(t \mapsto KT^*(t)\) is not continuous in the operator norm topology at \(t = s\).

**Remark 2.3.** The strong continuity of \(t \mapsto T(t)\) and \(K \in \mathcal{S}_p\) together imply that \(t \mapsto T(t)K\) and \(t \mapsto KT^*(t)\) are \(\mathcal{S}_p\)-norm continuous. However, it does not follow that \(t \mapsto KT(t)\) and \(t \mapsto T^*(t)K\) are \(\mathcal{S}_p\)-norm continuous.
unless additional conditions are placed on $T(t)$. For example, if we assume that $T(t) = S(t)$ is a $C_0$-semigroup over $\mathcal{H}$, then $t \mapsto S^*(t)$ is also strongly continuous since we are assuming $\mathcal{H}$ is a Hilbert space (for a proof see [43]). The problem resides in the fact that the involution map $K \mapsto K^*$ is not continuous in the strong operator topology. The map $K \mapsto K^*$ is continuous in the weak and norm operator topologies (see [44] for a proof). Thus in general, strong continuity of $t \mapsto T(t)$ does not imply strong continuity of $t \mapsto T^*(t)$. In this case we can not imply the $\mathcal{H}_p$-norm continuity of the mappings $t \mapsto T^*(t)K$ and $t \mapsto K(T^*)^*(t) \equiv KT(t)$. Consider the following counterexample.

Example 2.3.1 (A Counterexample). Let $\ell^2(\mathbb{N})$ be the Hilbert space of complex-valued, square summable sequences $\{a_k\}_{k=1}^{\infty}$ with inner product $\langle \{a_k\}, \{b_k\} \rangle_{\ell^2(\mathbb{N})} = \sum_{k=1}^{\infty} a_k b_k$. Consider the right shift operator $S_n$ on $\ell^2(\mathbb{N})$ defined by

$$S_n(a_1, a_2, \ldots) = (0, 0, \ldots, 0, a_1, a_2, a_3, \ldots).$$

The adjoint $S_n^*$ is given by

$$S_n^*(a_1, a_2, \ldots) = (a_{n+1}, a_{n+2}, a_{n+3}, \ldots),$$

and therefore $S_n^*$ converges strongly to zero, but $S_n$ does not since $\|S_n \{a_k\}\|_{\ell^2(\mathbb{N})} = \|\{a_k\}\|_{\ell^2(\mathbb{N})}$.

Define $R : [0, 1] \mapsto \mathcal{L}(\ell^2(\mathbb{N}))$ as follows. First consider the sequence of points $t_n = 1 - 2^{-n}$ and the open intervals $I_n = (t_n, t_{n+1})$, for $n = 0, 1, 2, \ldots$. On each $I_n$, consider the sequences of continuous functions $\alpha_n : I_n \mapsto [0, 1]$ and $\beta_n : I_n \mapsto [0, 1]$ such that

$$\alpha_n(t_n) = 1, \quad \alpha_n(t_{n+1}) = 0,$$

$$\beta_n(t_n) = 0, \quad \beta_n(t_{n+1}) = 1.$$

Let $R(0) = I$, $R(1) = 0$ and $R(t_n) = S_n^*$ for $n = 1, 2, 3, \ldots$ and define $t \mapsto R(t)$ restricted to each interval $I_n$ by

$$R(t) \big|_{I_n} = \alpha_n(t)S_n^* + \beta_n(t)S_{n+1}^*.$$

It is easy to check that $R(t) \in \mathcal{L}(\ell^2(\mathbb{N}))$ for all $t \in [0, 1]$, since $\|R(t)\| \leq 2$ because $\|S_n\| = \|S_n^*\| = 1$ and $\text{Range}(\alpha_n) = \text{Range}(\beta_n) = [0, 1]$. Even more,
the mapping \( t \mapsto R(t) \) is continuous in the operator norm topology only for \( t \in [0, 1) \). If \( t \) and \( s \) belong to some \( I_n \), then
\[
\|R(t) - R(s)\| \leq |\alpha_n(t) - \alpha_n(s)| + |\beta_n(t) - \beta_n(s)|,
\]
which implies the continuity in each \( I_n \) since each \( \alpha_n \) and \( \beta_n \) are continuous and a similar inequality can be used to establish the continuity at each \( t_n \). On the other hand, for \( t = 1 \), \( R(1) = 0 \), and \( t_n \to 1 \) as \( n \to \infty \), and \( \|R(t_n)\| = \|S_n^*\| = 1 \neq 0 \). Thus, \( t \mapsto R(t) \) is discontinuous in the operator norm topology at \( t = 1 \). However, if \( s_n \) is any monotone sequence such \( s_n \to 1 \), and \( \{a_k\} \in \ell^2(\mathbb{N}) \), then
\[
\|R(s_n)\{a_k\}\|_{\ell^2(\mathbb{N})} \leq \|S_n^*\{a_k\}\|_{\ell^2(\mathbb{N})} + \|S_n^*\{a_k\}\|_{\ell^2(\mathbb{N})} \to 0,
\]
as \( n \to \infty \). Thus, \( t \mapsto R(t) \) is a strongly continuous mapping for all \( t \in [0, 1] \). However, \( t \mapsto R^*(t) \) is strongly continuous only on \( t \in [0, 1) \). Strong continuity in \([0, 1)\) is implied by the continuity in operator norm of \( t \mapsto R(t) \).

Since \( R^*(1) = 0 \) and
\[
\|R^*(t_n)\{a_k\}\|_{\ell^2(\mathbb{N})} = \|S_n\{a_k\}\|_{\ell^2(\mathbb{N})} = \|\{a_k\}\|_{\ell^2(\mathbb{N})},
\]
the sequence \( \|R^*(t_n)\{a_k\}\|_{\ell^2(\mathbb{N})} \) does not converge to zero and \( t \mapsto R^*(t) \) it is not strongly continuous at \( t = 1 \). It is easy to see that \( t \mapsto R^*(t) \) is weakly continuous on \([0, 1]\). Therefore, by Proposition 6 above, there is a rank one operator \( K \) such that \( t \mapsto R^*(t)K \) is not strongly continuous at \( t = 1 \) and \( t \mapsto K(R^*(t))^*KR(t) \) is discontinuous in operator norm at \( t = 1 \).

The following result is a natural extension of Proposition 5 and will be used extensively in the rest of the text and in the proof of what we will call “The Smoothing Lemma” in the next section.

**Proposition 7.** Let \( T : R^+ \to \mathcal{L}(\mathcal{H}) \) be a strongly continuous mapping and let \( K(\cdot) \in \mathcal{C}(R; \mathscr{I}_p) \), for some \( 1 \leq p \leq \infty \). Then, \( t \mapsto T(t)K(t) \) and \( t \mapsto K(t)T^*(t) \) belong to \( \mathcal{C}(R^+, \mathscr{I}_p) \).

**Proof.** If \( t \) and \( t_0 \) are in \([0, \tau]\) with \( \tau > 0 \) arbitrary, then \( T(t)K(t) \) and \( T(t_0)K(t_0) \) belong to \( \mathscr{I}_p \). Again, the triangle inequality yields
\[
\|T(t)K(t) - T(t_0)K(t_0)\|_p \leq \|T(t)\| \|K(t) - K(t_0)\|_p + \|T(t)K(t_0) - T(t_0)K(t_0)\|_p.
\]
Since $K(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{I}_p)$ and $t \mapsto \|T(t)\|$ is uniformly bounded in $[0, \tau]$ (by the Uniform Boundedness Theorem), the first term in the right hand side goes to zero, as $t \to t_0$. The second term goes to zero by Proposition 5. Also, since $\|K^*(t)\|_p = \|K(t)\|_p$ the mapping $t \mapsto K^*(t)$ belongs to $\mathcal{C}(\mathbb{R}^+; \mathcal{I}_p)$ and hence $t \mapsto T(t)K^*(t)$ and $t \mapsto K^*(t)T^*(t)$ are both $\mathcal{I}_p$-norm continuous on $[0, \tau]$. Since $\tau > 0$ is arbitrary, the result holds on $\mathbb{R}^+$. \hfill \Box

The previous propositions have stronger conclusions in the case where $t \mapsto T(t)$ is not just a strongly continuous mapping but a $C_0$-semigroup of linear operators over some Hilbert space $\mathcal{H}$. In this case, both $t \mapsto T(t)$ and $t \mapsto T^*(t)$ are strongly continuous. In fact, $T^*(t)$ is also a $C_0$-semigroup of linear operators over $\mathcal{H}$ as we can see in [43]. We summarize this in the following proposition.

**Proposition 8.** Let $S(t)$ be a $C_0$-semigroup and let $K(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{I}_p)$, for some $1 \leq p \leq \infty$. Then, the mappings

i) $t \mapsto S(t)K(t)$

ii) $t \mapsto S^*(t)K(t)$

iii) $t \mapsto K(t)S(t)$

iv) $t \mapsto K(t)S^*(t)$

all belong to $\mathcal{C}(\mathbb{R}^+; \mathcal{I}_p)$.

**Proof.** Since $S(t)$ is a $C_0$-semigroup over a Hilbert space, both mappings $t \mapsto S(t)$ and $t \mapsto S^*(t)$ are strongly continuous. The conclusion follows directly from Proposition 7. \hfill \Box

The previous “smoothing results” allow us to conclude important continuity properties in the integral Riccati equation in some special cases.

**Proposition 9.** Let $t \mapsto \Sigma(t)$ be a strongly continuous operator valued mapping, $S(t)$ a $C_0$-semigroup and $\Sigma_0, BB^*, C^*C \in \mathcal{I}_p$. Then

$$t \mapsto S(t)\Sigma_0S^*(t),$$

is $\mathcal{I}_p$-norm continuous and

$$\int_0^t S(t-s)(BB^* - \Sigma(s)(C^*C)\Sigma(s))S^*(t-s) \, ds,$$

is a well defined Bochner integral.
Proof. Since \( \Sigma_0 \in \mathcal{A}_p \) and \( t \mapsto S(t) \) and \( t \mapsto S^*(t) \) are strongly continuous, it follows from Proposition 8 that the mappings \( t \mapsto S(t)\Sigma_0 \) and \( t \mapsto S(t)\Sigma_0 S^*(t) \) are \( \mathcal{A}_p \)-norm continuous for \( t \in [0, s] \).

Since \( BB^* \in \mathcal{A}_p \), and \( t \mapsto S(t) \) and \( t \mapsto S^*(t) \) are strongly continuous, then Proposition 7 implies that \( s \mapsto S(t-s)BB^* \) and \( s \mapsto S(t-s)BB^* S^*(t-s) \) are \( \mathcal{A}_p \)-norm continuous for \( s \in [0, t] \). The same argument proves that \( s \mapsto \Sigma(s)(C^*C)\Sigma(s) \) and hence \( s \mapsto S(t-s)\Sigma(s)(C^*C)\Sigma(sST^*(t-s) \) are also \( \mathcal{A}_p \)-norm continuous for \( s \in [0, t] \). Therefore, the integrand belongs to \( \mathcal{C}([0, t]; \mathcal{A}_p) \) and hence it is Bochner measurable and Bochner integrable. \( \square \)

At this point we have proven that if \( S(t) \) is a \( C_0 \)-semigroup, and \( K(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{A}_p) \), then all the mappings \( (KS)(\cdot), (SK)(\cdot), (KS^*)(\cdot) \) and \( (S^*K)(\cdot) \) belong to \( \mathcal{C}([0, \tau]; \mathcal{A}_p) \). This implies that multiplication by operators in \( \mathcal{A}_p \) raises the smoothness of the these products. In particular, although a \( C_0 \)-semigroup \( S(t) \) is only strongly continuous, the mappings \( (KS)(\cdot), (SK)(\cdot), (KS^*)(\cdot) \) and \( (S^*K)(\cdot) \) will belong to \( \mathcal{C}([0, \tau]; \mathcal{A}_p) \).

The next step is to show that multiplication by operators in \( \mathcal{A}_p \) improves the convergence properties of approximating semigroups. In particular, we consider a sequence \( \{S_n(t)\} \) of \( C_0 \)-semigroups converging strongly (as \( n \to \infty \)) to a \( C_0 \)-semigroup \( S(t) \) uniformly on \([0, \tau] \). The following Lemma was proven for the case \( p = 2 \) by A. Germani, et al. in [28]. We will extend the results to any \( p \) satisfying \( 1 \leq p \leq \infty \) and to an arbitrary strongly continuous mapping \( t \mapsto T(t) \).

**Lemma 2 (The Smoothing Lemma).** Let \( \{T_n(t)\} \) be a sequence of strongly continuous \( \mathcal{L}(\mathcal{H}) \)-valued functions and strongly convergent to \( t \mapsto T(t) \) uniformly in \( t \in [0, \tau] \) (i.e., the mapping \( t \mapsto T_n(t)x \) is continuous for each \( n \in \mathbb{N} \) and each \( x \in \mathcal{H} \), and \( \|T_n(t)x - T(t)x\| \to 0 \) uniformly in \( t \in [0, \tau] \) as \( n \to \infty \)). Assume that \( 1 \leq p \leq \infty \). If \( K \) is a compact set in \( \mathcal{A}_p \), then

\[
\sup_{t \in [0, \tau]} \|T_n(t)K - T(t)K\|_p \to 0,
\]

and

\[
\sup_{t \in [0, \tau]} \|KT_n^*(t) - KT^*(t)\|_p \to 0,
\]

both uniformly in \( K \in \mathcal{K} \), as \( n \to \infty \).
Proof. The Uniform Boundedness Principle implies that $\|T_n(t)\| \leq M$ for some $M > 0$ and for all $n \in \mathbb{N}$ and $t \in [0, \tau]$. This in turn also implies that $\|T(t)\| \leq M$. For each $t \in [0, \tau]$ we have that $\|T(t)x\| \leq \sup_{n \in \mathbb{N}} \|T_n(t)x\| \leq M\|x\|$ and hence $\|T(t)\| \leq \sup_{n \in \mathbb{N}} \|T_n(t)\| \leq M$.

If $K \in \mathcal{J}_p$, for each $t \in [0, \tau]$ and $n \in \mathbb{N}$, then $T_n(t)K$ and $T(t)K$ belong to $\mathcal{J}_p$ since the latter space is a double-sided ideal on $\mathcal{L}(\mathcal{H})$. We bound their difference by

$$
\|T_n(t)K - T(t)K\|_p \leq \|T_n(t)K\|_p + \|T(t)K\|_p 
\leq 2M \|K\|_p.
$$

Since $K \in \mathcal{K}$ and $\mathcal{K}$ is compact, it is bounded. Therefore, $\|T_n(t)K - T(t)K\|_p$ is uniformly bounded for all $t \in [0, \tau]$ and all $K \in \mathcal{K}$.

Define the functionals $J_n : \mathcal{K} \mapsto \mathbb{R}$ by

$$
J_n(K) = \sup_{t \in [0, \tau]} \|(T_n - T)(t)K\|_p.
$$

From above, we see that $J_n : \mathcal{K} \mapsto \mathbb{R}$ is uniformly bounded in $n \in \mathbb{N}$ and $K \in \mathcal{K}$.

Each mapping $t \mapsto T_n(t)$ is strongly continuous and hence $t \mapsto T(t)$ is strongly continuous since it is the strong limit of $\{T_n(\cdot)\}$ uniformly in $t \in [0, \tau]$. This follows from the inequality

$$
\|T(t)x - T(s)x\| \leq \|T(t)x - T_n(t)x\| + \|T(s)x - T_n(s)x\| + \|T_n(t)x - T_n(s)x\|,
$$

and the fact that each $t \mapsto T_n(t)x$ is continuous in $[0, \tau]$ and $\|T(t)x - T_n(t)x\| \to 0$ as $n \to \infty$, uniformly in $t \in [0, \tau]$.

Proposition 5 implies that $t \mapsto (T_n - T)(t)K$ is $\mathcal{J}_p$-norm continuous if $K \in \mathcal{J}_p$. For the sake of brevity, we define $\|F(\cdot)\|_p = \sup_{t \in [0, \tau]} \|F(t)\|_p$, for $F(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{J}_p)$. Thus, if $K_1$ and $K_2$ are arbitrary elements of $\mathcal{K}$, we have that

$$
|J_n(K_1) - J_n(K_2)| = \|\|(T_n - T)(\cdot)K_1\|_p - \|(T_n - T)(\cdot)K_2\|_p\| 
\leq \|\|(T_n - T)(\cdot)(K_1 - K_2)\|_p\| 
\leq 2M\|K_1 - K_2\|_p.
$$

Hence, for each $n \in \mathbb{N}$, $K \mapsto J_n(K)$ is a uniformly continuous mapping on the compact set $\mathcal{K}$ and therefore attains its maximum over $\mathcal{K}$, i.e.,

$$
\sup_{K \in \mathcal{K}} J_n(K) = J_n(\hat{K}^n),
$$
for some $\hat{K}^n \in \mathcal{K}$.

Since $J_n$ is uniformly bounded in $\mathcal{K}$, define

$$\epsilon = \lim_{n \to \infty} \left( \sup_{K \in \mathcal{K}} J_n(K) \right) = \lim_{n \to \infty} J_n(\hat{K}^n),$$

where $\{\hat{K}^n\}_{n=1}^{\infty}$ is the sequence of maximizers defined above. Then, there is a subsequence $J_{n_j}(\hat{K}^{n_j})$ for such that $J_{n_j}(\hat{K}^{n_j}) \to \epsilon$ as $j \to \infty$. Without loss of generality, suppose that $\epsilon = \lim_{n \to \infty} J_n(\hat{K}^n)$. Also, since $\mathcal{K}$ is compact, the sequence $\{\hat{K}^n\}_{n=1}^{\infty}$ contains a convergent subsequence, and for the sake of brevity suppose $\hat{K}^n \to \hat{K}$ as $n \to \infty$, for some $\hat{K} \in \mathcal{K}$. Since we have already established the inequality $|J_n(\hat{K}^n) - J_n(\hat{K})| \leq 2M\|\hat{K}^n - \hat{K}\|_p$, then it follows that

$$\epsilon = \lim_{n \to \infty} J_n(\hat{K}).$$

Now we prove that $\epsilon = 0$. Assume first that $\hat{K}$ is of rank one, defined by $\hat{K}x = \langle \psi, x \rangle \phi$ (for some $\psi$ and $\phi$, and all $x$ in $\mathcal{H}$). If $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ and $B \neq 0$, we have

$$J_n(\hat{K}) = \sup_{t \in [0, \tau]} \| (T_n - T)(t) \hat{K} \|_p$$

$$= \sup_{t \in [0, \tau]} \left( \sup_{B \text{ finite rank}} \left| \frac{\text{Tr} \left( B(T_n - T)(t) \hat{K} \right)}{\| B \|_q} \right| \right)$$

$$\leq \sup_{t \in [0, \tau]} \left( \sup_{B \text{ finite rank}} \left| \frac{\sum_{n=1}^{\infty} |\langle \phi_n, B(T_n - T)(t) \hat{K} \phi_n \rangle|}{\| B \|_q} \right| \right)$$

$$= \sup_{t \in [0, \tau]} \left( \sup_{B \text{ finite rank}} \left| \frac{\sum_{n=1}^{\infty} |\langle \psi, \phi_n \rangle| \| \langle \phi_n, B(T_n - T)(t) \varphi \rangle \|}{\| B \|_q} \right| \right)$$

$$\leq \sup_{t \in [0, \tau]} \left( \sup_{B \text{ finite rank}} \left| \frac{\sum_{n=1}^{\infty} |\langle \psi, \phi_n \rangle|^2}{\| B \|_q} \right| \right)^{1/2} \left( \sum_{n=1}^{\infty} |\langle \phi_n, B(T_n - T)(t) \varphi \rangle|^2 \right)^{1/2}$$

$$= \sup_{t \in [0, \tau]} \left( \sup_{B \text{ finite rank}} \left| \frac{\| \psi \| \| B(T_n - T)(t) \varphi \|}{\| B \|_q} \right| \right)$$

$$\leq \| \psi \| \sup_{t \in [0, \tau]} \left( \sup_{B \text{ finite rank}} \left| \frac{\| B \|_q \| (T_n - T)(t) \varphi \|}{\| B \|_q} \right| \right)$$

$$\leq \| \psi \| \sup_{t \in [0, \tau]} \| (T_n - T)(t) \phi \|. $$
This implies \( J_n(\hat{K}) \to 0 \) as \( n \to \infty \) because \( T_n(t)\phi \to T(t)\phi \) uniformly in \( t \in [0, \tau] \) as \( n \to \infty \). Next, suppose that \( \hat{K} = K_1 + K_2 \), such that \( \lim_{n \to \infty} J_n(K_1) = 0 \) and \( K_2 \) is of rank one. Since \( J_n(K_1 + K_2) \leq J_n(K_1) + J_n(K_2) \), we observe that \( \lim_{n \to \infty} J_n(\hat{K}) = 0 \) and hence this is valid for all \( \hat{K} \) of finite rank. Finally, suppose that \( \hat{K} \in \mathcal{I}_p \). Then, there is a sequence \( \{K_m\}_{m=1}^{\infty} \) of finite rank operators such that \( \|\hat{K} - K_m\|_p \to 0 \) as \( m \to \infty \), and

\[
J_n(\hat{K}) \leq J_n(\hat{K} - K_m) + J_n(K_m)
\]

\[
\leq 2M\|\hat{K} - K_m\|_p + J_n(K_m).
\]

Hence \( \lim_{n \to \infty} J_n(\hat{K}) \leq 2M\|\hat{K} - K_m\|_p \), for any \( m \in \mathbb{N} \) and therefore \( \epsilon = \lim_{n \to \infty} J_n(\hat{K}) = 0 \) for any \( \hat{K} \in \mathcal{I}_p \). Thus we have proven that

\[
\epsilon = \lim_{n \to \infty} \sup_{K \in \mathcal{K}} J_n(K) = \lim_{n \to \infty} \sup_{K \in \mathcal{K}} \|T_n(t)K - T(t)K\|_p = 0.
\]

In order to prove the second part of the initial statement, define \( \mathcal{K}^* = \{K^* : K \in \mathcal{K}\} \). Then \( \mathcal{K}^* \) is also a compact subset of \( \mathcal{I}_p \). Therefore, \( \|T_n(t)K^* - T(t)K^*\|_p \to 0 \) uniformly in \( K \in \mathcal{K} \) and \( t \in [0, \tau] \), but \( \|T_n(t)K^* - T(t)K^*\|_p = \|(T_n(t)K^* - T(t)K^*)^*\|_p = \|KT_n^*(t) - KT^*(t)\|_p \) and so \( \sup_{[0, \tau]} \|KT_n^*(t) - KT^*(t)\|_p \to 0 \) uniformly in \( K \in \mathcal{K} \). This completes the proof.

It is well known that if \( \{T_n(t)\} \) is a sequence of strongly continuous \( \mathcal{L}(\mathcal{H}) \)-valued functions and strongly convergent to \( t \mapsto T(t) \) uniformly in \( t \in [0, \tau] \), this does not imply (in general) that the sequence of mappings \( \{T_n^*(t)\} \) is strongly convergent to \( t \mapsto T^*(t) \) uniformly in \( t \in [0, \tau] \). This assertion even fails in the case of \( C_0 \)-semigroups (see [1] for a counterexample). However, in the case where one has convergence of the dual maps, we have the following result.

**Lemma 3 (The Smoothing Lemma for Dual Convergence).** Let \( \{T_n(t)\} \) and \( \{T_n^*(t)\} \) be sequences of \( \mathcal{L}(\mathcal{H}) \)-valued functions, strongly continuous and strongly convergent to the maps \( t \mapsto T(t) \) and \( t \mapsto T^*(t) \), respectively and uniformly in \( t \in [0, \tau] \). Suppose also that \( 1 \leq p \leq \infty \) and let \( \mathcal{K} \) be a compact set in \( \mathcal{I}_p \). Then as \( n \to \infty \)

i) \( \sup_{K \in \mathcal{K}} \left( \sup_{t \in [0, \tau]} \|T_n(t)K - T(t)K\|_p \right) \to 0 \)

ii) \( \sup_{K \in \mathcal{K}} \left( \sup_{t \in [0, \tau]} \|KT_n^*(t) - KT^*(t)\|_p \right) \to 0 \)
iii) \( \sup_{K \in K} \left( \sup_{t \in [0, \tau]} \| KT_n(t) - KT(t) \|_p \right) \to 0 \)

iv) \( \sup_{K \in K} \left( \sup_{t \in [0, \tau]} \| T_n^*(t) K - T^* K(t) \|_p \right) \to 0 \)

Proof. The proof follows immediately by the application of the previous Lemma.

2.4 The Input Map \( t \mapsto B(t) \)

We are interested in studying input mappings of the form \( B : I \mapsto \mathcal{L}(X, \mathcal{H}) \), for some real interval (commonly \( I = [0, \tau] \) or \( I = \mathbb{R}^+ = [0, \infty) \)) where \( X \) and \( \mathcal{H} \) are complex separable Hilbert spaces. Since \( X \) and \( \mathcal{H} \) may not be the same, we need to define the spaces \( \mathcal{I}_p \) for operators in \( \mathcal{L}(X, \mathcal{H}) \).

2.4.1 The Classes \( \mathcal{I}_p(X, \mathcal{H}) \)

Suppose \( A \in \mathcal{L}(X, \mathcal{H}) \), and let \( A = U|A| \) be its polar decomposition. In particular, \(|A| \in \mathcal{L}(X)\) and \( U \in \mathcal{L}(X, \mathcal{H})\) are the unique operators such that \(|A| \geq 0\) and \( U \) is a partial isometry with \( \text{Ker} \ U = \text{Ker} \ |A| \). Since \(|A| \geq 0\), then for any \( 1 \leq p < \infty \) we observe that \(|A|^p \geq 0\). This follows by the functional calculus, \( \sigma(|A|) \subset [0, \infty) \) then \( f(\lambda) = \lambda^p \geq 0 \) defined as \( f : \sigma(|A|) \mapsto \mathbb{C} \) satisfies \( f(\lambda) \geq 0 \) and hence \( f(|A|) = |A|^p \geq 0 \). Therefore \( \text{Tr} \ (|A|^p) = \sum_{n=1}^\omega \langle \phi_n, |A|^p \phi_n \rangle \geq 0 \), with \( \omega \leq \infty \), is independent (and could be finite or infinite) of the chosen orthonormal basis \( \{ \phi_n \}_{n=1}^\omega \) of \( X \).

**Definition 6.** Let \( X \) and \( \mathcal{H} \) be separable complex Hilbert spaces. The set of all operators \( A \in \mathcal{L}(X, \mathcal{H}) \) such that \( \text{Tr} \ (|A|^p) < \infty \) is denoted by \( \mathcal{I}_p(X, \mathcal{H}) \). That is, \( A \in \mathcal{I}_p(X, \mathcal{H}) \) if and only if \( |A| \in \mathcal{I}_p(X) \).

The sets \( \mathcal{I}_p(X, \mathcal{H}) \) are linear vector spaces. In fact, they are Banach Spaces when one uses \( \text{Tr} \ (|A|^p) < \infty \) to define a norm. Although this result seems to be well known, we could not find a proof so we include the follow result for completeness.

**Proposition 10.** The space \( \mathcal{I}_p(X, \mathcal{H}) \) with norm defined by

\[
\|A\|_{\mathcal{I}_p(X, \mathcal{H})} \triangleq \|A\|_{\mathcal{I}_p(X)} = \left( \text{Tr} \ (|A|^p) \right)^{1/p}
\]

is a Banach Space.
Proof. First we establish \( \mathcal{I}_p(X, \mathcal{H}) \) is a linear space. Let \( A_1 \) and \( A_2 \) be in \( \mathcal{I}_p(X, \mathcal{H}) \) and \( A_i = U_i|A_i| \) be their polar decompositions. It follows that \( |A_i| \in \mathcal{I}_p(X) \). Let \( A_1 + A_2 = V|A_1 + A_2| \) be the polar decomposition of \( A_1 + A_2 \). Consequently, \( |A_1 + A_2| = V^*U_1|A_1| + V^*U_2|A_2| \) and since \( V^*U_1 \in \mathcal{L}(X) \) we have that \( |A_1 + A_2| \in \mathcal{I}_p(X) \). This implies that \( A_1 + A_2 \in \mathcal{I}_p(X, \mathcal{H}) \). Also, if \( \alpha \in \mathbb{C} \), then \( |\alpha A_1| = |\alpha||A_1| \in \mathcal{I}_p(X) \), which implies that \( \alpha A_1 \in \mathcal{I}_p(X, \mathcal{H}) \) and this proves that \( \mathcal{I}_p(X, \mathcal{H}) \) is a linear space.

Next we establish that (2.6) defines a norm on \( \mathcal{I}_p(X, \mathcal{H}) \). By definition we have \( \|A\|_{\mathcal{I}_p(X, \mathcal{H})} \geq 0 \) and if \( \|A\|_{\mathcal{I}_p(X, \mathcal{H})} = 0 \) then \( |A| = 0 \) which yields \( A = 0 \). Also since \( |\alpha A| = |\alpha||A| \) then it follows that \( \|A\|_{\mathcal{I}_p(X, \mathcal{H})} = |\alpha|\|A\|_{\mathcal{I}_p(X, \mathcal{H})} \). Finally, \( |A_1 + A_2| = V^*U_1|A_1| + V^*U_2|A_2| \) and each \( V, U_1 \) and \( U_2 \) is a partial isometry. This is equivalent to \( \|V^*U_i\| \leq 1 \) for \( i = 1, 2 \) and hence \( \|A_1 + A_2\|_{\mathcal{I}_p(X)} \leq \|A_1\|_{\mathcal{I}_p(X)} + \|A_2\|_{\mathcal{I}_p(X)} \). We conclude that \( \|A_1 + A_2\|_{\mathcal{I}_p(X, \mathcal{H})} \leq \|A_1\|_{\mathcal{I}_p(X, \mathcal{H})} + \|A_2\|_{\mathcal{I}_p(X, \mathcal{H})} \) and hence \( (\text{Tr} (|A|^p))^{1/p} \) defines a norm.

In order to establish completeness, assume that \( \{A_n\}_{n=1}^\infty \) is a Cauchy sequence in \( \mathcal{I}_p(X, \mathcal{H}) \). If \( A_m - A_n = V_{mn}|A_m - A_n| \) is the polar decomposition of \( A_m - A_n \), then we have

\[
\|A_m - A_n\|_{\mathcal{L}(X, \mathcal{H})} \leq \|V_{mn}\|_{\mathcal{L}(X, \mathcal{H})}\|A_m - A_n\|_{\mathcal{L}(X)} \\
\leq \|A_m - A_n\|_{\mathcal{L}(X)} \\
\leq \|A_m - A_n\|_{\mathcal{I}_p(X)} \\
= \|A_m - A_n\|_{\mathcal{I}_p(X, \mathcal{H})}.
\]

Thus, \( \{A_n\}_{n=1}^\infty \) is a Cauchy sequence in \( \mathcal{L}(X, \mathcal{H}) \) and since this space is complete, it is convergent. Furthermore, since each \( A_n \) is compact, the limit is a compact operator, i.e., \( \lim_{n \to \infty} A_n = A \in \mathcal{I}_\infty(X, \mathcal{H}) \) in operator norm. This implies that \( |A_n| \to |A| \) in operator norm as \( n \to \infty \). The same holds for \( |A_n|^p \to |A|^p \) for \( p \in \mathbb{N} \) and we can use the continuous functional calculus to extend this to \( 1 \leq p < \infty \). All these claims follow from the continuity of \( A \mapsto f(A) \) when \( A \geq 0 \) and \( f \) is continuous on the right hand complex semiplane \( \{ z \in \mathbb{C} : \text{Re} z \geq 0 \} \) (see for example Halmos book [31]). First consider \( f(\lambda) = \lambda^{1/2} \) and \( B_n = A_n^*A_n \) which is self-adjoint and clearly \( B_n \to B = A^*A \) if \( A_n \to A \) and hence, \( f(B_n) = |A_n| \to |A| = f(B) \). Secondly, consider the continuous function \( g(\lambda) = \lambda^p \) on \( \mathbb{C} \) for \( p \geq 1 \); since \( |A_n| \geq 0 \) and \( |A| \geq 0 \) and \( |A_n| \to |A| \) in norm, \( |A_n|^p \to |A|^p \) in norm.

Therefore, for any \( \phi \in \mathcal{H} \), \( \langle \phi, |A|^p \phi \rangle \to \langle \phi, |A|^p \phi \rangle \) as \( n \to \infty \). Also, let \( \{\phi_n\}_{n=1}^\infty \) be an orthonormal basis of \( \mathcal{H} \). Since \( \{A_n\}_{n=1}^\infty \) is a Cauchy sequence
in $\mathcal{S}_p(X, \mathcal{H})$, then for any $N < \infty$,

$$\left( \sum_{k=1}^{N} \langle \phi_k, |A_n|^p \phi_k \rangle \right)^{1/p} \leq \| |A_n||_{\mathcal{S}_p(X, \mathcal{H})} = \|A_n\|_{\mathcal{S}_p(X, \mathcal{H})} \leq \sup_{n \in \mathbb{N}} \| |A_n||_{\mathcal{S}_p(X, \mathcal{H})} < \infty$$

where $M \triangleq \sup_{n \in \mathbb{N}} \| |A_n||_{\mathcal{S}_p(X, \mathcal{H})} < \infty$. Taking the limit as $n \to \infty$, we have

$$\left( \sum_{k=1}^{N} \langle \phi_k, |A|^p \phi_k \rangle \right)^{1/p} \leq M < \infty,$$

for any $N \in \mathbb{N}$. Therefore $\text{Tr} (|A|^p) < \infty$, i.e., $A \in \mathcal{S}_p(X, \mathcal{H})$. \qed

For $p = \infty$ we denote by $\mathcal{S}_\infty(X, \mathcal{H})$ the Banach space of compact operators in $\mathcal{L}(X, \mathcal{H})$ under the usual operator norm $\|A\|_\infty = \|A\|_{\mathcal{S}(X, \mathcal{H})} = \|A\|$. In the case in which $X = \mathcal{H}$, we use $\mathcal{S}_p(\mathcal{H})$ or $\mathcal{S}_p$ as usual when the space is understood. We need to extend several previous results to the space $\mathcal{S}_p(X, \mathcal{H})$.

**Proposition 11.** Let $A \in \mathcal{S}_p(X, \mathcal{H})$ where $X$ and $\mathcal{H}$ are separable complex Hilbert spaces and $1 \leq p \leq \infty$. Then $A \in \mathcal{S}_r(X, \mathcal{H})$ for all $p \leq r \leq \infty$. Also $A^* \in \mathcal{S}_p(\mathcal{H}, X)$ and $\|A\|_{\mathcal{S}_p(X, \mathcal{H})} = \|A^*\|_{\mathcal{S}_p(\mathcal{H}, X)}$. If $A \in \mathcal{S}_{2p}(X, \mathcal{H})$ then $AA^* \in \mathcal{S}_p(\mathcal{H})$ and $\|AA^*\|_{\mathcal{S}_p(\mathcal{H})} \leq \|A\|_{\mathcal{S}_{2p}(X, \mathcal{H})} \|A^*\|_{\mathcal{S}_{2p}(\mathcal{H}, X)}$.

**Proof.** If $A \in \mathcal{S}_p(X, \mathcal{H})$, then $|A| \in \mathcal{S}_p(X)$. Hence $|A| \in \mathcal{S}_r(X, \mathcal{H})$ for any $p \leq r \leq \infty$ and this implies $A \in \mathcal{S}_r(X, \mathcal{H})$.

Let $A = U|A| \in \mathcal{L}(X, \mathcal{H})$ be the polar decomposition of $A$, then the polar decomposition of the adjoint $A^* \in \mathcal{L}(\mathcal{H}, X)$ is given by $A^* = U^*|A^*|$ where $|A^*| = U|A|U^*$ (for a proof see [24]). Since $P_1 = U^*U \in \mathcal{L}(\mathcal{H})$ is a projection onto $(\text{Ker } U)^\perp = (\text{Ker } |A|)^\perp = (\text{Range } |A|)^\perp$, then $U^*U|A| = |A|$ and hence $|A^*|^p = U|A|^pU^*$ for $p \in \mathbb{N}$. By the continuous functional calculus $|A^*|^p = U|A|^pU^*$ holds for any $1 \leq p < \infty$.

Now, let $\{\phi_n\}_{n=1}^{\infty}$ be the orthonormal basis of $\mathcal{H}$ given by eigenvectors of $|A^*| \in \mathcal{L}(\mathcal{H})$. This is possible because $|A^*|$ is compact and self-adjoint since $A^*$ is compact. Then $\{\psi_n\}_{n=1}^{\infty} = \{U^*\phi_n\}_{n=1}^{\infty}$ is an orthonormal set (not necessarily a basis) in $X$. Since $\phi_n \in \text{Range}(|A^*|)$ and $P_2 = UU^* \in \mathcal{L}(X)$ is a
projection onto $\text{Range}(|A^*|)$, we observe that $\langle \psi_n, \psi_m \rangle_X = \langle UU^*\phi_n, \phi_m \rangle_{\mathcal{H}} = \langle \phi_n, \phi_m \rangle_{\mathcal{H}} = \delta_{n,m}$. Therefore,

$$\text{Tr} \left( |A^*|^p \right) = \sum_{n} \langle \phi_n, U|A|^pU^*\phi_n \rangle_{\mathcal{H}} = \sum_{n} \langle \psi_n, |A|^p\psi_n \rangle_X \leq \text{Tr} \left( |A|^p \right) < \infty.$$  

Interchanging the roles of $A$ and $A^*$ in the proof, we obtain the reverse inequality $\text{Tr} \left( |A|^p \right) \leq \text{Tr} \left( |A^*|^p \right)$. This implies that $\|A\|_{\mathcal{S}_p(X, \mathcal{H})} = \|A^*\|_{\mathcal{S}_p(\mathcal{H}, X)}$.

Now suppose that $A \in \mathcal{S}_{2p}(X, \mathcal{H})$. We observe that $|AA^*| = AA^* = U|A|^{2p}U^* \geq 0$. Then $|AA^*|^p = U|A|^{2p}U^*$ for $p \in \mathbb{N}$ (since $U^*U|A| = |A|$ as we used before) and the continuous functional calculus extends this to any $1 \leq p < \infty$. Therefore

$$\text{Tr}(AA^*|^p) = \sum_{n} \langle \phi_n, U|A|^{2p}U^*\phi_n \rangle_{\mathcal{H}} = \sum_{n} \langle \psi_n, |A|^{2p}\psi_n \rangle_X \leq \text{Tr}(|A|^{2p}) < \infty.$$  

Finally, $\text{Tr} \left( |A|^{2p} \right) = \|\|A\| |A|^p\|_{\mathcal{S}_2(X)} \leq \|\|A\| |A|^p\|_{\mathcal{S}_2(X)} \|A\|_{\mathcal{S}_2(X)}\|A\|_{\mathcal{S}_2(X)}$ and using the previous identities we have $\|\|A\|_{\mathcal{S}_2(X)} = (\text{Tr} \left( |A|^{2p} \right))^{1/2} = \|\|A\|_{\mathcal{S}_{2p}(X, \mathcal{H})} = \|A\|_{\mathcal{S}_{2p}(X, \mathcal{H})} \|A^*\|_{\mathcal{S}_{2p}(\mathcal{H}, X)} \| A^* \|_{\mathcal{S}_{2p}(\mathcal{H}, X)} \| A \|_{\mathcal{S}_{2p}(X, \mathcal{H})}$.

Therefore $\text{Tr}(AA^*|^p) \leq \|A\|^p_{\mathcal{S}_{2p}(X, \mathcal{H})} \|A^*\|_{\mathcal{S}_{2p}(\mathcal{H}, X)}^p$ or

$$\|AA^*\|_{\mathcal{S}_p(\mathcal{H}, X)} \leq \|A\|_{\mathcal{S}_{2p}(X, \mathcal{H})} \|A^*\|_{\mathcal{S}_{2p}(\mathcal{H}, X)}.$$  

\[ \square \]

### 2.4.2 Properties of $t \mapsto BB^*(t)$

In this section, we provide conditions on the map $t \mapsto B(t)$ so that $t \mapsto BB^*(t)$ is $\mathcal{S}_p$-valued, Bochner measurable and locally integrable.

**Lemma 4.** Let $X$ and $\mathcal{H}$ be separable complex Hilbert spaces, $I$ be a real interval (bounded or unbounded) and let $B : I \mapsto \mathcal{S}_{2p}(X, \mathcal{H})$ with $1 \leq p \leq \infty$.

i. If $B(\cdot) \in L^2_{\text{loc}}(I; \mathcal{S}_{2p}(X, \mathcal{H}))$ then $BB^*(\cdot) \in L^1_{\text{loc}}(I; \mathcal{S}_p(\mathcal{H}))$.

ii. If $B(\cdot) \in \mathcal{C}(I; \mathcal{S}_{2p}(X, \mathcal{H}))$, then $BB^*(\cdot) \in \mathcal{C}(I; \mathcal{S}_p(\mathcal{H}))$.

**Proof.** We first prove the measurability $BB^*(\cdot)$. Since $B(t) \in \mathcal{S}_{2p}(X, \mathcal{H})$ for each $t \in I$, then $B^*(t) \in \mathcal{S}_{2p}(\mathcal{H}, X)$ and $BB^*(t) \equiv B(t)B^*(t) \in \mathcal{S}_p(\mathcal{H})$. Then $BB^* : I \mapsto \mathcal{S}_p(\mathcal{H})$. Since $B : I \mapsto \mathcal{S}_{2p}(X, \mathcal{H})$ is measurable, there is a sequence of simple $\mathcal{S}_{2p}(X, \mathcal{H})$-valued functions $B_n(t) = \sum_{k=1}^{n} b_k(n)X_{E_k(n)}(t)$
for \( t \in I \) with \( b_k(n) \in \mathcal{L}_2p(X, \mathcal{H}) \) and \( E_k(n) \subset I \) measurable sets for 
\( 1 \leq k \leq n \in \mathbb{N} \) satisfying \( \|(B - B_n)(t)\|_{\mathcal{L}_2p(X, \mathcal{H})} \to 0 \) a.e. in \( t \in I \) as \( n \to \infty \).

Note that \( B_n B_n^*(t) \triangleq B_n(t) B_n^*(t) \) is a simple \( \mathcal{I}_p(\mathcal{H}) \)-valued function and

\[
\| (B B^* - B_n B_n^*)(t) \|_{\mathcal{I}_p(\mathcal{H})} \leq \| B(B^* - B_n^*)(t) \|_{\mathcal{I}_p(\mathcal{H})} + \| (B - B_n)B_n^*(t) \|_{\mathcal{I}_p(\mathcal{H})}
\]

\[
= \| (B - B_n)B^*(t) \|_{\mathcal{I}_p(\mathcal{H})} + \| (B - B_n)B_n^*(t) \|_{\mathcal{I}_p(\mathcal{H})}
\]

\[
\leq \| (B - B_n)(t) \|_{\mathcal{L}_2p(X, \mathcal{H})} \left( \| B^*(t) \|_{\mathcal{I}_2p(\mathcal{H}, X)} + \| B_n^*(t) \|_{\mathcal{I}_2p(\mathcal{H}, X)} \right).
\]

Hence \( \| (B B^* - B_n B_n^*)(t) \|_{\mathcal{I}_p(\mathcal{H})} \to 0 \) a.e. in \( t \in I \) as \( n \to \infty \) and this implies that \( B B^* : I \mapsto \mathcal{I}_p(\mathcal{H}) \) is measurable.

i. To prove the first claim we note that if \( B(\cdot) \in L^2_{loc}(I; \mathcal{L}_2p(X, \mathcal{H})) \), then

\[
\int_C \| B(t) \|_{\mathcal{L}_2p(X, \mathcal{H})}^2 \, dt < \infty,
\]

for any compact interval \( C \subset I \). The inequality \( \| B(t)B^*(t) \|_{\mathcal{I}_p(\mathcal{H})} \leq \| B(t) \|_{\mathcal{L}_2p(X, \mathcal{H})} \| B^*(t) \|_{\mathcal{L}_2p(X, \mathcal{H})} = \| B(t) \|_{\mathcal{L}_2p(X, \mathcal{H})}^2 \) implies that

\[
\int_C \| BB^*(t) \|_{\mathcal{I}_p(\mathcal{H})} \, dt \leq \int_C \| B(t) \|_{\mathcal{L}_2p(X, \mathcal{H})}^2 \, dt < \infty.
\]

Hence, \( BB^*(\cdot) \in L^1_{loc}(I; \mathcal{I}_p(\mathcal{H})) \).

ii. To establish the second claim, let \( t \) and \( s \) belong to some compact interval \( C \subset I \). It follows that

\[
\| BB^*(t) - BB^*(s) \|_{\mathcal{I}_p(\mathcal{H})}
\]

\[
\leq \| (B(t) - B(s))B^*(t) \|_{\mathcal{I}_p(\mathcal{H})} + \| (B(t) - B(s))B^*(s) \|_{\mathcal{I}_p(\mathcal{H})}
\]

\[
\leq \| B(t) - B(s) \|_{\mathcal{L}_2p(X, \mathcal{H})} \left( \| B^*(t) \|_{\mathcal{I}_2p(\mathcal{H}, X)} + \| B^*(s) \|_{\mathcal{I}_2p(\mathcal{H}, X)} \right)
\]

\[
\leq M \| B(t) - B(s) \|_{\mathcal{L}_2p(X, \mathcal{H})},
\]

where \( M = \sup_{t,s \in C} \left( \| B^*(t) \|_{\mathcal{I}_2p(\mathcal{H}, X)} + \| B^*(s) \|_{\mathcal{I}_2p(\mathcal{H}, X)} \right) < \infty \). Hence, if \( B(\cdot) \in C(I; \mathcal{L}_2p(X, \mathcal{H})) \), then \( BB^*(\cdot) \in C(I; \mathcal{I}_p(\mathcal{H})) \).

\( \square \)
In many realistic control problems the number of inputs and sensed outputs are typically finite (and often small in numbers). In this case, the previous results become stronger when $X$ is finite dimensional.

**Proposition 12.** Let $X$ be a finite dimensional complex Hilbert space and $\mathcal{H}$ a separable complex Hilbert space. If $A \in \mathcal{L}(X, \mathcal{H})$, then $A \in \mathcal{I}_1(X, \mathcal{H})$.

**Proof.** If $A = U|A|$ is the polar decomposition of $A$, then $|A| \in \mathcal{L}(X)$. Since $X$ is finite dimensional, $|A| \in \mathcal{I}_1(X)$ and this implies that $A \in \mathcal{I}_1(X, \mathcal{H})$.

As expected, for the same $X$ and $\mathcal{H}$ as above, the same result holds for any operator $A \in \mathcal{L}(\mathcal{H}, X)$. In this case $A \in \mathcal{I}_1(\mathcal{H}, X)$, since $A$ is of finite rank.

**Lemma 5.** Let $X$ be a finite dimensional complex Hilbert space, $\mathcal{H}$ be a separable complex Hilbert space and $I$ be a real interval (bounded or unbounded).

1. If $B(\cdot) \in L^1_{\text{loc}}(I; \mathcal{L}(X, \mathcal{H}))$, then $BB^*(\cdot) \in L^1_{\text{loc}}(I; \mathcal{I}_1(\mathcal{H}))$.

2. If $B(\cdot) \in C(I; \mathcal{L}(X, \mathcal{H}))$, then $BB^*(\cdot) \in C(I; \mathcal{I}_1(\mathcal{H}))$.

**Proof.** We first prove that for each $1 \leq p < \infty$ there is an $c > 0$ such that $\|A\|_{\mathcal{L}_p(X, \mathcal{H})} \leq c\|A\|_{\mathcal{L}(X, \mathcal{H})}$ for each $A \in \mathcal{L}(X, \mathcal{H})$. Since all norms are equivalent in finite dimensions and $\mathcal{L}(X)$ is finite dimensional there is a $c > 0$ such that $\|A\|_{\mathcal{L}_p(X)} \leq c\|A\|_{\mathcal{L}(X)}$ for all $A \in \mathcal{L}(X)$. Since $U^*A = |A|$ (where $A = U|A|$ is the polar decomposition of $A$), then $\|A\|_{\mathcal{L}(X)} \leq \|A\|_{\mathcal{L}(X, \mathcal{H})}$ this implies that $\|A\|_{\mathcal{L}_p(X, \mathcal{H})} \leq m\|A\|_{\mathcal{L}(X, \mathcal{H})}$. Since by definition $\|A\|_{\mathcal{L}_p(X, \mathcal{H})} = \|A\|_{\mathcal{L}_p(X)}$, the claimed result follows.

The previous proposition implies $B(\cdot)$ is $\mathcal{I}_1(X, \mathcal{H})$-valued and hence also $\mathcal{I}_2(X, \mathcal{H})$-valued, since $\mathcal{I}_1(X, \mathcal{H}) \hookrightarrow \mathcal{I}_2(X, \mathcal{H})$. We now prove $B(\cdot)$ measurable as a $\mathcal{I}_2(X, \mathcal{H})$-valued function. Since $B : I \mapsto \mathcal{L}(X, \mathcal{H})$ is Bochner measurable, then there is a sequence of simple functions $B_n : I \mapsto \mathcal{L}(X, \mathcal{H})$ such that $\|B(t) - B_n(t)\|_{\mathcal{L}(X, \mathcal{H})} \to 0$ a.e. in $t \in I$ as $n \to \infty$.

Since $\|B(t) - B_n(t)\|_{\mathcal{L}_2(X, \mathcal{H})} \leq c\|B(t) - B_n(t)\|_{\mathcal{L}(X, \mathcal{H})}$ for some $c > 0$, we observe that $\|B(t) - B_n(t)\|_{\mathcal{L}_2(X, \mathcal{H})} \to 0$ a.e. in $t \in I$ as $n \to \infty$. This implies $B : I \mapsto \mathcal{I}_2(X, \mathcal{H})$ is Bochner measurable.

1. If $B(\cdot) \in L^2_{\text{loc}}(I; \mathcal{L}(X, \mathcal{H}))$, since $\|B(t)\|_{\mathcal{L}_2(X, \mathcal{H})} \leq c\|B(t)\|_{\mathcal{L}(X, \mathcal{H})}$ for each $t \in I$, then it follows that $B(\cdot) \in L^2_{\text{loc}}(I; \mathcal{I}_2(X, \mathcal{H}))$. Therefore, by Lemma 4 we observe that $BB^*(\cdot) \in L^1_{\text{loc}}(I; \mathcal{I}_1(\mathcal{H})).$
ii. If \( B(\cdot) \in \mathcal{C}(I; \mathcal{L}(X, \mathcal{H})) \), since \( \|B(t)\|_{\mathcal{L}(X, H)} \leq c\|B(t)\|_{\mathcal{L}(X, H)} \) for some \( c > 0 \) and for each \( t \in I \), we observe that \( B(\cdot) \in \mathcal{C}(I; \mathcal{L}(X, \mathcal{H})) \) and this implies by Lemma 3 that \( BB^*(\cdot) \in \mathcal{C}(I; \mathcal{L}(\mathcal{H})) \).

\[ \Box \]

### 2.5 The Output Map \( t \mapsto C(t) \)

For the case of \( p \) sensors, the operator \( C(t) \) in (1.9) is given by

\[
(C(t)\varphi) = \begin{pmatrix}
(C_1(t)\varphi) \\
(C_2(t)\varphi) \\
\vdots \\
(C_n(t)\varphi)
\end{pmatrix},
\]

for \( \varphi(\cdot) \in L^2(\Omega) \), where

\[
(C_i(t)\varphi) = \int_{\Omega} K_i(t,x)\varphi(x) \, dx,
\]

for kernels \( K_i : I \times \Omega \rightarrow \mathbb{C} \), for \( i = 1, 2, \ldots, m \). We assume that \( K_i(t, \cdot) \in L^2(\Omega) \) for each \( t \in I \), so that \( C_i(t) \in \mathcal{L}(L^2(\Omega), \mathbb{C}) \) and \( C(t) \in \mathcal{L}(L^2(\Omega), \mathbb{C}^p) \) for each \( t \in I \).

**Proposition 13.** If the map \( t \mapsto K_i(t, \cdot) \) belongs to \( L^\infty_{\text{loc}}(I; L^2(\Omega)) \) for \( i = 1, 2, \ldots, p \) where \( I \) is a real (bounded or unbounded) interval, then \( (C^*C)(\cdot) \in L^\infty_{\text{loc}}(I; \mathcal{L}(L^2(\Omega))) \).

**Proof.** Since \( t \mapsto K_i(t, \cdot) \in L^\infty_{\text{loc}}(I; L^2(\Omega)) \), we have

\[
M = \operatorname{ess sup}_{t \in I} \|K_i(t, \cdot)\|_{L^2(\Omega)} < \infty
\]

and there are sequences \( \{K_i^k\}_{k=1}^\infty \) of simple \( L^2(\Omega) \)-valued functions \( K_i^k(t, x) = \sum_{j=1}^k P_i^j(k,x)X_{I_i^j(k)}(t) \) that converge point-wise a.e. to \( K_i \) with \( P_i^j(k, \cdot) \in L^2(\Omega) \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq k \). Define the sequence \( \{C_i^k(\cdot)\}_{k=1}^\infty \) by

\[
(C_i^k(t)\varphi) = \int_{\Omega} K_i^k(t,x)\varphi(x) \, dx = \sum_{j=1}^k \left( \int_{\Omega} P_i^j(k,x)\varphi(x) \, dx \right) X_{I_i^j(k)}(t),
\]

for \( \varphi(\cdot) \in L^2(\Omega) \), where
for \( \varphi(\cdot) \in L^2(\Omega) \). For each \( i, t \mapsto C_i^k(t) \) is a simple \( \mathcal{L}(L^2(\Omega), C) \)-valued function and satisfies
\[
\left| \left( C_i(t) - C_i^k(t) \right) \varphi \right| \leq \int_{\Omega} \left| K_i(t, x) - K_i^k(t, x) \right| |\varphi(x)| \, dx \\
\leq \| K_i(t, \cdot) - K_i^k(t, \cdot) \|_{L^2(\Omega)} \| \varphi(\cdot) \|_{L^2(\Omega)}.
\]
Therefore, \( \| C_i(t) - C_i^k(t) \|_{\mathcal{L}(L^2(\Omega), C)} \leq \| K_i(t, \cdot) - K_i^k(t, \cdot) \|_{L^2(\Omega)} \) and hence each \( t \mapsto C_i(t) \) is a Bochner measurable \( \mathcal{L}(L^2(\Omega), C) \)-valued function with domain \( I \). The same inequality shows that \( C_i(\cdot) \in L^{\infty}_{\text{loc}}(I, \mathcal{L}(L^2(\Omega); C)) \). This implies that \( C(\cdot) \in L^{\infty}_{\text{loc}}(I, \mathcal{L}(L^2(\Omega); C^p)) \) and \( C^*(\cdot) \in L^{\infty}_{\text{loc}}(I, \mathcal{L}(C^p; L^2(\Omega))) \).

Since \( C^*(\cdot) \in L^{\infty}_{\text{loc}}(I, \mathcal{L}(C^p; L^2(\Omega))) \) and \( C^p \) is finite dimensional we have that \( C^*(\cdot) \in L^2_{\text{loc}}(I, \mathcal{L}(C^{\infty}; L^2(\Omega))) \). If \( [a, b] \subset I \), then
\[
\int_{[a,b]} \| C^*(t) \|_{\mathcal{L}(C^{\infty}; L^2(\Omega))}^2 \, dt \leq (b - a) \left( \text{ess sup}_{t \in [a, b]} \| C^*(t) \|_{\mathcal{L}(C^{\infty}; L^2(\Omega))} \right)^2.
\]
Therefore, \( C^*(\cdot) \in L^2_{\text{loc}}(I, \mathcal{L}(C^p; L^2(\Omega))) \) and Lemma 5 implies
\[
C^* C(\cdot) \in L^{\infty}_{\text{loc}}(I; \mathcal{A}(L^2(\Omega))
\]
which completes the proof. \( \Box \)

### 2.5.1 The Stationary Network Case

Assume that one has \( p \) sensor-platforms fixed in \( \overline{\Omega} \) compact, each with a sensor capable of measuring a weighted average value of the process of interest. We will denote the position of a sensor with a “bar” on top of the variable. For example, the position of the first sensor we will be denoted by “\( \bar{x}_1 \)”. We consider two important examples.

**Example 2.5.1.** Let \( \bar{x}_i \in \overline{\Omega} \) and suppose the sensor measures an average value of each \( \varphi(\cdot) \in L^2(\Omega) \) within a fixed radius \( \delta > 0 \) of the location \( \bar{x}_i \in \overline{\Omega} \). In this case
\[
C_i(\bar{x}_i) \varphi(\cdot) = \int_{\Omega} \chi_i(x, \bar{x}_i) \varphi(x) \, dx,
\]
where \( \chi_i(x, y) = 1 \) if \( \|x - y\|_{\mathbb{R}^n} < \delta \) and \( \chi_i(x, y) = 0 \) otherwise.

**Example 2.5.2.** Let \( \bar{x}_i \in \overline{\Omega} \), \( k > 0 \) and \( K(x) = e^{-k\|x - \bar{x}_i\|_{\mathbb{R}^n}^2} \), so that
\[
C_i(\bar{x}_i) \varphi(\cdot) = \int_{\Omega} e^{-k\|x - \bar{x}_i\|_{\mathbb{R}^n}^2} \varphi(x) \, dx.
\]
Here \( C_i(\bar{x}_i) \) is a Gaussian-type kernel sensor.
The general form of the output operator when \( p \) sensors are placed in \( \overline{\Omega} \) is given by

\[
(C(t, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\varphi) = \begin{pmatrix}
(C_1(t, \bar{x}_1)\varphi) \\
(C_2(t, \bar{x}_2)\varphi) \\
\vdots \\
(C_n(t, \bar{x}_n)\varphi)
\end{pmatrix},
\]

for \( \varphi(\cdot) \in L^2(\Omega) \), with

\[
(C_i(t, \bar{x}_i)\varphi) = \int_{\Omega} K_i(t, x, \bar{x}_i)\varphi(x) \, dx.
\]

In applications, if the sensor is moved from the position \( \bar{x}_i \) to a position \( \bar{x}_i + \Delta \bar{x}_i \) with \( \|\Delta \bar{x}_i\|_{\mathbb{R}^n} \ll 1 \) we expect that the measurement at \( \bar{x}_i \) is close to the one in \( \bar{x}_i + \Delta \bar{x}_i \). This motivates the following definition.

**Definition 7** (Continuity w.r.t. Location). Let \( I \) be a real interval and \( X_0 \) a compact subset of \( \overline{\Omega} \). Also, let \( C : I \times X_0^p \mapsto \mathcal{L}(L^2(\Omega); \mathbb{C}^p) \), be of the form \((2.7)\). We say that \( C \) is continuous with respect to location on \( X_0 \) if there is a continuous function \( g : \mathbb{R}^+ \mapsto \mathbb{R}^+ \) such that \( g(0) = 0 \) and

\[
\|K_i(t, \cdot, y) - K_i(t, \cdot, z)\|_{L^2(\Omega)} \leq g(\|y - z\|_{\mathbb{R}^n}), \quad \forall t \in I \quad \forall y, z \in X_0,
\]

and for \( i = 1, 2, \ldots, p \). Here, \( t \mapsto K_i(t, \cdot, x) \in L^2_{loc}(I; L^2(\Omega)) \) for \( i = 1, 2, \ldots, p \) and for each \( x \in X_0 \), \( K_i \) is the kernel of the integral representation for \( C_i \) in \((2.8)\).

Since the solution of Riccati equation can be regarded a function of the mapping \( t \mapsto C^*C(t, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \) we are interested in properties of the set \( \{ t \mapsto C^*C(t, \bar{x}) : \text{ for } \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in X_0^n \} \), where \( X_0 \) is some compact set of interest that belongs to \( \overline{\Omega} \).

**Lemma 6.** Let \( X_0 \subset \overline{\Omega} \) be compact, \( I = [0, \tau] \) and \( C : I \times X_0^p \mapsto \mathcal{L}(L^2(\Omega); \mathbb{C}^p) \) be of the form \((2.7)\). In addition, suppose that each \( C \) is Lipschitz continuous with respect to location on \( X_0 \) (see definition 7). Then, the set \( \mathcal{F} \) defined as

\[
\mathcal{F} = \{ C^*C(\cdot, \bar{x}) \in L^\infty(I; \mathcal{S}_1(L^2(\Omega))) : \text{ for } \bar{x} \in X_0^n \},
\]

is compact in \( L^\infty(I; \mathcal{S}_1(L^2(\Omega))) \).
Proof. Since $I = [0, \tau]$ and $C$ is continuous w.r.t. location on $X_0$ for each $\bar{x} \in X_0^p$, we observe that $(C^*C)(\cdot, \bar{x}) \in L^\infty(I; \mathcal{J}_1(L^2(\Omega)))$ by Proposition 13. This follows since $K_i(\cdot, \bar{x}) \in L^\infty(I, L^2(\Omega))$ for each $\bar{x} \in X_0 \subset \Omega$ and $i = 1, 2, \ldots, p$.

Since $L^\infty([0, \tau]; \mathcal{J}_1(L^2(\Omega)))$ is a metric space, it is enough to prove that the set is sequentially compact, i.e., that each sequence has a convergent subsequence. Let $\{S_k\}_{k=1}^\infty$ be a sequence in $\mathcal{F}$. Then $S_k = (C^*C)(\cdot, \bar{x}_k)$ and since $X_0^p$ is compact, there is a convergent subsequence of $\{\bar{x}_k\}_{k=1}^\infty$. Without loss of generality, let $\bar{x}_k = (\bar{x}_k^1, \bar{x}_k^2, \ldots, \bar{x}_k^p) \to \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p) \in X_0^p$. Also, since $C$ is continuous with respect to location on $X_0$, we have

$$\|(C(t, \bar{x}) - C(t, \bar{x}_k))\varphi(\cdot)\|_{C^0} \leq \sum_{i=1}^p \int_\Omega |K_i(t, x, \bar{x}_i) - K_i(t, x, \bar{x}_k^i)|\varphi(x) \, dx \leq \sum_{i=1}^p \|K_i(t, \cdot, \bar{x}_i) - K_i(t, \cdot, \bar{x}_k^i)\|_{L^2(\Omega)}\|\varphi(\cdot)\|_{L^2(\Omega)} \leq \sum_{i=1}^p g(\|\bar{x}_i - \bar{x}_k^i\|_{C^0})\|\varphi(\cdot)\|_{L^2(\Omega)}.$$ 

This implies, since $g(0) = 0$ and $g$ is continuous, that

$$\|(C(t, \bar{x}) - C(t, \bar{x}_k))\|_{L^\infty(I; \mathcal{J}_1(L^2(\Omega), C^0))} \to 0.$$ 

The map $t \mapsto C(t, \bar{x})$ is Bochner measurable as a $\mathcal{L}(L^2(\Omega), C^p)$-valued mapping (see Proposition 13), and then

$$\|(C(\cdot, \bar{x}) - C(\cdot, \bar{x}_k))\|_{L^\infty(I; \mathcal{L}(L^2(\Omega), C^p))} \to 0.$$ 

Since $C^p$ is finite dimensional, we have $\|(C(t, \bar{x}) - C(t, \bar{x}_k))\|_{\mathcal{J}_1(L^2(\Omega), C^p)} \leq c\|(C(t, \bar{x}) - C(t, \bar{x}_k))\|_{\mathcal{L}(L^2(\Omega), C^p)}$ for some $c > 0$ (see the proof of Lemma 5). Consequently, it follows that $\|(C(\cdot, \bar{x}) - C(\cdot, \bar{x}_k))\|_{L^\infty(I; \mathcal{J}_1(L^2(\Omega), C^p))} \to 0$, since $t \mapsto C(t, \bar{x})$ is also $\mathcal{J}_1(L^2(\Omega), C^p)$-valued and measurable.

On the other hand, $(C^*C)(\cdot, \bar{x}) \in \mathcal{F}$ and by the properties of Proposition 13 we have

$$\|(C^*C)(t, \bar{x}) - C^*C)(t, \bar{x}_k)\|_{\mathcal{J}_1(L^2(\Omega))} \leq$$

$$= \|C^*(t, \bar{x})\|_{\mathcal{J}_1(C^p, L^2(\Omega))}\|C(t, \bar{x}) - C(t, \bar{x}_k)\|_{\mathcal{J}_1(L^2(\Omega), C^p)} + \|C(t, \bar{x}_k)\|_{\mathcal{J}_1(L^2(\Omega), C^p)}\|C^*(t, \bar{x}) - C^*(t, \bar{x}_k)\|_{\mathcal{J}_1(C^p, L^2(\Omega))}.$$
Therefore,
\[
\|(C^*C)(t, \bar{x}) - (C^*C)(t, \bar{x}_k)\|_{\mathcal{F}_1(L^2(\Omega))} \leq \\
\left(\|C(t, \bar{x})\|_{\mathcal{F}_1(L^2(\Omega), C^p)} + \|C(t, \bar{x}_k)\|_{\mathcal{F}_1(L^2(\Omega), C^p)}\right)\|C(t, \bar{x}) - C(t, \bar{x}_k)\|_{\mathcal{F}_1(L^2(\Omega), C^p)}
\]
and we observe that the the factor between parenthesis is uniformly bounded in \( n \in \mathbb{N} \) as elements in \( L^\infty(\mathcal{I}; \mathcal{F}_1(L^2(\Omega), C^p)) \). Hence, \( \|(C^*C)(t, \bar{x}) - (C^*C)(t, \bar{x}_k)\|_{L^\infty(\mathcal{I}; \mathcal{F}_1(L^2(\Omega)))} \to 0 \) which proves the compactness of \( \mathcal{F} \) in \( L^\infty(\mathcal{I}; \mathcal{F}_1(L^2(\Omega))) \).

**Remark 2.4.** The condition that states that there exist a continuous \( g : \mathbb{R}^+ \mapsto \mathbb{R}^+ \) satisfying \( g(0) = 0 \) and
\[
\|K_i(t, \cdot, y) - K_i(t, \cdot, y)\|_{L^2(\Omega)} \leq g(\|y - z\|_{\mathbb{R}^n}),
\]
for all \( t \in \mathcal{I} \) and all \( y, z \in \overline{\Omega} \) is satisfied by the kernels we considered in the Examples 2.5.1 and 2.5.2. To establish these claims we begin with the first example.

**i.** For Example 2.5.1 let \( y, z \in \overline{\Omega} \). Then
\[
\int_{\mathbb{R}^n} |\chi_{\delta}(x, y) - \chi_{\delta}(x, z)| \, dx = \int_{\mathbb{R}^n} |\chi_{\delta}(x, 0) - \chi_{\delta}(x, z - y)| \, dx \\
= 2m \{B_\delta(z - y) \setminus B_\delta(0)\},
\]
where \( m \) refers to the Lebesgue measure in \( n \) dimensions and \( B_\delta(x) \) is the closed \( n \)-dimensional ball of radius \( \delta \) centered at \( x \). The function \( \Omega \ni w \mapsto m \{B_\delta(w) \setminus B_\delta(0)\} \) depends only on the norm of \( w \), i.e., \( G(\|w\|_{C^p}) = 2m \{B_\delta(w) \setminus B_\delta(0)\} \). The function \( G \) is clearly monotonically increasing and satisfies \( G(0) = m\emptyset = 0 \). The continuity of \( x \mapsto G(x) \) can be easily checked using the Lebesgue Dominated Convergence Theorem on the integral representation above. Since \( |\chi_{\delta}(x, y) - \chi_{\delta}(x, z)|^2 = |\chi_{\delta}(x, y) - \chi_{\delta}(x, z)| \), then \( \int_{\Omega} |\chi_{\delta}(x, y) - \chi_{\delta}(x, z)| \, dx \leq m(\Omega)(\int_{\mathbb{R}^n} |\chi_{\delta}(x, y) - \chi_{\delta}(x, z)| \, dx)^{1/2} \) which yields
\[
\int_{\Omega} |\chi_{\delta}(x, y) - \chi_{\delta}(x, z)|^2 \, dx \leq m(\Omega)(G(\|y - z\|_{\mathbb{R}^n}))^{1/2}.
\]
ii. For Example 2.5.2 we consider \( K(x, \bar{x}_i) = e^{-k \|x-\bar{x}_i\|_2^2} \) for some \( k > 0 \). Since \( t \mapsto e^{-kt} : [0, \infty) \mapsto [0, 1] \) is Lipschitz continuous, it follows that
\[
|K(x, y) - K(x, z)| \leq C \|x - y\|_R^2 - \|x - z\|_R^2.
\]
The Law of Cosines yield
\[
\|x - y\|_R^2 - \|x - z\|_R^2 = \|y - z\|_R^2 - 2\|y - z\|_R \|x - z\|_R \cos(\theta_z),
\]
for some \( \theta_z \in [-\pi, \pi] \) and since \( \text{diam}(\Omega) = \sup_{v, w \in \Omega} \|v - w\|_R < \infty \), it follows that
\[
\|x - y\|_R^2 - \|x - z\|_R^2 \leq 3 \text{diam}(\Omega) \|y - z\|_R.
\]
Thus, we have
\[
\int_{\Omega} |K(x, y) - K(x, z)|^2 \, dx \leq \left( 9C^2 (\text{diam}(\Omega))^2 m(\Omega) \right) \|y - z\|_R^2.
\]

Finally, we note that it is straightforward to show that if the kernels \( K_i \) do not depend on \( t \), then the set
\[
\mathcal{F} = \{ C^* C(\bar{x}) \in \mathcal{I}(L^2(\Omega)) : \text{ for } \bar{x} \in X_0^* \},
\]
is compact.

We now consider the mobile sensor network problem where we have \( p \) sensor-platforms (vehicles) moving in \( \Omega \), each with a sensor capable of measuring a weighted average value of the process within an fixed range \( \delta > 0 \) of the location of the platform. We do not consider the impact on the platform velocity on the sensor.

### 2.5.2 The Moving Network Case

Here, we assume that \( I = [0, \tau] \) for some fixed \( \tau > 0 \) and that that \( \Omega \subset \mathbb{R}^3 \) is open and bounded. Thus, \( \bar{\Omega} \) is compact. Let \( \bar{x}_i(t) \in \bar{\Omega} \) for each \( t \in I \) be the position of the \( i \)th sensor at time \( t \). In this case the general form of the output map for a moving sensor platform is given by
\[
C_i(t, \bar{x}_i(t)) \varphi(\cdot) = \int_{\Omega} K_i(t, x, \bar{x}_i(t)) \varphi(x) \, dx,
\]
where $K_i : I \times \overline{\Omega} \times \overline{\Omega} \mapsto \mathbb{R}$. For every continuous curve $t \mapsto \bar{x}_i(t)$ in $\overline{\Omega}$ we will assume that the map $t \mapsto K_i(t, \cdot, \bar{x}_i(t))$ is Bochner measurable as a $L^2(\Omega)$-valued map on the interval $I = [0, \tau]$ and also essentially bounded, i.e., $t \mapsto K_i(t, \cdot, \bar{x}_i(t))$ belongs to $L^\infty(I; L^2(\Omega))$.

As noted previously, in real problems, trajectories are determined by vehicles that are often governed by nonlinear controlled ordinary differential equations. We will assume that each of the vehicles satisfy the following hypotheses.

The sensor trajectories $t \mapsto \bar{x}(t)$ are given by $\bar{x}(t) = M \bar{\theta}(t)$ where $M \in \mathbb{R}^{3 \times 6}$ is a constant matrix, $\bar{\theta}(t) \in \mathbb{R}^6$ for each $t \in [0, \tau]$. Also, the maps $t \mapsto \bar{\theta}(t)$ are outputs to a system of controlled ordinary differential equations

\[
\begin{align*}
\dot{\bar{\theta}}(t) &= f(t, \bar{\theta}(t), u(t)); \\
\bar{\theta}(0) &= \bar{\theta}_0;
\end{align*}
\]

with $f \in C^1(\mathbb{R}^{1+6+q}; \mathbb{R}^6)$, $\bar{\theta}_0 \in \Theta_0$ with $\Theta_0$ compact and $M \Theta_0 = X_0 \subset \overline{\Omega}$ and $u(\cdot) \in \mathcal{U}$, where

\[
\mathcal{U} = \{ u : u \text{ is measurable and } u(t) \in \Gamma \subset \mathbb{R}^3 \text{ for all } t \in I = [0, \tau] \},
\]

and $\Gamma$ is compact. We make the following standard assumptions.

**Ha)** The response satisfies $\bar{\theta}(t, \bar{\theta}_0, u) \in \Theta_1$, with $\Theta_1$ compact and $M \Theta_1 = \overline{\Omega}$, for all $u(\cdot) \in \mathcal{U}, \bar{\theta}_0 \in \Theta_0$ and all $t \in I = [0, \tau]$.

**Hb)** The set $V(\theta, t) = \{ f(t, \theta, u)/u \in \Gamma \}$ is convex for each fixed $(\theta, t)$.

The previous hypotheses are the usual hypotheses required for the attainability set to be compact (see [10]) and to vary continuously with respect to $t \in [0, \tau]$. Note also that **Ha** implies that $\bar{\theta}(t, \bar{\theta}_0, u)$ is uniformly bounded in $([0, \tau], \Theta_0, \mathcal{U})$. This condition implies that there is an $m > 0$ such that $f(t, \bar{\theta}(t, \bar{\theta}_0, u), u) \leq m$ for $t \in [0, \tau]$, $\bar{\theta}_0 \in \Theta_0$ and $u \in \mathcal{U}$ for any fixed $\tau > 0$.

Since we have $p$ sensors, we denote $\bar{x}(t, \bar{\theta}_0, u(\cdot))$ as the $p$-dimensional vector with $\bar{x}_i(t, \bar{\theta}_0, u(\cdot))$ for $i = 1, 2, \ldots, p$ as elements, where $\bar{\theta}_0 = (\bar{\theta}_1^0, \bar{\theta}_2^0, \ldots, \bar{\theta}_p^0) \in \Theta_0^p$, and $u(\cdot) = (u_1(\cdot), u_2(\cdot), \ldots, u_p(\cdot))$ such that $u_i(\cdot) \in \mathcal{U}$ for $i = 1, 2, \ldots, p$. We suppose that each $t \mapsto \bar{x}_i(t, \bar{\theta}_0^i, u_i(\cdot))$ is defined $\bar{x}_i(t, \bar{\theta}_0^i, u_i(\cdot)) = M \bar{\theta}_i(t, \bar{\theta}_0^i, u_i(\cdot))$ where $\bar{\theta}_i$ a solution to a controlled
differential equation of the type described above and with initial conditions
\( \theta_i(0) = \theta_i^0 \). Then, we have
\[
(C(t, \bar{x}(t, \bar{\theta}_0, u(\cdot))) \varphi) = \begin{pmatrix}
(C_1(t, \bar{x}_1(t, \bar{\theta}_0^1, u_1(\cdot))) \varphi) \\
(C_2(t, \bar{x}_2(t, \bar{\theta}_0^2, u_2(\cdot))) \varphi) \\
\vdots \\
(C_p(t, \bar{x}_n(t, \bar{\theta}_0^n, u_n(\cdot))) \varphi)
\end{pmatrix} \in C^p, \quad (2.10)
\]
with
\[
C_i(t, \bar{x}_i(t, \bar{\theta}_0^i, u_i(\cdot))) \varphi(\cdot) = \int_{\Omega} K_i(t, x, \bar{x}_i(t, \bar{\theta}_0^i, u_i(\cdot))) \varphi(x) \, dx. \quad (2.11)
\]
We denote \( \bar{x}(t, \bar{\theta}_0, u(\cdot)) = M\bar{\theta}(t, \bar{\theta}_0, u(\cdot)) \), where \( M\bar{\theta}(t, \bar{\theta}_0, u(\cdot)) \) is shorthand for \( \{M\bar{\theta}(t, \bar{\theta}_0^i, u_i(\cdot))\}_{i=1}^p \).

**Lemma 7.** Let \( I = [0, \tau] \) and suppose that \( C \) is continuous with respect to location on \( \Omega \) (see definition 7). Then, the set \( \mathcal{F} \) defined by
\[
\mathcal{F} = \{ C^\ast C(\cdot, M\bar{\theta}(\cdot, \bar{\theta}_0, u)) \in L^\infty(I; \mathcal{I}_1(L^2(\Omega))) : \text{ for } \bar{\theta}_0 \in \Theta_0^p \text{ and } u \in U^p \},
\]
is compact in \( L^\infty([0, \tau]; \mathcal{I}_1(L^2(\Omega))) \).

If the kernels \( K_i \) do not depend explicitly on \( t \), so that \( K_i(\cdot, y) \in L^2(\Omega) \) for each \( y \in \Omega \), then the set \( \mathcal{F} \) defined as
\[
\mathcal{F} = \{ C^\ast C(M\bar{\theta}(\cdot, \bar{\theta}_0, u)) \in \mathcal{C}(I; \mathcal{I}_1(L^2(\Omega))) : \text{ for } \bar{\theta}_0 \in \Theta_0^p \text{ and } u \in U^p \},
\]
is compact in \( \mathcal{C}([0, \tau]; \mathcal{I}_1(L^2(\Omega))) \).

**Proof.** We present a proof for the case of one moving sensor. The generalization to multiple sensors is straightforward. Suppose all previously hypotheses hold. For any sequence of \( \bar{\theta}_0^k(\cdot, \bar{\theta}_0^k, u_k(\cdot)) \) with \( \bar{\theta}_0^k, u_k(\cdot) \in (\Theta_0, U) \), there is a subsequence (that we will also call \( \bar{\theta}_0^k(\cdot, \bar{\theta}_0^k, u_k(\cdot)) \)) such that
\[
\sup_{t \in [0, \tau]} \| \bar{\theta}(t, \bar{\theta}_0^k, u(\cdot)) - \bar{\theta}(t, \bar{\theta}_0^k, u_k(\cdot)) \|_{R^n} \to 0,
\]
as \( k \to \infty \), for some \( \bar{\theta}_0 \in \Theta_0 \) and some \( u(\cdot) \in U \) (for a proof when see [40]).

Let \( \bar{x}(t) = \bar{x}(t, \bar{\theta}_0, u(\cdot)) \) and \( \bar{x}^k(t) = \bar{x}(t, \bar{\theta}_0^k, u_k(\cdot)) \) for \( k \in \mathbb{N} \). Since \( C \) is continuous with respect to location on \( \Omega \), we have the same inequality obtained in the proof of Lemma 6. In particular,
\[
|(C(t, \bar{x}(t)) - C(t, \bar{x}^k(t))) \varphi(\cdot)| \leq g(\|M(\bar{\theta}(t) - \bar{\theta}_0^k(t))\|_{R^n})\|\varphi(\cdot)\|_{L^2(\Omega)}.
\]
which implies that \( \|(C(t, \bar{x}(t)) - C(t, \bar{x}^k(t)))\|_{L^2(\Omega), C} \to 0 \) for any \( t \in I \) because \( g \) is continuous and \( g(0) = 0 \). This also implies that \( \|(C(\cdot, \bar{x}(\cdot)) - C(\cdot, \bar{x}^k(\cdot)))\|_{L^2(\Omega), C} \to 0 \) because \( \|\bar{\theta}(t) - \bar{\theta}^k(t)\|_{\mathbb{R}^n} \to 0 \) uniformly in \( t \in I \) as \( k \to \infty \) and by Proposition 13 (note that we have assumed that the Kernel \( t \mapsto K(t, \cdot, \bar{x}(t)) \) of integral representation of the output map \( C \) belongs to \( L^\infty(I; L^2(\Omega)) \)). Since \( C \) is finite dimensional we have that

\[
\|(C(t, \bar{x}(t)) - C(t, \bar{x}^k(t)))\|_{\mathcal{F}_1(L^2(\Omega), C)} \leq c\|C(t, \bar{x}(t)) - C(t, \bar{x}^k(t))\|_{L^2(\Omega), C}
\]

for some \( c > 0 \) (see the proof of Lemma 5). Hence

\[
\|(C(\cdot, \bar{x}(\cdot)) - C(\cdot, \bar{x}^k(\cdot)))\|_{L^\infty(I; \mathcal{F}_1(L^2(\Omega), C))} \to 0,
\]

as \( k \to \infty \) since \( t \mapsto C(t, \bar{x}) \) is also \( \mathcal{F}_1(L^2(\Omega), C) \)-valued and measurable.

Clearly \((C^*C)(\cdot, \bar{x}(\cdot)) \in \mathcal{F}\) and following the proof of Lemma 6 we have

\[
\|(C^*C)(t, \bar{x}(t)) - (C^*C)(t, \bar{x}^k(t))\|_{\mathcal{F}_1(L^2(\Omega))} \leq \left(\|C(t, \bar{x}(t))\|_{\mathcal{F}_1(L^2(\Omega), C)} + \|C(t, \bar{x}^k(t))\|_{\mathcal{F}_1(L^2(\Omega), C)}\right)\|C(t, \bar{x}(t)) - C(t, \bar{x}^k(t))\|_{\mathcal{F}_1(L^2(\Omega), C)}.
\]

Taking the ess sup\( t \in I \) we observe that the term in the parentheses is uniformly bounded in \( n \in \mathbb{N} \) and hence we have

\[
\|(C^*C)(\cdot, \bar{x}(\cdot)) - (C^*C)(\cdot, \bar{x}^k(\cdot))\|_{L^\infty(I; \mathcal{F}_1(L^2(\Omega)))} \to 0,
\]

which proves the compactness of \( \mathcal{F} \).

If the kernel \( K \) does not depend explicitly on \( t \), then the inequality

\[
\|K(\cdot, y) - K(\cdot, z)\|_{L^2(\Omega)} \leq g(\|y - z\|_{\mathbb{R}^n}) \quad \text{for any } y, z \in \overline{\Omega}
\]

implies that \( t \mapsto C(\bar{x}(t)) \) is continuous as we now prove. It follows that

\[
\|(C(\bar{x}(t)) - C(\bar{x}(s)))\phi(\cdot)\| \leq \|K(\cdot, \bar{x}(t)) - K(\cdot, \bar{x}(s))\|_{L^2(\Omega)}\|\varphi(\cdot)\|_{L^2(\Omega)} \leq g(\|M(\bar{\theta}(t) - \bar{\theta}(s))\|_{\mathbb{R}^n})\|\varphi(\cdot)\|_{L^2(\Omega)},
\]

which implies \( \|C(\bar{x}(t)) - C(\bar{x}(s))\|_{\mathcal{F}_1(L^2(\Omega), C)} \leq g(\|M(\bar{\theta}(t) - \bar{\theta}(s))\|_{\mathbb{R}^n}). \) Since \( C \) is finite dimensional we have that

\[
\|C(\bar{x}(t)) - C(\bar{x}(s))\|_{\mathcal{F}_1(L^2(\Omega), C)} \leq mg(\|M(\bar{\theta}(t) - \bar{\theta}(s))\|_{\mathbb{R}^n})
\]

for some constant \( m > 0 \), independent of \( t, s \in I \). Since \( t \mapsto \bar{\theta}(t) \) is a continuous trajectory, this implies that \( C(\bar{x}(\cdot)) \in \mathcal{C}(I; \mathcal{F}_1(L^2(\Omega), C)) \) and the same inequalities as before prove the compactness of \( \mathcal{F} \) in this topology. \( \square \)
Chapter 3
The Integral Riccati Equation

In this section we focus on the Riccati integral equation

$$\Sigma(t) = S(t)\Sigma_0S^*(t) + \int_0^t S(t-s)(BB^* - \Sigma(s)(C^*C)(s)\Sigma(s))S^*(t-s)ds. \quad (3.1)$$

Unlike much of the existing literature in which this equation is considered in the mild sense, we shall interpret (3.1) by employing the Bochner integral with operator-valued integrand. This is suitable for our applications. The advantage of considering the integrand in equation (3.1) as Bochner integrable is that it can be approximated by step functions. Therefore, its integral (for a fixed $t$) can be uniformly approximated by finite sums of operators and this could be applied to the development of numerical methods.

3.1 Properties of the Mapping $\gamma$

We will define the right hand side of (3.1) as $\gamma(\Sigma(\cdot))$ and hence fixed points of $\gamma$ are solutions to the integral Riccati equation. We will now prove that $\gamma$ is a well defined function in the appropriate spaces. Throughout this section we assume that $H$ be a separable complex Hilbert space, $I = [0, \tau]$ or $I = \mathbb{R}^+ = [0, \infty)$ and $1 \leq p \leq \infty$.

**Theorem 3.** Let $H$ be a separable complex Hilbert space and assume

(i) $S(t)$ is a $C_0$-semigroup over $H$;

(ii) $\Sigma_0 \in \mathcal{I}_p$;
(iii) $F(\cdot) \in L^1_{\text{loc}}(I; \mathcal{I}_p)$;
(iv) $G(\cdot) \in L^\infty_{\text{loc}}(I; \mathcal{L}(\mathcal{H}))$;

hold. If $\Sigma(\cdot) \in L^2_{\text{loc}}(I; \mathcal{I}_{2p})$, then for all $t \in I$ the mapping
\[
s \mapsto S(t-s)(F - \Sigma G \Sigma)(s)S^*(t-s)
\]
is Bochner integrable as a $\mathcal{I}_p$-valued mapping on $[0,t]$ and $\gamma(\Sigma)(\cdot)$ defined by
\[
\gamma(\Sigma(\cdot))(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)(F - \Sigma G \Sigma)(s)S^*(t-s) \, ds, \quad (3.2)
\]
is a well defined function $\gamma : L^2_{\text{loc}}(I; \mathcal{I}_{2p}) \rightarrow \mathcal{C}(I; \mathcal{I}_p)$. Moreover, since

$\gamma(\Sigma) \in \mathcal{C}(I; \mathcal{I}_p)$,

it follows that $\mathcal{C}(I; \mathcal{I}_p)$ is an $\gamma$-invariant subspace of $L^2_{\text{loc}}(I; \mathcal{I}_{2p})$.

If in addition, $G(\cdot)$ satisfies the stronger condition
\[(iv') \quad G(\cdot) \in L^\infty_{\text{loc}}(I; \mathcal{I}_p),
\]
and $\Sigma(\cdot) \in L^2_{\text{loc}}(I; \mathcal{L}(\mathcal{H}))$, then $\gamma(\Sigma)(\cdot) \in \mathcal{C}(I; \mathcal{I}_p)$. In this case,

$\gamma \left( L^2_{\text{loc}}(I; \mathcal{L}(\mathcal{H})) \right) \subset \mathcal{C}(I; \mathcal{I}_p)$,

and $\mathcal{C}(I; \mathcal{I}_p)$ is a $\gamma$-invariant subspace of $L^2_{\text{loc}}(I; \mathcal{L}(\mathcal{H}))$.

Before giving the proof, we note a couple of important consequences of the previous result. Since $\mathcal{I}_p \subset \mathcal{I}_{2p} \subset \mathcal{I}_\infty$ for any $1 \leq p \leq \infty$, we observe that $\mathcal{C}(I; \mathcal{I}_p) \subset L^2_{\text{loc}}(I; \mathcal{I}_{2p})$. For $p = 1$ this implies that $\mathcal{C}(I; \mathcal{I}_1)$ is continuously embedded in $L^2(I; \mathcal{I}_2)$ (with $I$ compact) and the latter is a Hilbert space. Therefore, if we can find a locally square integrable, Hilbert-Schmidt valued solution of the Riccati equation, that function is trace class-valued and continuous in trace norm. The other very important feature to observe is that if (iv) holds, then it is not possible to define $\gamma$ over $L^2_{\text{loc}}(I; \mathcal{L}(\mathcal{H}))$. The reason for this is that $S(t)$ is a general $C_0$-semigroup (and not necessarily norm continuous for $t > 0$). That is, $t \mapsto S(t)$ is Bochner measurable as a $\mathcal{L}(\mathcal{H})$-valued mapping if and only if it is operator-norm continuous for $t > 0$ (see [32]).
Proof of Theorem 3. Since $S(t)$ is a $C_0$-semigroup of linear operators on the Hilbert space $\mathcal{K}$, then $S^*(t)$ is also a $C_0$-semigroup on the same Hilbert space $\mathcal{K}$ and the map $t \mapsto S^*(t)$ is strongly continuous. Since $\Sigma_0 \in \mathcal{I}_p$, we have $S(\cdot)\Sigma_0 \in \mathcal{C}(I; \mathcal{I}_p)$, and since $t \mapsto S^*(t)$ is strongly continuous it follows that $S(\cdot)\Sigma_0 S^*(\cdot) \in \mathcal{C}(I; \mathcal{I}_p)$ by Proposition 8.

Suppose that $t \in I$ is fixed. We begin by proving that the mapping $s \mapsto S(t-s)F(s)S^*(t-s)$ is Bochner measurable on $[0,t]$. Suppose first that $F(\cdot)$ is a characteristic function, i.e., $F(s) = f \chi_E(s)$ with $f \in \mathcal{I}_p$ and $E \subseteq [0,t]$ measurable. Hence, $S(t-s)F(s)S^*(t-s) = S(t-s)fS^*(t-s)\chi_E(s)$ is Bochner measurable (for $s \in [0,t]$) since it is the product of a $\mathcal{I}_p$-valued continuous function and a scalar measurable function. By linearity, $s \mapsto S(t-s)F(s)S^*(t-s)$ is measurable for any $F : [0,t] \mapsto \mathcal{I}_p$ which is simple. If $F(\cdot) \in L^1(I; \mathcal{I}_p)$, it is measurable and there is a sequence of simple functions $F_n(\cdot)$ such that $\|F(s) - F_n(s)\|_p \rightarrow 0$ a.e. for $s \in [0,t]$ as $n \rightarrow \infty$. Since $S(t)$ is a $C_0$-semigroup, there is an $M_t$ such that $\|S(t-s)\| \leq M_t$ for all $s \in [0,t]$ and we have

$$\|S(t-s)F(s)S^*(t-s) - S(t-s)F_n(s)S^*(t-s)\|_p \leq M_t^2 \|F(s) - F_n(s)\|_p.$$ 

Therefore, we conclude that $s \mapsto S(t-s)F(s)S^*(t-s)$ is Bochner measurable on $[0,t]$ as a $\mathcal{I}_p$-valued function, for any Bochner measurable function $F : I \mapsto \mathcal{I}_p$ since is the sequence of measurable functions $s \mapsto S(t-s)F_n(s)S^*(t-s)$.

Suppose again that $t \in I$ is fixed and that (iv') holds. Based on the above paragraph, to prove that the mapping $s \mapsto S(t-s)\Sigma(s)G(s)\Sigma(s)S^*(t-s)$ is Bochner measurable for $s \in [0,t]$ as a $\mathcal{I}_p$-valued function, we only need to prove that $s \mapsto \Sigma(s)G(s)\Sigma(s)$ is Bochner measurable for $\Sigma(\cdot) \in L^2_{\text{loc}}(I; \mathcal{J}_{2p})$ (or $\Sigma(\cdot) \in L^2_{\text{loc}}(I; \mathcal{L}(\mathcal{K}))$ when (iv') holds). First consider $\sigma_1, \sigma_2 \in \mathcal{J}_{2p},$ $c \in \mathcal{L}(\mathcal{K})$ and $E_1, E_2, E_3$ measurable subsets of $[0,t]$, then

$$(\sigma_1 \chi_{E_1}(s))(c\chi_{E_2}(s))\langle \sigma_2 \chi_{E_3}(s) \rangle = (\sigma_1 c \sigma_2) \chi_{\bigcap_{i=1}^3 E_i}(s)$$

is Bochner measurable as a $\mathcal{I}_p$-valued function. We conclude this because $\sigma_1 c \sigma_2 \in \mathcal{J}_p$ ($\sigma_1 \in \mathcal{J}_{2p}$, we have $\sigma_1 c \in \mathcal{I}_p$ and hence $(\sigma_1 c)\sigma_2 \in \mathcal{J}_{\frac{1}{p} + \frac{1}{p}} \equiv \mathcal{I}_p$, since $\sigma_2 \in \mathcal{J}_{2p}$) and $E_1 \cap E_2 \cap E_3$ is measurable (the same holds if $\sigma_1 \in \mathcal{L}(\mathcal{K})$ and $c \in \mathcal{I}_p$). By the distributive law, $s \mapsto \Sigma(s)G(s)\Sigma(s)$ is $\mathcal{I}_p$-valued, Bochner measurable when $s \mapsto \Sigma(s)$ is a simple $\mathcal{I}_{2p}$-valued, and $s \mapsto G(s)$ is a simple $\mathcal{L}(\mathcal{K})$-valued (or when $s \mapsto \Sigma(s)$ is simple $\mathcal{L}(\mathcal{K})$-valued, and $s \mapsto G(s)$ is simple $\mathcal{I}_p$-valued). If $\Sigma(\cdot) \in L^2_{\text{loc}}(I; \mathcal{J}_{2p})$ and
If \( \| \) there are sequences of simple functions \( \Sigma_n(\cdot) \) and \( \Sigma_n(\cdot) \) \( \text{valued (with fixed} \) \( ) \) \( \text{of the previous paragraph.} \)

We have proven that the integrand in the definition of the operator \( \gamma \) is Bochner measurable. Now we prove that the integrand is locally Bochner integrable\(^1\). Recall that if \( A_1 \in \mathcal{A}_p, A_2 \in \mathcal{J}_{2p} \) and \( A \in \mathcal{L}(\mathcal{H}) \), this implies that \( A_i, AA_i \in \mathcal{J}_{ip} \) for \( i = 1, 2 \) and \( \| A_i \|_{ip} \) and \( \| AA_i \|_{ip} \) are bounded above by \( \| A \| \| A_i \|_{ip} \). By using these properties of \( \mathcal{A}_p \) and \( \mathcal{J}_{2p} \), we obtain the inequality

\[
\int_0^t \| S(t-s)F(s)^*S(t-s) \|_p \, ds \leq M_t^2 \int_0^t \| F(s) \|_p \, ds
= M_t^2 \| F(\cdot) \|_{L^1([0,t];\mathcal{J}_p)}.
\]

\(^1\)This fact follows immediately, since \( f \in L^1(I;X) \) iff \( f \) is Bochner measurable and \( \int_I \| f(t) \|_X \, dt < \infty \).
If case (iv) holds and $\Sigma(\cdot) \in L^2_{loc}(I; \mathcal{S}_p)$, then we have
\[
\int_0^t \|S(t-s)(\Sigma G \Sigma)(s)S^*(t-s)\|_p \, ds \leq M^2_t \int_0^t \|\Sigma G \Sigma(s)\|_p \, ds \\
\leq M^2_t \int_0^t \|G(s)\|_p^2 \|\Sigma(s)\|_p^2 \, ds \\
\leq M^2_t \|G(\cdot)\|_{L^\infty([0,t]; \mathcal{S}(\mathcal{H}))} \|\Sigma(\cdot)\|_p \|\Sigma(\cdot)\|_{L^2([0,t]; \mathcal{S}_p)}^2,
\]
and if (iv') holds and $\Sigma(\cdot) \in L^2_{loc}(I; \mathcal{L}(\mathcal{H}))$, then
\[
\int_0^t \|S(t-s)(\Sigma G \Sigma)(s)S^*(t-s)\|_p \, ds \leq M^2_t \int_0^t \|\Sigma G \Sigma(s)\|_p \, ds \\
\leq M^2_t \|G(\cdot)\|_{L^\infty([0,t]; \mathcal{L}(\mathcal{H}))} \|\Sigma(\cdot)\|_p \|\Sigma(\cdot)\|_{L^2([0,t]; \mathcal{L}(\mathcal{H}))}.
\]
This implies the local Bochner integrability of the integrands in any case.

We have established the local integrability of the integrand in the definition [3.2] of $\gamma$. Now we prove that the integral defines a continuous $I_p$-valued function. Assume that $I = [0, \tau]$ with $\tau > 0$ arbitrary. Define $H : I \times I \mapsto \mathcal{S}_p$ by
\[
H(t,s) = \chi_{[0,t]}(s) \left( S(t-s)(F - \Sigma G \Sigma)(s)S^*(t-s) \right),
\]
so that
\[
H(t,s) = \begin{cases} 
S(t-s)(F - \Sigma G \Sigma)(s)S^*(t-s), & 0 \leq s < t \leq \tau; \\
0, & 0 \leq t \leq s \leq \tau.
\end{cases}
\]
We have that $\gamma(\Sigma)(t) = S(t)\Sigma_0 S^*(t) + \int_0^t H(t,s) \, ds$ when $t \in [0, \tau]$.

It follows from above that for each fixed $t \in I = [0, \tau]$, the mapping $s \mapsto H(t,s)$ is Bochner measurable in $[0, \tau]$. In addition, we have the bound
\[
\|H(t,s)\|_p \leq M^2_t \|F - \Sigma G \Sigma\|_p.
\]
Thus, the right hand side is integrable and independent of $t$. To complete the proof we show that
\[
\lim_{t_n \to t} \|H(t_n, s) - H(t, s)\|_p = 0,
\]
a.e. in \( s \in [0, \tau] \). To prove this, select \( s \) fixed \( s \in (0, \tau) \) and observe \((F - \Sigma G \Sigma)(s) \in \mathcal{J}_p\). Hence \( t \mapsto S(t-s)(F - \Sigma G \Sigma)(s) S^*(t-s) \) is \( \mathcal{J}_p \)-continuous for \( t \in (s, \tau) \) by Proposition \[\text{□}\] and then \( t \mapsto H(t, s) \) is \( \mathcal{J}_p \)-continuous for \( t \in (s, \tau) \). If \( t \in (0, s) \), then \( H(t, s) = 0 \). Hence, \( H(t_n, s) \to H(t, s) \) as \( t_n \to t \) a.e. in \( s \in [0, \tau] \). Finally let \( t_n \to t \), by Dominated Convergence Theorem we observe that \( \int_0^t H(t_n, s) \, ds \to \int_0^t H(t, s) \, ds \), i.e., the mapping

\[
t \mapsto \int_0^t S(t-s)(F - \Sigma G \Sigma)(s) S^*(t-s) \, ds,
\]

is continuous in \( \mathcal{J}_p \)-norm on \( t \in [0, \tau] \) with \( \tau > 0 \) arbitrary.

We have proven that if \( \Sigma(\cdot) \in L^2([0, \tau]; \mathcal{J}_{2p}) \) when (iv) holds (or \( \Sigma(\cdot) \in L^2([0, \tau]; \mathcal{L}(\mathcal{H})) \) when (iv') holds), we conclude that \( \gamma(\Sigma)(\cdot) \in C([0, \tau]; \mathcal{J}_p) \).

Since \( \tau > 0 \) is arbitrary, the conclusion of the theorem follows. \[\square\]

### 3.2 Solutions for Uniformly Continuous Semigroups

We will prove in this section that there are solutions to \( \Sigma = \gamma(\Sigma) \) in the case when \( S(t) \) is a uniformly continuous semigroup. We will pursue a new direct approach to the Riccati equation which is based specifically in solutions of the integral operator valued equation

\[
\Sigma(t) = \Sigma_0 - \int_0^t \Sigma(s) \Sigma^*(s) \, ds,
\]

which can be solved explicitly because of its relation to the analyticity of the Resolvent \( R_\lambda(\Sigma_0) \).

Since we will be relying heavily on spectral methods for the following proofs, it is assumed that \( \mathcal{H} \) is a complex Hilbert space. It is also useful to keep in mind the following definitions that were previously stated in Definition \[\text{□}\].

We say that an operator \( A \in \mathcal{L}(\mathcal{H}) \) is

\(\text{(a)}\) non-negative if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \),

\(\text{(b)}\) positive if \( \langle Ax, x \rangle > 0 \) for all nonzero \( x \in \mathcal{H} \),

\(\text{(c)}\) strictly positive if there is a \( c > 0 \) such that \( \langle Ax, x \rangle \geq c \|x\|^2 \) for all \( x \in \mathcal{H} \).
We write \( A \geq 0, \ A > 0 \) and \( A \gg 0 \) if \( A \) is non-negative, positive or strictly positive, respectively.

Since we assume that \( \mathcal{H} \) is a complex Hilbert space, \( A \geq 0 \) implies that \( A^* = A \). This is virtue of the polarization identity (see [47] or [44] for a proof). We observe that \( A \gg 0 \) implies that \( A > 0 \) and this implies that \( A \geq 0 \), and that if \( A \geq 0 \) and \( A \) is invertible, then \( A \gg 0 \). Also, if \( A \geq 0 \), then \( \sigma(A) \subset [0, \infty) \) and if \( A \gg 0 \), then \( \sigma(A) \subset [c, \infty) \) for some \( c > 0 \).

**Proposition 14.** Let \( \mathcal{H} \) be a complex Hilbert space and suppose that \( \Sigma_0 \in \mathcal{L}(\mathcal{H}) \) satisfies \( \Sigma_0 \geq 0 \). Then, the equation

\[
\Sigma(t) = \Sigma_0 - \int_0^t \Sigma(s)\Sigma^*(s) \, ds,
\]

has a unique solution in the space \( \mathcal{C}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \), and the solution is given by

\[
\Sigma(t) = \Sigma_0(I + t\Sigma_0)^{-1},
\]

which satisfies \( \Sigma^*(t) = \Sigma(t) \geq 0 \) for \( t \in \mathbb{R}^+ \), and \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \| \leq \| \Sigma_0 \| \).

If, in addition, \( \Sigma_0 \in \mathcal{I}_p \), then \( \Sigma(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{I}_p) \) and we observe \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \|_p \leq \| \Sigma_0 \|_p \).

**Proof.** We observe that since \( \Sigma_0 \geq 0 \), for each \( t \in \mathbb{R}^+ \), \( t\Sigma_0 \geq 0 \) and hence \( \sigma(t\Sigma_0) \subset [0, \infty) \). Therefore \( -1 \in \rho(t\Sigma_0) \) and, in terms of the resolvent \( R_\lambda(t\Sigma_0) = (\lambda - t\Sigma_0)^{-1} \), we have that \( (I + t\Sigma_0)^{-1} = -R_{-1}(t\Sigma_0) \) exists as a bounded linear operator and even more, \( \|(I + t\Sigma_0)^{-1}\| = \|R_{-1}(t\Sigma_0)\| \leq 1 \), for all \( t \in \mathbb{R}^+ \). The last inequality follows because for a non-negative operator \( A \), it is the case that \( \|R_\lambda(A)\| \leq \frac{1}{|\lambda|} \) for \( \lambda < 0 \).

We will know follow the scheme for the (usual) proof of First Resolvent Identity \[^2\] We observe that

\[
(I + t\Sigma_0) - (I + s\Sigma_0) = (t - s)\Sigma_0.
\]

Which implies that

\[
(I + s\Sigma_0)^{-1} - (I + t\Sigma_0)^{-1} = -(s - t)(I + t\Sigma_0)^{-1}\Sigma_0(I + s\Sigma_0)^{-1},
\]

\[^2\] For \( A \in \mathcal{L}(\mathcal{H}) \), and \( \lambda, \mu \in \rho(A) \), we observe that

\[
R_\lambda(A) - R_\mu(A) = (\lambda - \mu)R_\lambda(A)R_\mu(A),
\]

the proof follows the same idea we are using and can be found in [27].
from which we can prove that \( t \mapsto (I + t\Sigma_0)^{-1} \) is uniformly continuous in the operator norm for \( t \in \mathbb{R}^+ \). The uniform continuity follows from the estimate
\[
\|(I + s\Sigma_0)^{-1} - (I + t\Sigma_0)^{-1}\| \leq |t - s|\|\Sigma_0\| \|(I + t\Sigma_0)^{-1}\||(I + s\Sigma_0)^{-1}\| \\
\leq |t - s|\|\Sigma_0\|.
\]

We now prove that \( t \mapsto (I + t\Sigma_0)^{-1} \) is continuously (strongly) differentiable in the operator norm topology as a mapping from \( t \in (0, \infty) \) with values in \( \mathcal{L}(\mathcal{H}) \) and with derivative \( \frac{d}{dt}(I + t\Sigma_0)^{-1} = \Sigma_0((I + t\Sigma_0)^{-1})^2 \). This result follows from the fact that \( \Sigma_0 \) commutes with \( (I + t\Sigma_0)^{-1} \) and
\[
\frac{d}{dt}(I + t\Sigma_0)^{-1} = \lim_{s \to t} \frac{(I + s\Sigma_0)^{-1} - (I + t\Sigma_0)^{-1}}{s - t} \\
= -(I + t\Sigma_0)^{-1}\Sigma_0(I + t\Sigma_0)^{-1} \\
= -\Sigma_0((I + t\Sigma_0)^{-1})^2.
\]

Finally, we observe that if we define \( \Sigma(t) = \Sigma_0(I + t\Sigma_0)^{-1} \), and if \( \Sigma_0 \geq 0 \), then since \( (I + t\Sigma_0)^{-1} \geq 0 \) and commutes with \( \Sigma_0 \)
\footnote{Both claims are direct consequences of the continuous functional calculus: Let \( \phi_{\Sigma_0} : \mathcal{C}(\sigma(\Sigma_0); \mathbb{C}) \to \mathcal{L}(\mathcal{H}) \) be the unique algebraic \( * \)-homomorphism (of the functional calculus associated with \( \Sigma_0 \geq 0 \)) that takes complex-valued continuous functions with domain \( \sigma(\Sigma_0) \subset [0, \infty) \) to bounded linear bounded operators. Then, \( f(\lambda) = \frac{1}{1+t^2\lambda} \) is positive on \([0, \infty)\) for each \( t \in \mathbb{R}^+ \) and hence \( \phi_{\Sigma_0}(f) = (I + t\Sigma_0)^{-1} \geq 0 \). Also if \( g(\lambda) = \lambda \), then \( \phi_{\Sigma_0}(g) = \Sigma_0 \). Hence \( \phi_{\Sigma_0}(f)\phi_{\Sigma_0}(g) = \phi_{\Sigma_0}(fg) = \phi_{\Sigma_0}(gf) = \phi_{\Sigma_0}(g)\phi_{\Sigma_0}(f) \).}

we satisfy that \( \Sigma^*(t) = \Sigma(t) \geq 0 \) for \( t \in \mathbb{R}^+ \). Even more, since \( t \mapsto \Sigma(t) \) is continuously (strongly) differentiable and verifies
\[
\frac{d}{dt} \Sigma(t) = -\Sigma_0^2((I + t\Sigma_0)^{-1})^2 = -\Sigma(t)\Sigma^*(t).
\]

By properties of the Riemann integral \( \int_0^t \frac{d}{ds} \Sigma(s) \, ds = \Sigma(t) - \Sigma(0) = \Sigma(t) - \Sigma_0 \) and hence \( t \mapsto \Sigma(t) \) verifies the original integral equation \( (3.3) \).

Now we prove uniqueness. Assume that \( \tilde{\Sigma}(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \) satisfies \( (3.3) \). Then the difference \( \Sigma - \tilde{\Sigma} \), is bounded for \([0, \tau] \) with \( \tau > 0 \) arbitrary,
as
\[ \| \Sigma(t) - \tilde{\Sigma}(t) \| \leq \int_0^t \| \Sigma(s)\Sigma^*(s) - \tilde{\Sigma}(s)\tilde{\Sigma}^*(s) \| \, ds \]
\[ \leq \int_0^t \| \Sigma(s) - \tilde{\Sigma}(s) \| \| \Sigma^*(s) \| + \| \tilde{\Sigma}(s) \| \| \Sigma^*(s) - \tilde{\Sigma}^*(s) \| \, ds \]
\[ \leq \left( \sup_{t \in [0, r]} \| \Sigma(t) \| + \sup_{t \in [0, r]} \| \tilde{\Sigma}(t) \| \right) \int_0^t \| \Sigma(s) - \tilde{\Sigma}(s) \| \, ds. \]

Then by Grönwall’s inequality, it follows \( \tilde{\Sigma}(t) = \Sigma(t) \) for any compact interval \( I \) in \( \mathbb{R}^+ \).

Finally, if \( \Sigma_0 \in \mathcal{A}_p \) with \( 1 \leq p \leq \infty \), then \( \Sigma(t) = \Sigma_0(I + t\Sigma_0)^{-1} \in \mathcal{A}_p \) for all \( t \in \mathbb{R}^+ \) because \( \mathcal{A}_p \) is a double-sided *-ideal. Also, from the inequality
\[ \| \Sigma_0(I + s\Sigma_0)^{-1} - \Sigma_0(I + t\Sigma_0)^{-1} \|_p \leq \| \Sigma_0 \|_p \| (I + s\Sigma_0)^{-1} - (I + t\Sigma_0)^{-1} \|, \]
t \mapsto \Sigma(t) \) is continuous in the \( \mathcal{A}_p \)-norm, and the same argument used before to prove the differentiability of the mapping \( t \mapsto \Sigma(t) \) in operator norm can be used now to prove differentiability in the \( \mathcal{A}_p \)-norm for \( 1 \leq p < \infty \). Note that the case \( p = \infty \) refers to the operator norm one. Uniqueness in this case follows from the continuous embedding \( \mathcal{C}(I; \mathcal{A}_p) \hookrightarrow \mathcal{C}(I; \mathcal{L}(\mathcal{H})) \), when \( I \) is a compact interval. Since \( \Sigma(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{A}_p) \) then \( \Sigma(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \), and we observe \( \| \Sigma(t) \| \leq \| \Sigma(t) \|_p \) for all \( t \in \mathbb{R}^+ \) which implies \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \| \leq \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \|_p. \)

Finally, if \( \Sigma_0 \in \mathcal{A}_p \), for \( 1 \leq p \leq \infty \), we observe that
\[ \| \Sigma(t) \|_p = \| \Sigma_0(I + t\Sigma_0)^{-1} \|_p \leq \| \Sigma_0 \|_p \| (I + t\Sigma_0)^{-1} \| \leq \| \Sigma_0 \|_p. \]
Hence \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \|_p = \| \Sigma_0 \|_p. \) If \( \Sigma_0 \in \mathcal{L}(\mathcal{H}) \) is not compact, then \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \| = \| \Sigma_0 \|. \)

The first natural extension of the previous result is the following.

**Proposition 15.** Let \( \mathcal{H} \) be a complex Hilbert space, and suppose that \( E \in \mathcal{L}(\mathcal{H}) \) and \( \Sigma_0 \in \mathcal{A}_p \) are both non-negative. Then there is a unique solution \( \Sigma(\cdot) \) of
\[ \Sigma(t) = \Sigma_0 - \int_0^t \Sigma(s)E\Sigma^*(s) \, ds, \]
in \( \mathcal{C}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \) which also belongs to \( \mathcal{C}(\mathbb{R}^+; \mathcal{A}_p) \). Moreover, \( \Sigma^*(t) = \Sigma(t) \geq 0 \) for all \( t \in \mathbb{R}^+ \) and \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \|_p = \| \Sigma_0 \|_p. \)
Even more, $\Sigma(\cdot)$ is given by

$$\Sigma(t) = \Sigma_0 (I + tE\Sigma_0)^{-1} = (I + t\Sigma_0E)^{-1}\Sigma_0,$$

(3.6)

for each $t \in \mathbb{R}^+$. 

Also, if $\{E_n\}^\infty_{n=1}$ is a sequence in $\mathcal{L}(\mathcal{H})$, such that $E_n \geq 0$ and $E_n \to E$ as $n \to \infty$, then the sequence of solutions $\{\Sigma_n(\cdot)\}^\infty_{n=1}$, converges to $\Sigma(\cdot)$ uniformly in $\mathcal{S}_p$-norm in compact intervals $[0,\tau] \subset \mathbb{R}^+$.

Proof. We will first prove that when $E \in \mathcal{L}(\mathcal{H})$ is strictly positive, then there is a solution $\Sigma(\cdot)$ of the equation 3.6 that belongs to $C(\mathbb{R}^+; \mathcal{S}_p)$ and satisfies $\Sigma(t) \geq 0$ for $t \in \mathbb{R}^+$ and $\sup_{t \in \mathbb{R}^+} \|\Sigma(t)\|_p = \|\Sigma_0\|_p$. Recall that a bounded operator is strictly positive if there is a constant $c > 0$, such that $\langle \phi, E\phi \rangle \geq c \|\phi\|_2$ for all $\phi \in \mathcal{H}$. We will also denote this by $E \succ 0$.

Let $E \succ 0$, then we observe that $E = \sqrt{E}\sqrt{E}$, where $\sqrt{E}$ is also strictly positive and therefore invertible. Since $E^* = E \succ 0$, then $\sigma(E) \subset [c,\infty)$, for some $c > 0$. By the Spectral Mapping Theorem, $\sigma(\sqrt{E}) = \sigma(E)^{1/2} \subset [\sqrt{c},\infty)$, and so $0 \in \rho(\sqrt{E})$ which implies that $(\sqrt{E})^{-1}$ exists as a bounded linear operator. Define $\Sigma(t) = (\sqrt{E})^{-1}\Sigma(t)(\sqrt{E})^{-1}$, where the map $t \mapsto \Sigma(t)$ is the solution of (3.5) with $E = I$ and initial condition $\sqrt{E}\Sigma_0\sqrt{E}$. That is

$$\hat{\Sigma}(t) = \sqrt{E}\Sigma_0\sqrt{E} - \int_0^t \hat{\Sigma}(s)\Sigma^*(s) \, ds,$$

(3.7)

whose existence and uniqueness in $\mathcal{C}(\mathbb{R}^+; \mathcal{S}_p)$ is given by the previous result. Also it is non-negative (and hence self-adjoint) for each $t \in \mathbb{R}^+$ since $\sqrt{E}\Sigma_0\sqrt{E} \in \mathcal{S}_p$ is non-negative. Therefore, by definition of the map $t \mapsto \Sigma(t)$, it follows that it is point-wise self-adjoint and non-negative and $\mathcal{S}_p$-norm continuous. Even more, it satisfies

$$\sqrt{E}\Sigma(t)\sqrt{E} = \sqrt{E}\Sigma_0\sqrt{E} - \int_0^t \sqrt{E}\Sigma(s)\sqrt{E}\sqrt{E}\Sigma^*(s)\sqrt{E} \, ds$$

$$= \sqrt{E} \left( \Sigma_0 - \int_0^t \Sigma(s)\sqrt{E}\Sigma^*(s) \, ds \right) \sqrt{E},$$

and since $\sqrt{E}$ is invertible; it also follows that

$$\Sigma(t) = \Sigma_0 - \int_0^t \Sigma(s)\left(\sqrt{E}\sqrt{E}\right)\Sigma^*(s) \, ds,$$
which proves the desired result since \( \sqrt{E} \sqrt{E} = E \) in \( t \in \mathbb{R}^+ \). We are left to prove, in this case, that \( \sup_{t \in [0, T]} \| \Sigma(t) \|_p = \| \Sigma_0 \|_p \). For this matter, recall that \( \Sigma^*(t) = \Sigma(t) \geq 0 \) and that \( \Sigma(t) \in \mathcal{S}_p \) if \( 0 \leq \Sigma_0 \in \mathcal{S}_p \), for all \( t \in \mathbb{R}^+ \), and that \( t \mapsto \Sigma(t) \) satisfies, for each \( \phi \in \mathcal{H} \),

\[
\frac{d}{dt} \langle \phi, \Sigma(t) \phi \rangle = - \langle \left( \Sigma(t) \phi \right), E \left( \Sigma(t) \phi \right) \rangle ,
\]

i.e., \( t \mapsto \langle \phi, \Sigma(t) \phi \rangle \geq 0 \) is decreasing since \( E \gg 0 \). Hence, \( \langle \phi, \Sigma_0 \phi \rangle \geq \langle \phi, \Sigma(t) \phi \rangle \geq 0 \). Then, for \( p = \infty \), this implies that \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \| = \| \Sigma_0 \| \).

4Since \( t \mapsto \Sigma(t) \) verifies \( \Sigma(s) = \Sigma_0 - \int_0^s \Sigma(s) E \Sigma^*(s) \) ds, then for each \( \phi \in \mathcal{H} \), the mapping \( t \mapsto \langle \phi, \Sigma(t) \phi \rangle \) satisfies \( \langle \phi, \Sigma(t) \phi \rangle = \langle \phi, \Sigma_0 \phi \rangle - \int_{t_0}^t \langle \Sigma(s) \phi, E \Sigma^*(s) \phi \rangle \) ds. Then, the result follows by applying the differential operator \( \frac{d}{dt} \) and recognizing \( \Sigma^*(t) = \Sigma(t) \).

5If \( A \geq 0 \), then \( \| A \| = \sup_{\| x \|=1} \langle x, A x \rangle \) (see for example [53].

6Explicitly, we refer to the following result: if \( A \in \mathcal{S}_p \), with \( 1 \leq p < \infty \) and \( \{ \phi_n \}_{n=1}^\omega \) (with \( \omega \leq \infty \)) is some orthonormal system in \( \mathcal{H} \), then

\[
\left( \sum_{n=1}^\omega \| \phi_n, A \phi_n \|_p \right)^{1/p} \leq \| A \|_p .
\]

And for \( p > 1 \), equality holds if and only if \( A \) is normal and \( A = \sum_{n=1}^\omega a_j \langle \cdot, \phi_j \rangle \phi_j \), with \( a_j = \langle A \phi_j, \phi_j \rangle \).
Thus \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \|_1 \leq \| \Sigma_0 \|_1 \). In summary, so far we have proven that if \( E \gg 0 \) and \( 0 \leq \Sigma_0 \in \mathcal{I}_p \), then there is a solution \( t \mapsto \Sigma(t) \) of the Riccati equation [3.6] that satisfies \( 0 \leq \Sigma(t) \in \mathcal{I}_p \), and \( \Sigma(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{I}_p) \), with the bound \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \|_p = \| \Sigma_0 \|_p \).

If \( E^* = E \geq 0 \) but it is not strictly positive, then \( E_n = E + \frac{1}{n} I \) is strictly positive and self-adjoint for all \( n \in \mathbb{N} \) and \( \| E - E_n \| = 1/n \to 0 \) as \( n \to \infty \). Now, we consider a sequence \( \{ \Sigma_n(\cdot) \} \) of solutions

\[
\Sigma_n(t) = \Sigma_0 - \int_0^t \Sigma_n(s) E_n \Sigma_n^*(s) \, ds, \tag{3.8}
\]

and first we note that they are uniformly bounded as \( \sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}^+} \| \Sigma_n(t) \|_p = \| \Sigma_0 \|_p \) by virtue of all we have done in the previous paragraph. Then we observe by re-arranging terms that

\[
\Sigma_n(s) E_n \Sigma_n^*(s) - \Sigma_m(s) E_m \Sigma_m^*(s) = \\
\Sigma_n(s) (E_n - E_m) \Sigma_n^*(s) + \Sigma_n(s) E_m (\Sigma_n^* - \Sigma_m^*)(s) + (\Sigma_n - \Sigma_m)(s) E_m \Sigma_m^*(s).
\]

Also, we see that the sequence \( \{ E_n \}_{n=1}^\infty \) is uniformly bounded \( \| E_n \| \leq C \), with \( C = \| E \| + 1 \), for all \( n \in \mathbb{N} \), then

\[
\| \Sigma_n(s) E_n \Sigma_n^*(s) - \Sigma_m(s) E_m \Sigma_m^*(s) \|_p \leq \\
\| \Sigma_0 \|_p^2 \| E_n - E_m \| + 2C \| \Sigma_0 \|_p \| (\Sigma_n - \Sigma_m)(s) \|_p.
\tag{3.9}
\]

Therefore, using (3.8), we have

\[
\| (\Sigma_n - \Sigma_m)(t) \|_p \leq \left( t \| \Sigma_0 \|_p^2 \| E_n - E_m \| \right) + \\
2C \| \Sigma_0 \|_p \int_0^t \| (\Sigma_n - \Sigma_m)(s) \|_p \, ds,
\]

and by Gronwäll’s inequality, in any interval \([0, \tau]\), we observe

\[
\sup_{t \in [0, \tau]} \| (\Sigma_n - \Sigma_m)(t) \|_p \leq \left( \tau \| \Sigma_0 \|_p^2 e^{2C\tau \| \Sigma_0 \|_p} \right) \| E_n - E_m \|.
\]

This implies that \( \{ \Sigma_n(\cdot) \} \) is a Cauchy sequence in the Banach space \( \mathcal{C}([0, \tau]; \mathcal{I}_p) \), since \( \{ E_n \} \) is an operator norm convergent sequence to \( E \). Therefore, \( \Sigma_n(\cdot) \to \Sigma(\cdot) \), for some \( \Sigma(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{I}_p) \), in the topology of
supremum $I_p$-norm. Also the inequality in [3.9] implies $\Sigma_n(s)E_n\Sigma_n^*(s) \to \tilde{\Sigma}(s)E\tilde{\Sigma}^*(s)$ in $I_p$-norm and uniformly in $s \in [0, \tau]$, and hence

$$\tilde{\Sigma}(t) = \lim_{n \to \infty} \Sigma_n(t) = \Sigma_0 - \lim_{n \to \infty} \int_0^t \Sigma_n(s)E_n\Sigma_n^*(s) \, ds = \Sigma_0 - \int_0^t \lim_{n \to \infty} \Sigma_n(s)E_n\Sigma_n^*(s) \, ds = \Sigma_0 - \int_0^t \bar{\Sigma}(s)E\bar{\Sigma}^*(s) \, ds.$$

The non-negativity property follows since $\langle \phi, \bar{\Sigma}(t)\phi \rangle = \lim_{n \to \infty} \langle \phi, \Sigma_n(t)\phi \rangle \geq 0$. The self-adjointness follows from the non-negativity and the fact that $\mathcal{H}$ is a complex Hilbert space. Also, since $\sup_{t \in [0, \tau]} \|\Sigma(t)\|_p = \|\Sigma_0\|_p$, then $\sup_{t \in [0, \tau]} ||\bar{\Sigma}(t)||_p = ||\Sigma_0||_p$. The uniqueness of the solution follows directly.

Suppose $\tilde{\Sigma}(\cdot) \in C([0, \tau]; \mathcal{L}(\mathcal{H}))$ is another solution of the integral Riccati equation of interest, then

$$||\Sigma - \tilde{\Sigma}(t)|| \leq \int_0^t ||\Sigma(s)E\Sigma^*(s) - \tilde{\Sigma}(s)E\tilde{\Sigma}^*(s)|| \, ds \leq \int_0^t ||(\tilde{\Sigma} - \bar{\Sigma})(s)E\Sigma^*(s)|| + ||\tilde{\Sigma}(s)E(\Sigma^* - \bar{\Sigma}^*)(s)|| \, ds \leq ||E|| \left( \sup_{t \in [0, \tau]} ||\tilde{\Sigma}(t)|| + ||\bar{\Sigma}(t)|| \right) \int_0^t ||\tilde{\Sigma} - \bar{\Sigma}(s)|| \, ds,$$

and an application of the Gronwall’s inequality implies that $\tilde{\Sigma}(t) = \bar{\Sigma}(t)$ for all $t \in [0, \tau]$. Finally, since $\tau > 0$ is arbitrary, and $\sup_{t \in [0, \tau]} ||\tilde{\Sigma}(t)||_p = ||\Sigma_0||_p$, then $t \mapsto \bar{\Sigma}(t)$ is the unique solution of the integral Riccati equation of interest in $C(\mathbb{R}^+; \mathcal{L}(\mathcal{H}))$ when $E \geq 0$ and $0 \leq \Sigma_0 \in I_p$. In addition, it verifies to be point-wise positive (and hence self-adjoint) and belongs to $C(\mathbb{R}^+; I_p)$.

The same approximation argument is directly extended to any sequence $\{E_n\}_{n=1}^\infty$ approaching any $E \geq 0$.

Finally, we know that in the case $E \gg 0$, $\Sigma(t) = (\sqrt{E})^{-1}\hat{\Sigma}(t)(\sqrt{E})^{-1}$,
then
\[
\Sigma(t) = (\sqrt{E})^{-1}
\left(\sqrt{E}\Sigma_0\sqrt{E} \left( I + t\sqrt{E}\Sigma_0\sqrt{E} \right)^{-1}\right) (\sqrt{E})^{-1}
\]
\[
= \Sigma_0 \sqrt{E} \left( I + t\sqrt{E}\Sigma_0\sqrt{E} \right)^{-1} (\sqrt{E})^{-1}
\]
\[
= \Sigma_0 \left( \sqrt{E}(I + t\sqrt{E}\Sigma_0\sqrt{E}) (\sqrt{E})^{-1} \right)^{-1}
\]
\[
= \Sigma_0 (I + tE\Sigma_0)^{-1}.
\]

If \( E \) is not strictly positive, then consider \( E_n = E + \frac{1}{n}I \) which satisfies \( E_n > 0 \) and \( E_n \to E \). Since \( I + tE_n\Sigma_0 \to I + tE\Sigma_0 \) and \( \| (I + tE_n\Sigma_0)^{-1} \| \leq 1 \) for all \( n \in \mathbb{N} \), then \( (I + tE_n\Sigma_0)^{-1} \to (I + tE\Sigma_0)^{-1} \) and so \( \Sigma(t) = \Sigma_0 (I + tE\Sigma_0)^{-1} \) is a solution, for all \( E \geq 0 \), of
\[
\Sigma(t) = \Sigma_0 - \int_{t_0}^{t} \Sigma(s)E\Sigma^*(s) \, ds.
\]

An analogous argument proves that \( \Sigma(t) = (I + t\Sigma_0E)^{-1}\Sigma_0 \) also.

Remark 3.1. If for some sequence \( \{E_n\}_{n=1}^{\infty} \) in \( \mathcal{L}(\mathcal{H}) \), \( E_n \geq 0 \) and \( E_n \to E \) in operator norm, then approximation in compact subsets of \( \mathbb{R}^+ \) is somehow the best result possible for it is not possible to assure convergence in \( \mathbb{R}^+ \) even though the sequence \( \{\Sigma_n(\cdot)\} \) is uniformly bounded. Let \( \{\phi_n\}_{n=1}^{\infty} \) be an orthonormal basis of \( \mathcal{H} \), and let \( \Sigma_0 \) be of rank one and defined as \( \Sigma_0x = \langle \phi_1, x \rangle \phi_1 \) for all \( x \in \mathcal{H} \). Consider \( E = 0 \) and \( E_n = \frac{1}{n}I \), then \( \Sigma(t) = \Sigma_0 \) and \( \Sigma_n(t) = \Sigma_0 (I + \frac{t}{n}\Sigma_0)^{-1} = \frac{1}{1 + t/n}\Sigma_0 \), and then \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) - \Sigma_n(t) \|_p = \| \Sigma_0 \|_p \sup_{t \in \mathbb{R}^+} | 1 - \frac{1}{1 + t/n} | = \| \Sigma_0 \|_p \) and we never observe convergence in the whole \( \mathbb{R}^+ \), unless \( \Sigma_0 = 0 \), in which case \( \Sigma(t) = \Sigma_n(t) = 0 \) for all \( t \in \mathbb{R}^+ \). 

\[\text{All the calculations are easy to follow if we use “matrix” notation for the problem. Then,}
\]
\[
\Sigma_0 = \begin{pmatrix}
1 & 0 & \cdots \\
0 & 0 & \cdots \\
\vdots & \ddots & \ddots
\end{pmatrix}, \quad I + tE_n\Sigma_0 = \begin{pmatrix}
1 + \frac{t}{n} & 0 & \cdots \\
0 & 1 & \cdots \\
\vdots & \ddots & \ddots
\end{pmatrix}.
\]
Now, the previous Lemma has a useful consequence but we first need to define step functions for our purposes.

**Definition 8 (Step Function).** Let $T : I \mapsto X$ where $I$ is a real interval (finite or infinite) and $X$ is a Banach space. The mapping $t \mapsto T(t)$ is called a *step function* if there is a finite partition of $I$ of pairwise disjoint intervals $\{I_n\}_{n=0}^N$ (that is, each $I_n$ is an interval, $\cup_{n=0}^N I_n = I$ and $I_n \cap I_m = \emptyset$ if $n \neq m$), such that $t \mapsto T(t)$ is constant in each $I_n$, for $n = 0, 1, \ldots, N$.

Let $t \mapsto E(t)$ be a step $\mathcal{L}(\mathcal{H})$-valued function, self-adjoint and non-negative for each $t \in \mathbb{R}^+$ of the form $E(t) = \sum_{n=1}^N e_n \chi_{I_n}(t)$ with $I_n = [t_{n-1}, t_n)$ for $n = 1, 2, \ldots, N - 1$, $I_N = [t_{N-1}, \infty)$, and $[0, \infty) = \cup_n I_n$. Then when $\Sigma_0 \in \mathcal{J}_p$, there is a unique solution $\Sigma(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{J}_p)$ of

$$\Sigma(t) = \Sigma_0 - \int_0^t (\Sigma E \Sigma^*)(s) \, ds,$$

such that $\Sigma^*(t) = \Sigma(t) \geq 0$ for all $t \in \mathbb{R}^+$ and $\sup_{t \in \mathbb{R}^+} \|\Sigma(t)\|_p = \|\Sigma_0\|_p$. All follows by induction. First, for $t \in I_1$, the previous results are applicable and there is a unique mapping $t \mapsto \Sigma(t)$ that verifies all the wanted conclusions, and it implies that $\Sigma(t_1^-) = \Sigma_0 - \int_0^{t_1} (\Sigma E \Sigma^*)(s) \, ds \geq 0$. If $\Sigma(t_{n-1}^-) \geq 0$, then for $t \in I_n = [t_{n-1}, t_n)$ we observe that the equation

$$\Sigma(t) = \Sigma_0 - \int_0^t (\Sigma E \Sigma^*)(s) \, ds$$

$$= \left(\Sigma_0 - \int_0^{t_{n-1}} (\Sigma E \Sigma^*)(s) \, ds\right) - \int_{t_{n-1}}^t (\Sigma E \Sigma^*)(s) \, ds$$

$$= \Sigma(t_{n-1}^-) - \int_{t_{n-1}}^t (\Sigma E \Sigma^*)(s) \, ds$$

and then

$$(I + tE_n \Sigma_0)^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{1 + \frac{t}{\pi}} & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 1 \end{pmatrix},$$

which implies that $\Sigma_0 (I + tE_n \Sigma_0)^{-1} = \frac{1}{1 + \frac{t}{\pi}} \Sigma_0$.
has a unique solution satisfying all the claimed properties and satisfying
\( \Sigma(t_{n-1}) = \Sigma(t_n^-) \). Therefore \( \Sigma(t_n^-) \geq 0 \). The bound \( \sup_{t \in \mathbb{R}^+} \| \Sigma(t) \|_p = \| \Sigma_0 \|_p \) follows from the fact that \( \| \Sigma(t_n^-) \|_p \leq \| \Sigma_0 \|_p \) for all \( n = 1, 2, \ldots, N \) and the elementary result of the previous proposition. Continuity of the mapping \( t \mapsto \Sigma(t) \) follows directly from the integral representation above. Also, it follows immediately that the same result holds for any step function \( t \mapsto E(t) \) and not only of the form we have used. Now that we know that the previous result holds when \( t \mapsto E(t) \) is a step function, we can generalize things even more as we will do in the next proposition. First, we need a couple of definitions.

**Definition 9 (Increasing Monotonicity).** Let \( t \mapsto T(t) \) be a mapping defined as \( T : I \mapsto L(\mathcal{H}) \), where \( I \subset \mathbb{R} \) is some interval and \( \mathcal{H} \) is a Hilbert space. We say that \( t \mapsto T(t) \) is monotonically increasing if \( T(t_1) \leq T(t_2) \) whenever \( t_1 \leq t_2 \) and \( t_1, t_2 \in I \).

Monotonically increasing mappings have an important role in the Riccati equation derived from optimal control and observability problems. For example, in the case when \( T(t) = I \) is the semigroup in the Riccati integral equation, then in the aforementioned equation there is a term of the form \( \int_0^t B B^*(s) \, ds \) which is a monotonically increasing mapping.

**Definition 10 (Regulated Function).** A mapping \( T : I \mapsto X \) where \( I \) is a real interval (finite or infinite) and \( X \) is a Banach space is called a regulated function if in any compact interval in \( I \), the map \( t \mapsto T(t) \) is a uniform limit of step functions. We will denote this by \( T(\cdot) \in \text{Reg}(I;X) \).

The first observation we can make is that every step \( X \)-valued function is regulated and every continuous \( X \)-valued function is also regulated. Regulated functions are widely used in modern theory of integration, and it is easy to check that a regulated function is Bochner integrable. The sum of two regulated functions is also a regulated function, and the product of two regulated functions is also a regulated function in the case when \( X \) is a Banach algebra. An important fact is that if \( X \) is an infinite dimensional Banach space, then for \( 1 \leq p < \infty \), the step \( X \)-valued functions are dense in \( L^p(I;X) \) but they are not dense in \( L^\infty(I;X) \). Consider \( X \) to be a separable Hilbert space with \( \{ \phi_n \}_{n=1}^\infty \) as an orthonormal basis. Let also \( I = [0,1] \) and \( I_n = (2^{-n}, 2^{-(n-1)}) \). Then, consider the function \( f(t) = \sum_{n=1}^\infty \phi_n \chi_{I_n}(t) \) which is not a step function. Also, \( f(\cdot) \in L^\infty(I;X) \) for being Bochner measurable.
as it is countably valued and bounded by 1. Then, if \( t \mapsto g(t) \) is a step function, we observe \( \sup_{t \in I} \| f(t) - g(t) \| \geq 1 \). The same idea can be extended to any infinite dimensional Banach space \( X \).

Following Bourbaki, we have a characterization of \( \text{Reg}(I; X) \) when \( I \) is a real interval and \( X \) a Banach space. A mapping \( t \mapsto T(t) \) belongs to \( \text{Reg}(I; X) \) if and only if for every \( t \in I \), \( T(t^-) \) and \( T(t^+) \) exist and the right-hand limit on the left endpoint and the left-hand limit on the right endpoint exist when these points belong to \( I \). This implies that the points of discontinuity of \( t \mapsto T(t) \) in \( I \) are at most, countable.

If \( I \subset \mathbb{R} \) is a closed interval (bounded or unbounded) and \( X \) is a Banach space, then the space \( \text{Reg}(I; X) \) is a Banach space with norm \( \| T(\cdot) \|_{\text{Reg}(I; X)} = \sup_{t \in I} \| T(t) \|_X \). If \( I \) is compact interval, then the step \( X \)-valued functions are dense in \( \text{Reg}(I; X) \).

Let \( I \subset \mathbb{R} \) be a closed interval (bounded or unbounded). Every continuous function on \( I \) can be uniformly approximated by step functions in compact intervals. Therefore every continuous function is regulated, but not every regulated function is continuous. Hence, if \( I \) is compact, we have the following continuous embeddings

\[
\mathcal{C}(I; X) \hookrightarrow \text{Reg}(I; X) \hookrightarrow L^\infty(I; X),
\]

when \( X \) is a Banach space. Each of the inclusions is “proper” when \( X \) is infinite dimensional and then \( \mathcal{C}(I; X) \) is a closed subspace of \( \text{Reg}(I; X) \) and this latter one is a closed subspace of \( L^\infty(I; X) \).

**Lemma 8.** Let \( \mathcal{H} \) be a complex separable Hilbert space. Suppose that \( t \mapsto E(t) \) is a point-wise non-negative, step \( L(\mathcal{H}) \)-valued function with domain \( \mathbb{R}^+ = [0, \infty) \). Also, let \( t \mapsto D(t) \) be a monotonically increasing, right-continuous, step \( \mathcal{I}_p \)-valued function which is also point-wise non-negative and has domain \( \mathbb{R}^+ \). Then, there is a unique solution \( \Sigma(\cdot) \) of

\[
\Sigma(t) = D(t) - \int_0^t \Sigma(s)E(s)\Sigma^*(s) \, ds,
\]

in the space of regulated \( L(\mathcal{H}) \)-valued functions \( \text{Reg}(\mathbb{R}^+; L(\mathcal{H})) \) which also belongs to the space of regulated \( \mathcal{I}_p \)-valued functions \( \text{Reg}(\mathbb{R}^+; \mathcal{I}_p) \). Moreover, \( \Sigma^*(t) = \Sigma(t) \geq 0 \) for all \( t \in \mathbb{R}^+ \), and

\[
\sup_{t \in [0, \tau]} \| \Sigma(t) \|_p \leq \sup_{t \in [0, \tau]} \| D(t) \|_p = \| D(\tau) \|_p,
\]

for any \( \tau > 0 \).
Without losing generality, suppose that \( \bigcup_{n=1}^{N} I_n = [0, \infty) \), and \( I_n = [t_{n-1}, t_n) \) for \( n = 1, 2, \ldots, N - 1 \) and \( I_N = [t_{N-1}, \infty) \), and that

\[
D(t) = \sum_{n=1}^{N} D_n \chi_{I_n}(t), \quad E(t) = \sum_{n=1}^{N} E_n \chi_{I_n}(t),
\]

where \( E_n^* = E_n \geq 0 \), and \( D_n^* = D_n \geq 0 \) with \( D_1 \leq D_2 \leq \cdots \leq D_N \).

We will follow by induction over the sets \( X_n = \bigcup_{k=1}^{n} I_k \). In the first interval \( X_1 = I_1 \), we observe that there is unique \( \mathcal{L}(\mathcal{H}) \)-valued continuous solution \( t \mapsto \Sigma_1(t) \) of

\[
\dot{\Sigma}_1(t) = D_1 - \int_{0}^{t} \Sigma_1(s) E_1 \Sigma_1^*(s) \, ds,
\]

which also verifies \( \Sigma_1(\cdot) \in \mathcal{C}(I_1; \mathcal{H}) \), and that it also satisfies \( \Sigma_1^*(t) = \Sigma_1(t) \geq 0 \) and \( \| \Sigma_1(t) \|_p \leq \| D_1 \|_p \) for \( t \in I_1 \).

Suppose that \( \Sigma_n(\cdot) \in \text{Reg}(\bigcup_{k=1}^{n} I_k; \mathcal{H}) \) with \( \Sigma_n^*(t) = \Sigma_n(t) \geq 0 \) and \( \| \Sigma_n(t) \|_p \leq \| D_n \|_p \) for all \( t \in \bigcup_{k=1}^{n} I_k \) and in this latter interval \( \Sigma_n(\cdot) \) solves the integral Riccati equation. Then, by Proposition 15, on the interval \( I_{n+1} \), there is a unique, \( \mathcal{L}(\mathcal{H}) \)-valued and norm continuous, solution \( t \mapsto \hat{\Sigma}(t) \) (that is also \( \mathcal{I}_p \)-norm continuous and point-wise non-negative) of the equation

\[
\dot{\hat{\Sigma}}(t) = \left( D_{n+1} - \int_{0}^{t_n} \Sigma_n(s) E(s) \Sigma_n^*(s) \, ds \right) - \int_{t_n}^{t} \hat{\Sigma}(s) E_{n+1} \hat{\Sigma}^*(s) \, ds.
\]

This follows since

\[
0 \leq \Sigma_n(t_n) = D_n - \int_{0}^{t_n} \Sigma_n(s) E(s) \Sigma_n^*(s) \, ds \leq D_{n+1} - \int_{0}^{t_n} \Sigma_n(s) E(s) \Sigma_n^*(s) \, ds,
\]

because \( D_n \leq D_{n+1} \). Then we can define the mapping \( t \mapsto \Sigma_{n+1}(t) \) as

\[
\Sigma_{n+1}(t) = \begin{cases} 
\Sigma_n(t), & t \in X_n = \bigcup_{k=1}^{n} I_k; \\
\hat{\Sigma}(t), & t \in I_{n+1}.
\end{cases}
\]

which is defined on \( X_{n+1} = \bigcup_{k=1}^{n} I_k \bigcup I_{n+1} \) and satisfies the desired properties.

The bound

\[
\sup_{t \in [0, \tau]} \| \Sigma(t) \|_p \leq \sup_{t \in [0, \tau]} \| D(t) \|_p = \| D(\tau) \|_p = \| D_N \|_p,
\]
follows from the previous paragraph and \( \|D_1\|_p \leq \|D_2\|_p \leq \cdots \leq \|D_N\|_p \). The latter chain of inequalities is obtained because \( D_1 \leq D_2 \leq \cdots \leq D_N \) (see the proof of Proposition 15).

**Remark 3.2.** The solution \( t \mapsto \Sigma(t) \) that we’ve constructed in the proof is right continuous and has discontinuities only where \( t \mapsto D(t) \) has discontinuities. Hence, in this case \( t \mapsto \Sigma(t) \) has only a finite number of discontinuities.

**Remark 3.3.** The increasing monotonicity condition of \( t \mapsto D(t) \) is not superfluous to ensure the non-negativity of the solution to the Riccati equation. Consider \( D_1 \geq 0 \) and \( D_2 \geq 0 \) of rank one, defined by \( D_1 x = \langle \phi_1, x \rangle \phi_1 \) and \( D_2 x = \langle \phi_2, x \rangle \phi_2 \) for all \( x \in \mathcal{H} \), where \( \{\phi_n\}_{n=1}^\infty \) is an orthonormal basis of \( \mathcal{H} \). Note that either \( D_1 \leq D_2 \) nor \( D_1 \geq D_2 \) holds. Let \( D(t) = D_1 \chi_{[0,1)}(t) + D_2 \chi_{[1,\infty)}(t) \), and suppose that there is a Regulated solution to \( \Sigma(t) = D(t) - \int_0^t (\Sigma \Sigma^*) (s) \, ds \) which verifies \( \Sigma^*(t) = \Sigma(t) \geq 0 \). All the previous Propositions imply that if \( t \in [0,1) \), then \( \Sigma(t) = D_1(I + tD_1)^{-1} = \frac{1}{1+t} D_1 \), hence \( \Sigma(1^-) = D_1 - \int_0^1 (\Sigma \Sigma^*) (s) \, ds = \frac{1}{2} D_1 \). Now, if \( t \in [1,\infty) \), then \( t \mapsto \Sigma(t) \) satisfies

\[
\Sigma(t) = D_2 - \int_0^t (\Sigma \Sigma^*) (s) \, ds \\
= D_2 - \int_0^1 (\Sigma \Sigma^*) (s) \, ds - \int_1^t (\Sigma \Sigma^*) (s) \, ds \\
= D_2 - \frac{1}{2} D_1 - \int_1^t (\Sigma \Sigma^*) (s) \, ds,
\]

then,

\[
\langle \phi_1, \Sigma(t) \phi_1 \rangle = -\frac{1}{2} \langle \phi_1, D_1 \phi_1 \rangle - \int_1^t \langle \phi_1, (\Sigma \Sigma^*) (s) \phi_1 \rangle \, ds \\
= -\frac{1}{2} - \int_1^t \|\Sigma(s) \phi_1\|^2 \, ds \\
\leq -\frac{1}{2},
\]

Hence \( t \mapsto \Sigma(t) \) is not non-negative on \( [1,\infty) \).

**Lemma 9.** Let \( \mathcal{H} \) be a complex separable Hilbert space. Suppose that \( E(\cdot) \in L^\infty_{loc}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \), \( D(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{L}_p) \), and also that both are point-wise non-negative. In addition, suppose that \( t \mapsto D(t) \) is monotonically increasing.
Then there is a unique solution of
\[
\Sigma(t) = D(t) - \int_0^t \Sigma(s)E(s)\Sigma^*(s) \, ds,
\]
in $\mathcal{C}(\mathbb{R}^+; \mathcal{L}([\mathcal{H}]))$ which also belongs to $\mathcal{C}(\mathbb{R}^+; \mathcal{C}_p)$. Even more, the solution $t \mapsto \Sigma(t)$ satisfies that $\Sigma^*(t) = \Sigma(t) \geq 0$ for all $t \in \mathbb{R}^+$ and
\[
\sup_{t \in [0, \tau]} \|\Sigma(t)\|_p \leq \|D(\tau)\|_p,
\]
for any $\tau > 0$.

Proof. Since $t \mapsto D(t)$ is continuous in each interval $[0, \tau]$ with $\tau > 0$ arbitrary, then there is a sequence $\{D_n(\cdot)\}_{n=1}^\infty$ of step $\mathcal{C}_p$-valued functions $D_n(t) = \sum_{k=1}^n d_k(n)\chi_{I_k(n)}(t)$, such that
\[
\sup_{t \in [0, \tau]} \|D(t) - D_n(t)\|_p \to 0,
\]
as $n \to \infty$. Since $t \mapsto D(t)$ is monotonically increasing, then for each $n \in \mathbb{N}$ we can choose the $d_k(n)$ such that $0 \leq d_1(n) \leq d_2(n) \leq \cdots \leq d_n(n)$ \footnote{Consider the restriction of $t \mapsto D(t)$ to $\mathcal{C}([0, \tau]; \mathcal{C}_p)$; since $[0, \tau]$ is compact, there is finite strictly increasing sequence $0 = t_0 < t_1 < \cdots < t_N = \tau$, such that $\sup_{t \in [0, \tau]} \|D(t) - \sum_{n=1}^N D(t_n)\chi_{[t_{n-1}, t_n]}(t)\|_p \leq \epsilon$.}
Without loss of generality, we can choose $I_k = [t_k, t_{k-1})$ for $k = 1, \ldots, n-1$ and $I_n = [t_{n-1}, \tau]$.

Since $E(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}([\mathcal{H}]))$, then the restriction of $t \mapsto E(t)$ to the interval $[0, \tau]$ with $\tau > 0$ arbitrary, belongs $L^\infty([0, \tau]; \mathcal{L}([\mathcal{H}]))$ and the latter is continuously embedded in $L^1([0, \tau]; \mathcal{L}([\mathcal{H}]))$. We now observe that $E(\cdot) \in L^1([0, \tau]; \mathcal{L}([\mathcal{H}]))$ \footnote{Actually, the restriction of $t \mapsto E(t)$ to the interval $[0, \tau]$ is what belongs to $L^1([0, \tau]; \mathcal{L}([\mathcal{H}]))$.}
This implies, since step functions are dense in $L^1([0, \tau]; \mathcal{L}([\mathcal{H}]))$, that there is a sequence of step functions $E_n(t) = \sum_{k=1}^n e_k(n)\chi_{I_k(n)}(t)$ with domain $[0, \tau]$ and with $e_k(n) \geq 0$ for all $1 \leq k \leq n$ and $n \in \mathbb{N}$, such that
\[
\int_0^{\tau} \|E(t) - E_n(t)\| \, dt \to 0,
\]
as $n \to \infty$. Without loss of generality, for each $n \in \mathbb{N}$, suppose that $\bigcup_{k=1}^n I_k(n) = [0, \tau]$, where $I_k(n) = [t_{k-1}(n), t_k(n))$ for $k = 1, 2, \ldots, n-1$ and $I_n(n) = [t_{n-1}(n), \tau]$. 

Then, the sequence \( \{\Sigma_n(\cdot)\} \) of mappings solutions of

\[
\Sigma_n(t) = D_n(t) - \int_0^t \Sigma_n(s)E_n(s)\Sigma_n^*(s) \, ds,
\]

(3.14)
belong to the \( \text{Reg}([0, \tau]; \mathcal{H}) \) and since \( \tau > 0 \) is arbitrary, \( \Sigma_n(\cdot) \in \text{Reg}(\mathbb{R}^+; \mathcal{H}) \) (and hence each \( t \mapsto \Sigma_n(t) \) is Bochner measurable). Also, we observe \( \Sigma_n^*(t) = \Sigma_n(t) \geq 0 \) for all \( t \in \mathbb{R}^+ \) and \( n \in \mathbb{N} \), as proven in the previous Lemma.

We will prove that the sequence \( \{\Sigma_n(\cdot)\} \) is a Cauchy sequence in \( \text{Reg}([0, \tau]; \mathcal{H}) \) (using the sup\(_{t \in [0,\tau]} \| \cdot \|_p \) norm) for any \( \tau > 0 \). By Lemma 8, for each \( n \in \mathbb{N} \), we observe

\[
\sup_{t \in [0,\tau]} \|\Sigma_n(t)\|_p \leq \|D_n(\tau)\|_p.
\]

Since \( \{D_n(\cdot)\} \) is uniformly convergent, it is bounded, and hence the sequence \( \{\Sigma_n(\cdot)\} \) is uniformly bounded, i.e., there is an \( c_1(\tau) > 0 \) such that \( \|\Sigma_n(t)\|_p \leq c_1(\tau) \) holds for all \( n \in \mathbb{N} \), all \( t \in [0,\tau] \). We also observe the following familiar bound

\[
\|\Sigma_n(s)E_n(s)\Sigma_n^*(s) - \Sigma_m(s)E_m(s)\Sigma_m^*(s)\|_p \leq \|\Sigma_n(s)(E_n - E_m)(s)\Sigma_m^*(s)\|_p + \|\Sigma_n(s)E_m(s)(\Sigma_n^*(s) - \Sigma_m^*(s))\|_p + \|(\Sigma_n(s) - \Sigma_m(s))E_m(s)\Sigma_m^*(s)\|_p,
\]

and then

\[
\|\Sigma_n(s)E_n(s)\Sigma_n^*(s) - \Sigma_m(s)E_m(s)\Sigma_m^*(s)\|_p \leq c_1(\tau)^2\|(E_n - E_m)(s)\|_p + c_1(\tau)(\|\Sigma_m(s)E_m(s)\|_p + \|\Sigma_n(s)E_n(s)\|_p)\|(\Sigma_n - \Sigma_m)(s)\|_p.
\]

(3.15)

Therefore, we can bound the difference \( (\Sigma_m - \Sigma_n)(t) \) as follows:

\[
\|(\Sigma_m - \Sigma_n)(t)\|_p \leq \left( \sup_{t \in [0,\tau]} \|(D_m - D_n)(t)\|_p + c_1(\tau)^2\|(E_m - E_n)(\cdot)\|_{L^1([0,\tau]; \mathcal{L}(\mathcal{H}))} \right) + \int_0^t c_1(\tau)(\|\Sigma_m(s)E_m(s)\|_p + \|\Sigma_n(s)E_n(s)\|_p)\|(\Sigma_n - \Sigma_m)(s)\|_p \, ds,
\]
where we have used the fact that $(\Sigma_n E_n)(\cdot) \in L^1([0, \tau]; \mathcal{L}_p)$. Finally, we invoke the generalized Grönwall’s Lemma\footnote{The generalization of Grönwall’s Lemma to measurable functions reads as follows: let $z : [0, \tau] \to \mathbb{R}$ be a measurable function that satisfies $0 \leq z(t) \leq Z + \int_0^t h(s) z(s) \, ds$, a.e. in $t \in [0, \tau]$, for some constant $Z$ and some non-negative integrable function $h$. Then $0 \leq z(t) \leq Z \left( 1 + \int_0^t h(r) e^{\int_0^r h(s) \, ds} \, dr \right)$, a.e. in $t \in [0, \tau]$.} to show

$$\sup_{t \in [0, \tau]} \| (\Sigma_m - \Sigma_n)(t) \|_p \leq Z(m, n) \left( 1 + \sup_{t \in [0, \tau]} \int_0^t h(r, m, n) e^{\int_0^r h(s, m, n) \, ds} \, dr \right),$$

where $Z(m, n) = \sup_{t \in [0, \tau]} \| (D_m - D_n)(t) \|_p + c_1(\tau)^2 \| (E_m - E_n)(\cdot) \|_{L^1([0, \tau]; \mathcal{L}(\mathcal{X}))}$ and $h(t, m, n) = c_1(\tau) \left( \| \Sigma_m(s) E_m(s) \|_p + \| \Sigma_n(s) E_n(s) \|_p \right)$. Now, let $I$ be a compact interval inside $[0, \tau]$. Then

$$\int_I h(t, m, n) \, dt = c_1(\tau) \int_I \left( \| \Sigma_m(t) E_m(t) \|_p + \| \Sigma_n(t) E_n(t) \|_p \right) \, dt;$$

$$\leq c_1(\tau) \int_I \left( \| \Sigma_m(t) \|_p \| E_m(t) \| + \| \Sigma_n(t) \|_p \| E_n(t) \| \right) \, dt;$$

$$\leq c_1(\tau)^2 \int_I \left( \| E_m(t) \| + \| E_n(t) \| \right) \, dt;$$

but $\| (E - E_n)(\cdot) \|_{L^1([0, \tau]; \mathcal{L}(\mathcal{X}))} \to 0$ as $n \to \infty$, and hence there is a $c_2(\tau) \geq 0$, such that $\int_0^\tau \| E_n(t) \| \, dt \leq c_2(\tau)$ for all $n \in \mathbb{N}$. Then

$$\int_I h(t, m, n) \, dt \leq 2c_1(\tau)^2 c_2(\tau).$$

Therefore, our inequality for $\sup_{t \in [0, \tau]} \| (\Sigma_m - \Sigma_n)(t) \|_p$ becomes

$$\sup_{t \in [0, \tau]} \| (\Sigma_m - \Sigma_n)(t) \|_p \leq Z(m, n) \left( 1 + 2c_1(\tau)^2 c_2(\tau) \tau e^{2c_1(\tau)^2 c_2(\tau) \tau} \right),$$
and hence \( \{ \Sigma_n(\cdot) \} \) is a Cauchy sequence in \( \text{Reg}([0, \tau]; \mathcal{A}_p) \) since \( Z(m, n) \to 0 \) uniformly as \( m \) and \( n \) increase. Since the latter space is a Banach space under the sup norm, there is a \( \Sigma(\cdot) \in \text{Reg}([0, \tau]; \mathcal{A}_p) \) such that \( \sup_{t \in [0, \tau]} \| (\Sigma - \Sigma_n)(t) \|_p \to 0 \) as \( n \to \infty \). Since \( \tau > 0 \) is arbitrary, then \( \Sigma(\cdot) \in \text{Reg}(\mathbb{R}^+; \mathcal{A}_p) \) because it is the uniform limit of regulated functions on each compact interval in \( \mathbb{R}^+ \). Also,

\[
\sup_{t \in [0, \tau]} \| \Sigma(\cdot) \|_p = \lim_{n \to \infty} \sup_{t \in [0, \tau]} \| \Sigma_n(t) \|_p \leq \lim_{n \to \infty} \| D_n(\tau) \|_p = \| D(\tau) \|_p
\]

We will prove now that \( t \mapsto \Sigma(t) \) solves the integral Riccati equation of interest. The inequality in (3.15) together with the fact that \( \sup_{t \in [0, \tau]} \| (\Sigma - \Sigma_n)(t) \|_p \to 0 \) as \( n \to \infty \) allow us to observe that \( t \mapsto \Sigma(t)E^*(s) \) is Bochner measurable and that \( \int_I \| (\Sigma(t)E^* - \Sigma_n(t)E_n^* \|_p \) \( \to \) \( 0 \) as \( n \to \infty \) for any compact interval \( I \) in \([0, \tau] \). Then,

\[
\Sigma(t) = \lim_{n \to \infty} \Sigma_n(t) = \lim_{n \to \infty} D_n(t) - \lim_{n \to \infty} \int_0^t \Sigma_n(s)E_n(s)\Sigma_n^*(s) \, ds,
\]

\[
= D(t) - \int_0^t \Sigma(s)E(s)\Sigma^*(s) \, ds,
\]

for any \( t \in [0, \tau] \) \(^{11}\) but since \( \tau > 0 \) is arbitrary, then \( t \mapsto \Sigma(t) \) solves the integral Riccati equation of interest for any \( t \in \mathbb{R}^+ \).

The fact that \( \Sigma(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{A}_p) \) follows directly by the application of the vector-valued version of the Lebesgue Dominated Convergence Theorem, and its self-adjointness and non-negativity by the uniform convergence of \( t \mapsto \Sigma_n(t) \) to \( t \mapsto \Sigma(t) \) in any compact interval in \( \mathbb{R}^+ \).

If there is another solution \( \bar{\Sigma}(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \) of the Riccati integral equation then the difference \( (\Sigma - \bar{\Sigma})(\cdot) \), on each interval \([0, \tau] \), is bounded as,

\[
\| (\Sigma - \bar{\Sigma})(t) \| \leq \int_0^t \left( \| \Sigma(s)E(s) \| + \| \bar{\Sigma}(s)E(s) \| \right) \| (\Sigma - \bar{\Sigma})(s) \| \, ds,
\]

where \( \max(\sup_{t \in [0, \tau]} \| \Sigma(t) \|, \sup_{t \in [0, \tau]} \| \bar{\Sigma}(t) \| \) \( \leq \) \( c_1 \). Then the generalized Grönwall’s Lemma implies that \( \sup_{t \in [0, \tau]} \| (\Sigma(t) - \bar{\Sigma}(t)) \| = 0 \) and this implies that \( \Sigma(\cdot) = \bar{\Sigma}(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{A}_p) \) since \( \tau > 0 \) was arbitrary.

\(^{11}\) We have also used here that since \( E(\cdot) \in L^\infty([0, \tau]; \mathcal{L}(\mathcal{H})) \) and \( \Sigma(\cdot) \in \text{Reg}([0, \tau]; \mathcal{A}_p) \), then \( (\Sigma E)(\cdot) \in L^\infty([0, \tau]; \mathcal{A}_p) \) and hence \( (\Sigma E \Sigma^*)(\cdot) \in L^\infty([0, \tau]; \mathcal{A}_p) \) and therefore \( (\Sigma E \Sigma^*)(\cdot) \in L^1([0, \tau]; \mathcal{A}_p) \).
The previous result implies the following. Consider the equation
\[ \Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)(BB^* - \Sigma(C^*C)\Sigma)(s)S^*(t-s)\, ds, \]
with \( S(t) = I \). Then, the previous equation is
\[ \Sigma(t) = \left( \Sigma_0 + \int_0^t (BB^*)(s) \, ds \right) - \int_0^t (\Sigma(C^*C)\Sigma^*)(s) \, ds. \] (3.16)

If \( \Sigma_0^* = \Sigma_0 \geq 0 \), \( BB^*(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{A}_p) \) and \( C^*C(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \) (with \( BB^*(\cdot) \) and \( C^*C(\cdot) \) point-wise non-negative), then \( t \mapsto \Sigma_0 + \int_0^t (BB^*)(s) \, ds \) is a monotone increasing non-negative mapping and hence we observe a unique solution \( t \mapsto \Sigma(t) \) of the Riccati equation (3.16) in \( C(\mathbb{R}^+; \mathcal{A}_p) \). In addition, \( \Sigma(\cdot) \) is point-wise self-adjoint, non-negative, and bounded on compact intervals \([0, \tau]\) by
\[ \sup_{t \in [0, \tau]} \| \Sigma(t) \| \leq \| \Sigma_0 \|_p + \| BB^*(\cdot) \|_{L^1([0, \tau]; \mathcal{A}_p)}. \]

Now, we need to turn our attention to the case then \( S(t) \) is a different semigroup of linear operators than the identity. We can prove the following.

**Theorem 4.** Let \( \mathcal{H} \) be a complex separable Hilbert space, \( S(t) \) be a uniformly continuous semigroup on \( \mathcal{H} \) such that \( \| S(t) \| \leq M e^{\omega t} \) for \( t \in \mathbb{R}^+ \), and suppose that

1. \( \Sigma_0 \in \mathcal{A}_p \) and \( \Sigma_0 \geq 0 \);
2. \( F(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{A}_p) \), where \( t \mapsto F(t) \) is point-wise self-adjoint and non-negative;
3. \( G(\cdot) \in L^\infty_{\text{loc}}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \), where \( t \mapsto G(t) \) is point-wise self-adjoint and non-negative.

Then, the equation
\[ \Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)(F - \Sigma GS)(s)S^*(t-s)\, ds, \]
has a unique solution in $L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{A}_p)$, which verifies also to belong to $\mathcal{C}(\mathbb{R}^+; \mathcal{A}_p)$ and $\Sigma^*(t) = \Sigma(t) \geq 0$ for $t \in \mathbb{R}^+$. More over, we have

$$
\|\Sigma(t)\|_p \leq M^2 e^{2\omega t} \left( \|\Sigma_0\|_p + M^2 e^{2\omega t} \int_0^t \|F(s)\|_p \, ds \right),
$$

for $t \in \mathbb{R}^+$.

Proof. Since $S(t)$ is uniformly continuous, $S(t) = e^{At}$ for some $A \in \mathcal{L}(\mathcal{H})$, which implies that we can embed $S(t)$ and $S^*(t)$ in groups of linear operators. So $S(t) = e^{At}$ and $S^*(t) = e^{A^*t}$ for $t \in \mathbb{R}$. Then the maps $t \mapsto S(t)$ and $t \mapsto S^*(t)$ have $\mathbb{R}$ as domain, are continuous in operator norm and satisfy the group property: $S(t)S(s) = S(t+s)$ and $S^*(t)S^*(s) = S^*(t+s)$ for all $-\infty < s, t < \infty$.

Since $F(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{A}_p)$, then $\hat{F}(t) \triangleq S(-t)F(t)S^*(-t)$ for $t \in \mathbb{R}^+$, satisfies that $\hat{F}(\cdot) \in L^1([0, \tau]; \mathcal{A}_p)$ for any $\tau > 0$. The measurability follows immediately since $t \mapsto S(-t)$ and $t \mapsto S^*(-t)$ are norm continuous, and the local integrability follows from the bound $\|\hat{F}(t)\|_p \leq M^2 e^{2\omega t}\|F(t)\|_p$.

Similarly, since $G(\cdot) \in L^\infty_{\text{loc}}(\mathbb{R}^+; \mathcal{L}(\mathcal{H}))$, then $\hat{G}(t) \triangleq S(t)G(t)S^*(t)$ for $t \in \mathbb{R}^+$ satisfies that $\hat{G}(\cdot) \in L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))$ for arbitrary $\tau > 0$. We also observe that $\hat{F}^*(t) = \hat{F}(t) \geq 0$ and $\hat{G}^*(t) = \hat{G}(t) \geq 0$. Hence, the equation

$$
\Pi(t) = \left( \Sigma_0 + \int_0^t \hat{F}(s) \, ds \right) - \int_0^t \Pi(s)\hat{G}(s)\Pi^*(s) \, ds,
$$

by Lemma 9, has a unique solution in $\mathcal{C}([0, \tau]; \mathcal{L}(\mathcal{H}))$, that also belongs to $\mathcal{C}([0, \tau]; \mathcal{A}_p)$, and such that $\Pi^*(t) = \Pi(t) \geq 0$. This follows since $t \mapsto \Sigma_0 + \int_0^t \hat{F}(s) \, ds$ is a monotonic, point-wise self-adjoint and non-negative mapping. We also observe $\|\Pi(t)\|_p \leq \|\Sigma_0\|_p + \int_0^t \|\hat{F}(s)\|_p \, ds$.

Define $\Sigma(t) = S(t)\Pi(t)S^*(t)$. By definition $\Sigma(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{A}_p)$ by Proposition 8 and it is also point-wise non-negative and self-adjoint since $\Pi^*(t) = \Pi(t) \geq 0$. Then, this implies that
\[ \Sigma(t) = S(t)\Pi(t)S^*(t) \]
\[ = S(t)\left(\Sigma_0 + \int_0^t \hat{F}(s) \, ds\right)S^*(t) - S(t) \int_0^t \Pi(s)\hat{G}(s)\Pi(s) \, ds \quad S^*(t) \]
\[ = S(t)\Sigma_0S^*(t) + \int_0^t S(t)\hat{F}(s)S^*(t) \, ds - \int_0^t S(t)\Pi(s)\hat{G}(s)\Pi(s)S^*(t) \, ds \]
\[ = S(t)\Sigma_0S^*(t) + \int_0^t S(t-S)F(s)S^*(-S)S^*(t) \, ds \]
\[ - \int_0^t S(t)S(-S)\Sigma(s)S^*(-S)S^*(s)G(s)S(s)S(-S)\Sigma(s)S^*(-S)S^*(t) \, ds \]
\[ = S(t)\Sigma_0S^*(t) + \int_0^t S(t-s)F(s)S^*(t-s) \, ds \]
\[ - \int_0^t S(t-s)\Sigma(s)G(s)\Sigma(s)S^*(t-s) \, ds, \]

this is \( t \mapsto \Sigma(t) \) satisfies the desired Riccati equation on \( t \in [0, \tau] \) with \( \tau > 0 \) arbitrary.

Suppose there is another solution \( t \mapsto \tilde{\Sigma}(t) \in C([0, \tau]; L_p) \) to this equation. Since the adjoint map \( A \mapsto A^* \) is a bounded (conjugate) linear map on \( L(H) \), it follows that \( \left( \int_0^t Y(s) \, ds \right)^* = \int_0^t Y^*(s) \, ds \) for any Bochner integrable \( L(H) \)-valued function \( Y(\cdot) \). This implies, since \( \Sigma_0, G(\cdot) \) and \( F(\cdot) \) are point-wise non-negative (and hence self-adjoint) that \( t \mapsto \tilde{\Sigma}^*(t) \in C([0, \tau]; L_p) \) solves the same Riccati equation. A direct application of Gronwall’s lemma over the difference \( \| (\Sigma - \Sigma^*)(\cdot) \| \) implies that \( \Sigma(\cdot) \) is point-wise self-adjoint. Then define \( t \mapsto \tilde{\Pi}(t) = S(-t)\tilde{\Sigma}(t)S^*(-t) \in C([0, \tau]; L_p) \) and this a solution to the equation \( (3.17) \). Since \( t \mapsto \Pi(t) \) was the unique solution to \( (3.17) \), hence \( \tilde{\Pi}(t) = \Pi(t) \), and then \( \tilde{\Sigma}(t) = \Sigma(t) \) since \( t \mapsto S(t) \) and \( t \mapsto S^*(t) \) are invertible for each \( t \in \mathbb{R} \).

Suppose there is another solution \( t \mapsto \tilde{\Sigma}(t) \) of the Riccati equation belonging to \( L^2([0, \tau]; L_{2p}) \). Then, by Theorem 3, \( \tilde{\Sigma}(\cdot) \in C([0, \tau]; L_p) \) and then \( \tilde{\Sigma}(t) = \Sigma(t) \) for all \( t \in [0, \tau] \) by the previous paragraph.

The inequality \( \| \Pi(t) \|_p \leq \| \Sigma_0 \|_p + \int_0^t \| \hat{F}(s) \|_p \, ds \) immediately leads to
\[ \| \Sigma(t) \|_p \leq M^2e^{2\omega t}\left( \| \Sigma_0 \|_p + M^2e^{2\omega t} \int_0^t \| F(s) \|_p \, ds \right), \]
for \( t \in \mathbb{R}^+ \), since \( \| \Sigma(t) \|_p \leq M^2 e^{2\omega t} \| \Pi(t) \|_p \) and \( \| \hat{F}(t) \|_p \leq M^2 e^{2\omega t} \| F(t) \|_p \).

It seems we are one step away from proving existence and uniqueness of the \( \mathcal{I}_p \)-norm continuous solution to the integral Riccati equation when \( S(t) \) is a \( C_0 \)-semigroup. But several difficulties arise if we try to apply the same idea in the previous proofs for this case. Fortunately, we can overcome this problem, by using the aid of local and approximation results as we will prove subsequently.

### 3.3 An Approximation Result

In Theorem 4, we have proven that

\[
\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)(F - \Sigma G\Sigma)(s)S^*(t-s) \, ds,
\]

(3.18)

has a unique solution in \( \mathcal{C}(\mathbb{R}^+; \mathcal{I}_p) \) when \( F(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{I}_p) \), \( G(\cdot) \in L^\infty_{\text{loc}}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \) and \( S(t) \) is a uniformly continuous semigroup. Also, by Theorem 3 if \( \Sigma(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathcal{I}_p) \), then \( \gamma(\Sigma) \) is well defined when \( S(t) \) is a \( C_0 \)-semigroup. Now, we will prove that if we have a solution to the equation (3.18) when \( S(t) \) is a \( C_0 \)-semigroup, then under certain hypotheses we will be able to approximate this solution by solutions to equation (3.18) when \( S_n(t) \) is a sequence of uniformly continuous semigroups.

**Theorem 5 (First Approximation Theorem).** Suppose that \( S(t) \) is a \( C_0 \)-semigroup of linear operators over a separable complex Hilbert space \( \mathcal{H} \), and suppose also that \( \{S_n(t)\} \) is a sequence of uniformly continuous semigroups over the same Hilbert space \( \mathcal{H} \) that satisfy, for each \( x \in \mathcal{H} \),

\[
\| S(t)x - S_n(t)x \| \to 0 \quad \text{and} \quad \| S^*(t)x - S^*_n(t)x \| \to 0,
\]

as \( n \to \infty \), uniformly in compact intervals. Suppose also the following.

(i) \( \Sigma_0 \geq 0 \) and the sequence \( \{\Sigma_n\}_{n=1}^\infty \) are in \( \mathcal{I}_p \), \( \Sigma_n \geq 0 \) for all \( n \in \mathbb{N} \) and \( \| \Sigma_0 - \Sigma_n \|_p \to 0 \) as \( n \to \infty \).

(ii) \( BB^*(\cdot) \) and the sequence \( \{D_n(\cdot)\}_{n=1}^\infty \) are in \( L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{I}_p) \), \( BB^*(t) \geq 0 \) and \( D_n(t) \geq 0 \) for all \( t \in \mathbb{R}^+ \) and all \( n \in \mathbb{N} \) and satisfy

\[
\int_0^T \| BB^*(t) - D_n(s) \|_p \, ds \to 0,
\]
for any fixed \( \tau > 0 \) and as \( n \to \infty \).

(iii) \( C^*C(\cdot) \) and the sequence \( \{E_n(\cdot)\}_{n=1}^{\infty} \) are in \( L^\infty(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \), \( C^*C(t) \geq 0 \) and \( E_n(t) \geq 0 \) for all \( t \in \mathbb{R}^+ \) and all \( n \in \mathbb{N} \) and satisfy

\[
\text{ess sup}_{t \in [0, \tau]} \|C^*C(t) - E_n(t)\| \to 0,
\]

for any fixed \( \tau > 0 \) and as \( n \to \infty \).

Then, if \( \Sigma(\cdot) \in \mathcal{C}([0, a], \mathcal{I}_p) \) is a solution of

\[
\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)(BB^* - \Sigma(C^*C)\Sigma)S^*(t-s)\, ds,
\]

for some \( a > 0 \) and if \( \Sigma_n(\cdot) \in \mathcal{C}(\mathbb{R}^+, \mathcal{I}_p) \) is the sequence of solutions of

\[
\Sigma_n(t) = S_n(t)\Sigma_0 S_n^*(t) + \int_0^t S_n(t-s)(D_n - \Sigma_n E_n\Sigma_n)S_n^*(t-s)\, ds,
\]

we observe that

\[
\sup_{t \in [0, a]} \|\Sigma(t) - \Sigma_n(t)\|_p \to 0,
\]

as \( n \to \infty \).

Proof. First note that since \( S_n(t) \) is a uniformly continuous semigroup for each \( n \in \mathbb{N} \), the sequence of solutions of the Riccati equation \( t \mapsto \Sigma_n(t) \) belong to \( \mathcal{C}(\mathbb{R}^+, \mathcal{I}_p) \) according to Theorem 4.

Suppose \( \tau > 0 \) is fixed. The convergence of the sequences \( \{\Sigma_0^{n}\}_{n=1}^{\infty} \), \( \{D_n(\cdot)\}_{n=1}^{\infty} \) and \( \{E_n(\cdot)\}_{n=1}^{\infty} \) (in the respective norms) imply that they are bounded, and hence there are positive numbers \( \sigma_0, b_\tau \) and \( c_\tau \) such that

\[
\sigma_0 = \sup_{n \in \mathbb{N}} \|\Sigma_0^n\| \geq \|\Sigma_0\|_p,
\]

\[
b_\tau = \sup_{n \in \mathbb{N}} \|D_n(\cdot)\|_{L^1([0, \tau]; \mathcal{I}_p)} \geq \|BB^*(\cdot)\|_{L^1([0, \tau]; \mathcal{I}_p)}
\]

\[
c_\tau = \sup_{n \in \mathbb{N}} \|E_n(\cdot)\|_{L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))} \geq \|C^*C(\cdot)\|_{L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))}.
\]

Also, the Uniform Boundedness Principle implies that there is a constant \( M_\tau \) such that

\[
\sup(\|S(t)\|, \|S_n(t)\|) \leq M_\tau,
\]
where the sup ranges in all \( n \in \mathbb{N} \) and all \( t \in [0, \tau] \) (and of course the same bound is valid for \( S^*(t) \) and \( S_n^*(t) \)).

The sequence \( \{\Sigma_n(\cdot)\}_{n=1}^\infty \) is bounded, as we observed in Theorem 4 as

\[
\|\Sigma_n(t)\|_p \leq M^2_{\tau} \left( \|\Sigma_0^n\|_p + M^2_{\tau} \int_0^\tau \|D_n(s)\|_p \, ds \right)
\]

\[
\leq M^2_{\tau}(\sigma_0 + M^2_{\tau}b_\tau),
\]

in \( t \in [0, \tau] \). We define \( \rho_\tau = M^2_{\tau}(\sigma_0 + M^2_{\tau}b_\tau) \).

We first prove that \( S_n(t)\Sigma_0^nS_n^*(t) \rightarrow S(t)\Sigma_0S^*(t) \) in the sup norm for \( C([0, \tau]; \mathcal{I}_p) \). We have the following bound

\[
\|S_n(t)\Sigma_0^nS_n^*(t) - S(t)\Sigma_0S^*(t)\|_p \leq (3.19) \]

\[
\|S_n(t)(\Sigma_0^n - \Sigma_0)S_n^*(t)\|_p + \|S_n(t)\Sigma_0(S_n^*(t) - S^*(t))\|_p + \|(S_n(t) - S(t))\Sigma_0S^*(t)\|_p.
\]

The first term in the right hand side satisfies

\[
\|S_n(t)(\Sigma_0^n - \Sigma_0)S_n^*(t)\|_p \leq M^2_\tau \|\Sigma_0^n - \Sigma_0\|_p,
\]

and then converges to zero since \( \|\Sigma_0^n - \Sigma_0\|_p \rightarrow 0 \). The second term in the right hand side of inequality (3.19) satisfies

\[
\|S_n(t)\Sigma_0(S_n^*(t) - S^*(t))\|_p \leq M_\tau \sup_{t \in [0, \tau]} \|\Sigma_0(S_n^*(t) - S^*(t))\|_p,
\]

and by the Smoothing Lemma (Lemma 2 in Page 26), goes to zero as \( n \rightarrow \infty \) and uniformly in \( t \in [0, \tau] \) because \( \Sigma_0 \in \mathcal{I}_p \) and due to the strong convergence of \( S_n^*(t) \) to \( S^*(t) \). Since \( t \mapsto S(t) \) is strongly continuous and \( \Sigma_0 \in \mathcal{I}_p \), then by Proposition 5 we observe \( t \mapsto \Sigma_0S^*(t) \) is \( \mathcal{I}_p \)-norm continuous which implies that the set \( \{\Sigma_0S^*(t)/ t \in [0, \tau] \} \) is compact in the \( \mathcal{I}_p \)-norm topology. Then again by The Smoothing Lemma, we observe

\[
\sup_{t \in [0, \tau]} \|(S_n(t) - S(t))\Sigma_0S^*(t)\|_p \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Therefore

\[
\sup_{t \in [0, \tau]} \|S_n(t)\Sigma_0^nS_n^*(t) - S(t)\Sigma_0S^*(t)\|_p \rightarrow 0,
\]

as \( n \rightarrow \infty \).

Next, we prove that the mapping \( t \mapsto \int_0^t S_n(t - s)D_n(s)S_n^*(t - s) \, ds \) converges to \( t \mapsto \int_0^t S(t - s)BB^*(s)S^*(t - s) \, ds \) in the sup \( \mathcal{I}_p \)-norm. Both mappings are elements of \( C([0, \tau]; \mathcal{I}_p) \) as we have proven in the first part.
of the proof of Theorem 6. Then, we observe the following bound on the integrands

$$\|S(t - s)BB^*(s)S^*(t - s) - S_n(t - s)D_n(s)S_n^*(t - s)\|_p \leq \|S_n(t - s)(D_n - BB^*)(s)S_n^*(t - s)\|_p + \|S_n(t - s)BB^*(s)(S_n^* - S^*)(t - s)\|_p + \|S_n(t - s)S^*(t - s)BB^*(s)S^*(t - s)\|_p.$$  

(3.20)

For the first term in the right hand side, we observe that

$$\sup_{t \in [0,\tau]} \int_0^t \|S_n(t - s)(D_n - BB^*)(s)S_n^*(t - s)\|_p \, ds \leq \sup_{t \in [0,\tau]} \int_0^t \|S_n(t - s)\|_p \|(D_n - BB^*)(s)\|_p \|S_n^*(t - s)\|_p \, ds \leq M_1^2 \sup_{t \in [0,\tau]} \int_0^t \|(D_n - BB^*)(s)\|_p \, ds \leq M_1^2 \|\|D_n - BB^*(\cdot)\|_{L^1([0,\tau];\mathcal{F}_p)}\|.$$ 

Hence, it goes to zero by the initial hypotheses. For the second term in the right hand side of the inequality in (3.20), we proceed as follows. Since $BB^*(\cdot) \in L^1([0,\tau];\mathcal{F}_p)$, it can be approximated with simple $\mathcal{F}_p$-valued functions. Suppose that $F(\cdot)$ is simple. Then, we have the following bound

$$\|S_n(t - s)BB^*(s)(S_n^* - S^*)(t - s)\|_p \leq \|S_n(t - s)(BB^* - F)(s)(S_n^* - S^*)(t - s)\|_p + \|S_n(t - s)F(s)(S_n^* - S^*)(t - s)\|_p \leq 2M_2^2 \|(BB^* - F)(s)\|_p + M_2 \|F(s)(S_n^* - S^*)(t - s)\|_p.$$ 

We know that $F$ is of the form $F(t) = \sum_{k=1}^N f_k \chi_{E_k}(t)$, with a finite number of nonzero $f_k \in \mathcal{F}_p$. The set $\mathcal{K}_F = \{f_k/_{1 \leq k \leq N}\}$ is compact in the topology of $\mathcal{F}_p$. Therefore

$$\sup_{s \in [0,t]} \|F(s)(S_n^* - S^*)(t - s)\|_p \leq \sup_{f_k \in \mathcal{K}_F} \sup_{s \in [0,t]} \|f_k(S_n^* - S^*)(t - s)\|_p \leq \sup_{f_k \in \mathcal{K}_F} \sup_{s \in [0,t]} \|f_k(S_n^* - S^*)(t - s)\|_p,$$

and the right hand side goes to zero by virtue of the Smoothing Lemma (Lemma 2). Then, in order to clarify things, let $\epsilon > 0$ be arbitrary, and choose
a simple $\mathcal{I}_p$-valued function $F_\epsilon$ such that $\|(BB^*-F_\epsilon)(\cdot)\|_{L^1([0,\tau];\mathcal{I}_p)} < \frac{\epsilon}{4M^2}$. Also there is an $N(\epsilon) > 0$ such that if $n \geq N(\epsilon)$, then $\|K(S_n^*-S^*)(t)\|_p < \frac{\epsilon}{2M^2}$ uniformly in $t \in [0, \tau]$ and $K \in \mathcal{K}_{\mathcal{I}_p}$. Therefore

$$\sup_{t \in [0, \tau]} \int_0^t \|S_n(t-s)BB^*(s)(S_n^*-S^*)(t-s)\|_p \, ds \leq$$

$$\leq \sup_{t \in [0, \tau]} \left( 2M^2 \int_0^t \|(BB^*-F_\epsilon)(s)\|_p \, ds + M^2 \int_0^t \|F(s)(S_n^*-S^*)(t-s)\|_p \, ds \right)$$

$$\leq 2M^2\|(BB^*-F_\epsilon)(\cdot)\|_{L^1([0,\tau];\mathcal{I}_p)} + \tau M^2 \sup_{K \in \mathcal{K}_{\mathcal{I}_p}, t \in [0, \tau]} \|K(S_n^*-S^*)(t)\|_p$$

$$< \epsilon.$$  

The same argument shows that for any $\epsilon > 0$, there is an $N(\epsilon) > 0$ such that if $n \geq N(\epsilon)$, we observe

$$\sup_{t \in [0, \tau]} \int_0^t \|(S_n-S)(t-s)BB^*(s)(S^*)(t-s)\|_p \, ds < \epsilon.$$  

Since $\epsilon > 0$ is arbitrary, this proves that the mapping $t \mapsto \int_0^t S_n(t-s)D_n(s)S_n^*(t-s) \, ds$ converges to $t \mapsto \int_0^t S(t-s)BB^*(s)S^*(t-s) \, ds$ in the $\mathcal{C}([0, \tau]; \mathcal{I}_p)$ norm.

Let $I = [0, \tau] = [0, a]$. We observe (when $0 \leq s \leq t \in I$) the following bound

$$\|S(t-s)(\Sigma(C^*C)\Sigma)(s)S^*(t-s) - S_n(t-s)(\Sigma_nE_n\Sigma_n)(s)S_n^*(t-s)\|_p \leq$$

$$\|S_n(t-s)(\Sigma_nE_n\Sigma_n - \Sigma(C^*C)\Sigma)(s)S_n^*(t-s)\|_p +$$

$$\|S_n(t-s)(\Sigma(C^*C)\Sigma)(s)(S_n^*-S^*)(t-s)\|_p +$$

$$\|(S_n-S)(t-s)(\Sigma(C^*C)\Sigma)(s)S^*(t-s)\|_p.$$  

Since $\Sigma(\cdot) \in \mathcal{C}(I; \mathcal{I}_p)$ and $C^*C(\cdot) \in L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))$, then it is straightforward to observe that $\Sigma(C^*C)\Sigma(\cdot) \in L^\infty(I; \mathcal{I}_p)$ and therefore $(\Sigma(C^*C)\Sigma)(\cdot) \in L^1(I; \mathcal{I}_p)$. Hence it can be approximated by simple $\mathcal{I}_p$-valued functions. Therefore, as we proved before, for each $\epsilon > 0$, there is an $N(\epsilon)$ such that if $n \geq N(\epsilon)$, then

$$\sup_{t \in [0, \tau]} \int_0^t \|S_n(t-s)(\Sigma(C^*C)\Sigma)(s)(S_n^*-S^*)(t-s)\|_p \, ds < \epsilon,$$
and
\[ \sup_{t \in [0, \tau]} \int_0^t \| (S_n - S)(t - s)(\Sigma(C^*C)\Sigma)(s)S^*(t - s) \|_p \, ds < \epsilon. \]

As we did before, suppressing “(t)” for the sake of brevity, we observe the following bound:
\[ \| \Sigma(C^*C)\Sigma - \Sigma_n E_n \Sigma_n \|_p \leq \| \Sigma - \Sigma_n \|_p \left( \| C^*C \|\| \Sigma \|_p + \| \Sigma_n \|_p \| C^*C \| + \| C^*C - E_n \| \| \Sigma_n \|_p^2 \right). \]

We know that \( \sup_{t \in [0, \tau]} \| C^*C(t) \| \leq c_\tau, \sup_{t \in [0, \tau]} \| \Sigma_n(t) \|_p \leq \rho \tau \) and we define \( \hat{\rho} = \sup_{t \in [0, \tau]} \| \Sigma(t) \|_p \) and \( \rho = \max(\rho_\tau, \hat{\rho}) \). Hence
\[ \int_0^t \| S_n(t - s)(\Sigma_n E_n \Sigma_n - \Sigma(C^*C)\Sigma)(s)S^*_n(t - s) \|_p \, ds \leq M^2 c_\tau \rho \int_0^t \| (\Sigma - \Sigma_n)(s) \|_p \, ds + \tau \rho^2 \| (C^*C - E_n)(\cdot) \|_{L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))}. \]

Finally, define the following functions
\[ h_1(n) = \sup_{t \in [0, \tau]} \| S_n(t)\Sigma_0^*S_n^*(t) - S(t)\Sigma_0 S^*(t) \|_p, \]
\[ h_2(n) = \sup_{t \in [0, \tau]} \int_0^t \| S_n(t - s)(BB^* - D_n)(s)S^*_n(t - s) \|_p \, ds, \]
\[ h_3(n) = \sup_{t \in I} \left( \int_0^t \| S_n(t - s)(\Sigma(C^*C)\Sigma)(s)(S^*_n - S^*)(t - s) \|_p \, ds + \int_0^t \| (S_n - S)(t - s)(\Sigma(C^*C)\Sigma)(s)S^*(t - s) \|_p \, ds \right), \]
\[ h_4(n) = \tau \rho^2 \| (C^*C - E_n)(\cdot) \|_{L^\infty ([0, \tau]; \mathcal{L}(\mathcal{H}))}. \]

We have thus far shown that \( \lim_{n \to \infty} h_i(n) = 0 \), independently of \( t \in I \), for \( i = 1, 2, 3 \) and \( \lim_{n \to \infty} h_4(n) = 0 \), by initial hypotheses. Therefore, since \( \Sigma(\cdot) \) and \( \Sigma_n(\cdot) \) satisfy the Riccati equation, the difference is bounded as
\[ \| (\Sigma - \Sigma_n)(t) \|_p \leq h(n) + 2M^2 c_\tau \rho \int_0^t \| (\Sigma - \Sigma_n)(s) \|_p \, ds, \]
in \( t \in I \) and with \( h(n) = \sum_{k=1}^{4} h_k(n) \). Then a direct application of Grönwall’s Lemma implies that

\[
\sup_{t \in I} \| (\Sigma - \Sigma_n)(t) \|_p \leq h(n)e^{2M_2e_{c\rho}m(I)},
\]

where \( m(I) \) is the measure of \( I = [0, a] = [0, \tau] \) and because \( n \to h(n) \) is independent of \( t \). Finally, since \( \lim_{n \to \infty} h(n) = 0 \), the Theorem is proved.

### 3.4 Solutions for \( C_0 \)-Semigroups

Suppose that \( S(t) \) is a \( C_0 \)-semigroup, \( F(\cdot) \in L_1^{loc}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \) and \( G(\cdot) \in L_1^{loc}(\mathbb{R}^+; \mathcal{L}(\mathcal{H})) \). In addition suppose there is a solution \( \Sigma(\cdot) \in \mathcal{C}([0, a]; \mathcal{L}) \), of

\[
\Sigma(t) = S(t)\Sigma_0S^*(t) + \int_0^t S(t-s)(F - \Sigma G)(s)S^*(t-s) \, ds. \tag{3.21}
\]

Also, assume that \( \{S_n(t)\}_{n=1}^{\infty} \) is a sequence of uniformly continuous semigroups such that \( S_n(t) \) and \( S_n^*(t) \) converge strongly, and uniformly in \( t \in [0, a] \), to \( S(t) \) and \( S^*(t) \) as \( n \to \infty \), respectively. Then, Theorem 5 implies that the sequence of solutions \( \{\Sigma_n(\cdot)\}_{n=1}^{\infty} \) in \( \mathcal{C}([0, a]; \mathcal{L}_p) \) of

\[
\Sigma_n(t) = S_n(t)\Sigma_0S_n^*(t) + \int_0^t S_n(t-s)(F - \Sigma_n G\Sigma_n)(s)S_n^*(t-s) \, ds, \tag{3.22}
\]

satisfies \( \sup_{t \in [0, a]} \| \Sigma(t) - \Sigma_n(t) \|_p \to 0 \) as \( n \to \infty \). Naturally, we ask what the behavior of the sequence \( \{\Sigma_n(\cdot)\}_{n=1}^{\infty} \) is if we omit the hypothesis of the existence of a solution \( \Sigma(\cdot) \in \mathcal{C}([0, a]; \mathcal{L}_p) \) to (3.21). The answer is that the sequence converges to a mapping that will be proven to be the solution to the Riccati equation (3.21). We will carry out the proof on the following Theorem.

An interesting feature of the following result is that, under certain conditions, there is a unique solution to (3.21) in \( L^2([0, \tau]; \mathcal{L}_2) \) that also belongs to \( \mathcal{C}([0, \tau]; \mathcal{L}_1) \). This is an useful result for approximation purposes because \( L^2([0, \tau]; \mathcal{L}_2) \) is a Hilbert space and \( \mathcal{C}([0, \tau]; \mathcal{L}_1) \) is not.
Theorem 6. Let $\mathcal{H}$ be a separable complex Hilbert space, $I = [0, \tau]$ or $I = \mathbb{R}^+$, $S(t)$ be a $C_0$-semigroup on $\mathcal{H}$, and suppose that

(i) $\Sigma_0 \in \mathcal{S}_p$ and $\Sigma_0 \geq 0$;

(ii) $BB^*(\cdot) \in L^1_{loc}(I; \mathcal{S}_p)$, with $BB^*(t) \geq 0$ for $t \in I$;

(iii) $C^*C(\cdot) \in L^\infty_{loc}(I; \mathcal{L}(\mathcal{H}))$, with $C^*C(t) \geq 0$ for $t \in I$.

Then, the equation

$$\Sigma(t) = S(t) \Sigma_0 S^*(t) + \int_0^t S(t-s)(BB^* - \Sigma(C^*C)\Sigma)(s)S^*(t-s) \, ds,$$

where the integral is a Bochner integral, has a unique solution in the space $L^2_{loc}(I; \mathcal{S}_{2p})$, and even more the solution belongs to $C(I; \mathcal{S}_p)$ and is point-wise self-adjoint and non-negative.

Proof. The following argument will be a modification of the one of Da Prato in [9] and [20]. The idea of the proof consists in an application of the Contraction Mapping Principle to prove existence and uniqueness locally, and then making use of the non-negativity of the solution to extend existence and uniqueness to the entire interval $I$.

Using the definition of the map $\gamma$ in Theorem 3, we can write the Riccati equation as $\Sigma = \gamma(\Sigma)$. Since $\gamma : L^2_{loc}(I; \mathcal{S}_{2p}) \mapsto \mathcal{C}(I; \mathcal{S}_p)$, it is enough to search for fixed points of $\gamma$ in the latter space, because $\mathcal{C}(I; \mathcal{S}_p) \subset L^2_{loc}(I; \mathcal{S}_{2p})$.

Let $\tau > 0$ be such that $[0, \tau] \subset I$. If we call $b_\tau = \|BB^*(\cdot)\|_{L^1([0,\tau]; \mathcal{S}_p)}$ and $c_\tau = \|(C^*C)(\cdot)\|_{L^\infty([0,\tau]; \mathcal{L}(\mathcal{H}))}$ and we consider that $\sup_{t \in [0,\tau]} \|\Sigma(t)\|_p \leq \rho$, we have

$$\|\gamma(\Sigma)(t)\|_p \leq \|S(t) \Sigma_0 S^*(t)\|_p +$$

$$\int_0^t \|S(t-s)(BB^* - \Sigma(C^*C)\Sigma)(s)S^*(t-s)\|_p \, ds$$

$$\leq M^2_\tau \left(\|\Sigma_0\|_1 + b_\tau + \tau c_\tau \sup_{t \in [0,\tau]} \|\Sigma(t)\|_p^2\right)$$

$$\leq M^2_\tau \left(\|\Sigma_0\|_p + b_\tau + \tau c_\tau \rho^2\right),$$
where \( \|S(t)\| = \|S^*(t)\| \leq M_\tau \) for all \( t \in [0, \tau] \). \( M_\tau \geq 1 \) exist since \( S(t) \) and \( S^*(t) \) are \( C_0 \)-semigroups. Hence, taking the supremum over \([0, \tau]\),

\[
\sup_{t \in [0, \tau]} \|\gamma(\Sigma)(t)\|_p \leq M_\tau^2 \left( \|\Sigma_0\|_p + b_\tau + \tau c_\tau \rho^2 \right).
\]

Now, let \( \Lambda_i(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{S}_p) \) and \( \sup_{t \in [0, \tau]} \|\Lambda_i(t)\|_p \leq \rho \) for \( i = 1, 2 \), then we obtain the following bounds for the difference \( (\gamma(\Lambda_1) - \gamma(\Lambda_2))(t) \) when \( t \in [0, \tau] \) as

\[
\|\gamma(\Lambda_1)(t) - \gamma(\Lambda_2)(t)\|_p \leq \\
\leq \int_{0}^{t} \|S(t-s)(\Lambda_2(C^*C)\Lambda_2 - \Lambda_1(C^*C)\Lambda_1)(s)S^*(t-s)\|_p \, ds \\
\leq M_\tau^2 \int_{0}^{t} \|((\Lambda_2 - \Lambda_1)(C^*C)\Lambda_2 + \Lambda_1(C^*C)(\Lambda_2 - \Lambda_1))(s)\|_p \, ds \\
\leq M_\tau^2 \int_{0}^{t} \|((\Lambda_2 - \Lambda_1)(C^*C)\Lambda_2)(s)\|_p + \|((\Lambda_1(C^*C)(\Lambda_2 - \Lambda_1))(s)\|_p \, ds \\
\leq M_\tau^2 c_\tau \int_{0}^{t} \|\Lambda_2(s)\|_p \|\Lambda_2(s)\|_p + \|\Lambda_1(s)\|_p \|\Lambda_2 - \Lambda_1(s)\|_p \, ds \\
\leq M_\tau^2 c_\tau \tau \sup_{s \in [0, \tau]} \left( \|\Lambda_1(s)\|_p + \|\Lambda_2(s)\|_p \right) \sup_{t \in [0, \tau]} \|\Lambda_2 - \Lambda_1(s)\|_p.
\]

hence

\[
\sup_{t \in [0, \tau]} \|\gamma(\Lambda_1) - \gamma(\Lambda_2)(t)\|_p \leq 2M_\tau^2 c_\tau \tau \rho \sup_{t \in [0, \tau]} \|\Lambda_1 - \Lambda_2(t)\|_p.
\]

Then define \( \beta \) and pick \( \rho \) and \( 0 < s \leq \tau \) such that

\[
\beta = M_\tau^2 (\|\Sigma_0\|_p + b_\tau); \quad b_\tau + s \rho^2 c_\tau \leq \beta; \\
\rho = 2M_\tau^2 \beta; \quad 2s \rho M_\tau^2 c_\tau \leq \frac{1}{2};
\]

which is always possible (if \( \|\Sigma_0\|_p = 0 \) use \( M_\tau > 1 \) ). Then the mapping \( \gamma \) defines a contraction on the ball

\[
\mathbf{B}_{s, \rho} = \left\{ F(\cdot) \in \mathcal{C}([0, s]; \mathcal{S}_p) \mid \sup_{t \in [0, s]} \|F(t)\|_p \leq \rho \right\},
\]

and then the equation \( \Sigma = \gamma(\Sigma) \) defines an unique solution on \( \mathbf{B}_{s, \rho} \) by the Contraction Mapping Theorem.
Since $S(t)$ is a $C_0$-semigroup over $\mathcal{H}$, there is a sequence $\{S_n(t)\}_{n=1}^{\infty}$ of uniformly continuous semigroups over the same Hilbert space $\mathcal{H}$ that satisfy that for each $x \in \mathcal{H}$,

$$\|S(t)x - S_n(t)x\| \to 0 \quad \text{and} \quad \|S^*(t)x - S_n^*(t)x\| \to 0,$$

as $n \to \infty$, uniformly in $t \in [0, \tau]$ \footnote{Since $S(t)$ is a $C_0$-semigroup over a Hilbert space $\mathcal{H}$, then $S^*(t)$ also is a $C_0$-semigroup over $\mathcal{H}$ and even more there are $M \geq 1$ and $\omega > 0$ such that $\|S^*(t)\| = \|S(t)\| \leq Me^{\omega t}$ for $t \in \mathbb{R}^+$. Then, let $A_n$ be the Yosida approximant of the infinitesimal generator $A$ of $S(t)$. That is $A_n = nAR_n(A) = n^2R_n(A) - n \in \mathcal{L}(\mathcal{H})$ with $n \in (\omega, \infty) \cap \mathbb{N}$ where $R_n(A) = (n - A)^{-1}$. It is a well-known result that the sequence of uniformly continuous semigroups $S_n(t) = e^{tA_n}$ satisfies $\|S(t)x - S_n(t)x\| \to 0$ as $n \to \infty$ for any $x \in \mathcal{H}$ and uniformly on compact intervals. Since $S^*(t)$ is also a $C_0$-semigroup and with generator $A^*$; the Yosida approximant $A^*_n = nA^*R_n(A^*) = n^2R_n(A^*) - n$ is well defined for $n \in (\omega, \infty) \cap \mathbb{N}$ and $\|S^*(t)x - e^{tA^*_n}x\| \to 0$ as $n \to \infty$ for any $x \in \mathcal{H}$ and uniformly on compact intervals. We observe that $(A_n)^* = n^2((n - A)^{-1})^* - n = n^2(n - A^*)^{-1} - n = A_n^*$, which implies that $S_n^*(t) = (e^{tA_n})^* = e^{t(A_n)^*} = e^{tA^*_n}$ and the assertion is proved.} Then, the sequence $\{\Sigma_n(\cdot)\}_{n=1}^{\infty}$ of solutions of

$$\Sigma_n(t) = S_n(t)\Sigma_0S_n^*(t) + \int_0^t S_n(t - r)(BB^* - \Sigma(C^*C)\Sigma)(r)S_n^*(t - r) \, dr,$$

belongs to $\mathcal{C}(\mathbb{R}^+; \mathcal{S}_p)$ and satisfies $\Sigma_n^*(t) = \Sigma_n(t) \geq 0$ for $t \in \mathbb{R}^+$ by Theorem 4 (page 68). Without loss of generality, suppose that $\sup_n \|S_n(t)\| \leq M_r$ for $t \in [0, \tau]$, where $M_r > 0$ was chosen such that $\|S(t)\| \leq M_r$ for $t \in [0, \tau]$. Therefore by the First Approximation Theorem (Theorem 5),

$$\sup_{t \in [0,s]} \|\Sigma(t) - \Sigma_n(t)\|_p \to 0,$$

as $n \to \infty$, which implies that $\Sigma^*(t) = \Sigma(t) \geq 0$. Since $\Sigma^*(t) = \Sigma(t) \geq 0$ for $t \in [0, s]$ and solves the integral Riccati equation in this interval, we observe that for any $\phi \in \mathcal{H}$ and $t \in [0, s]$

$$0 \leq \langle \phi, \Sigma(t)\phi \rangle = \langle \phi, S(t)\Sigma_0S^*(t)\phi \rangle + \int_0^t \langle \phi, S(t - r)BB^*(r)S^*(t - r)\phi \rangle - \langle \phi, S(t - r)(\Sigma(C^*C)\Sigma)(r)S^*(t - r)\phi \rangle \, ds$$

$$\leq \langle \phi, S(t)\Sigma_0S^*(t)\phi \rangle + \int_0^t \langle \phi, S(t - s)BB^*(r)S^*(t - s)\phi \rangle \, ds.$$
That is
\[ 0 \leq \langle \phi, \Sigma(t) \phi \rangle \leq \left\langle \phi, \left( S(t) \Sigma_0 S^*(t) + \int_0^t S(t-s) BB^*(r) S^*(t-s) \, ds \right) \phi \right\rangle, \]
and this latter inequality implies (as proven in the proof of Proposition 15 in page 53) that
\[ \| \Sigma(s) \|_p \leq \| S(s) \Sigma_0 S^*(s) + \int_0^s S(s-r) BB^*(r) S^*(s-r) \, dr \|_p \leq M_2^2 (\| \Sigma_0 \|_p + b \tau), \]
(note that this inequality could have been obtained by observing that \( \Sigma_n(s) \) also satisfies it by Theorem 4) and this allows us to repeat the contraction argument on the interval \([s, 2s] \subset [0, \tau]\) and then the First Approximation Theorem proves that \( \sup_{t \in [0, 2s]} \| \Sigma(t) - \Sigma_n(t) \|_p \to 0 \) as \( n \to \infty \). Hence again \( \Sigma^*(t) = \Sigma(t) \geq 0 \) on \( t \in [s, 2s] \) and we can again use the same argument on \([2s, 3s], [3s, 4s], \ldots \).

\[ \text{Remark 3.4.} \text{ We know now that} \]
\[ \Sigma(t) = S(t) \Sigma_0 S^*(t) + \int_0^t S(t-s) (BB^* - \Sigma(C^*C) \Sigma)(s) S^*(t-s) \, ds, \]
(3.23)

has a unique solution \( \Sigma(\cdot) \) in \( C([0, \tau]; \mathcal{J}_p) \) under Theorem hypotheses. Suppose that \( BB^*(\cdot) \in C([0, \tau]; \mathcal{J}_p) \) and that \( C^*C(\cdot) \in C([0, \tau]; \mathcal{L}(\mathcal{H})) \). Let \( A \) be the infinitesimal generator of the \( C_0 \)-semigroup \( S(t) \) over the complex separable Hilbert space \( \mathcal{H} \). Since \( \mathcal{H} \) is reflexive, \( S^*(t) \) is a \( C_0 \)-semigroup with generator \( A^* \) (see [43]). Let \( x, y \in \mathcal{D}(A^*) \), and then \( \Sigma(\cdot) \) satisfies
\[ \langle \Sigma(t)x, y \rangle = \langle \Sigma_0 S^*(t)x, S^*(t)y \rangle + \int_0^t \langle (BB^* - \Sigma(C^*C) \Sigma)(s) S^*(t-s)x, S^*(t-s)y \rangle \, ds. \]

Therefore, \( t \mapsto \langle \Sigma(t)x, y \rangle \) is differentiable and a simple computation with the Leibniz integral rule (see [9] for a proof when \( BB^* \) and \( C^*C \) are constant mappings) shows that
\[ \frac{d}{dt} \langle \Sigma(t)x, y \rangle = \]
\[ \langle A^*y, \Sigma(t)x \rangle + \langle \Sigma(t)x, A^*y \rangle + \langle BB^*(t)x, y \rangle - \langle \Sigma(t)(C^*C)(t) \Sigma(t)x, y \rangle, \]
with $\langle \Sigma(0)x, y \rangle = \langle \Sigma_0x, y \rangle$. Therefore, any solution in $C([0, \tau]; \mathcal{I}_p)$ of the integral Riccati equation (3.23) is a weak solution of the differential equation

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A^* + BB^*(t) - \Sigma(t)(C^*C)(t)\Sigma(t),$$

(3.24)

with initial condition $\Sigma(0) = \Sigma_0$. Conversely, any weak solution to this equation can be proven to be a mild solution to the integral Riccati equation (3.23) (See [9] for a proof for constant mappings $BB^*$ and $C^*C$. The extension for $BB^*(\cdot) \in C([0, \tau]; \mathcal{I}_p)$ and $C^*C(\cdot) \in C([0, \tau]; \mathcal{L}(\mathcal{H}))$ is straightforward). Since the unique solution of this latter equation in the space $C([0, \tau]; \mathcal{I}_p)$ is also a mild solution, these two are equivalent. Therefore, under the hypotheses of Theorem 6 and when $BB^*(\cdot) \in C([0, \tau]; \mathcal{I}_p)$ and $C^*C(\cdot) \in C([0, \tau]; \mathcal{L}(\mathcal{H}))$, any weak solution to (3.24) is $\mathcal{I}_p$-valued continuous solution of the integral Riccati equation (3.23).

### 3.5 Existence of Minimizers of

$$\int_0^\tau \text{Tr} \left( Q(t)\Sigma(t) \right) \, dt$$

We now prove that Problem(\mathcal{P}), described in the introduction of this thesis, has a solution. Although the stationary network case is an special case of the mobile sensor network case, we present the proofs separately for completeness. We will assume throughout this section that $\mathcal{H} = L^2(\Omega)$.

**Theorem 7.** Consider the case of a stationary network of $p$ sensors in the positions: $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p) = \bar{x}$ in some compact set $X_0$ inside $\Omega$. Suppose that the output map $(t, \bar{x}) \mapsto C(t, \bar{x})$ satisfies the conditions of Lemma 6. Also assume that $BB^*(\cdot) \in L^1([0, \tau]; \mathcal{I}_1)$ with $BB^*(t) \geq 0$ for $t \in [0, \tau]$, $S(t)$ is a $C_0$-semigroup, $0 \leq \Sigma_0 \in \mathcal{I}_1$ and that $(t, \bar{x}) \mapsto \Sigma(t, \bar{x})$ is the unique solution in $C([0, \tau]; \mathcal{I}_1)$ to

$$\Sigma(t, \bar{x}) = S(t)\Sigma_0S^*(t) + \int_0^t S(t-s)(BB^*(s) - (\Sigma(C^*C)S)(t, \bar{x}))S^*(t-s) \, ds.$$  

(3.25)

In addition, let $Q(\cdot) \in L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))$ with $Q(t) \geq 0$ for $t \in [0, \tau]$ and $J : X_0^p \mapsto \mathbb{R}$ be defined as

$$J(\bar{x}) = \int_0^\tau \text{Tr} \left( Q(t)\Sigma(t, \bar{x}) \right) \, dt.$$
Then, there is a $x_{min} \in X_0^p$ such that

$$\inf_{x \in X_0^p} J(x) = J(x_{min}),$$

i.e., Problem $(P)$ has a solution.

**Proof.** Since all the Hypotheses of Lemma 6 are met, we observe that the set $\mathcal{F} = \{ C^*(\cdot, x) \in L^\infty([0, \tau]; \mathcal{A}_1) : x \in X_0^p \}$ is compact. Then, $C^*(\cdot, x) \in L^\infty([0, \tau]; \mathcal{A}_1)$ for each $x \in X_0^p$ and by Theorem 6, we observe that there is a unique solution $\Sigma(\cdot, x) \in \mathcal{C}([0, \tau]; \mathcal{A}_1)$ to equation (3.26) for each $x \in X_0^p$.

Since $\Sigma(t, x) \in \mathcal{A}_1$ for each $(t, x) \in [0, t] \times X_0^p$, then $Q(t)\Sigma(t, x) \in \mathcal{A}_1$ and $\text{Tr} (Q(t)\Sigma(t, x))$ is well defined. It is also a non-negative real number since $Q(t)$ and $\Sigma(t, x)$ are non-negative for each $(t, x) \in [0, t] \times X_0^p$ (see [30] for a proof).

Also, Theorem 5 implies that the map $t \mapsto \Sigma(t, x)$ solution to the integral Riccati equation varies continuously in the $\sup_{t \in [0, \tau]} \mathcal{A}_1$-norm with respect to $C^*(\cdot, x) \in \mathcal{F}$. If $\Sigma_1(\cdot), \Sigma_2(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{A}_1)$, then we have the inequality

$$\left| \int_0^\tau \text{Tr} (Q(t)\Sigma_1(t)) \, dt - \int_0^\tau \text{Tr} (Q(t)\Sigma_2(t)) \, dt \right| \leq \tau \| Q(\cdot) \|_{L^\infty([0, \tau]; \mathcal{A}(\mathcal{A}_1))} \sup_{t \in [0, \tau]} \| (\Sigma_1 - \Sigma_2)(t) \|_1.$$ 

Which implies that the map $\Sigma(\cdot) \mapsto \int_0^\tau \text{Tr} (Q(t)\Sigma(t)) \, dt$ is uniformly continuous in $\mathcal{C}([0, \tau]; \mathcal{A}_1)$. Hence by composition of continuous mappings $C^*(\cdot, x) \mapsto \int_0^\tau \text{Tr} (Q(t)\Sigma(t, x)) \, dt$ is continuous over the compact set $\mathcal{F}$. Then

$$\inf_{x \in X_0^p} J(x) = J(x_{min}),$$

for some $x_{min} \in X_0^p$. \hfill $\square$

Now we extend the previous theorem to the mobile network case.

**Theorem 8.** Consider now the case of a moving sensor network of $p$ sensors with trajectories given by $x(\cdot, \Theta_0, u(\cdot)) = \{ \bar{x}_i(\cdot, \bar{\Theta}_0, u_i(\cdot)) \}_{i=1}^p$, where $\bar{x}_i(t, \bar{\Theta}_0, u_i(\cdot)) = M\bar{\Theta}_0(t, \bar{\Theta}_0, u_i(\cdot))$ for all $t \in [0, \tau]$ and $t \mapsto \bar{\Theta}_0(t, \bar{\Theta}_0, u_i(\cdot))$ is a solution to a controlled ordinary differential equation of the type described in Section 2.5.2. Suppose that the initial set $\Theta_0$ and the admissible control set $U$ are also of the type described in Section 2.5.2.
Suppose that the output map \((t, \theta_0, u(\cdot)) \mapsto C(t, x(t, \theta_0, u(\cdot)))\) satisfies the conditions of Lemma 7. Also assume that \(BB^* (\cdot) \in L^1([0, \tau]; \mathcal{A}_1)\) with \(BB^* (t) \geq 0\) for \(t \in [0, \tau]\), \(S(t)\) is a \(C_0\)-semigroup, \(0 \leq \Sigma_0 \in \mathcal{A}_1\) and that \((t, \theta_0, u(\cdot)) \mapsto \Sigma(t, \theta_0, u(\cdot))\) is the unique solution in \(C([0, \tau]; \mathcal{A}_1)\) to
\[
\Sigma(t, \theta_0, u(\cdot)) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t - s)(BB^*(s) - (\Sigma(C^*C)\Sigma)(t, \theta_0, u(\cdot)))S^*(t - s)\, ds,
\]
for each \((\theta_0, u(\cdot)) \in \Theta^p_0 \times \mathcal{U}^p\). In addition, let \(Q(\cdot) \in L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))\) with \(Q(t) \geq 0\) for \(t \in [0, \tau]\) and \(J : \Theta^p_0 \times \mathcal{U}^p \mapsto \mathbb{R}\) be defined as
\[
J(\theta_0, u(\cdot)) = \int_0^\tau \text{Tr}(Q(t)\Sigma(t, \theta_0, u(\cdot)))\, dt.
\]
Then, there is \((\theta_0^{\text{min}}, u^{\text{min}}(\cdot)) \in \Theta^p_0 \times \mathcal{U}^p\) such that
\[
\inf_{\theta_0 \in \mathcal{X}^p_0, u(\cdot) \in \mathcal{U}^p} J(\theta_0, u(\cdot)) = J(\theta_0^{\text{min}}, u^{\text{min}}(\cdot)),
\]
i.e., Problem \((\mathcal{P})\) has a solution.

**Proof.** The set
\[
\mathcal{F} = \{ C^*C(\cdot, M\bar{\theta}(\cdot, \theta_0, u(\cdot))) \in L^\infty([0, \tau]; \mathcal{A}_1) : \theta_0 \in \Theta^p_0 \text{ and } u(\cdot) \in \mathcal{U}^p \},
\]
is compact in \(L^\infty([0, \tau]; \mathcal{A}_1)\) by Lemma 7 where \(M\bar{\theta}(\cdot, \theta_0, u(\cdot))\) is shorthand for the set
\[
\{ M\bar{\theta}_i(\cdot, \theta_0^i, u_i(\cdot)) \}_{i=1}^p = \{ \bar{x}_i(\cdot, \theta_0^i, u_i(\cdot)) \}_{i=1}^p,
\]
hence \(M\bar{\theta}(\cdot, \theta_0, u(\cdot)) = \bar{x}(\cdot, \theta_0, u(\cdot))\).

Theorem 5 implies that the map \(t \mapsto \Sigma(t, \theta_0, u(\cdot))\) varies continuously in the sup\(_{t \in [0, \tau]}\) \(-\mathcal{A}_1\)-norm with respect to \(C^*C(\cdot, \theta_0, \theta_0, u(\cdot)) \in \mathcal{F}\). Also, as we proved in Theorem 7, the map \(\Sigma(\cdot) \mapsto \int_0^\tau \text{Tr}(Q(t)\Sigma(t))\, dt\) is uniformly continuous in \(C([0, \tau]; \mathcal{A}_1)\) and \(J(\theta_0, u(\cdot))\) is well defined over \(\Theta^p_0 \times \mathcal{U}\). Therefore,
\[
C^*C(\cdot, \bar{x}(\cdot, \theta_0, u(\cdot))) \mapsto \int_0^\tau \text{Tr}(Q(t)\Sigma(t, \bar{x}(t, \theta_0, u(\cdot))))\, dt
\]
is continuous over the compact set \( \mathcal{F} \). Then, there is \( (\bar{\theta}_0^{\text{min}}, u_{\text{min}}(\cdot)) \in \Theta_p^0 \times U_p^0 \) such that
\[
\inf_{\theta_0 \in \Theta_p^0, u(\cdot) \in U_p^0} J(\bar{\theta}_0, u(\cdot)) = J(\bar{\theta}_0^{\text{min}}, u_{\text{min}}(\cdot)).
\]

\[\square\]

### 3.6 The Gradient of \( \Sigma \) w.r.t. the Map \( t \mapsto C^*C(t) \)

We are interested in using a gradient type algorithm over the trajectories of the sensors. For this matter, we will first prove that the solution of the integral Riccati equation, as a function of the mapping \( t \mapsto C^*C(t) \), is Fréchet differentiable. We first need a lemma.

**Lemma 10.** Let \( \mathcal{H} \) be a complex separable Hilbert space and \( S(t) \) be a \( C_0 \)-semigroup over \( \mathcal{H} \). Suppose that \( G(\cdot) \) and \( \Sigma(\cdot) \) belong to \( X = C([0, \tau]; \mathcal{J}_1) \). Then the equation \( \Lambda = \hat{\gamma}(\Lambda) \) has a unique solution in \( L(X) \), where \( \hat{\gamma} : L(X) \mapsto L(X) \) and \( \hat{\gamma}(\Lambda) \) is defined by

\[
(\hat{\gamma}(\Lambda)K)(t) = -\int_0^t S(t-s)((\Lambda K)G\Sigma + \Sigma(G\Lambda K) + \Sigma K\Sigma)(s)S^*(t-s) \, ds,
\]

for all \( K(\cdot) \in X \).

**Proof.** We will use the re-normalization technique first described in \cite{10}. Define \( X_\lambda \) (with \( \lambda > 0 \)) to be the set of all trace class continuous mappings \( t \mapsto F(t) \) and domain \([0, \tau]\) with the norm \( \| \cdot \|_{1,\lambda} \) defined as

\[
\| F(\cdot) \|_{1,\lambda} = \sup_{t \in [0,\tau]} e^{-\lambda t} \| F(t) \|_1.
\]

Consider that \( X \) is normed with \( \| F(\cdot) \|_1 = \sup_{t \in [0,\tau]} \| F(t) \|_1 \). It is obvious that \( X \) and \( X_\lambda \) coincide element-wise, and we can prove that they coincide topologically. We observe the equivalency of the norms of \( X_\lambda \) and \( X \):

\[
e^{-\lambda \tau} \| F(\cdot) \|_1 \leq \| F(\cdot) \|_{1,\lambda} \leq \| F(\cdot) \|_1,
\]

and even more, the spaces \( L(X_\lambda) \) and \( L(X) \) are also topologically equivalent and

\[
e^{-\lambda \tau} \| \Lambda \|_{L(X)} \leq \| \Lambda \|_{L(X_\lambda)} \leq e^{\lambda \tau} \| \Lambda \|_{L(X)}.
\]
If $\Lambda \in \mathcal{L}(X)$, then it follows immediately that $\hat{\gamma}(\Lambda)$ is also a linear operator acting on $X$. Also, let $\|S(t)\| \leq M_s$ for $t \in [0, \tau]$, and $m = \max \left(\|G(\cdot)\|_1, \|\Sigma(\cdot)\|_1\right)$. Then

$$\|\left(\hat{\gamma}(\Lambda)K\right)(\cdot)\|_1 \leq \tau M^2 m^2 \|K(\cdot)\|_1 \left(2\|\Lambda\|_{\mathcal{L}(X)} + 1\right).$$

Hence $\hat{\gamma}(\Lambda) \in \mathcal{L}(X)$.

Let $\Lambda_1, \Lambda_2 \in \mathcal{L}(X)$ and $K(\cdot) \in X$. Then by the definition of the norm $\|\cdot\|_{1,\lambda}$ we observe

$$\left\|\left(\hat{\gamma}(\Lambda_1) - \hat{\gamma}(\Lambda_2)\right)K\right\|_1 \leq 2M^2 m^2 \int_0^t \|((\Lambda_1 - \Lambda_2)K)(s)\|_1 ds$$

$$\leq 2M^2 m^2 \int_0^t \|((\Lambda_1 - \Lambda_2)K)(s)\|_1 e^{-\lambda s}e^{\lambda s} ds$$

$$\leq 2M^2 m^2 \|((\Lambda_1 - \Lambda_2)K)(\cdot)\|_{1,\lambda} \int_0^t e^{\lambda s} ds$$

$$\leq \frac{2M^2 m^2}{\lambda} \|((\Lambda_1 - \Lambda_2)K)(\cdot)\|_{1,\lambda} e^{\lambda t}$$

$$\leq \frac{2M^2 m^2}{\lambda} \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(X,\lambda)} \|K(\cdot)\|_{1,\lambda} e^{\lambda t}$$

since $\int_0^t e^{\lambda s} ds = \frac{e^{\lambda t} - 1}{\lambda} < \frac{e^{\lambda t}}{\lambda}$ for $\lambda > 0$. Therefore

$$e^{-\lambda t}\left\|\left(\hat{\gamma}(\Lambda_1) - \hat{\gamma}(\Lambda_2)\right)K\right\|_1 \leq \frac{2M^2 m^2}{\lambda \lambda} \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(X,\lambda)} \|K(\cdot)\|_{1,\lambda},$$

which implies by taking the sup over $t \in [0, \tau]$ that

$$\left\|\left(\hat{\gamma}(\Lambda_1) - \hat{\gamma}(\Lambda_2)\right)K\right\|_{1,\lambda} \leq \frac{2M^2 m^2}{\lambda} \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(X,\lambda)} \|K(\cdot)\|_{1,\lambda}.$$  

Finally

$$\left\|\left(\hat{\gamma}(\Lambda_1) - \hat{\gamma}(\Lambda_2)\right)\right\|_{\mathcal{L}(X,\lambda)} \leq \frac{2M^2 m^2}{\lambda} \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(X,\lambda)}. \quad (3.28)$$

Suppose that $\lambda > 2M^2 m^2$. Then from Equation 3.28 we satisfy that the mapping $\hat{\gamma} : \mathcal{L}(X,\lambda) \mapsto \mathcal{L}(X,\lambda)$ is a contraction, and by the contraction mapping principle there is a unique solution to $\Lambda = \hat{\gamma}(\Lambda)$ in $\mathcal{L}(X,\lambda)$. Since $\|\cdot\|_{\mathcal{L}(X)}$ and $\|\cdot\|_{\mathcal{L}(X,\lambda)}$ are equivalent norms, $\Lambda = \hat{\gamma}(\Lambda)$ has only one solution in in $\mathcal{L}(X)$. \qed
We now prove that the solution of the integral Riccati equation \( \Sigma(\cdot) \) is Fréchet differentiable with respect to the mapping \( t \mapsto C^*C(t) \).

**Theorem 9.** Let \( \mathcal{H} \) be a complex separable Hilbert space and \( S(t) \) be a \( C_0 \)-semigroup over \( \mathcal{H} \). Suppose that \( 0 \leq \Sigma_0 \in \mathcal{I}_1 \) and \( F(\cdot) \in L^1([0, \tau]; \mathcal{I}_1) \). Let \( \mathcal{D} \) be an open set of \( X = C([0, \tau]; \mathcal{I}_1) \), then \( \Sigma(\cdot, G) \in X \) the unique solution of

\[
\Sigma(t, G) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s) \left( F - \Sigma(G)\Sigma(G) \right) (s) S^*(t-s) \, ds,
\]

with \( t \in [0, \tau] \) is Fréchet differentiable with respect to \( G(\cdot) \in \mathcal{D} \). The Fréchet derivative of \( \Sigma(\cdot; G) \) with respect to \( G \) is denoted by \( \Lambda(G) \in \mathcal{L}(X) \) and is equal to the unique solution of

\[
(\Lambda(G)h)(t) = -\int_0^t S(t-s) \left( (\Lambda(G)h)G\Sigma(G) + \Sigma(G)G(\Lambda(G)h) + \Sigma(G)h\Sigma(G) \right) (s) S^*(t-s) \, ds,
\]

for all \( h(\cdot) \in X \) and all \( t \in [0, \tau] \).

**Proof.** If \( F(\cdot) \in X \), we define \( ||F(\cdot)||_1 = \sup_{t \in [0, \tau]} ||F(t)||_1 \). By the continuity of the map \( G \mapsto \Sigma(\cdot, G) \) obtained in Theorem \( 5 \), we know that \( ||\Sigma(\cdot, G + h) - \Sigma(\cdot, G)||_1 = O(||h(\cdot)||_1) \), and therefore

\[
a(G, h) := ||(\Sigma(G + h)h\Sigma(G + h) - \Sigma(G)h\Sigma(G)) (\cdot)||_1 = O(||h(\cdot)||_1^2).
\]

Also since \( \Lambda(G) \in \mathcal{L}(X) \), we satisfy that

\[
b(G, h) := ||(\Sigma(G + h) - \Sigma(G)) G(\Lambda(G)h) (\cdot)||_1 = O(||h(\cdot)||_1^2).
\]

By direct calculation we observe that

\[
\Sigma(t, G + h) - \Sigma(t, G) - (\Lambda(G)h)(t) =
\]

\[
-\int_0^t S(t-s) \left[ \Sigma(G + h)(G + h)\Sigma(G + h) - \Sigma(G)G\Sigma(G) + (\Lambda(G)h)G\Sigma(G) - \Sigma(G)G(\Lambda(G)h) + \Sigma(G)h\Sigma(G) \right] (s) S^*(t-s) \, ds,
\]
and that
\[
\Sigma(t, G + h) - \Sigma(t, G) - (\Lambda(G)h)(t) = \\
- \int_0^t S(t - s) \left[ \left( \Sigma(G + h) - \Sigma(G) - (\Lambda(G)h) \right) G\Sigma(G) + \\
\Sigma(G + h) \left( \Sigma(G + h) - \Sigma(G) - (\Lambda(G)h) \right) \right. \\
\left. + \left( \Sigma(G + h) - \Sigma(G) \right) G(\Lambda(G)h) + \\
\Sigma(G + h) h \Sigma(G + h) - \Sigma(G) h \Sigma(G) \right] (s) S^*(t - s) \, ds.
\]
Therefore if \( z(t, G, h) = \| \Sigma(t, G + h) - \Sigma(t, G) - (\Lambda(G)h)(t) \|_1 \), then
\[
z(t, G, h) = 2M^2_\tau \rho \| G(\cdot) \|_1 \int_0^t z(s, G, h) \, ds + \tau M^2_\tau \left( a(h, G) + b(h, G) \right).
\]
By the Grönwall’s Lemma we observe that
\[
\frac{\| \Sigma(\cdot, G + h) - \Sigma(\cdot, G) - (\Lambda(G)h)(\cdot) \|_1}{\| h(\cdot) \|_1} \leq \tau M^2_\tau \left( a(h, G) + b(h, G) \right) e^{2\tau M^2_\tau \rho \| G(\cdot) \|_1},
\]
\[i.e., \]
\[
\frac{\| \Sigma(\cdot, G + h) - \Sigma(\cdot, G) - (\Lambda(G)h)(\cdot) \|_1}{\| h(\cdot) \|_1} = O(\| h(\cdot) \|_1),
\]
since \( a(h, G) + b(h, G) = O(\| h(\cdot) \|_1^2) \). This implies, as claimed, that \( \Sigma'(G) = \Lambda(G) \), where the derivative is taken in the Fréchet sense.

The previous result proves that we can consider the Fréchet derivative of \( \Sigma \) w.r.t. the mapping \( t \mapsto C^*C(t) \). We now want to prove that in the case of moving sensors, we can take the derivative with respect to the controls.

We will denote a position at time \( t \) with input control \( u \) as \( \bar{x}(t, u) \). Then we can regard \( \bar{x} \) as a mapping from \( L^2([0, \tau]; \mathbb{R}) \) to \( C([0, \tau]; \Omega) \). Let
\[
\bar{x}(t, u) = e^{At} \bar{x}_0 + \int_0^t e^{A(t-s)}bu(s) \, ds.
\]
Then the Fréchet derivative of \( \bar{x} \) with respect to \( u \) satisfies \( D_u \bar{x} \in \mathcal{L}(L^2([0, \tau]; \mathbb{R}), C([0, \tau]; \Omega)) \) and it’s given by
\[
(D_u \bar{x}h)(t) = \int_0^t e^{A(t-s)}bh(s) \, ds.
\]
Also, let $G(t, \bar{x}(t)) = C^*C(t, \bar{x}(t))$ where the output map comes from one moving sensor. $G$ is a mapping from $C([0, \tau]; \overline{\Omega})$ to $C([0, \tau]; \mathcal{A}_1(L^2(\Omega)))$. Then,

$$G(t, \bar{x}(t))\varphi(\cdot) = K(t, x, \bar{x}(t)) \int_\Omega K(t, y, \bar{x}(t))\varphi(y) \, dy,$$

and the regularity of $\bar{x}(t) \mapsto G(t, \bar{x}(t))$ is guaranteed by the smoothness of $K$, in which case, following the chain rule, we can compute $D_u\Sigma(\cdot, G(\cdot, \bar{x}(t, u)))$. 
Chapter 4
Galerkin type Approximation Scheme

4.1 Introduction
In this chapter, we discuss the approximation of $I_p$-valued solutions to the integral Riccati equation. We will make use of previous results by [20], [21] and [28]. Although, these references treat Hilbert-Schmidt operators or bounded operators, we will extend these results to $I_p$ for $1 \leq p \leq \infty$.

4.2 Technical Results and Standard Assumptions
Let, for each $n \in \mathbb{N}$, $P_n$ be the projection from our initial Hilbert space $\mathcal{H}$ onto a finite dimensional Hilbert space $\mathcal{V}_n$ such that $\mathcal{V}_n \subset \mathcal{H}$ and $\mathcal{V}_n \subset \mathcal{D}(A)$, where the sequence $P_n^*P_n$ converges strongly to the identity and $[\mathcal{N}(P_n)]^\perp \subset \mathcal{D}(A)$ for each $n \in \mathbb{N}$. Since $P_n$ are projections onto $\mathcal{V}_n \subset \mathcal{H}$, the norm $\|P_n\|$ is uniformly bounded.

Then $A_n = P_nAP_n^*$ is a bounded linear operator and the infinitesimal generator of the uniformly continuous semigroup $S_n(t) = e^{A_n t}$ on $\mathcal{H}$ where $A$ is the infinitesimal generator of a semigroup $S(t)$ on $\mathcal{H}$. Consider the conditions:

H1) There are $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_n(t)\| \leq Me^{\omega t}$. 

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There is a dense subset \( D \subset \mathcal{H} \) such that \( D \subset \mathcal{D}(A) \) and if \( x \in D \), then \( A_n x \to Ax \) as \( n \to \infty \) and there is a complex number \( \lambda_0 \), with \( \Re \lambda_0 > \omega \) such that \( (\lambda_0 - A)D = \mathcal{H} \).

If \( \mathbf{H1} \) and \( \mathbf{H2} \) are satisfied, then

\[
\| S(t)x - S_n(t)x \| \to 0,
\]

for each \( x \in \mathcal{H} \) as \( n \to \infty \) and uniformly on compact intervals in \( t \) by an application of Trotter-Kato Theorem (see [5] and [43]).

A particular case is when \( P_n \) is an orthogonal projection, in which case \( P_n^* = P_n \) and \( P_n \to I \) strongly. In this case, we observe the following result.

**Proposition 16.** Let \( \{ P_n \}_{n=1}^\infty \) be a sequence of orthogonal projectors over a complex separable Hilbert space \( \mathcal{H} \) that converge strongly to the identity, \( 0 \leq \Sigma_0 \in \mathcal{I}_p \), \( F(\cdot) \in L^1(I;\mathcal{I}_p) \) and \( G(\cdot) \in \mathcal{C}(I;\mathcal{I}_1) \) where \( I = [0, \tau] \) for some \( \tau > 0 \). Then

1. \( P_n \Sigma_0 P_n \in \mathcal{I}_p \) and \( \| \Sigma_0 - P_n \Sigma_0 P_n \|_p \to 0 \) as \( n \to \infty \).
2. The map \( t \mapsto P_n F(t)P_n \) belongs to \( L^1(I;\mathcal{I}_p) \) and

\[
\int_I \| (F - P_n FP_n)(s) \|_p \, ds \to 0,
\]

as \( n \to \infty \).

3. The map \( t \mapsto P_n G(t)P_n \) belongs to \( \mathcal{C}(I;\mathcal{I}_1) \) and

\[
\sup_{t \in I} \| (G - P_n GP_n)(t) \|_1 \to 0,
\]

as \( n \to \infty \).

**Proof.**

1. Let \( \Sigma_0 \geq 0 \) be of rank one, so \( \Sigma_0 x = \langle \varphi, x \rangle \varphi \) and define \( \varphi_n = P_n \varphi \).

Then \( P_n \Sigma_0 P_n x = \langle \varphi_n, x \rangle \varphi_n \) since \( P_n^* = P_n \), for all \( x \in \mathcal{H} \). Then,

\[
(\Sigma_0 - P_n \Sigma_0 P_n)x = (\varphi, x) \varphi - (\varphi_n, x) \varphi_n = (\varphi - \varphi_n, x) \varphi + (\varphi_n, x) (\varphi - \varphi_n),
\]

and then

\[
| \text{Tr} (B(\Sigma_0 - P_n \Sigma_0 P_n)) | \leq \sum_{k=1}^\infty | \langle \varphi - \varphi_n, \phi_k \rangle | | \langle \phi_k, B \varphi \rangle | +
| \langle \varphi_n, \phi_k \rangle | | \langle \phi_k, B(\varphi - \varphi_n) \rangle | +
| \langle \varphi_n, \varphi \rangle | | \langle \phi_k, B(\varphi - \varphi_n) \rangle | \leq \| \varphi - \varphi_n \| \| B \varphi \| + \| \varphi_n \| \| B(\varphi - \varphi_n) \| \leq \| B \|_p \left( \| \varphi - \varphi_n \| \| \varphi \| + \| \varphi_n \| \| \varphi - \varphi_n \| \right)
\]
therefore
\[
\|\Sigma_0 - P_n \Sigma_0 P_n\|_p = \sup_{B \neq 0 \text{ of finite rank}} \frac{|\operatorname{Tr}(B(\Sigma_0 - P_n \Sigma_0 P_n))|}{\|B\|_q} \\
\leq \|\varphi - \varphi_n\| \|\varphi\| + \|\varphi_n\| \|\varphi - \varphi_n\|.
\]
Then \(\|\Sigma_0 - P_n \Sigma_0 P_n\|_p \rightarrow 0\) as \(n \rightarrow \infty\), since \(\varphi_n \rightarrow \varphi\). If \(\Sigma_0\) is of finite rank, the same result follows easily. If \(0 \leq \Sigma_0 \in \mathcal{I}_p\), there is a sequence of finite rank operators \(\{\Sigma_0^n\}_{n=1}^\infty\) such that \(\|\Sigma_0 - \Sigma_0^n\|_p \rightarrow 0\) as \(m \rightarrow \infty\), then
\[
\|\Sigma_0 - P_n \Sigma_0 P_n\|_p \leq \|\Sigma_0 - \Sigma_0^n\|_p + \|P_n(\Sigma_0 - \Sigma_0^n)P_n\|_p \\
\leq 2 \|\Sigma_0 - \Sigma_0^n\|_p + \|\Sigma_0^n - P_n \Sigma_0^n P_n\|_p,
\]
taking then \(\limsup_{n \rightarrow \infty} \|\Sigma_0 - P_n \Sigma_0 P_n\|_p \leq 2 \|\Sigma_0 - \Sigma_0^n\|_p\) and hence \(\|\Sigma_0 - P_n \Sigma_0 P_n\|_p \rightarrow 0\) as \(n \rightarrow \infty\) since \(\|\Sigma_0 - \Sigma_0^n\|_p \rightarrow 0\) as \(m \rightarrow \infty\).

ii. Since \(P_n\) is not time dependent, it is elementary to check that \(P_n F(\cdot) P_n \in L^1(I; \mathcal{I}_p)\). Let \(F(t)\) be a step function \(F(t) = \sum_{k=1}^q f_k \chi_{I_k}(t)\), then
\[
\int_I \|(F - P_n F P_n)(t)\|_p dt \leq \sum_{k=1}^q \|f_k - P_n f_k P_n\|_p m(I_k),
\]
and from the previous result we observe that \(\int_I \|(F - P_n F P_n)(t)\|_p \rightarrow 0\) as \(n \rightarrow \infty\). Since, step functions are dense in \(L^1(I; \mathcal{I}_p)\) the result will follows for any \(F(\cdot) \in L^1(I; \mathcal{I}_p)\). Let \(\{F_m(\cdot)\}_{m=1}^\infty\) be a sequence of step functions in \(L^1(I; \mathcal{I}_p)\) such that \(\|(F - F_m(\cdot))\|_{L^1(I; \mathcal{I}_p)} \rightarrow 0\) as \(m \rightarrow \infty\).

Also,
\[
(F - P_n F P_n)(t) = (F - F_m(t)) + P_n(F - F_m(t))P_n + (F_m - P_n F_m P_n)(t),
\]
and then
\[
\|(F - P_n F P_n)(t)\|_p \leq 2\|(F - F_m(t))\|_p + \|(F_m - P_n F_m P_n)(t)\|_p, \tag{4.2}
\]
for \(t \in I\), since \(\|P_n\| \leq 1\). Hence,
\[
\int_I \|(F - P_n F P_n)(t)\|_p dt \leq 2\|(F - F_m(\cdot))\|_{L^1(I; \mathcal{I}_p)} + \int_I \|(F_m - P_n F_m P_n)(t)\|_p dt.
\]
Therefore, \( \lim_{n \to \infty} \int_I \| (F - P_n F P_n)(t) \|_p \, dt \leq 2 \| (F - F_m)(\cdot) \|_{L^1(I ; \mathcal{F}_p)} \), but \( \| (F - F_m)(\cdot) \|_{L^1(I ; \mathcal{F}_p)} \to 0 \) as \( m \to \infty \) from which the result follows.

iii. The continuity of the map \( t \mapsto P_n G(t) P_n \) follows from Proposition [7]. Since \( I = [0, \tau] \) is compact, step functions are dense in \( \mathcal{C}(I ; \mathcal{F}_p) \). So if \( G(\cdot) \) is a step function \( G(t) = \sum_{k=1}^q g_k \chi_{t_k}(t) \), then

\[
\sup_{t \in I} \| (G - P_n G P_n)(t) \|_p \leq \sum_{k=1}^q \| g_k - P_n g_k P_n \|_p,
\]

and hence from i. it follows that \( \sup_{t \in I} \| (G - P_n G P_n)(t) \|_p \to 0 \) as \( n \to \infty \). The density of step functions in \( \mathcal{C}(I ; \mathcal{F}_p) \) implies that there is a sequence of step functions \( \{ G_m(\cdot) \}_{m=1}^\infty \) in \( \mathcal{C}(I ; \mathcal{F}_p) \), such that \( \sup_{t \in I} \| (G - G_m)(t) \|_p \to 0 \) as \( m \to \infty \). The inequality in (4.2), shows that

\[
\sup_{t \in I} \| (G - P_n G P_n)(t) \|_p \leq 2 \sup_{t \in I} \| (G - G_m)(t) \|_p + \sup_{t \in I} \| (G_m - P_n G_m P_n)(t) \|_p.
\]

Then, \( \lim_{n \to \infty} \sup_{t \in I} \| (G - P_n G P_n)(t) \|_p \leq 2 \sup_{t \in I} \| (G - G_m)(t) \|_p \).

Since \( \sup_{t \in I} \| (G - G_m)(t) \|_p \to 0 \), the initial claim follows.

\[ \square \]

**The Convection-Diffusion Operator case**

Let \( \Omega \subset \mathbb{R}^n \) be an open bounded domain with boundary \( \partial \Omega \) of Lipschitz class (for example, the open unit cube in \( \mathbb{R}^n \) has Lipschitz class boundary). Consider the differential operator of order 2

\[
A(x,D) = -\epsilon^2 \Delta + \sum_{|\alpha| \leq 1} a_\alpha(x) D^\alpha,
\]

with \( \epsilon > 0 \) and where \( \Delta = D^{(2,0,0,\ldots,0)} + D^{(0,2,0,\ldots,0)} + \cdots + D^{(0,0,\ldots,0,2)} = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} \) is the Laplacian operator on \( \Omega \) and the functions \( x \mapsto a_\alpha(x) \) are smooth complex values functions on \( \overline{\Omega} \). Since \( \epsilon^2 > 0 \), \( A(x,D) \) is strongly elliptic of order 2 (see [50] or [43]). We define \( A \) as \( Ax = A(x,D)x \) for each \( x \in H^2(\Omega) \cap H^1_0(\Omega) \) and denote \( \partial(-A) = H^2(\Omega) \cap H^1_0(\Omega) \). The operator \( -A \)
generates a $C_0$-semigroup $S(t) = e^{-At}$ over $L^2(\Omega)$ (see [50]) and the unique solution to
\[
\frac{\partial u(t, x)}{\partial t} + A(x, D)u(t, x) = 0, \quad \text{for } t > 0 \text{ and } x \in \Omega \\
u(t, x) = 0, \quad \text{for } t \geq 0 \text{ and } x \in \partial \Omega \\
u(0, x) = u_0(x), \quad \text{for } u_0(\cdot) \in L^2(\Omega),
\]

is given by
\[
u(t, x) = (S(t)u_0)(x).
\]

It is a well known fact that the Laplacian defined as
\[
\Delta : H^2(\Omega) \cap H^1_0(\Omega) \mapsto L^2(\Omega),
\]
has eigenvalues $\{\lambda_k\}_{k=1}^\infty$ that can be arranged in decreasing order $0 \geq \lambda_1 \geq \lambda_2 \geq \cdots$ such $\lambda_k \to -\infty$ as $k \to \infty$. Also, the eigenspaces are finite-dimensional and we can choose the eigenfunctions $\{\phi_k(\cdot)\}_{k=1}^\infty$ to be an orthonormal basis of $L^2(\Omega)$ and they are of class $C^\infty(\Omega)$.

Define then
\[
V_n = \text{span} \{\phi_1, \phi_2, \ldots, \phi_n\},
\]
and let $P_n$ be the orthogonal projector from $L^2(\Omega)$ to $V_n$. Clearly, $V_n \in \mathcal{D}(-A)$ and $P_n^*P_n = P_n^2 = P_n \to I$ strongly as $n \to \infty$ since
\[
\|(I - P_n)\psi\|^2 = \sum_{k=n+1}^\infty |\langle \phi_k, \psi \rangle|^2 \to 0, \quad \text{as } n \to \infty,
\]
and
\[
(N(P_n))^\perp = (\text{span} \{\phi_{n+1}, \phi_{n+2}, \ldots\})^\perp \\
= \text{span} \{\phi_1, \phi_2, \ldots, \phi_n\} \\
= V_n \\
\subset \mathcal{D}(-A).
\]

A well known result is that there is a $\hat{\lambda}_0 \geq 0$ (given by the Gårding's inequality) such that $-A_{\hat{\lambda}_0} = -(A + \hat{\lambda}_0I)$ is the infinitesimal generator of a $C_0$-semigroup of contractions in $L^2(\Omega)$ (see [13]), i.e., $-A_{\hat{\lambda}_0} \in \mathcal{G}(1, 0)$. Then $-A = -A_{\hat{\lambda}_0} + \hat{\lambda}_0I$ and since $\hat{\lambda}_0I$ is a bounded operator and $\|\hat{\lambda}_0I\| = \hat{\lambda}_0$, 

−A ∈ G(1, ˆ{λ}_0). Therefore, since V_n ∈ D(A), A_n = P_n A P_n satisfies −A_n ∈ G(1, ˆ{λ}_0) (see [3] for a proof). This implies that the hypothesis H1 is satisfied, since

$$\|S_n(t)\| \leq e^{\hat{\lambda}_0 t} \text{ for all } n \in \mathbb{N} \text{ and all } t \geq 0,$$

where S_n(t) is the uniformly continuous semigroup generated by −A_n. Even more ∥S(t)∥ ≤ e^{\hat{\lambda}_0 t} for all t ≥ 0 where S(t) is the semigroup generated by −A.

We are now left to prove the Hypothesis H2. Let D be given by finite linear combinations of the \{φ_k\}_{k=1}^\infty; that is

$$D = \text{span}\{φ_1, φ_2, \ldots\}.$$  

Since \{φ_k\}_{k=1}^\infty is an orthonormal basis of L^2(Ω), D is dense in L^2(Ω). Let x ∈ D, then x = \sum_{k=1}^{N} \langle φ_k, x \rangle φ_k for some N < ∞. Now let n ≥ N. Then

$$\|Ax - A_n x\| \leq \sum_{k=1}^{N} |\langle φ_k, x \rangle| \|Aφ_k - A_n φ_k\|$$

$$\leq \sum_{k=1}^{N} |\langle φ_k, x \rangle| \|Aφ_k - P_n Aφ_k\| \rightarrow 0$$

as n → ∞ since P_n → I strongly as n → ∞ and N < ∞. Since −A ∈ G(1, ˆ{λ}_0), the last condition of H2 states that there is a complex number λ_0 with \text{Re} \ λ_0 > ˆ{λ}_0 such that \((\hat{\lambda}_0 + A)D = L^2(Ω)\). We follow almost word for word to Pazy’s analysis (see [43]) on Parabolic Equations. We observe that A(x, D) is strongly elliptic of order 2 with smooth coefficients x ↦⇒ a_α(x) on Ω. If we integrate by parts, we see that for every λ ∈ C, \(⟨(\lambda + A(x, D))u, v⟩_0\) can be extended to a continuous sesquilinear form \(⟨u, v⟩ \mapsto B(u, v)\) on \(H^1_0(Ω) \times H^1_0(Ω)\). If \text{Re} \ λ ≥ ˆ{λ}_0, then it follows from Garding’s inequality that this form is coercive. We can then apply the Lax-Milgram lemma to derive the existence of a unique solution \(u(\cdot) ∈ H^1_0(Ω)\) (it can actually be proven that u(·) ∈ H^2(Ω)) of the boundary value problem

\((\lambda + A(x, D))u = f,\)

for every f(·) ∈ L^2(Ω) and \text{Re} \ λ ≥ ˆ{λ}_0. Hence, given any f(·) ∈ L^2(Ω) there is a u ∈ H^1_0(Ω) such that B(u, v) = ⟨f, v⟩_0 for all v(·) ∈ H^1_0(Ω). Since D
is dense in $L^2(\Omega)$, there is a sequence $u_n \in D$ such that $u_n \to u$ in $H^1_0(\Omega)$ sense. Hence $B(u_n, v) \to \langle f, v \rangle$ as $n \to \infty$ for any $v(\cdot) \in H^1_0(\Omega)$, i.e.,

$$\text{Re} \lambda \geq \hat{\lambda}_0.$$

Since $H^1$ and $H^2$ are satisfied, we observe that

$$\|S(t)x - S_n(t)x\| \to 0$$

as $n \to \infty$ for each $x \in L^2(\Omega)$ and uniformly in compact intervals where $S(t)$ is the $C_0$-semigroup generated by $-A$ and $S_n(t)$ are the uniformly continuous semigroup generated by $P_n(-A)P_n$ for $n = 1, 2, \ldots$.

Since $A(x, D) = -\epsilon^2 \Delta + \sum_{|\alpha| \leq 1} a_\alpha(x)D^\alpha$, its formal adjoint $A^*(x, D)$ is defined (see [43]) by

$$A^*(x, D)u = -\epsilon^2 \Delta u + \sum_{|\alpha| \leq 1} D^\alpha(a_\alpha(x)u),$$

and it is also strongly elliptic of order 2. Since the infinitesimal generator of our semigroup $-A$ is defined as $Ax = A(x, D)x$ for each $x \in H^1_0(\Omega) \cap H^2(\Omega)$, its adjoint can be proven to be $A^*x = A^*(x, D)x$ for each $x \in H^1_0(\Omega) \cap H^2(\Omega)$ (for a proof, see Pazy’s book [43]). Therefore, exactly the same analysis that was carried out before can be applied to this case to imply that

$$\|S^*(t)x - S^*_n(t)x\| \to 0$$

as $n \to \infty$, for each $x \in L^2(\Omega)$ and uniformly in compact intervals where $S^*(t)$ is the $C_0$-semigroup generated by $-A^*$ and $S^*_n(t)$ are the uniformly continuous semigroup generated by $P_n(-A^*)P_n$ for $n = 1, 2, \ldots$.

A similar approach, for a much wider class of parabolic systems and for a general abstract approximation framework, was first developed by Banks and Kunisch in [4]. This approach satisfies the hypotheses, in most cases, of the finite element approach. Finally we can prove convergence of the approximation scheme.

**Theorem 10 (Second Approximation Theorem).** Let $X$ be a complex separable Hilbert space and let $Y$ be a complex finite dimensional Hilbert space. Let $S(t)$ be the $C_0$-semigroup over $\mathcal{H} = L^2(\Omega)$ generated by the strongly elliptic operator $-A$ previously described and let $S_n(t)$ be the sequence generated by $-A_n = P_n(-A)P_n$. Suppose also that
(i) $0 \leq \Sigma_0 \in \mathcal{S}_p(\mathcal{H})$.

(ii) $B(\cdot) \in L^2([0, \tau]; \mathcal{S}_2(\mathcal{X}, \mathcal{H}))$.

(iii) $C(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{L}(\mathcal{H}, \mathcal{Y}))$.

Then, $\Sigma(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{S}_p(\mathcal{H}))$, the unique solution of

$$
\Sigma(t) = S^*(t)\Sigma_0 S(t) + \int_0^t S^*(t-s)(BB^* - \Sigma(s)(C^*C)(s)\Sigma(s))S(t-s) \, ds,
$$

and the sequence of solutions $\Sigma_n(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{S}_p(\mathcal{H}))$ of

$$
\Sigma_n(t) = S^*_n(t)(P_n\Sigma_0 P_n)S_n(t) + \int_0^t S^*_n(t-s)\left((P_nBB^*P_n) - \Sigma_n(P_nC^*CP_n)\Sigma_n\right)(s)S_n(t-s) \, ds,
$$

satisfy

$$
\sup_{t \in [0, \tau]} \|\Sigma(t) - \Sigma_n(t)\|_p \to 0 \quad (4.3)
$$
as $n \to \infty$.

Proof. Hypothesis (ii) and (iii) imply that $BB^*(\cdot) \in L^1([0, \tau]; \mathcal{S}_p(\mathcal{H}))$ and that $C^*C(\cdot) \in \mathcal{C}([0, \tau]; \mathcal{S}_1(\mathcal{H}))$ by Lemma 4 and Lemma 5. The existence and uniqueness of the mappings $t \mapsto \Sigma(t)$ and $t \mapsto \Sigma_n(t)$ are given by Theorem 6.

By Proposition 16, we observe that $\|\Sigma_0 - P_n\Sigma_0 P_n\|_p \to 0$, $\|(BB^* - P_nBB^*P_n)\|_{L^1([0, \tau]; \mathcal{S}_p)} \to 0$ and $\|(C^*C - P_nC^*CP_n)\|_{\mathcal{C}([0, \tau]; \mathcal{S}_1)} \to 0$ as $n \to \infty$. These are the hypotheses required to apply the First Approximation Theorem (Theorem 5) which implies the claimed result.

Finally, we have to address conditions on the sequence $\{Q_n(\cdot)\}_{n=1}^\infty$ and $Q(\cdot)$ under which we can observe that

$$
\int_0^\tau \text{Tr} (Q_n\Sigma_n)(t) \, dt \to \int_0^\tau \text{Tr} (Q\Sigma)(t) \, dt.
$$

Corollary 4.2.1. Assume the hypotheses of Theorem 10 with $p = 1$ and suppose that the sequence $\{Q_n(\cdot)\}_{n=1}^\infty$ and $Q(\cdot)$ are in $L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))$. Let $\{\Sigma_n(\cdot)\}_{n=1}^\infty$ and $\Sigma(\cdot)$ be the ones in the aforementioned Theorem. Therefore,

$$
\int_0^\tau \text{Tr} (Q_n\Sigma_n)(t) \, dt \to \int_0^\tau \text{Tr} (Q\Sigma)(t) \, dt,
$$

if $\|(Q - Q_n)(\cdot)\|_{L^\infty([0, \tau]; \mathcal{L}(\mathcal{H}))} \to 0$ as $n \to \infty$. 

Proof. We observe the inequality
\[
\| (Q \Sigma - Q_n \Sigma_n)(t) \|_1 \leq \| Q(t) \| \| (\Sigma - \Sigma_n)(t) \|_1 + \| \Sigma_n(t) \|_1 \| (Q - Q_n)(t) \|. 
\]
The initial hypotheses imply that \( \sup_{t \in [0, \tau]} \| (\Sigma - \Sigma_n)(t) \|_1 \to 0 \) and \( \| (Q - Q_n)(t) \|_{L^\infty([0, \tau]; L(\mathcal{H}))} \) as \( n \to \infty \). Also \( \sup_{t \in [0, \tau]} \| \Sigma_n(t) \|_1 \leq c \) for some \( c > 0 \) uniformly in \( n \in \mathbb{N} \), therefore
\[
\left| \int_0^\tau \text{Tr} (Q \Sigma) (t) \, dt - \int_0^\tau \text{Tr} (Q_n \Sigma_n) (t) \, dt \right| \leq \\
\tau \left( \| Q(\cdot) \|_{L^\infty([0, \tau]; L(\mathcal{H}))} \sup_{t \in [0, \tau]} \| (\Sigma - \Sigma_n)(t) \|_1 + c \| (Q - Q_n)(\cdot) \|_{L^\infty([0, \tau]; L(\mathcal{H}))} \right),
\]
and the claimed result follows. \( \square \)
Chapter 5

Numerical Implementation

5.1 2D Problems

We will consider
\[
\frac{\partial T}{\partial t} = \epsilon^2 \Delta T + \left( a_x \frac{\partial T}{\partial x} + a_y \frac{\partial T}{\partial y} \right) + b(x, y)\eta(t),
\]
on the open unit square \( \Omega \) and with \( T(t,x,y) = 0 \) if \((x,y) \in \partial \Omega \). We will assume that \( \eta \) is a real Wiener process. For a sensor at position \((x_0,y_0)\), the output is
\[
h(t) = \int_{\Omega} K(x-x_0, y-y_0)T(t, x, y) \, dx \, dy + \nu(t),
\]
where \( \nu \) is a real Wiener process that is uncorrelated with \( \eta \).

The functional to minimize is, in this case,
\[
J(x_0, y_0) = \int_0^1 \text{Tr} \left( \Sigma_{(x_0,y_0)}(t) \right) \, dt,
\]
where \( \Sigma_{(x_0,y_0)} \) refers to the solution of the Riccati equation where the output map is determined by the sensor in position \((x_0,y_0)\).

The orthonormal set of eigenfunctions of the Laplacian \( \Delta \) in the unit square are given by \( \psi_{m,n}(x,y) = 2 \sin(\pi mx) \sin(\pi ny) \). We order them using only one parameter first according to its associated eigenvalue \( \lambda_{m,n} = -\pi^2(m^2 + n^2) \). In the case of two functions sharing the same eigenvalue (e.g.,
ψ_{1,3} and ψ_{3,1}), we put the one with the highest \( m \) first. Therefore, we define the sequence \( \{φ_n\}_{n=1}^{∞} \) as

\[
φ_1(x, y) = ψ_{1,1}(x, y),
φ_2(x, y) = ψ_{2,1}(x, y),
φ_3(x, y) = ψ_{1,2}(x, y),
φ_4(x, y) = ψ_{2,2}(x, y),
\vdots
\]

Let \( P_n \) be the orthogonal projector onto \( \text{span}\{φ_1, φ_2, \cdots, φ_n\} \). Since \( ⟨φ_i, φ_j⟩ = δ_{ij} \), then the matrix representation \( [A_n] ∈ \mathbb{R}^{n×n} \) of the approximation \( A_n = P_nAP_n \) is given by

\[
[A_n]_{ij} = ε^2 ⟨φ_i, Δφ_j⟩_{L^2(Ω)} + a_x⟨φ_i, \frac{∂}{∂x}φ_j⟩_{L^2(Ω)} + a_y⟨φ_i, \frac{∂}{∂y}φ_j⟩_{L^2(Ω)},
\]

where \( [A_n]_{ij} \) is the \( i \) row and \( j \) column element of \( [A_n] \). These can be computed exactly. We observe that

\[
⟨φ_i, Δφ_j⟩ = −δ_{ij}π^2λ_i,
\]

where \( λ_i \) is the eigenvalue associated with \( φ_i \). Also for \( α, β ∈ \mathbb{N} \), we observe

\[
∫_0^1 \sin(παx) \cos(πβx) \, dx = \begin{cases} 0, & α = β; \\ \frac{α((-1)^α+β−1)}{β^2−α^2}, & α ≠ β. \end{cases}
\]

Using this, we can obtain exact expressions for \( ⟨φ_i, \frac{∂}{∂x}φ_j⟩ \) and \( ⟨φ_i, \frac{∂}{∂y}φ_j⟩ \).

The output map, for one sensor, is defined \( C : L^2(Ω) → \mathbb{R} \) is given by \( Cφ = ∫_Ω c(x, y)φ(x, y) \, dx \, dy \), where \( c(x, y) = K(x − x_0, y − y_0) \) and \( (x_0, y_0) \) is the position of the sensor. The approximation \( C_n \) of \( C \), is going to be computed as

\[
C_nφ = ∫_Ω c(x, y)(P_nφ)(x, y) \, dx \, dy.
\]

Therefore, its matrix representation is given by

\[
[C_n] = (∫_Ω c(x)φ_1(x) \, dx \quad ∫_Ω c(x)φ_2(x) \, dx \quad \cdots \quad ∫_Ω c(x)φ_n(x) \, dx),
\]

where \( x = (x, y) \). Since \( C^* : \mathbb{R} → L^2(Ω) \) is given by \( C^*a = ac(x, y) \), it is elementary to observe that \( C_n^*C_n = P_nC^*CP_n \).
The input map, in this case, is defined as \( B : \mathbb{R} \mapsto L^2(\Omega) \) and given by \( Ba = b(x,y)a \). Then, its adjoint, \( B^* \), satisfies \( B^* : L^2(\Omega) \mapsto \mathbb{R} \) and it is given by \( B^* \varphi = \int_{\Omega} b(x,y) \varphi(x,y) \, dx \, dy \). The matrix representation of the approximation \((BB^*)_n = P_nBB^*P_n\) is then given by

\[
[(BB^*)_n] = \begin{pmatrix}
  f(1,1) & f(2,1) & \cdots & f(1,n) \\
  f(2,1) & f(2,2) & \cdots & f(2,n) \\
  \vdots & \vdots & \ddots & \vdots \\
  f(n,1) & f(n,2) & \cdots & f(n,n)
\end{pmatrix},
\]

where

\[
f(i, j) = \left( \int_{\Omega} b(x,y) \phi_i(x,y) \, dx \, dy \right) \left( \int_{\Omega} b(x,y) \phi_j(x,y) \, dx \, dy \right).
\]

### 5.1.1 The Approximate Riccati Equation

In order to approximate the (weak) solution to

\[
\dot{\Sigma} = A\Sigma + \Sigma A^* + BB^* - \Sigma C^*C \Sigma,
\]

we solve the differential matrix Riccati equation for

\[
\frac{d}{dt}[\Sigma_n] = [A_n][\Sigma_n] + [\Sigma_n][A_n]^* + [(BB^*)_n] - [\Sigma_n][C_n]^*[C_n][\Sigma_n].
\]

This equation is approximated using the implicit Euler method. The method is chosen because we observe convergence using a relatively large time step. Then,

\[
\frac{[\Sigma_{n+1}^k] - [\Sigma_n^k]}{h} = [A_n][\Sigma_{n+1}^k] + [\Sigma_{n+1}^k][A_n]^* + [(BB^*)_n] - [\Sigma_{n+1}^k][C_n]^*[C_n][\Sigma_{n+1}^k],
\]

where \( h > 0 \) is the time step and \( \Sigma_n^k \simeq \Sigma_n(kh) \). Re-arranging terms, we observe that

\[
\left( h[A_n] - \frac{1}{2} \right) [\Sigma_{n+1}^k] + [\Sigma_{n+1}^k] \left( h[A_n] - \frac{1}{2} \right)^* - [\Sigma_{n+1}^k](\sqrt{h}[C_n])^*(\sqrt{h}[C_n])[\Sigma_{n+1}^k] + \\
+ \left( h[(BB^*)_n] + [\Sigma_n^k] \right) = 0.
\]
Therefore, given $[\Sigma^k_n]$, we can compute $[\Sigma^{k+1}_n]$ by solving this algebraic Riccati equation. For this matter we iterate the MATLAB function `care` using $\Sigma^0_n = 0$.

The approximation of the functional is done as $\int_0^1 \text{Tr}(\Sigma(t))dt \simeq h(\text{Tr}(\Sigma^1) + \text{Tr}(\Sigma^2) + \cdots + \text{Tr}(\Sigma^{10}))$.

### 5.1.2 2D Stationary Sensor Problem

For this problem we will consider $\epsilon^2 = 0.01$.

$$K(x, y) = e^{-5(x^2+y^2)}.$$ 

The functional to minimize is in this case

$$J(x_0, y_0) = \int_0^1 \text{Tr}(\Sigma_{(x_0, y_0)}(t)) dt,$$

where $\Sigma_{(x_0, y_0)}$ refers to the solution of the Riccati equation where the output map is determined by the sensor in position $(x_0, y_0)$.

#### Uniform Noise and Zero Convective Term

We use $b(x, y) = 10$ and $a_x = a_y = 0$. The number of eigenfunctions we use to generate the plots is 20. The integrals involved in the matrix approximations $[(BB^*)_n]$ and $[C_n]$ are computed with relative tolerances of $10^{-6}$ and $10^{-5}$ respectively. The results can be observed in Figure 5.1.

The time step for the implicit Euler’s method is $h = 0.1$. The minimizer in this case is found exactly at the point $(x_0, y_0) = (0.5, 0.5)$.

Using only 4 eigenfunctions we observe a variation of less than 10% in the approximation of the functional $J(\Sigma) = \int_0^1 \text{Tr}(\Sigma(t)) dt$ with respect to the case with 20 eigenfunctions. In this case we also observe that the minimizer is in the same place as in the 20 eigenfunctions case. The results between 12 and 20 eigenfunctions have no significant difference. The difference of the results in the cases between 20 and 33 eigenfunctions is even less significant.

#### Non-uniform Noise and Zero Convective Term

We use $b(x, y) = 10 + 20e^{-10((x-0.2)^2+(y-0.2)^2)}$ and $a_x = a_y = 0$. The number of modes used to create the plots is 20. The integrals involved in the matrix
(a) $J(x,y) = \int_0^1 \text{Tr} \left( \Sigma_{x,y}(t) \right) \, dt$

(b) $J(x,y) = \int_0^1 \text{Tr} \left( \Sigma_{x,y}(t) \right) \, dt$ Top view

Figure 5.1: $J(x,y)$ for $b = 10$ and $a_x = a_y = 0$. 
approximates \([(BB^*)_n]\) and \([C_n]\) are computed with relative tolerances of \(10^{-6}\) and \(10^{-5}\) respectively. The results can be observed in Figure 5.2.

The time step for the implicit Euler method is \(h = 0.1\). The minimizer in this case is found at the point \((x_0, y_0) \simeq (0.38, 0.38)\).

As in the previous case, using only 4 eigenfunctions of the Laplacian we observe a variation of less than 10% in the approximation of the functional \(J(\Sigma) = \int_0^1 \text{Tr}(\Sigma(t))\, dt\) with respect to the case involving 20 eigenfunctions. In this case, the minimizer is at the point \((x_0, y_0) \simeq (0.42, 0.42)\). As the number of eigenfunctions used increases, the minimizer moves from this point to the point \((0.38, 0.38)\).

We observe that the minimizer has been displaced from the center, to a point in between \((0.5, 0.5)\) and the “nosiest” place in the square at \((0.2, 0.2)\).

Also as in the previous case, the results using between 12 and 20 eigenfunctions have no significant difference. The difference of the results in the cases between 20 and 33 eigenfunctions is even less significant.

**Uniform Noise and Non-zero Convective Term**

We use \(b(x, y, z) = 10\) and \(a_x = a_y = 5\). The number of modes used to create the plots is 20. The integrals involved in the matrix approximations \([(BB^*)_n]\) and \([C_n]\) are computed with relative tolerances of \(10^{-6}\) and \(10^{-5}\) respectively. The results can be observed in Figure 5.3.

The time step for the implicit Euler’s method is \(h = 0.1\). The minimizer in this case is found at the point \((x_0, y_0) \simeq (0.57, 0.57)\).

There is 20% difference between the results using 4 and 20 eigenfunctions of the Laplacian. The results using between 16 and 20 eigenfunctions have no significant difference. The difference of the results in the cases between 20 and 33 eigenfunctions is even less significant. In order to observe convergence in this case with non-zero convective term, we require more eigenfunctions than in the cases with a zero convective term. This feature will also be present in the 3D stationary sensor case.

We observe that the minimizer has been displaced from the center to a point upstream. In this case, since \(a_x = a_y = 5\), the flow has the direction vector \((-5, -5)\). Hence, we observe that the global minimizer is moved from \((0.5, 0.5)\) against the direction of the flow.
(a) $J(x, y) = \int_0^1 \text{Tr} (\Sigma_{x,y}(t)) \, dt$

(b) $J(x, y) = \int_0^1 \text{Tr} (\Sigma_{x,y}(t)) \, dt$

Figure 5.2: $J(x, y)$ for $b(x, y) = 10 + 40e^{-5((x-0.1)^2+(y-0.1)^2)}$ and $\alpha_x = \alpha_y = 0$
(a) \( J(x, y) = \int_0^1 \text{Tr} (\Sigma_{x,y}(t)) \, dt \)

(b) \( J(x, y) = \int_0^1 \text{Tr} (\Sigma_{x,y}(t)) \, dt \)

Top view

Figure 5.3: \( J(x, y) \) for \( b = 10 \) and \( a_x = a_y = 5 \).
5.1.3 2D Mobile Sensor Network Problem

We will consider 3 sensors located initially at the points \((0.6, 0.4), (0.5, 0.5)\) and \((0.4, 0.6)\) and their trajectories are given by the integral equations

\[
\bar{x}_i(t, u) = \begin{pmatrix} x^0_i \\ y^0_i \end{pmatrix} + \int_0^t e^{A(t-s)} b u_i(s) \, ds
\]

where

\[
A = \begin{pmatrix} -1 & 0.3 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1.5 \\ -1 \end{pmatrix}.
\]

These trajectories \(t \mapsto \bar{x}(t, u)\) are not given as outputs of a controlled differential equation of the type we require to prove the existence of minimizers (see Section 2.5.2). However, it is simply observe that these trajectories are determined by the initial value problems

\[
\frac{d}{dt} \left( \bar{x}_i(t) - \begin{pmatrix} x^0_i \\ y^0_i \end{pmatrix} \right) = A \left( \bar{x}_i(t) - \begin{pmatrix} x^0_i \\ y^0_i \end{pmatrix} \right) + b u_i(t)
\]

\[
\left( \bar{x}_i(0) - \begin{pmatrix} x^0_i \\ y^0_i \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

If the sensor is at position \((x(t), y(t))\), the output is \(t \mapsto (h_1(t), h_2(t), h_3(t))\) where each \(h_i\) is

\[
h_i(t) = \int_{\Omega} K(x - x(t), y - y(t)) T(t, x, y) \, dx \, dy + \nu_i(t).
\]

For this problem we will consider

\[
\epsilon^2 = 0.01
\]

\[
K(x, y) = e^{-20(x^2+y^2)}.
\]

The functional to minimize is in this case

\[
J(u_1, u_2, u_3) = \int_0^{10} \text{Tr} \left( \Sigma_{(u_1, u_2, u_3)}(t) \right) \, dt,
\]

where \(\Sigma_{(u_1, u_2, u_3)}\) refers to the solution of the Riccati equation where the output map is determined by the moving sensors with controls \((u_1, u_2, u_3) \in L^2([0, 1]) \times L^2([0, 1]) \times L^2([0, 1]).

In order to approximate a local minimizer we will use a gradient descent method for this problem:
1. Start with the control with some choice $u^0(t) = (u_1^0(t), u_2^0(t), u_3^0(t))$.

2. Update the control as follows

$$u^{n+1}(t) = u^n(t) - \alpha_n J'(u^n)(t),$$

where $J'(u)$ is the gradient of $J$ at $u$ and $\alpha_n$ is chosen if possible as

$$\alpha_n = \arg \min \alpha J(u^n - \alpha J'(u^n)),$$

and stop if $J(u^{n+1})$ is not decreased by at least 2% with respect to $J(u^n)$.

The termination condition for the algorithm does not involve any decrease condition on the gradient $J'$. This is because, in this case, there are no conditions that ensure that $J'(u^n) \to 0$ as $n \to \infty$. The computation of $\alpha_n = \arg \min \alpha J(u^n - \alpha J'(u^n))$, is done using “brute force”. First $J(u^n - \alpha J'(u^n))$ is evaluated for $\alpha = 0, 0.1, \ldots, 0.9, 1$. If the minimum is reached at $\alpha = 0$, then the search is done over $\alpha = 0, 0.01, \ldots, 0.09, 0.1$ and so on. On the other hand, if the minimum is achieved at $\alpha = 1$, then the search is done over $\alpha = 1, 1.1, \ldots, 1.9, 2$.

As we used before, the approximation to the solution of the Riccati equation is given by

$$\frac{d}{dt} \Sigma_t(n) = [A_n][\Sigma_t(n)] + [\Sigma_t(n)][A_n]^* + [(BB^*)_n] - [\Sigma_t(n)][C_n(t)]^*[C_n(t)][\Sigma_t(n)],$$

with $\Sigma(0) = 0$ and where $[A_n], (BB^*)_n$ and $t \mapsto [C_n(t)]$ are the matrix representations of the approximations to the operators $A$ and $BB^*$ and the operator valued function $t \mapsto C(t)$, respectively. The approximation to the derivative $D_{C,C}\Sigma$ is computed using the sensitivity equation

$$\frac{d}{dt} \Lambda_t(n) = [A_n][\Lambda_t(n)] + [\Lambda_t(n)][A_n]^* - [\Lambda_t(n)][C_n(t)]^*[C_n(t)][\Sigma_t(n)] - [\Sigma_t(n)][\Sigma_t(n)]$$

$$- [\Sigma_t(n)][C_n(t)]^*[C_n(t)][\Lambda_t(n)],$$

where $\Lambda_t(n)(0) = 0$ and $\Sigma_t(n)(t)$ is the solution to the approximated Riccati equation. Both equations are approximated using the implicit Euler method with time step $h = 0.1$. 
The Fréchet derivative of \( \bar{x}(t,u) \) with respect to \( u \), is a map \( D_u \bar{x} \in \mathcal{L}(L^2([0,\tau]),\mathcal{C}([0,\tau];\Omega)) \) given by

\[
(D_u \bar{x}h)(t) = \int_0^t e^{A(t-s)}bh(s) \, ds,
\]

for each \( h \in L^2([0,\tau]) \). The map \( C^*C : \mathcal{C}([0,\tau];\Omega) \mapsto \mathcal{C}([0,\tau];\mathcal{F}_1) \), is given by

\[
(C^*C \varphi)(x) = K(x - \bar{x}(t)) \int_\Omega K(y - \bar{x}(t))\varphi(y) \, dy.
\]

Since, we use \( K(x) = ae^{b||x||^2_{\Omega}} \) for some \( a > 0 \) and \( b > 0 \), then \( D_u \bar{x}(t,u) \) is well defined as a Fréchet derivative. Therefore, we observe that \( D_n C^*C \in \mathcal{L}(\mathcal{C}([0,\tau];\mathcal{F}_1)) \) is well defined as the Fréchet derivative of \( C^*C \) with respect to \( \bar{x} \). Hence \( D_n C^*C(\bar{x}(t,u)) \in \mathcal{L}(L^2([0,\tau]);\mathcal{C}([0,\tau];\mathcal{F}_1)) \) is well defined. If we define \( H(u) = D_n C^*C(\bar{x}(t,u)) \), for \( h(\cdot) \in L^2([0,\tau]) \), the matrix form elements \([H_n(u)h]_{ij}\) of the approximation to \( H(u)h \) are given by

\[
\langle \phi_i, (H(u)h)\phi_j \rangle(t) = e^{-\Lambda t}bh(s) \, ds.
\]

For the case of one sensor, \( J(u) = \int_0^\tau \text{Tr} \left( \Sigma(t) \right) \, dt \) has a Fréchet derivative \( J'(u) \in \mathcal{L}(L^2([0,\tau]);\mathcal{F}_1) \) and it is given by

\[
J'(u)h = \int_0^\tau \text{Tr} \left( \Lambda(t) \circ H(u)(t)h \right) \, dt.
\]

Hence, its approximation \((J'(u))_n\), is calculated as

\[
(J'(u))_n h = \int_0^\tau \text{Tr} \left( \Lambda_n(t) \circ H_n(u)(t)h \right) \, dt.
\]

Finally, after a tedious algebraic manipulation, we obtain

\[
(J'(u))_n h = \int_0^\tau \text{Tr} \left( R_n(t) \right) h(t) \, dt,
\]

for some \( R_n(t) \). We identify \((J'(u))_n\) with \( \text{Tr} \left( R_n(t) \right) \). The generalization for the case of three sensors is natural.
Uniform Noise and Zero Convective Term

We use \( b(x, y) = 10 \) and \( a_x = a_y = 0 \). The number of modes used is 16 and after 15 iterations the terminal condition is met. The initial and final controls are shown on Figure 5.4. The initial and final trajectories are shown in Figure 5.5, where the initial position of the sensors is marked by a small circumference.

Based on previous numerical results, in the case of one stationary sensor, the global minimizer is on the point \((0.5, 0.5)\). We should notice that the sensor with initial position in the center of the square, remains in this point for all \( t \) as we can observe on Figure 5.5. The other two trajectories, as we may expect, try to reach the center of the square.

Non-uniform Noise and Zero Convective Term

We use \( b(x, y) = 10 + 10e^{-5((x-0.1)^2+(y-0.9)^2)} \) and \( a_x = a_y = 0 \). The number of modes used is 16 and it takes 12 iterations until the termination criteria is met. The initial and final controls are shown on Figure 5.6. The initial and final trajectories of the sensors are shown in Figure 5.7, where the initial position of the sensors is marked with a small circumference.

Based on previous numerical results, in the case of one stationary sensor, the global minimizer is on a the point in between \((0.5, 0.5)\) and \((0.1, 0.9)\) (the “noisiest” place in the square). We observe that trajectories tend to a region in between these two points (see Figure 5.7).
(a) Initial constant controls \((u_1^0(t), u_2^0(t), u_3^0(t))\)

(b) Final controls after 15 iterations of steepest descent

Figure 5.4: Initial and final controls for \(b = 10\) and \(a_x = a_y = 0\).
Figure 5.5: Initial and final trajectories for $b = 10$ and $a_x = a_y = 0$. 
(a) Initial constant controls \((u_1^0(t), u_2^0(t), u_3^0(t))\)

(b) Final controls after 12 iterations of steepest descent

Figure 5.6: Initial and final controls for \(b(x, y) = 10 + 10e^{-5((x-0.1)^2+(y-0.9)^2)}\) and \(a_x = a_y = 0\).
Figure 5.7: Initial and final trajectories for $b(x, y) = 10 + 10e^{-5(x-0.1)^2+(y-0.9)^2}$ and $a_x = a_y = 0$. 

(a) Initial Sensor Trajectories

(b) Final Sensor Trajectories after 12 iterations of steepest descent
5.2 3D Stationary Sensor Problem

We will consider

$$\frac{\partial T}{\partial t} = \epsilon^2 \Delta T + \left( a_x \frac{\partial T}{\partial x} + a_y \frac{\partial T}{\partial y} + a_z \frac{\partial T}{\partial z} \right) + b(x, y, z) \eta(t),$$

on the open unit cube $\Omega$ and with $T(t, x, y, z) = 0$ if $(x, y, z) \in \partial \Omega$. We will suppose that $b$ is some continuous function and that $\eta$ is a real Wiener process. If the sensor is at position $(x_0, y_0, z_0)$, the output is defined by

$$h(t) = \int_{\Omega} K(x - x_0, y - y_0, z - z_0) T(t, x, y, z) \, dx \, dy \, dz + \nu(t),$$

where $\nu$ is another real Wiener process that is uncorrelated with respect to $\eta$. For this problem we will consider

$$\epsilon^2 = 0.01$$

$$K(x, y, z) = 10e^{-(x^2+y^2+z^2)}$$

The functional to minimize is in this case

$$J(x_0, y_0, z_0) = \int_0^1 \text{Tr} \left( \Sigma_{(x_0, y_0, z_0)}(t) \right) \, dt,$$

where $\Sigma_{(x_0, y_0, z_0)}$ refers to the solution of the Riccati equation where there output map is determined by the sensor in position $(x_0, y_0, z_0)$.

The approximation scheme is almost the same as the one described in the 2D example. The only difference is that in this case, the orthonormal set of eigenfunctions of the Laplacian $\Delta$ in the unit cube is given by

$$\psi_{l,m,n}(x, y, z) = 2^{3/2} \sin(\pi lx) \sin(\pi my) \sin(\pi nz).$$

We order them using only one parameter first according to its associated eigenvalue $\lambda_{l,m,n} = -\pi^2(l^2 + m^2 + n^2)$. In the case of two functions sharing the same eigenvalue (e.g. $\psi_{l,3,1}$ and $\psi_{3,1,1}$), we put first the one with the highest $l$. In the case, they share the same $l$, we order them according to the greater $m$ and so on. Therefore, we define the
sequence \( \{ \phi_n \} \) as

\[
\begin{align*}
\phi_1(x, y) &= \psi_{1,1,1}(x, y), \\
\phi_2(x, y) &= \psi_{2,1,1}(x, y), \\
\phi_3(x, y) &= \psi_{1,2,1}(x, y), \\
\phi_4(x, y) &= \psi_{1,1,2}(x, y), \\
\phi_5(x, y) &= \psi_{2,2,1}(x, y), \\
&\vdots
\end{align*}
\]

Using this sequence \( \{ \phi_n \} \), the approximation scheme is exactly the same as the one described before in the 2D case.

The integration of the Riccati equation is done by the implicit Euler’s method with time step \( h = 0.1 \) and the integral of the trace is done using trapezoidal integration.

**Uniform Noise and Zero Convective Term**

We use \( b(x, y, z) = 100 \) and \( a_x = a_y = a_z = 0. \) The number of eigenfunctions we use to generate the plots is 11. The integrals involved in the matrix approximates \((BB^*)_n\) and \([C_n]\) are computed with relative tolerances of \(10^{-6}\) and \(10^{-3}\) respectively. The results can be observed in Figures 5.8 and 5.9.

The time step for the implicit Euler’s method is \( h = 0.1 \). The minimizer in this case is found exactly at the point \((x_0, y_0, z_0) = (0.5, 0.5, 0.5)\) and is given by \( J(0.5, 0.5, 0.5) \approx 12 \). The maximum value of the functional is attained in all vertices and it is approximately 20. The results obtained with higher number of eigenfunctions (up to 33) have no significative difference with the ones obtained using 11 eigenfunctions.

**Non-uniform Noise and Zero Convective Term**

We use \( b(x, y, z) = 1 + 20e^{-5((x-0.2)^2+(y-0.2)^2+(z-0.2)^2)} \) and \( a_x = a_y = a_z = 0. \) The number of modes used is 11. The integrals involved in the matrix approximates \((BB^*)_n\) and \([C_n]\) are computed with relative tolerances of \(10^{-7}\) and \(10^{-3}\) respectively. The results can be observed in Figures 5.10 and 5.11.

The time step for the implicit Euler’s method is \( h = 0.1 \). The minimizer in this case is found approximately in the point \((x_0, y_0, z_0) = (0.4, 0.4, 0.4)\) and
Figure 5.8: $J(x, y, z)$ for $b = 100$ and $a_x = a_y = a_z = 0$. 
Figure 5.9: $J(x, y, z)$ for $b = 100$ and $a_x = a_y = a_z = 0$. Top view.
has value $J(0.4, 0.4, 0.4) \simeq 1$. The highest value attained by the functional is approximately 2. Note that the minimizer has been displaced from the center towards the location of the "noisiest" place in the cube, given by $(0.2, 0.2, 0.2)$.

The results obtained with higher number of eigenfunctions (up to 33) have no significative difference with the ones obtained using 11 eigenfunctions. Note that in this case we have increased the accuracy of the integrals of the approximation $[(BB^*)_n]$ from $10^{-6}$ to $10^{-7}$, due to the rapid decay of the function $b$.

Figure 5.10: $J(x, y, z)$ for $b(x, y, z) = 1 + 20e^{-5((x-0.2)^2+(y-0.2)^2+(z-0.2)^2)}$ and $a_x = a_y = a_z = 0$. 
Figure 5.11: $J(x, y, z)$ for $b(x, y, z) = 1 + 20e^{-5((x-0.2)^2 + (y-0.2)^2 + (z-0.2)^2)}$ and $a_x = a_y = a_z = 0$. Top view.
Uniform Noise and Non-zero Convective Term

We use $b(x, y, z) = 100$ and $a_x = a_y = a_z = 20$. The number of modes used is 33. The integrals involved in the matrix approximates $[(BB^*)_n]$ and $[C_n]$ are computed with relative tolerances of $10^{-6}$ and $10^{-4}$ respectively. The results can be observed in Figures 5.12 and 5.13. Several isosurfaces for this problem are shown on Figure 5.14.

The time step for the implicit Euler’s method is $h = 0.1$. The minimizer in this case is found approximately in the point $(x_0, y_0, z_0) = (0.65, 0.65, 0.65)$ and has value $J(0.65, 0.65, 0.65) \approx 42$. The highest value attained by the functional is approximately 177. Note that the minimizer has been displaced from the center to a location upstream.

In this case with a non-zero convective term, we require more eigenfunctions than in the previous ones (with a zero convective term) to observe convergence. This same phenomenon was observed in the 2D case. The results using 28 up to 33 eigenfunctions show no significative difference.

Computing Time

We computed the value of the functional $J(x, y, z)$ on the the points $(x, z, y) = (id, jd, kd)$ with $d = 0.05$ and $i, j, k = 1, 2, \ldots, 19$. The time taken on a 8 core machine, with a fully parallelized code, was approximately 6 hours for the case with zero convective term and 24 hours for the nonzero convective term.
Figure 5.12: $J(x, y, z)$ for $b = 100$ and $a_x = a_y = a_z = 20$. 
Figure 5.13: $J(x, y, z)$ for $b = 100$ and $a_x = a_y = a_z = 20$. Top view.
Figure 5.14: Isosurfaces of $J(x, y, z)$ for $b = 100$ and $a_x = a_y = a_z = 20$. 

(a) Isosurface for $J(x, y, z) \approx 43$

(b) Isosurface for $J(x, y, z) \approx 50$

(c) Isosurface for $J(x, y, z) \approx 61$

(d) Isosurface for $J(x, y, z) \approx 74$

(e) Isosurface for $J(x, y, z) \approx 87$

(f) Isosurface for $J(x, y, z) \approx 118$
5.2.1 Effects of “rough” integration and almost point-evaluation sensors

We will consider
\[
\frac{\partial T}{\partial t} = \epsilon^2 \Delta T + \left( a_x \frac{\partial T}{\partial x} + a_y \frac{\partial T}{\partial y} + a_z \frac{\partial T}{\partial z} \right) + b(x, y, z) \eta(t),
\]
if the sensor is at position \((x_0, y_0, z_0)\), the output is
\[
h(t) = \int_{\Omega} K(x - x_0, y - y_0, z - z_0) T(t, x, y, z) \, dx \, dy \, dz + \nu(t).
\]

For this problem we will consider
\[
\epsilon^2 = 0.01,
\]
\[
a_x = a_y = a_z = 20,
\]
\[
b(x, y, z) = 25,
\]
\[
K(x, y, z) = 30e^{-60(x^2+y^2+z^2)}.
\]

The functional to minimize is in this case
\[
J(x_0, y_0, z_0) = \int_0^{0.1} \text{Tr} \left( \Sigma_{(x_0,y_0,z_0)}(t) \right) \, dt,
\]
where \(\Sigma_{(x_0,y_0,z_0)}\) refers to the solution of the Riccati equation where there output map is determined by the sensor in position \((x_0, y_0, z_0)\).

Note that the gain \(K\) of the sensor, has a very rapid decay, and could be consider an approximation to a point-evaluation sensor. The number of modes used is 33. The integrals involved in the matrix approximates \([BB^*]_n\) and \([C_n]\) are computed with relative tolerances of \(10^{-6}\) and \(10^{-2}\) respectively.

Several iso-surfaces for this problem are shown in Figure 5.15. We observe, from the numerical results, that in this case the functional is not convex and that there are a several local minimizers. The shape of the iso-surfaces seem different from the ones obtained in the previous problem, however they have the symmetry expected for this problem. This could be caused by a combination of the rapid decay of the kernel \(K(x) = 30e^{-60(x^2+y^2+z^2)}\) and the fact that the relative set tolerance \(10^{-2}\) for the integrals involved in the approximation of \(C\) is relatively high. We should note that none of the theory described in this thesis applies to the point-evaluation case.
Figure 5.15: Isosurfaces of $J(x, y, z)$ for $b = 25$ and $a_x = a_y = a_z = 20$
Chapter 6

Conclusions

In this thesis we developed a rigorous mathematical framework for analyzing and approximating optimal sensor placement problems for distributed parameter systems and applied these results to PDE problems defined by the convection-diffusion equations. The mathematical problem is formulated as a distributed parameter optimal control problem with integral Riccati equations as constraints. In order to prove existence of the optimal sensor network and to construct a framework in which to develop rigorous numerical integration of the Riccati equations, we developed a theory based on Bochner integrable solutions of the Riccati equations. In particular, we focused on $\mathcal{I}_p$-valued continuous solutions of the Bochner integral Riccati equation. The basic problem was formulated as an optimal filtering problem in infinite dimensions. We presented new results concerning the smoothing effect achieved by multiplying a general strongly continuous mapping by operators in $\mathcal{I}_p$. These smoothing results were essential in order to prove the existence of Bochner integrable solutions of the Riccati integral equations. We also established that multiplication of continuous $\mathcal{I}_p$-valued functions improve continuity properties of strongly continuous mappings and specifically $C_0$-semigroups.

These smoothing results were also used to show that multiplication of operators in $\mathcal{I}_p$ enhanced convergence properties of approximating operators. In particular, Lemmas 2 and 3 imply that if $T_n(t)$ is a sequence of strongly continuous mappings strongly convergent to the map $T(t)$, then multiplication of operators in $\mathcal{I}_p$ produces convergence in the stronger topology defined by $\mathcal{I}_p$. In addition, the Smoothing Lemma produces dual convergence in the space $\mathcal{I}_p$. 

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We provided sufficient conditions for the maps $t \mapsto BB^*(t)$ and $t \mapsto C^*C(t)$ to be Bochner measurable and $\mathcal{S}_p$-valued. These conditions are easy to verify for the typical problem and are stated in terms of the properties of the input $t \mapsto B(t)$ and output $t \mapsto C(t)$ operators. These conditions can be relaxed and still yield stronger conclusions when either the input space or output space of $B(t)$ or $C(t)$, respectively, is finite dimensional.

An important new result is Theorem 6 which guarantees the existence of Bochner integrable $\mathcal{S}_p$-valued solutions to the Riccati integral equation. Moreover, these solutions are continuous when $S(t)$ is a $C_0$-semigroup. In order to prove Theorem 6 we combined the approximation result given in Theorem 5 with Theorem 4. When applied to uniformly continuous semigroups we obtained a direct approach to the Integral Riccati equation based on well-known properties of the resolvent operator.

In Section 3.5 we gave a general existence proof for the problem of minimizing the functional $J(\Sigma) = \int_0^T \text{Tr} \, Q(t) \Sigma(t) \, dt$. This result is applicable to a wide class of sensor network problems including both stationary and mobile sensor networks. We assumed that the dynamics of the networks are given by outputs of controlled ordinary differential equations and also that the initial position and velocity of the sensor belong to some compact set.

We developed a Galerkin type numerical scheme for approximating the solutions of the integral Riccati equation. This scheme was based on eigenfunctions of the Laplacian and we proved convergence of the approximating solutions in the $\mathcal{S}_p$-norm. This numerical method was used in all the numerical examples. Although not presented in this thesis, we also investigated other numerical methods such as finite element schemes but the numerical results were mixed. In 1D problems the finite element and Galerkin methods produced identical results, although the finite element scheme required more basis functions to achieve the same level of fidelity. In 2D and 3D problems the finite element scheme failed to converge. This could be due to not having a fine enough mesh to capture the required fidelity. However, we note that the finite element scheme fails to satisfy the assumptions required in our proof of convergence. This is an area that requires future investigation.

Section 5 contains several numerical examples for convection-diffusion problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. These examples include systems with uniformly and non-uniformly spatially distributed noise. We showed that the position of the minimizer (in the case of one stationary sensor) is highly dependent on the convection terms and on the spatial distribution of the noise. We also presented numerical results to illustrate how important it is to use high
fidelity numerical integrators when solving the Riccati equations. In particular, when low order methods are used, one sees local minimizers that are not real, the result of poor numerical integration.

We identified several future challenges. For example, although it is possible to prove existence of minimizers, it is difficult (although likely from the numerical results) to prove the uniqueness of these minimizers. Another interesting feature that was observed in the numerical examples is that the functional $J(x) = \int_0^1 \text{Tr} (\Sigma_x(t)) \, dt$ seems to be convex. There is no reason to expect this and it is not obvious how one could prove this type of result. Also, we know of no algorithms for which we can prove convergence (in this type of problem) of a sequence of controls to the optimal one when there are point-wise constraints. Convergence of the iterative algorithms seem to be possible in the unconstrained case. We should mention that when minimization is done on a Hilbert function space like $L^2([0,1];\mathbb{R})$, point-wise constraints like $|u(t)| \leq 1$ are hard (and sometimes impossible) to translate into a useful form for the minimization problem. Perhaps penalty function methods could be of use here. Finally, one of the biggest issues for future research is concerned with finding enhanced and fast computational methods for approximating the optimal controllers and the corresponding solutions to the Riccati equation.
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