INTERNAL RESONANCES IN VIBRATION ISOLATORS AND THEIR CONTROL USING PASSIVE AND HYBRID DYNAMIC VIBRATION ABSORBERS

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Keywords: Vibration Isolation, Noise Control, Internal Resonance, Dynamic Vibration Absorber, Passive Control, Hybrid Control

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Internal Resonances in Vibration Isolators and Their Control
Using Passive and Hybrid Dynamic Vibration Absorbers

Yu Du

(ABSTRACT)

Conventional isolation models deal with massless isolators, which tend to overestimate the isolator performance because they neglect the internal resonances (IRs) due to the inertia of the isolator. Previous researches on the IR problem is not adequate because they only discussed this problem in terms of vibration based on single degree-of-freedom (SDOF) models. These studies did not reveal the importance of the IRs, especially from the perspective of the noise radiation. This dissertation is novel compared to previous studies in the following ways: (a) a three-DOF (3DOF) model, which better represents practical vibration systems, is employed to investigate the importance of the IRs; (b) the IR problem is studied considering both vibration and noise radiation; and (c) passive and hybrid control approaches using dynamic vibration absorbers (DVAs) to suppress the IRs are investigated and their potential demonstrated.

The 3DOF analytical model consists of a rigid primary mass connected to a flexible foundation through three isolators. To include the IRs, the isolator is modeled as a continuous rod with longitudinal motion. The force transmissibility through each isolator and the radiated sound power of the foundation are two criteria used to show the effects and significance of the IRs on isolator performance. Passive and hybrid DVAs embedded in the isolator are investigated to suppress the IRs. In the passive approach, two DVAs are implemented and their parameters are selected so that the IRs can be effectively attenuated without significantly degrading the isolator performance at some
other frequencies that are also of interest. It is demonstrated that the passive DVA enhanced isolator performs much better than the conventional isolator in the high frequency range where the IRs occur. The isolator performance is further enhanced by inserting an active force pair between the two passive DVA masses, forming the hybrid control approach. The effectiveness and the practical potential of the hybrid system are demonstrated using a feedforward control algorithm. It is shown that this hybrid control approach not only is able to maintain the performance of the passive approach, but also significantly improve the isolator performance at low frequencies.
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Nomenclature

\(a, b, h\) Dimensions of plate
\(A, B\) Constants
\(A_d\) Displacement amplitude of the plate response
\(A_p\) Amplitude of acoustic pressure
\(c\) Speed of sound
\(c_i\) The wave speed in the isolator
\(\bar{c}_i\) The complex wave speed in the isolator
\(C\) Viscous damping coefficient
\(C_a\) DVA damping coefficient
\(Cc\) Critical damping coefficient
\(C(W_i)\) Cost function in the filtered-x LMS control algorithm
\(\text{CG}\) Center of gravity
\(d\) The spring coil diameter
\(d_k\) Output (discrete) of the primary path
\(D\) Bending stiffness of plate
\([D]\) Dynamic stiffness matrix of isolator
\(D_{ij}\) Term in dynamic stiffness matrix
\(D^d\) Diagonal term of the dynamic stiffness matrix of isolator
\(-D^o\) Off-diagonal term of the dynamic stiffness matrix of isolator
\(D_{\text{mean}}\) Mean diameter of coil spring
\(\text{DOF}\) Degree of freedom
\(\text{DVA}\) Dynamic vibration absorber
\(e_k\) Error signal (discrete) of the control system
<table>
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<tr>
<td>$E$</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>$\tilde{E}$</td>
<td>Complex modulus</td>
</tr>
<tr>
<td>$E_k$</td>
<td>Kinetic energy of the primary mass</td>
</tr>
<tr>
<td>EOM</td>
<td>Equation of motion</td>
</tr>
<tr>
<td>$f_0$</td>
<td>Undamped natural frequency</td>
</tr>
<tr>
<td>$f_t$</td>
<td>Transition frequency</td>
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<tr>
<td>$F_B$</td>
<td>Force at the bottom of the isolator</td>
</tr>
<tr>
<td>$F_c$</td>
<td>Active control force</td>
</tr>
<tr>
<td>$F_R$</td>
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<td>$F_T$</td>
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<td>FIR</td>
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<td>$G$</td>
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<td>HDVA</td>
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<tr>
<td>$I$</td>
<td>Sound intensity</td>
</tr>
<tr>
<td>IM</td>
<td>Intermediate mass</td>
</tr>
<tr>
<td>IR</td>
<td>Internal resonance</td>
</tr>
<tr>
<td>$J_{xx}$</td>
<td>Mass moment of inertia about $x$-axis</td>
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<tr>
<td>$J_{zz}$</td>
<td>Mass moment of inertia about $z$-axis</td>
</tr>
<tr>
<td>$k$</td>
<td>Acoustic wavenumber</td>
</tr>
<tr>
<td>$k_a$</td>
<td>DVA stiffness</td>
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<tr>
<td>$k_b$</td>
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<tr>
<td>$k_s$</td>
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\( \tilde{k}_x \) Complex spring stiffness
\( k_x \) Bending wavenumber in x-axis
\( k_z \) Bending wavenumber in z-axis
\( l \) Length of plate strip
\( L \) Isolator length
\( L_s \) Effective spring length of the coil spring
\( L_A \) A-weighting function
\( L_{PA} \) A-weighted sound power
\( L_{TOTAL} \) Total acoustic power
\( \text{LMS} \) Least mean square
\( m \) Primary mass of vibration system
\( m_a \) DVA mass
\( m_i \) Intermediate mass
\( m_s \) Mass per unit area
\( [M] \) Mass matrix
\( p \) Acoustic pressure
\( P \) Acoustic power
\( \Delta P \) Sound power reduction
\( \text{PDVA} \) Passive DVA
\( q_{mn} \) Modal amplitude of \((m, n)\)th mode
\( r \) Frequency ratio
\( S \) Isolator cross-sectional area
\( \text{SISO} \) Single input single output
\( t \) Time
\( T \) Vibration Transmissibility (general)
\( T_d \) Displacement Transmissibility
\( T_f \) Force Transmissibility
\( T_t \) Transition temperature
\( u \)  Deformation along the isolator axis
\( u_k \)  Control signal (discrete) of the control system
\( v(k_x, k_z) \)  Velocity wavenumber transform of plate
\( V(x, z) \)  Two-dimensional velocity distribution of plate
\( w \)  Displacement response of plate
\( \dot{w} \)  Velocity response of plate
\( W(z) \)  Transfer function of finite impulse response filter in \( z \)-domain
\( x \)  Displacement of the primary mass
\( \ddot{x} \)  Acceleration of the primary mass
\( x_k \)  Disturbance signal (discrete) of the control system
\( \hat{x}_k \)  Filtered-\( x \) signal
\( y_k \)  Output (discrete) of the secondary path
\( Y_B \)  Displacement at the bottom of the isolator
\( Y_{cg} \)  Displacement amplitude at the center of gravity
\( \dot{Y}_{cg} \)  Velocity amplitude of the translation at the center of gravity
\( Y_T \)  Displacement at the top of the isolator
\( \zeta \)  Damping ratio
\( \rho \)  Material density
\( \rho_a \)  Density of air
\( \eta \)  Material loss factor
\( \eta_a \)  Loss factor of DVA spring
\( \varepsilon \)  Coordinates along the isolator axis
\( \omega \)  Radian frequency of disturbance
\( \omega_n \)  Natural radian frequency
\( \Theta_x \)  Angular displacement amplitude about x-axis
\( \dot{\Theta}_x \)  Angular velocity amplitude about x-axis
Angular displacement amplitude about z-axis
Angular velocity amplitude about z-axis
Poisson’s ratio
Modal shape of (m, n)th mode
Mass ratio of the primary mass to the isolator’s mass
Step size used in the filtered-x LMS control algorithm
Chapter 1

Introduction

The idea of suppressing unwanted vibrations and noise has attracted numerous research efforts for many decades. Vibration isolators have been demonstrated as one of the most effective ways to attenuate the transmitted vibrations between mechanical components. Most design methodologies for vibration isolators assume that the isolator is massless. This assumption is acceptable at low frequencies; however, at high frequencies this massless isolator assumption will lead to over-estimation of the isolator performance. In the high frequency range (between several hundred to several thousand Hertz, the transmitted vibration as well as the resulting noise radiation is significantly increased due to the influence of the isolator’s inertia, i.e. the mass of the isolator.
As an introduction, this chapter describes the necessary background information putting in context the research conducted in this dissertation. First, the fundamentals of vibration isolation and noise reduction are presented. Then, the traditional massless isolator model and its limitations are discussed. It is followed by an introduction for the concept of the internal resonances (IRs) associated with the distributed mass of the vibration isolators, demonstration of the significance of these IRs and summary of some previous work on the IR problem. Then a review of the fundamentals of the dynamic vibration absorber (DVA) is presented. Finally, the chapter is concluded with the objectives and organization of this dissertation.

1.1 Vibration Isolation and Noise Radiation of Vibrating Structures

Vibration is the physical phenomenon of the repetitive motion of objects with respect to a reference or nominal position, i.e. the equilibrium position. Vibration exists everywhere in our daily lives. It can transmit from the source to the receiver in the form of displacement, velocity and acceleration. Although in some cases vibration can be useful and desirable (for example, vibrating conveyors), in many other cases it is harmful and generally undesirable. For example, vibration is a common cause for structural fatigue and unwanted noise; excessive vibration amplitude can lead to damage of mechanical systems or even destruction of buildings. Large magnitude acceleration may cause discomfort or even harm to humans. These potentially detrimental effects motivate engineers to find ways to reduce the vibration levels in structures.

Three approaches can be considered to control vibration levels. They are: (a) reducing the strength of the source, (b) modifying the dynamic characteristics of the receiver to reduce the ability of the structure to respond to the input energy, and (c) isolating the sensitive parts from the vibrating structures. The last approach is called vibration isolation. Two interconnected structural systems are said to be isolated from each other
when the transmission of vibration from one to the other is largely prevented. Although complete isolation is never possible, the transmission can be minimized if the interconnecting elements are sufficiently resilient. Such resilient interconnections constitute the *vibration isolators* or “anti-vibration mounts”. As indicated in Figure 1.1, the general vibration isolation problem involves a source of vibration, a receiver that should be protected from vibrations, and an isolator between the source and the receiver. For a given source and receiver, the isolation problem provides an isolator that reduces the vibrations of the receiver to acceptable levels [1]. One of the most commonly used measures of the isolator performance is the transmissibility, $T$. The transmissibility is defined as the ratio of the amplitude of the transmitted physical quantity at the receiver to the amplitude of the input physical quantity at the source. This physical quantity could be the displacement, velocity, acceleration, or force. For instance, in terms of the notations in Figure 1.1, the force transmissibility is given as

$$T_f = \frac{|F_R|}{|F_S|}$$  \hspace{1cm} (1.1)

Obviously, a good isolator results in a low transmissibility.

In Figure 1.1, if the receiver is sufficiently flexible, the vibrating surfaces of the receiver (excited by the transmitted force) will radiate noise to the surrounding fluid,
referred as structure-borne noise. Many noise sources encountered in practice are associated with this structural vibration (e.g. surface vibration). A common example is that of a passenger vehicle in which the engine is the noise source. The engine causes vibrations that are transmitted through the isolators to the whole car body structure which leads to interior radiated noise. This vibration and noise problem in vehicles becomes more important due to the tendency in the modern automotive industry to design lighter vehicles and the demand for more comfort. A good isolator can also attenuate the structure-borne noise radiation by reducing the strength of the surface vibration. Therefore, the radiated acoustic power from the receiver can also be used as another metric for quantifying the isolator’s performance.

1.2 Traditional Isolator Model

A commonly used model for a vibration isolation system is the traditional single-degree-of-freedom (SDOF) system model shown in Figure 1.2. It deals with a rigid mass $m$, representing a primary equipment, mounted on a rigid supporting structure via an isolator. For design purposes, the isolator is considered massless and is modeled as a viscous damper with damping coefficient $C$ and a spring with stiffness $k_s$. The values of the spring stiffness and the damping coefficient are also assumed to be constant in the frequency range of interest. This model can be found in many textbooks and papers on mechanical vibrations [2-4]. Two cases can be represented by this model, which are illustrated in Figures 1.2(a) and 1.2(b), respectively. In the first case, the primary mass is driven by an external force and the foundation is fixed while in the second case the base is assumed to vibrate with certain amplitude.
Figure 1.2: Traditional vibration isolation model: (a) force input, fixed base; (b) displacement input (vibrating base).

This SDOF spring-mass system has an undamped natural frequency or system resonance as follows

\[
f_0 = \frac{1}{2\pi} \sqrt{\frac{k_s}{m}}
\]  

(1.2)

According to the definition of transmissibility described in Section 1.1, the force transmissibility, \( T_f \), for the force input system and the displacement transmissibility, \( T_d \), for the displacement input system have the same form as expressed in Equation (1.3).

\[
T_f = T_d = \sqrt{\frac{1+(2r\zeta)^2}{(1-r^2)^2 + (2r\zeta)^2}}
\]  

(1.3)

In the above equation, \( r \) is the frequency ratio defined as the driving frequency to the system undamped natural frequency, \( \zeta = C/C_c \) is the damping ratio and \( C_c \) is the critical damping coefficient defined as

\[
C_c = 2\sqrt{k_sm}
\]  

(1.4)
Figure 1.3 shows, for example, the force transmissibility calculated from the traditional model in which the isolator mass is ignored. The transmissibility is shown for various values of the damping ratio $\zeta$ and as a function of frequency ratio $r$. Note from the figure that there is only one resonant peak corresponding to the system resonance. If the frequency ratio $r$ is greater than $\sqrt{2}$, the magnitude of the transmitted vibration, i.e. the force in this case, is smaller than the magnitude of the input excitation force. This region is usually referred as the isolation region. It is observed that for low damping, e.g. $\zeta \leq 0.1$, and frequencies well above the system resonant frequency, the transmissibility decreases at a rate of 12 dB per octave [5, 6]. For $r$ less than $\sqrt{2}$, the transmissibility is greater than one, i.e. the isolator amplifies the transmitted force. Near the resonance, the amplitude of the transmissibility is determined by the value of the damping ratio – the larger the value of $\zeta$, the smaller the value of transmissibility. However, this viscous damping effect is reversed in the isolation region. That is, increasing damping in the isolator is detrimental to the isolation performance in this region. From the traditional model, design guidelines for achieving good isolation for a given disturbance are: (a) select the isolator so as to yield the lowest disturbance’s frequency $> f_o \sqrt{2}$ (i.e. $r > \sqrt{2}$), and (b) reduce the isolator damping coefficient as much as possible so that the vibration is attenuated to the largest extent in the isolation region while the response near the resonance is limited to an acceptable level during system transient conditions.
Although the traditional model offers a wealth of information about vibration isolation and basic guidelines for the isolator design, it works well only at low frequencies. At high frequencies (over several hundred Hertz) the predictions from this massless model may be misleading due to the internal mass effects of the isolator that are omitted by this model.
1.3 Internal Resonances in Vibration Isolators

In practice, all isolators have distributed mass, which introduces dynamics into the isolator. At some frequencies, these dynamics will be emphasized by the resonant behavior of the elastic motion of the isolator. These are called the IRs of isolators.

1.3.1 Introduction to the IR Problem

The IRs in isolators can be considered as the resonances of a continuous vibration system. For example, if the isolator in Figure 1.2 is modeled as a continuous rod with only longitudinal motion, there will be numerous frequencies associated with the natural modes, or resonances, of the rod. These resonant frequencies depend on the shape, material properties, dimensions, and boundary conditions of the isolators.

The IRs can also be viewed from the perspective of wave propagation and are therefore called “wave effects” [7]. It is found that the IRs occur when the wavelength of the exciting vibration in the isolator is less than approximately ten times the isolator’s length [7]. Since the wavelength is inversely proportional to the frequency, the IRs occur at high frequencies.

It can be demonstrated that the IRs degrade the effectiveness of the isolator. Figure 1.4 shows the same SDOF system as in the traditional model (Figure 1.2) but with a realistic isolator. The isolator is modeled as a continuous elastic rod with density $\rho$ and structural damping characterized by the loss factor $\eta$ which has been used by previous researchers [4-7]. The isolator is fixed to the foundation or base thus the model can only examine the IR problem in terms of vibration. In Figure 1.4(b), the foundation is flexible to allow the study of the noise radiation as a metric to evaluate the IR problem. The study of the IR effect on the noise radiation is one of the main components in this research work.
Yu Du

Chapter 1. Introduction

As an illustration of the IR problem found in practical in vibration isolators, the force transmissibility of the system shown in Figure 1.4(a) is plotted in Figure 1.5. For comparison, the transmissibility for the same system calculated from the traditional massless model (Figure 1.2) is also shown in Figure 1.5. Compared with the massless model, the transmissibility curve of the “realistic” isolator shows the same peak at the system resonance as the massless isolator. In addition, there are also IRs at high frequencies ($r > 10$) in the case of the “realistic” isolator. That is, the transmissibility of the realistic isolator does not decrease monotonically with frequency after the system resonance, as it would for a massless isolator. The transmissibility for the portion of the curves between the IR peaks (the valleys) decreases at about 6 dB per octave instead of the standard value of 12 dB for the massless isolator [4]. This indicates that the traditional isolation model with a massless isolator significantly overestimates the isolation performance at high frequencies where the IRs occur.

Figure 1.4: SDOF isolation model with realistic isolator: (a) fixed base, commonly used by previous researchers; (b) flexible base considered in this research.
It is also noted that for the case shown in Figure 1.5, the two models yield almost the same prediction at low frequencies, e.g. $r < 5$, which explains the suitability of the massless model for the low frequency range. However, given the industry trend of increasingly complex pieces of equipment and machines operating at greater speeds and higher power ratings, the massless model is no longer suitable and the IR problem becomes a more important issue. In addition, the IRs are even more important when considering radiated noise, since industry is also striving for quieter products. The problem is exacerbated by the fact that noise in the frequency range where the IRs occur, such as from several hundred to 2 or 3 kilo Hertz, is most bothersome to humans. The significance of the IRs will be investigated in detail in Chapter 3.

Theoretically, the IRs exist in any type of isolators since they result from the distributed mass effect. The next three parts in this section give a brief review of previous research efforts on the IR problem in leaf springs, metal coil springs, and isolators made of elastomers.
1.3.2 IRs in Leaf Springs

Leaf springs work by their flexural elasticity and thus their IRs are associated with the resonant behavior of flexural vibrations. Ungar [8] presented a simple SDOF model to show the IRs in leaf springs. In his model, a primary mass was mounted at the center of the spring, the two ends of the spring were clamped to a rigid foundation and the system was driven by an external harmonic force acting on the primary mass. The leaf spring was modeled as a uniform beam. The damping effects were also included by considering a complex modulus of elasticity associated with the beam. Based on this model, the force
transmissibility curve was then investigated. This force transmissibility was defined as the ratio of the resulting shear force at the connection between the spring and the support to the excitation force. It was shown that the transmissibility peaks resulting from standing wave resonances greatly increase the transmissibility value predicted by the classical lumped parameter theory.

Ungar also compared the influence of the IRs on vibration isolation effectiveness of leaf (flexural vibration) and compression (longitudinal vibration) springs. In order to make those results comparable, he chose the same system frequency and mass ratio (primary mass to the isolator mass) for both systems. Compared with the compression spring, it was seen that (a) the IRs of the leaf spring occur at higher frequencies, (b) the leaf spring exhibits fewer IRs in a given frequency band, and (c) the envelope of the IR peaks is higher for a leaf spring than for a comparable compression spring. According to the comparison results (a) and (b), Ungar concluded that vibration isolators that deform primarily in flexure may work better than isolators that deform mostly in tension or compression. The IRs in flexural springs have lower density with respect to the frequency and appear at much higher frequencies that may not be of interest in practice. Although the IR level for a flexural spring is higher than that for a comparable compression spring, it can be damped to a large extent because greater amounts of damping can be incorporated more easily in practical flexural springs than in compression springs.

1.3.3 IRs in Metal Coil Springs

Metal coil springs, also known as helical springs, are commonly associated with IRs. Coil springs are extensively used in many industries because of their attractive features including more freedom on isolation design, low system natural frequencies of isolated systems, good performance under severe conditions, and long service life. Typical situations that use coil springs include engine valve springs and automotive suspension
springs. However, due to the long wavelength and low damping of metal materials, the IRs generally occur at lower frequencies (sometimes below 100 Hertz) with higher amplitudes. As a result, many research efforts had been spent to show the importance of the IRs in coil springs and to investigate effective methods to suppress the IRs in such springs.

Tomlinson [9] pointed out that for high frequency isolation design, the wave effects in the isolator have to be considered. This is especially necessary when metal springs are used. Using the “four-pole parameter matrix” method, he analytically demonstrated IRs in coil springs with free and mass loaded boundary conditions. Some experiments were also presented. These experimental results showed that the IRs due to the longitudinal vibrations of metal coil springs are more significant than those due to torsional vibrations. However, with the increasing spring size, these two different IRs tend to be equally important. For the mass loaded case, both the analytical and the experimental results showed that the first IR appears below 200 Hz and has almost the same amplitude as the system resonance. Tomlinson also proposed and tested a method to minimize the wave effects by using a parallel-mount isolator. This method employs a metal coil spring acting in parallel with a polymeric damping material. The polymeric material, which has a high loss factor, helps dissipate the IRs while the metal coil spring maintains the capability of supporting heavy components.

Lin et. al. [10] addressed the IR problem of valve springs using a specific automobile engine. At the valve spring resonances, the resonant oscillations cause the spring force to drop near the maximum valve lift, when the valve is returning and maximum spring force is required to provide the necessary acceleration for the valve. To alleviate this problem, they developed an optimization routine based on a previously available helical spring model. The basic concept was minimizing the amplitude of spring resonance by adjusting the diameter of the spring wire at different cross sections. In the case considered in this
reference, an energy reduction of about fifty percent was obtained at the spring resonant frequency and the engine working speed was also extended.

Compared to the valve springs, which are usually protected by the valve cover, a more visible place where coil springs are used is the vehicle suspension. Lee and Thompson [11] pointed out that the IRs lead to significant dynamic stiffening above a certain frequency. For an automotive suspension spring, this occurs at frequencies as low as about 40 Hz. They presented an efficient method for calculating the dynamic stiffness of a helical coil spring, based on Timoshenko’s beam theory and Frenet’s formulation for curved systems. This method can be easily implemented in a complicated vehicle model in order to help reveal the unwanted spring behavior due to the IRs in the design stage.

1.3.4 IRs in Viscoelastic Rubber Mounts

Viscoelastic isolators are quite common in all areas of everyday life. They are used throughout an automobile in engine mounts, transmission mounts, body mounts, electrical component mounts, and so forth. They are also used in appliances such as refrigerators, washing machines and computers, etc. The resilient property and relatively high internal damping of elastomers make them very popular as isolator material to help reduce the transmitted vibration and noise.

Unlike metal, elastomers’ properties are more likely to change with variation of operating conditions, such as frequency, temperature and strain. Since the effectiveness and IRs of elastomer isolators are greatly affected by their material properties, such as the modulus and loss factor, it is worthwhile to review the dynamic characteristics of the elastomers.
The variation of the material properties with strain amplitude inside the isolator was examined experimentally by Cole [12]. For large strain, the modulus decreases with increasing strain amplitude while this effect reversed for the loss factor. However, for strains less than 10%, the dynamic properties do not change perceptibly with the strain amplitude [13, 14]. In practice, most isolators work in this range where the deflection changes linearly with the external load.

The frequency and temperature dependency of elastomer properties were well documented by Cole [12], DeJong et. al. [14] and Snowdon [15-17]. The value of the dynamic modulus increases when the frequency increases or when the temperature decreases. However, the loss factor doesn’t change monotonically with frequency or temperature, i.e. there are frequency and temperature transitions. These quantities refer to the transition of the elastomers to a non-resilient glass-like state at very high frequency or sufficiently low temperature, e.g. –40°C [18]. Above and below the transition points, the loss factor decreases and increases with increasing frequency or temperature, respectively. This can be seen in Figures 1.6 and 1.7 that are extracted from reference [15] (Snowdon, 1968; the legends are modified in accordance with the notation used herein). In those figures, \( f_t \) and \( T_t \) are transition frequency and temperature, respectively. However, practical isolation problems are usually encountered under room temperature and at frequencies much lower than the transition frequency. In this range, shaded in Figure 1.7, the dynamic properties of the isolator typically change very slowly with both temperature and frequency [12, 16, 17]. Therefore, most researchers used constant properties to investigate the IRs in rubber mounts.

To the best of the author’s knowledge, almost all previous researchers used the system shown in Figure 1.4(a) to demonstrate the IRs. The rubber isolator was idealized as a “long-rod” that has mass characterized by the material density. Ungar and Dietrich [19] discussed the standing waves in resilient elements and their adverse effects on high
frequency vibration isolation. They qualitatively explained that the wave effects are more important in a heavier, larger isolator than in a lighter, smaller isolator of equal static stiffness.

Harrison, Sykes, and Martin [5] studied the wave effects in isolation mounts by evaluating the force transmissibility both analytically and experimentally. They concluded that the wave effects could increase the transmissibility of a mount in certain frequency ranges by as much as 20 dB above the transmissibility predicted by the massless isolator theory. A much slower decreasing rate of the transmissibility curve was also observed because of the IRs. They also mentioned that for practical mounts, wave effects are most detrimental in the most audible frequency range (500 to 1000 Hz). Although this was an implication that the IR problem should be also investigated in terms of the noise radiation, Harrison et. al. failed to do so in their work.

Figure 1.6: Dependence of (a) the dynamic modulus $E$ and (b) the damping factor $\eta$ of a rubberlike material upon frequency $f$ and temperature $T$. (J. C. Snowdon, Copyright 1968, Wiley).
The effects of the isolator properties, dimensions and the primary mass on the IRs were studied by Snowdon [15-17] and Harrison et. al. [5]. It was shown that the amplitude of the IRs is smaller if the primary mass is more massive and the resonant peak is lower with higher damping. Snowdon [17] pointed out that the wave effects would be observed at high frequencies when the mount dimensions become comparable with multiples of the half-wavelengths of the elastic waves traveling through the mount. Therefore, for a longer or softer (small modulus) isolator, the IRs occur at lower frequencies. Although Snowdon also noticed that the IRs could impair the isolator’s performance, he concluded that they are not always important because the relatively high damping in practical rubber isolators could attenuate these wave effects.

Since the material properties, such as modulus and damping, play important roles in the appearance of the IRs, it is useful to know the range of commonly used isolator material properties. Sykes [7] tabulated data about the dynamic mechanical properties for some rubber mount materials, which permits the estimation of the IRs from known wave velocities. The Young’s modulus usually ranges from 2 to 50 MPa and the loss factor ranges from 0.1 to 0.3. Other sources [20, 21] also show that the typical loss factor for
rubber material is 0.1; for metallic material such as aluminum and steel, the loss factor is very small with a range of 0.007-0.1.

Although researchers become aware of the IRs in rubber isolators many decades ago, previous researchers [9, 15-17] only suggested relying on the high internal damping of elastomer materials to attenuate the IRs. Although high damping can alleviate the IR problem, this does not necessarily mean that rubber materials with high damping are good for constructing isolators. This is because that (a) the wave velocity in many highly damped materials increases rapidly with frequency with the result that the isolator’s effectiveness may be decreased [7], and (b) typical highly damped elastomers exhibit poor returnability and greater drift than elastomers with medium and low damping levels, which limit their loading capacity [7, 22]. To the best of the author’s knowledge, besides the suggested damping treatment, no direct attempt has been made to suppress the IRs. However, Neubert [23] presented a study in which he found that some of the resonances of the axially excited bar can be suppressed using one or two dynamic absorbers or concentrated masses.

1.4 Control of IRs - Dynamic Vibration Absorbers

Since there is a clear lack of research in controlling IRs, one of the purposes of this dissertation is to explore effective ways to suppress them. A classical method for vibration or resonance control, as Neubert [23] used, is the dynamic vibration absorber (DVA). It is also used for the purpose of controlling noise radiation from vibrating structures. This section gives a brief review on the fundamentals of DVAs.

The DVA is an additional mass-spring-damper structure attached to a main vibration system. Figure 1.8 shows the schematic diagram of a DVA mounted on a primary mass, which represents sensitive equipments (the primary mass) protected from vibration. One
major effect of adding the DVA is that it changes the original system from a SDOF system to a two-degree-of-freedom (2DOF) system. Although this increases the number of design variables for the vibration isolation problem, it also offers more choices for controlling the primary system vibration. Specifically, when the parameters of the DVA are properly chosen, the motion of the primary mass can be minimized in order to meet the practical requirements. Unlike the isolator that works at high frequencies, a DVA works best when tuned to the exact disturbance frequency. Therefore, the natural frequency of the DVA is usually tuned to the resonant frequency of the primary system for broadband excitations, or selected to match the center frequency of the disturbance for the narrow-band isolation problem.

Based on the system shown in Figure 1.8, the DVA effectiveness is briefly explained by calculating the transmissibility of the isolator as shown in Figure 1.9. Additional details about DVA theory can be found in many studies [2, 24]. In Figure 1.9, the DVA frequency is tuned to the natural frequency of the primary mass. It is found that the resonance of the primary system is greatly attenuated by the DVA. This attenuation is accompanied by two new resonant peaks (corresponding to the 2DOF system) on either side of the original resonant frequency. The DVA damping effects are also shown in the same figure. High damping will weaken the DVA effectiveness at the tuning frequency but it help attenuate the amplitudes of the new peaks.
Figure 1.8: Dynamic vibration absorber mounted on a primary system.

Figure 1.9: Force transmissibility for the system with and without DVA.
Since its invention, the DVA performance has been continuously improved. Sun et. al. [25] gave an excellent review on design methodology and control strategy of DVA technology. For passive DVA consisting of only mass, spring and damping elements, optimization methods were developed and introduced in several papers and textbooks [2, 15, 24, 25]. These design optimization problems involve finding the best values for DVA mass, damping and tuning frequency such that the magnitudes of the new peaks are within the acceptable level while adequate attenuation is obtained at the tuning frequency.

The passive DVA has two obvious limitations. First, the effective frequency band is narrow. Second, its effectiveness is sensitive to the system parameters. If any change occurs in the natural frequency of the primary system, the pre-designed tuning frequency of the DVA and the disturbance frequencies, the DVA will not perform properly any more. To overcome this drawback, an absorber that can alter its own parameters, known as the adaptive absorber, with the variation of system parameters was invented [26-29]. Another approach for extending the working range of passive DVAs is the addition of an active force. The DVA integrated with active components is called the active or active/passive (hybrid) DVA. As shown in Figure 1.10(a), the active force is normally applied between the absorber mass and the primary structure. The characteristics of the active force is determined by a controller based on the error signal that is usually the primary system response including displacement, velocity, acceleration or any combination of them. Currently, there are several different kinds of control algorithms available for the DVA application, such as classical feedback control [29, 30, 31], neural network control [26], fuzzy logic control [32, 33] and adaptive feedforward control [34].

The effectiveness of the active or hybrid DVAs depends on the control force. To operate properly, active/hybrid DVAs may require large amount of active force at some frequencies, such as those that are particularly below the DV’As natural frequencies [35].
This is an important limitation to the application of active/hybrid DVAs since large control forces consume more power, an undesirable effect in practice. Therefore, some research efforts have been directed to reduce the magnitude of the control force while keeping an acceptable effectiveness of the DVA. Heilmann and Burdisso [36, 37] introduced a dual-reaction mass DVA as shown in Figure 1.10(b). Different from the traditional DVAs, this DVA has two reaction masses and the active force is applied between them instead of between the DVA mass and the primary structure as in the traditional configuration (Figure 1.10(a)). It was claimed that the dual-reaction mass DVA can achieve the same effectiveness as the traditional DVA while consuming only 50% of the power required by the traditional DVA.

Figure 1.10: Schematic of active/hybrid configurations of (a) traditional single-mass DVA and (b) dual-mass DVA.
1.5 Motivation and Objectives

Although the IR problem has been previously identified by some researchers, prior works have some limitations. First, previous researchers used a SDOF model, which is not adequate to demonstrate the significance of the IRs in practical isolation systems. In real life, a machine, such as a car engine, may require different mount arrangements for different installation conditions. A SDOF model cannot account for the isolator location effects. Moreover, it cannot predict the coupling effects of multiple isolators. Second, previous studies examined the effects of IRs on the vibration but not on the radiated noise of a system; no previous work has shown if and to what extent the vibration system becomes quieter or noisier when the characteristics of the IRs change. This is a significant limitation because the IRs usually appear at high frequencies (e.g. above 500 Hz), where radiated noise is particularly troublesome. Finally, most of the previous researchers concluded that the IR problem is not of primary concern for the vibration isolation design. Although some suggestions of using optimal design methods to suppress IRs and using viscoelastic material to compensate the IR effects in metallic springs are mentioned, for IRs in a viscoelastic elastomeric isolator, no active efforts for attenuating them have been reported. This could be because: (a) IRs of elastomer isolators occur at high frequencies at which enough isolation may be already achieved or the isolation performance may be not of interest [17], or (b) the relatively high damping of elastomer material attenuates the IRs effectively [7, 17]. However, given the fact that manufacturers in different industries tend to build much lighter equipments working at much higher speeds than their predecessors, especially when the noise is a sensitive issue, it becomes necessary to suppress the IRs in both metallic and elastomer isolators.

Motivated by the limitations of previous studies and necessities of the industry trends, there are two main objectives for this dissertation. The first goal is to gain insight into the physics and demonstrate the significance of the IRs, in particular, with respect to radiated
noise. For this purpose, an analytical model is developed. Different from the SDOF model that is commonly adopted in other studies, this model consists of a multi-DOF primary mass supported on a flexible foundation through multiple isolators. Therefore, it is more effective for representing a practical isolation problem. The flexible foundation makes it possible to examine the IR problem from the perspective of noise radiation. The significance of the IRs is assessed through two metrics: the force transmissibility of isolators and the acoustic power radiated by the foundation.

The second goal is to investigate and implement effective approaches to suppress IRs and therefore to further improve the isolator performance. Two approaches are considered in this study: passive and hybrid DVAs. Differing from the traditional applications, this study proposes to embed DVAs directly into the isolator. For the passive DVA, only spring, mass and damper elements are used. For the hybrid DVA, passive elements work together with the active force to help improve the DVA performance. The feedforward algorithm is selected to control the active force. To validate these approaches, experiments showing the passive and hybrid DVA performance are carried out.

In this study, both the analytical model and the associated experiments are developed for isolators made of elastomers, such as rubber. However, the results and technology obtained here have the potential to be applied to other types of isolators such as metallic springs.

1.6 Dissertation Organization

This chapter addresses the background of the IR problem and summarizes relevant previous research work that has been done previously. The rest contents of this dissertation are organized in the following manner.
Chapter 2 reviews vibration isolation theory and different isolator models. It discusses the development of the analytical model used in this study to investigate the effects of the IRs on isolator performance. The model development includes three parts. First, the development of the vibration system model including the practical isolator model for demonstrating the IRs is described. Second, the definition of the force transmissibility in a multiple DOF system is presented. Finally, the calculation of the acoustic power of the foundation is derived.

Chapter 3 presents an in-depth analysis of the effects of IRs on isolator’s performance and reveals the significance of the IR problem. Based on the 3DOF model developed in Chapter 2, a parametric study of the effects of variation of isolator properties on the IRs and isolator’s performance in the IR range is performed. The isolator properties that are considered include Young’s modulus and damping. The influence of the locations of isolators on the foundation and the foundation flexibility on the IRs are also investigated numerically.

Chapter 4 addresses the approach of using passive DVAs to suppress the IRs, hence to improve the isolator performance. The passive DVA performance is first investigated analytically in this chapter. Some experimental results are then included. Due to practical limitations and for simplicity, all the experiments in this study are based on a SDOF system. The experimental validations of the IRs in a commercial rubber isolator and their attenuation using passive DVAs are presented in this chapter.

Chapter 5 discusses the approach of using hybrid controlled DVAs to suppress the IRs. Based on the passive configuration, an active force is added to form the hybrid DVA enhanced isolator. An analytical model for this isolator is then developed. The
experimental setup used to investigate the hybrid approach and the test results are also presented.

Chapter 6 contains a summary of the main conclusions and contributions of this dissertation. In the end, some recommendations for future work are suggested.
Chapter 2

Development of Analytical Model

This chapter discusses the development of an analytical model to investigate the effects of the IRs on isolator performance. Unlike the SDOF system with rigid foundation generally used by previous researchers, the model in this study consists of multiple isolators working in parallel on a flexible foundation. Therefore, it more accurately represents practical vibration problems. In this study, the noise radiation is employed as another metric, in addition to the classical criteria – the transmissibility, to evaluate the isolator performance. This is because the foundation vibrates and radiates noise into the ambient fluid under the excitation of the transmitted forces through the isolators. The model development includes three parts. First, the development of the isolator model for both lumped and continuous systems is discussed. Second, the definition of the force
transmissibility in a multiple DOF system used in this study is presented. Third and last, the calculation of the acoustic power of the foundation is derived.

2.1 Vibration Isolator Model

This study considers the elastomer isolator working mainly under compression. As shown in Figure 2.1, the isolator has a cylindrical shape with length $L$ and cross-sectional area $S$. Traditionally, the isolator is modeled as an ideal spring element with pure stiffness and damping mechanism. In this dissertation, the lumped parameter massless model is used as a reference to show the isolator performance degradation due to the IRs. To include the IRs, the isolator needs to be modeled as a continuous system which takes its mass effects into account. In both models, only the longitudinal vibration along the isolator’s axis is considered.

Figure 2.1: Cylindrical elastomer isolator considered in this study.
2.1.1 Lumped Parameter Isolator Model

The lumped parameter isolator model is generally adopted by textbooks when explaining the classical vibration isolation theory. In this model, the isolator is characterized by two elements: a massless spring that has pure stiffness and a viscous (Figure 2.2(a)) or structural damping (Figure 2.2(b)). The spring stiffness $k_s$ is given by

$$k_s = \frac{ES}{L}$$  \hspace{1cm} (2.1)

where $E$ is the Young’s modulus of the isolator material.

In Figure 2.2(a), the viscous damping is denoted as a dashpot with a constant damping coefficient $C$ whose units are N·s/m. For a given primary mass $m$, there is a critical damping coefficient $C_c$ that is calculated by

$$C_c = 2\sqrt{k_s m}$$  \hspace{1cm} (2.2)

According to Equation (2.2), a nondimensional damping ratio $\zeta$ can be defined by

$$\zeta = \frac{C}{C_c} = \frac{C}{2m_p\omega_n}$$  \hspace{1cm} (2.3)

which is commonly used to characterize the viscous damping. Note from Equation (2.3) that the damping ratio is inversely proportional to the system’s undamped natural frequency $\omega_n$.

In Figure 2.2(b), the structural damping is taken into account by considering a complex spring stiffness $\tilde{k}_s$ defined by

$$\tilde{k}_s = k_s(1 + j\eta)$$  \hspace{1cm} (2.4)

where $j = \sqrt{-1}$ is the unit imaginary number and $\eta$ is the nondimensional loss factor associated with the isolator material property. Since the isolator stiffness is a function of
the Young’s modulus, the structural damping is considered by using the complex modulus $\tilde{E} = E(1 + j\eta)$ in Equation (2.1).

The viscous damping and structural damping (characterized by damping ratio and loss factor, respectively) function in different ways. It is concluded that the damping mechanism in a solid structure, such as the rubber isolator, is more effectively represented by the loss factor instead of the damping ratio [19]. Therefore, the structural loss factor, $\eta$, is used to represent the isolator’s internal damping in this dissertation. At the system resonance, the damping ratio and the loss factor have the following relationship:

$$\eta = 2\zeta$$  \hspace{1cm} (2.5)

Since the material damping ratio can be experimentally estimated using the “half-power” method in vibration testing, the loss factor can then be estimated according to Equation (2.5).

$$k_s = \frac{ES}{L}$$

$$\tilde{k}_s = k_s(1 + j\eta)$$

Figure 2.2: Isolator modeled as (a) massless spring and viscous damping, and (b) massless spring and structural damping.
2.1.2 Distributed Parameter Isolator Model

In the distributed parameter model, the isolator is considered as a continuous, thin, and uniform rod in axial motion as shown in Figure 2.3. The isolator has length $L$, cross-sectional area $S$, Young’s modulus $E$, and loss factor $\eta$. Its mass is taken into account by considering the material density $\rho$. The rod is subjected to two axial forces: $F_T$ acting at its top and $F_B$ acting at its bottom. The application of these forces will produce a longitudinal displacement distribution $u(\varepsilon, t)$. The relationship between the displacements and external forces at the ends of the isolator is given by:

$$
[D][u(L)] = [F_T] \quad \text{and} \quad [u(0)] = [F_B]
$$

(2.6)

where

$$
[D] = \begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
$$

(2.7)

is the dynamic stiffness matrix of the isolator. The terms in matrix $[D]$ can be obtained in closed form by one dimensional wave analysis.

The equation of motion for the axial vibration of this rod is given by [38]:

$$
\frac{\partial^2 u(\varepsilon, t)}{\partial \varepsilon^2} = \frac{1}{\tilde{c}_i^2} \frac{\partial^2 u(\varepsilon, t)}{\partial t^2}
$$

(2.8)

where $\tilde{c}_i$ is the complex wave speed of the isolator given as:

$$
\tilde{c}_i = \sqrt{\frac{E}{\rho}}
$$

(2.9)

Here, the isolator damping is included by using the complex modulus. The harmonic solution of the one-dimensional wave equation of Equation (2.8) has the form:
\( u(\varepsilon, t) = (A e^{-j_k \varepsilon} + B e^{j_k \varepsilon}) e^{j\omega t} \)  \hspace{1cm} (2.10)

where \( A \) and \( B \) are two constants that depend on the boundary conditions, \( \omega \) is the excitation angular frequency, and \( k_i = \omega / c_i \) is the wavenumber.

Now, the objective is to develop the expressions of the terms in matrix \([D]\), which is the continuous model including the mass effects of the isolator. According to Equation (2.6), the diagonal terms of this matrix represent the forces required to induce a unit displacement on the DOF where the force is applied, while the other DOF is fixed. The cross terms of this matrix represent the force required to keep the DOF where the force is applied fixed, while the other DOF undergoes a unit displacement. Because of the reciprocity principle, the two cross terms are the same; on the other hand, the two diagonal terms are also identical due to the geometrical symmetry of the isolator. Thus,
the elements in the matrix can be obtained by imposing the following boundary conditions:

\[
u(0, t) = 1 \cdot e^{j\omega t}
\]  

at \( \varepsilon = 0 \), i.e. the displacement is assumed to be unity and:

\[
u(L, t) = 0
\]  

at \( \varepsilon = L \), i.e. the displacement vanishes at the opposite end.

The constants \( A \) and \( B \) can be computed by substituting Equation (2.10) into Equations (2.11a, b), which yields

\[
A = \frac{j \sin(k, L) + \cos(k, L)}{j 2 \sin(k, L)}, \quad B = \frac{j \sin(k, L) - \cos(k, L)}{j 2 \sin(k, L)}
\]  

(2.12a, b)

On the other hand, based on the force continuity condition, the external force at each end of the rod should be equal to the internal force at the same end. That is:

\[
F_T = -\bar{E}S \frac{\partial u}{\partial \varepsilon} \bigg|_{\varepsilon=L}, \quad F_B = -\bar{E}S \frac{\partial u}{\partial \varepsilon} \bigg|_{\varepsilon=0}
\]  

(2.13a, b)

where the partial derivative of displacement \( u \) with respect to axial coordinate \( \varepsilon \) is evaluated from Equation (2.10) as

\[
\frac{\partial u}{\partial \varepsilon} = \left(-j Ak_i e^{-jk_i \varepsilon} + jB k_i e^{jk_i \varepsilon}\right) e^{j\omega t}
\]  

(2.14)

Substituting the expressions of \( A \) and \( B \) into Equation (2.14) and then evaluating Equations (2.13a, b) yields

\[
F_T = \left[-\bar{E}Sk_i / \sin(k, L)\right] e^{j\omega t}
\]  

(2.15a)

and

\[
F_B = \left[\bar{E}Sk_i \cot(k, L)\right] e^{j\omega t}
\]  

(2.15b)

The diagonal term in the dynamic stiffness matrix can then be calculated as
\[ D_{11} = D_{22} = F_b / u(0,t) = \tilde{E} Sk_i \cot(k_i L) = D^d \]  

Similarly, the off-diagonal term is obtained as
\[ D_{12} = D_{21} = F_r / u(0,t) = -\tilde{E} Sk_i / \sin(k_i L) = -D^o \]  

Therefore, the distributed parameter isolator model is characterized by the matrix
\[
[D] = \begin{bmatrix}
D^d & -D^o \\
-D^o & D^d
\end{bmatrix}
\]  

Once the continuous model is obtained, it is interesting to examine the predicted system natural frequency when the isolator mass is ignored, i.e. \( \rho = 0 \). This can be done by considering the free vibration of a SDOF system as shown in Figure 1.4(a). Using the isolator model given in Equation (2.17), the equations-of-motion (EOM) of this system are given by
\[
\begin{bmatrix}
m & 0 \\
0 & 0
\end{bmatrix}\begin{bmatrix}
\ddot{x} \\
x
\end{bmatrix} + \begin{bmatrix}
D^d & -D^o \\
-D^o & D^d
\end{bmatrix}\begin{bmatrix}
x \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

where \( m \) is the primary mass and \( x \) is the displacement of the primary mass. Since the foundation is rigid, there is only one valid EOM that describes the vibration of the primary mass:
\[ m\ddot{x} + D^d x = 0 \]  

Taking the Fourier transform of the above equation yields
\[ (-\omega^2 m + D^d) \tilde{x} = 0 \]  

The natural frequency, \( \omega_n \), of this system can be obtained by setting the coefficient of \( x \) in Equation (2.20) equal to 0. That is
\[ -\omega_n^2 m + D^d = -\omega_n^2 m_p + \tilde{E} Sk_i \cot(k_i L) = 0 \]  

After some mathematical manipulation and replacing the wavenumber \( k_i \) with the expression \( \omega_n \sqrt{\rho / \tilde{E}} \), Equation (2.21) becomes
\[
\frac{\bar{E}S}{m} \sqrt{\frac{\rho}{\bar{E}}} \cos(\omega_n L\sqrt{\frac{\rho}{\bar{E}}}) = \omega_n
\]

(2.22)

Generally, Equation (2.22) has infinite solutions for \(\omega_n\) when the density term, \(\rho\), is non-zero. The fundamental solution corresponds to the system resonance and other solutions correspond to the IRs. However, if the isolator density \(\rho\) approaches 0 (i.e. massless isolator), Equation (2.22) can be simplified as
\[
\frac{\bar{E}S}{mL} = \omega_n^2
\]

(2.23)

As expected, this result is exactly the same as the prediction from the massless model shown in Figure 2.2(b), i.e. the continuous model converges to the massless model when \(\rho\) is zero. Therefore, the continuous model is used as a more general isolator model herein. If a massless model is required for comparison purposes, it is obtained by assuming the isolator material density to be zero.

### 2.2 Vibration System Model

In this section, the isolator model is integrated with the primary mass and foundation to form the vibration system model. According to the objectives of this study, a multiple DOF system is preferred. Hence the vibration system model is developed for a 3DOF system. In the case where a SDOF model is convenient and preferred, the 3DOF model will be easily simplified to simulate a SDOF system.
2.2.1 Description of the 3DOF System

As shown in Figure 2.4, the 3DOF system consists of a primary mass connected to a flexible foundation by three isolators. The system is subjected to an external force with amplitude $F_0$ acting at an arbitrary location on the primary mass.

The primary mass is modeled as a rigid body with three DOFs, which are the translation in y- (vertical) direction and rotations about the x- and z- axes through its center of gravity (CG). To directly relate the motion of the primary mass with the displacements at the ends of the isolators, the three independent vertical translations, $Y_{Ti}$ (where $i = 1, 2, \text{ and } 3$) at the mounting points between the isolators and the primary mass are used as the three DOFs of the primary mass in the model.

Using the model in Equation (2.17), each isolator is considered as a “one dimensional” continuous rod that accounts for its own inertia. All connecting points between the isolators and the primary mass, and between the isolators and the foundation are assumed pinned. Therefore, each isolator transfers forces along its axis only.

The flexible foundation is modeled as a simply supported rectangular plate that can vibrate and radiate noise to the ambient fluid. The displacements at the connection points between the isolators and the foundation are denoted as $Y_{Bi}$. The transmitted forces through the isolators excite the plate that vibrates and radiates noise. Because these transmitted forces are considered as the only inputs to the plate and their strength depends on the isolator performance, the radiated noise from the plate is sensitive to changes in the isolator characteristics. Therefore, the acoustic power radiated from the foundation will be considered when examining the isolator’s IR effects from the perspective of the noise radiation.
The modeling approach and EOMs of the components and the system in Figure 2.4 are presented in the following sections. The model is developed in the frequency domain so both the force transmissibility and the radiated sound power are given as functions of frequency.

![Figure 2.4: Schematic plot of the 3DOF system consisting of a rigid primary mass with 3DOF connected to a flexible foundation through three isolators.](image)
2.2.2 Modeling Approach

As indicated earlier, the system shown in Figure 2.4 comprises three fundamental parts: the rigid primary mass, the continuous isolators and the simply supported foundation. The modeling approach adopted here is first decomposing the system into two sub-systems, modeling each of the separate sub-system and then assembling them together. In detail, the EOMs of the fully coupled system are derived by initially modeling the substructures consisting of (a) the primary mass and isolators, and (b) the foundation separately. The two substructures are then coupled by imposing conditions for continuity of forces and displacements at their interfaces, i.e. at the isolators’ attachments to the foundation.

Before developing the coupled system model, another problem associated with the selection of the DOFs should be discussed. Since the dynamic property of the primary mass is given by its mass and mass moment of inertia, it is normally straightforward to write the system EOMs with respect to the DOFs associated with the motion of the CG of the primary mass (Figure 2.4). In this case, the mass matrix is diagonal, i.e. the EOMs are dynamically uncoupled with respect to the mass matrix. However, as mentioned earlier, it is more convenient for the purpose of this study to use the three independent vertical translations, \(Y_{Ti}\), to describe the motion of the primary mass. In this case, the corresponding mass matrix in the system EOMs will have non-zero off-diagonal terms. This mass matrix can be derived according to the theory of conservation of the kinetic energy. In addition, to use the set of DOF comprising the three translations, the external force \(F_0\) (Figure 2.4) is also required to be split into three sub-forces acting at the top of each isolator through the primary mass. This results in a force vector. The following two sections describe the developments of this mass matrix and force vector.
2.2.3 Mass Matrix

The mass matrix corresponding to a given set of DOFs of a structure can be obtained according to the definition of the kinetic energy. Considering the primary mass in Figure 2.4, the coordinates’ origin is chosen to be at its CG. If the motion of this mass is represented using one translation at the CG, $Y_{cg}$, and two rotations $\Theta_x$ and $\Theta_z$ about x- and z- axes respectively, its kinetic energy can be written as

$$E_K = \frac{1}{2} \begin{bmatrix} \dot{Y}_{cg} \\ \dot{\Theta}_x \\ \dot{\Theta}_z \end{bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & J_{xx} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix} \begin{bmatrix} \dot{Y}_{cg} \\ \dot{\Theta}_x \\ \dot{\Theta}_z \end{bmatrix}$$

(2.24)

where $m$ is the primary mass, $J_{xx}$ and $J_{zz}$ are mass moments of inertia of the primary mass about x- and z- axes respectively. The superscript $T$ of the vector in Equation (2.24) denotes the transpose operation.

On the other hand, the motion of the primary mass can also be described using the three translations, $Y_{T1}$, $Y_{T2}$, and $Y_{T3}$, at the top of each isolator. According to their geometrical relationship and assuming the primary mass can be considered as a planer structure, i.e. its thickness is small compared to the lateral dimensions, the two sets of DOFs (with respect to different coordinates) can be related by Equation (2.25):

$$\begin{bmatrix} Y_{T1} \\ Y_{T2} \\ Y_{T3} \end{bmatrix} = [T_m] \begin{bmatrix} Y_{cg} \\ \Theta_x \\ \Theta_z \end{bmatrix}$$

(2.25)

where $[T_m]$ is given by

$$[T_m] = \begin{bmatrix} 1 & -L_{1z} & L_{1x} \\ 1 & -L_{2z} & L_{2x} \\ 1 & -L_{3z} & L_{3x} \end{bmatrix}$$

(2.26)

where $L_{xy}$ and $L_{yz}$ are the x- and z-direction distance vectors of the CG to the $r^{th}$ mounting point at which the displacement $Y_{Tr}$ appears. If the x- and z-coordinate of the three
mounting points and the CG are denoted in the form \((x_r, z_r)\) and \((x_{cg}, z_{cg})\), respectively, the distance vector is calculated as

\[ L_{rr} = x_r - x_{cg} \]  \hfill (2.27a)

and

\[ L_{rz} = z_r - z_{cg} \]  \hfill (2.27b)

Substituting Equation (2.25) into Equation (2.24) yields

\[
E_K = \frac{1}{2} \begin{bmatrix} \dot{Y}_{r1} \\ \dot{Y}_{r2} \\ \dot{Y}_{r3} \end{bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & J_{xx} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix}^{-1} \begin{bmatrix} \dot{Y}_{r1} \\ \dot{Y}_{r2} \\ \dot{Y}_{r3} \end{bmatrix} \]  \hfill (2.28)

Thus the 3-by-3 mass matrix with respect to the three translations at the mounting points is given as

\[ [M] = [T_m]^{-T} \begin{bmatrix} m & 0 & 0 \\ 0 & J_{xx} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix}^{-1} \]  \hfill (2.29)

### 2.2.4 Excitation Force Vector

In Figure 2.4, the external excitation of the primary mass is idealized as a concentrated force \(F_0\) acting at the position \((x_F, z_F)\). The force \(F_0\) can be split into three sub-forces that are applied at the mounting points between the isolators and the primary mass. These sub-forces are obtained as

\[
\begin{align*}
F_0 &= F_{T1} + F_{T2} + F_{T3} \\
-F_0L_{Fx} &= -F_{T1}L_{1x} - F_{T2}L_{2x} - F_{T3}L_{3x} \\
F_0L_{Fz} &= F_{T1}L_{1z} + F_{T2}L_{2z} + F_{T3}L_{3z}
\end{align*}
\]  \hfill (2.30a-c)

Equations (2.30a-c) are based on the static equilibrium condition of force and moment and show that the sub-forces, \(F_{Ti}\) \((i = 1, 2, 3)\), result in the same force \(F_0\) and moments \(F_0L_{Fx}\) and \(F_0L_{Fz}\). It is derived in Appendix A that Equations (2.30a-c) are valid for the
situation considered in this study where all the forces are dynamic forces. The three sub-
forces, \( F_{Ti} \), act at the junctions of the \( i \)th isolator and the primary mass. Variables \( L_{Fx} \) and \( L_{Fz} \) are the distance vectors of the CG to the external force \( F_0 \) in the x- and z-direction, respectively. They are calculated as

\[
L_{Fx} = x_F - x_{cg} \quad (2.31a)
\]

and

\[
L_{Fz} = z_F - z_{cg} \quad (2.31b)
\]

Rewriting Equation (2.30) in matrix form gives

\[
\begin{bmatrix}
F_{T1} \\
F_{T2} \\
F_{T3}
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
-L_{t1z} & -L_{t2z} & -L_{t3z} \\
L_{Fx} & L_{Fx} & L_{Fx}
\end{bmatrix}^{-1} \begin{bmatrix}
1 \\
-L_{Fx} \\
-L_{Fx}
\end{bmatrix} F_0 = [T_m^{-1}]^{-1} \begin{bmatrix}
1 \\
-L_{Fx} \\
-L_{Fx}
\end{bmatrix} F_0 = \{T_f\} F_0 \quad (2.32)
\]

where vector \( \{T_f\} \) is called the force transfer vector whose entries depend on the position between the isolators and the concentrated external force \( F_0 \). The 3-by-1 force vector \( (F_{T1}, F_{T2}, F_{T3})^T \) is used as the excitation of the primary mass in the system model.

### 2.2.5 Model of the 3DOF System

Figure 2.5 shows the substructure consisting of the primary mass and the isolators separated from the foundation. The plot on the top left corner of Figure 2.5 magnifies the intersection between the primary mass and isolator 1. In this detailed plot, the isolator is separated from the primary mass. The internal force acting between them is indicated by \( F_{e1} \). Similarly, the forces acting on the primary mass by isolator 2 and 3 will be denoted as \( F_{e2} \) and \( F_{e3} \), respectively. Using the notations in Figure 2.5, the EOM of the primary mass with respect to the three translations, \( Y_{T1} - Y_{T3} \), is written as

\[
-\omega^2 [M] \begin{bmatrix}
Y_{T1} \\
Y_{T2} \\
Y_{T3}
\end{bmatrix} = \begin{bmatrix}
F_{T1} \\
F_{T2} \\
F_{T3}
\end{bmatrix} - \begin{bmatrix}
F_{e1} \\
F_{e2} \\
F_{e3}
\end{bmatrix} \quad (2.33)
\]
Recall the isolator model (Equation 2.17), the reaction force $F_{ei}$, where $i = 1, 2, 3$, can be computed as

$$F_{ei} = D_i^d Y_{Ti} - D_i^o Y_{Bi}$$  \hspace{1cm} (2.34)$$

Substituting Equation (2.34) into Equation (2.33) gives

$$\begin{bmatrix}
-\omega^2[M] + \begin{bmatrix} D_1^d & 0 & 0 \\
0 & D_2^d & 0 \\
0 & 0 & D_3^d
\end{bmatrix} & \begin{bmatrix} Y_{T1} \\
Y_{T2} \\
Y_{T3}
\end{bmatrix} + \begin{bmatrix} -D_1^o & 0 & 0 \\
0 & -D_2^o & 0 \\
0 & 0 & -D_3^o
\end{bmatrix} & \begin{bmatrix} Y_{B1} \\
Y_{B2} \\
Y_{B3}
\end{bmatrix} = \begin{bmatrix} F_{T1} \\
F_{T2} \\
F_{T3}
\end{bmatrix}
\end{bmatrix}$$  \hspace{1cm} (2.35)$$

Also from the isolator model, the transmitted force through each isolator can be expressed as

$$\begin{align*}
-F_{B1} &= -D_1^o Y_{T1} + D_1^d Y_{B1} \\
-F_{B2} &= -D_2^o Y_{T2} + D_2^d Y_{B2} \\
-F_{B3} &= -D_3^o Y_{T3} + D_3^d Y_{B3}
\end{align*}$$  \hspace{1cm} (2.36)$$

Note that the transmitted force through the $i^{th}$ isolator, $F_{Bi}$, is the superposition of the effects of the three excitation forces, i.e. $F_{Bi} = \sum_{j=1}^{3} f_{ij}$, where $f_{ij}$ represents the transmitted force through isolator $i$ due to the external force $F_{Tj}$ (Figure 2.5).

Equations (2.35) and (2.36) can be combined together and written in matrix form by moving all forces to the right hand side and taking all displacements as unknowns. This results in

$$\begin{bmatrix}
-\omega^2[M] & \begin{bmatrix} D_1^d & 0 & 0 \\
0 & D_2^d & 0 \\
0 & 0 & D_3^d
\end{bmatrix} & \begin{bmatrix} \begin{bmatrix} -D_1^o & 0 & 0 \\
0 & -D_2^o & 0 \\
0 & 0 & -D_3^o
\end{bmatrix}
\end{bmatrix} & \begin{bmatrix} Y_{T1} \\
Y_{T2} \\
Y_{T3}
\end{bmatrix} = \begin{bmatrix} F_{T1} \\
F_{T2} \\
F_{T3}
\end{bmatrix}
\end{bmatrix}$$  \hspace{1cm} (2.37)$$
The above equation is the EOM of the primary mass – isolators substructure. The 6-by-6 matrix on the left hand side is called the dynamic stiffness matrix of the primary mass – isolators substructure and is represented by \([D_S]\) in the rest of this dissertation.

In Equation (2.37), the matrix \([D_S]\) can be evaluated after knowing the geometrical information and properties of the primary mass and the isolators. Furthermore, the three excitation forces \((F_{Ti})\) should be known, either from analysis or from experiments, for a given vibration problem. In addition to the six unknown displacements at the two ends of each isolator, there are three unknown transmitted forces, \(F_{Bi}\). To solve for all the nine unknowns, three additional equations are required. They are obtained from the model of the flexible foundation.
Figure 2.5: Illustration for separating the primary mass and isolators from the foundation.
Referring to Figure 2.5, the foundation is subject to three forces that are transmitted through the three isolators. According to the continuity conditions, the force acting on the plate by the isolator has the same amplitude as the force acting on the isolator by the plate. The displacements of the lower ends of the three isolators and the corresponding points on the foundation are equal. Therefore, analogous to the EOMs of the primary mass, the EOMs of the foundation can be written with respect to the three mounting points of the isolators as

\[
[D_f] \begin{bmatrix} Y_{b1} \\ Y_{b2} \\ Y_{b3} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} Y_{b1} \\ Y_{b2} \\ Y_{b3} \end{bmatrix} = \begin{bmatrix} F_{b1} \\ F_{b2} \\ F_{b3} \end{bmatrix}
\] (2.38)

where the 3-by-3 matrix \([D_f]\) is the dynamic stiffness matrix of the foundation. This matrix can be obtained as the inverse of the dynamic receptance matrix, \([R]\) [2]. That is:

\[
[D_f] = [R]^{-1} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}^{-1}
\] (2.39)

where the term \(R_{rs}\), for \(r, s = 1, 2, 3\), represents the displacement at \(r^{th}\) point due to a unit force applied at the \(s^{th}\) point on the foundation. The dynamic receptance matrix can be computed from the model of the foundation, which is a simply supported rectangular plate. Since the model for plate vibration is well known [39, 40], it will not be presented here. Detail about the plate model used in this dissertation can found in Appendix B.

Equation (2.38) expresses each of the transmitted forces as a linear combination of the three displacements at the ends of the three isolators weighted by \(D_{rs}\). These coefficients
are associated with the dynamics of the foundation. Substituting Equation (2.38) into Equation (2.37) yields

\[
\begin{bmatrix}
    D_s \quad [0] \\
    [0] \quad D_f
\end{bmatrix}
\begin{bmatrix}
    Y_{r1} \\
    Y_{r2} \\
    Y_{r3} \\
    Y_{b1} \\
    Y_{b2} \\
    Y_{b3}
\end{bmatrix}
= 
\begin{bmatrix}
    F_{r1} \\
    F_{r2} \\
    F_{r3} \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]

where \([0]\) denotes a 3-by-3 zero matrix. Equation (2.40) is the EOM for the fully coupled 3DOF system of Figure 2.4. It is used to calculate the displacements at the ends of each isolator. Later in this Chapter these displacements are used to compute the metrics for assessing the isolators’ performance.

### 2.2.6 Model of the SDOF System

Although the 3DOF model is primarily used in this study, an equivalent SDOF model is still worthy of development here. The SDOF model can be simply obtained by imposing the following two constraints on the 3DOF system: (a) the primary mass is allowed to move only in the y-direction (this happens if the rotational inertia of the primary mass is infinite); and (b) the three parallel-mounted isolators are combined into one isolator being connected at the center of gravity of the primary mass. Accordingly, there will be only one connection point between the isolator and the foundation.

Applying the above constraints to the 3DOF model, Equation (2.40) is then simplified into the SDOF model given by
The symbols in Equation (2.41) represent the same variables as in the 3DOF model except that \([D_s]\) is now a 2-by-2 matrix including the dynamic properties of the isolator where \(D_F\) is the scalar valued point dynamic stiffness of the foundation at the mounting point of the isolator, and \(F_{T1}\) equals \(F_0\) in this case. Translation \(Y_{T1}\) is the only motion of the primary mass in Equation (2.41). Furthermore, if the static stiffness of the single isolator in the SDOF system equals to the sum of the static stiffness of the three isolators used in parallel in the 3DOF system, the SDOF system is equivalent to the 3DOF system as long as the translational DOF of the primary mass is concerned. In other words, the two systems will have the same translational displacements. In this case, the SDOF model is referred to as the equivalent SDOF model in the rest of this dissertation.

In this study, the 3DOF model will be primarily used to investigate the IRs and their significance in practical isolation systems. The SDOF model will be used to examine the effects of the isolator properties on the IRs and also provide some guidance for the experiments since all the experiments are based on a SDOF setup. Moreover, the predictions from the equivalent SDOF model will be compared to those from the 3DOF model in Chapter 3. This comparison shows the advantages of the 3DOF model.

### 2.3 Isolator Performance Metrics

The isolator performance is evaluated through its force transmissibility and the radiated sound power of the foundation. These two metrics can be derived after solving for the unknown displacements at the ends of each isolator. Specifically, the unknown displacements \(Y_{Bi}\) and \(Y_{Ti}\) are calculated using Equation (2.40), for the 3DOF system, or Equation (2.41), for the SDOF system. The transmitted force \(F_{Bi}\) through the \(i^{\text{th}}\) isolator...
can be evaluated by substituting the corresponding displacements into the isolator model in Equation (2.6). To correlate variables, force $F_B$ and displacements $u(0)$ and $u(L)$ in Equation (2.6) represent the transmitted force $F_{Bi}$, the displacements at the ends of the $i^{th}$ isolator $Y_{Bi}$ and $Y_{Ti}$ in Figure (2.5), respectively. Since the input force $F_{Ti}$ is known, the force transmissibility is easy to calculate after knowing $F_{Bi}$. Furthermore, the velocity response of the foundation and the radiated sound power can also be calculated since $F_{Bi}$ is considered as the only excitation to the foundation. The definition and calculation of the force transmissibility is explained in this section, the derivation of the acoustic power of the foundation will be addressed in Section 2.3.

2.3.1 Force Transmissibility

Generally, the force transmissibility is defined as the amplitude ratio of the resulting transmitted force through the isolator to the corresponding input force acting on the primary mass. For a SDOF system, the force transmissibility associated with the single isolator is calculated as

$$T_{SDOF} = \frac{|F_B|}{|F_0|} \quad (2.42)$$

where $F_B$ is the transmitted force and $F_0$ is the input force.

For a multi-DOF system with multiple isolators, the force transmissibility is given by a matrix because there are coupling effects between different DOFs. For example, considering the 3DOF system shown in Figure 2.5, there are three input forces $F_{Tj}$, for $j = 1, 2$ and 3, acting on the top of each of the three isolators that are coupled by the primary mass; accordingly, each input force $F_{Tj}$ results in a transmitted force $f_{ij}$ through isolator $i$, for $i = 1, 2$, and 3. Therefore, the transmissibility can be expressed using a 3-by-3 matrix as shown in Equation (2.43). The entry at the $(i, j)$ position of this matrix denotes the
force transmitted to the plate at the bottom of isolator $i$ due to a unit input force applied at the top of isolator $j$. The transmissibility matrix is written as

$$[T_{MDOF}] = \begin{bmatrix}
    f_{11} & f_{12} & f_{13} \\
    F_{T1} & F_{T2} & F_{T3} \\
    f_{21} & f_{22} & f_{23} \\
    F_{T1} & F_{T2} & F_{T3} \\
    f_{31} & f_{32} & f_{33} \\
    F_{T1} & F_{T2} & F_{T3} \\
\end{bmatrix} \tag{2.43}$$

Due to the reciprocity principle, the matrix in Equation (2.43) is symmetric. Note that the terms in each row correspond to the transmitted forces through a particular isolator due to the three input forces acting from the primary mass. The summation of the terms in the $i^{th}$ row can be considered as a metric to describe the performance of the $i^{th}$ isolator, which is given as

$$T_i = \sum_{j=1}^{3} \frac{f_{ij}}{F_{Tj}} = \sum_{q=1}^{3} \frac{f_{qj}}{F_{Tq}} \tag{2.44}$$

The second equality in (2.44) is a result of the symmetry of the transmissibility matrix - the row summation is equivalent to the column summation. Since the $j^{th}$ column terms have the same denominator, which is the $j^{th}$ input force, their sum can be considered as the system transmissibility with respect to that force.

According to Equation (2.44), there are three transmissibilities for this 3DOF system, one for each of the three isolators. Therefore, measuring the performance of a particular isolator allows for a direct comparison between the transmissibility of this particular isolator and its counterpart in a SDOF system. Hence, this definition for calculating the force transmissibility in multi-DOF system is used in this study. In Equation (2.44), when
the three input forces $F_{Tj}$ are identical and of unit amplitude, the transmissibility of the $i^{th}$ isolator can be simplified as

$$T_i = \sum_{j=1}^{3} f_{ij}$$  \hspace{1cm} (2.45)

Referring to Figure 2.5, the summation of $f_{ij}$, for $j = 1, 2, 3$, is $F_{Bi}$ which is obtained from the isolator model as stated earlier. Therefore, the force transmissibility for the individual isolator $i$ in the 3DOF system is calculated as

$$T_i = |F_{Bi}|, \quad \text{when } |F_{T1}| = |F_{T2}| = |F_{T3}| = 1$$  \hspace{1cm} (2.46)

### 2.3.2 Radiated Noise

Noise radiation is often a problem associated with vibrations of solid structures. The radiated noise associated with vibrating solid structures is called *structure-borne noise*. A large number of practical systems that have structure-borne noise have a primary structure connecting to a supporting structure through single or multiple isolators. For example, a stamping machine supported on its base, an engine mounted on a car body or the hull of a ship, etc. All these examples can be represented using the model shown in Figure 2.4. Theoretically, both the primary mass (the stamping machine or the engine) and the flexible plate foundation (the machine base, the car body or the hull of a ship) can radiate noise into the ambient fluid because of their vibrating behavior. A good isolator transmits less energy into the supporting structure, which results in less noise radiation. Therefore, the radiated noise of the supporting structure can be used as another metric for evaluating the isolator’s performance. To some extent, these two metrics have similar physical meanings because if the force transmissibility is small, the excitation level to the supporting structure is small and so is the radiated noise. However, the radiated noise or *acoustic power* is normally denoted in the A-weighted decibel scale, which emphasizes the isolator performance in the most audible frequencies where the first several IRs
usually occur. Hence, the A-weighted acoustic power is used as a key measurement to investigate the influence of the IRs on the isolator performance.

As indicated previously, the supporting structure of the 3DOF vibration system (Figure 2.4) is modeled as a simply supported plate in this study. For simplicity, this plate is assumed to be embedded in an infinite baffle when its acoustic response is modeled. This section briefly discusses the calculation of the acoustic power radiated by this plate. More detailed information about the noise radiation of a planner structure can be found in Appendix C.

The acoustic power radiated by a simply supported plate embedded in an infinite baffle can be derived by means of wavenumber transform and is given as [41, 42]

\[
P(\omega) = \frac{\omega \rho_a}{8\pi^2} \int \int \frac{|v(k_x, k_z)|^2}{\sqrt{k^2 - k_x^2 - k_z^2}} dk_x dk_z
\]

(2.47)

where \( P \) is the sound power measured in watts, \( \rho_a \) is the density of the ambient medium, i.e. air, \( \rho_a = 1.2 \text{ kg/m}^3 \), \( \omega \) is the frequency of the external excitation, and \( v(k_x, k_z) \) is the two dimensional velocity wavenumber transform of the plate response. The velocity wavenumber transform is given as:

\[
v(k_x, k_z) = \int_{x=-\infty}^{x=\infty} \int_{z=-\infty}^{z=\infty} V(x, z) e^{j k_x x} e^{j k_z z} dx dz
\]

(2.48)

where \( V(x, z) \) is the velocity response of the plate. It can be calculated by substituting the transmitted forces \( F_{Bi} \) (obtained in Section 2.3.1) into the simply supported plate model and expressed as a linear combination of the plate’s modes as (See Appendix A)

\[
V(x, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Phi_{mn}(x) q_{mn}(\omega) \cdot j \omega
\]

(2.49)
where \( \Phi_{mn}(x,z) \) is the modal shape and \( q_{mn}(\omega) \) is the modal amplitude of the \((m, n)\)th mode of the plate.

For the simply supported plate, Equation (2.48) has a closed form given as

\[
v(k_x, k_z) = \frac{2j \omega \pi^2}{ab \sqrt{m_s ab}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn}^* \sum_{\nu=1}^{\infty} \left[ \frac{(-1)^m e^{i \nu k_x} - 1}{[k_x^2 - \left( \frac{m \pi}{a} \right)^2]} \right] \left[ \frac{(-1)^n e^{i \nu k_z} - 1}{[k_z^2 - \left( \frac{n \pi}{b} \right)^2]} \right]
\]  

(2.50)

where \( a, b \) are the lateral dimensions of the plate and \( m_s \) is the mass per unit area of the plate.

After substituting Equation (2.50) into Equation (2.47), the acoustic power can be calculated using a numerical technique such as the composite Simpson method. A detailed explanation on the calculation of the acoustic power can be found in Appendix D. To take into account of the characteristics of human ears, the radiated sound power is A-weighted as:

\[
L_{P,A} = 10 \cdot \log(P/10^{-12}) + L_A \quad \text{(dBA – ref. 10^{-12} Watts)}
\]

(2.51)

where \( L_A \) is the A-weighting function [44].

2.4 Summary

This chapter addresses the theoretical basis and model developments that are essential for investigating the IR problem in vibration isolators. First, both the lumped and distributed parameter isolator models were discussed. It was shown that the massless model can be achieved using the continuous model by setting the isolator density to zero. Second, a 3DOF vibration model consisting of three isolators was presented. This model can more effectively represent a practical problem than a SDOF model. The EOMs of the 3DOF system were developed with respect to the displacements at the ends of the three isolators. This representation for the EOMs is convenient to examine the isolators’
performance. Third, the force transmissibility of a specific isolator in the multi-DOF system was defined. This definition makes the results from the 3DOF model directly comparable with the previous results from SDOF model. Finally, the acoustic power radiated by the foundation was formulated. The noise radiation from the foundation will be used as an important metric for assessing the effects of the IRs since the A-weighted acoustic power emphasizes the frequency range where the IRs usually occur.
Chapter 3

Numerical Simulation

This chapter presents an in-depth numerical analysis of the IR effects on isolators’ performance and reveals the significance of the IR problem. As mentioned in Chapter 1, previous studies examined the IR problem only from the point of view of transmitted force using SDOF models. This dissertation explores the importance of IRs by examining both the force transmissibility and the sound radiated from the foundation using a 3DOF vibration model. In this chapter, the force transmissibility and the radiated sound power are computed for the system shown in Figure 2.4, which is termed the reference system. The baseline parameters of this reference system are selected according to the values in practical systems. Based on the reference system, a parametric study on the effects of variation of isolator properties on the IRs and isolators’ performance in the IR range is performed. The isolator properties that are considered in the parametric study include the
Young’s modulus, damping, etc. The influences of isolators’ locations on the foundation and the foundation’s flexibility on the IRs are also investigated numerically. In addition, a comparison is made between the predictions of the 3DOF and the SDOF models.

3.1 Selection of Parameters for the 3DOF Reference System

Refer to Figure 2.4, the reference system has a primary mass \( m = 27.8 \) kg. The moments of inertia about the \( x \)-axis and the \( z \)-axis of the primary mass are \( J_{xx} = 0.44 \) and \( J_{zz} = 0.82 \) kg.m\(^2\), respectively. An external force of amplitude \( F_0 \) is applied on the primary mass. To induce rotations of the primary mass, the force is applied off the center of gravity at \((x_F, z_F) = (0.1, 0.1)\) m. Figure 3.1 shows the top view of the reference system to illustrate the position relationships between the external force, the primary mass, the three isolators, and the foundation. In the figure, it is seen that the primary mass is supported at the center of the foundation through three isolators. Isolators 1 and 2 are symmetrical about \( z \)-axis and isolators 3 and 1, 2 are symmetrical about \( x \)-axis. The three isolators are identical with length 0.066 m and cross sectional area 0.00123 m\(^2\). They are made of viscoelastic material with density 1103 kg/m\(^3\), Young’s modulus 20 MPa, and loss factor 0.1. The foundation is a 1.5×1×0.02 m rectangular steel plate (the long side is parallel to the \( x \)-axis) with simply supported boundary conditions. The modulus of elasticity of the foundation is \( 2 \times 10^5 \) MPa, the density is 7800 kg/m\(^3\), the loss factor is 0.01, and the Poisson's ratio is 0.28.

The above system parameters were selected on the basis of information about practical systems. For example, the primary mass can represent a small engine. The selection of the isolator parameters was based on both the properties of a commercial rubber mount manufactured by Lord Corporation and the values presented in the bibliography [7, 18]. The parameters of the foundation were chosen in such a way that it simulates a typical flexible base, such as the body of a car.
In the numerical study the primary mass is subject to a band-limit white noise excitation with unit amplitude in the [0–3000] Hz frequency range. This disturbance induces three system resonances at 27.8, 37.5 and 45.6 Hz, which correspond to the translation, rotations about z- and x-axis of the primary mass, respectively. Two IRs at frequencies of 1018.5 and 2037.0 Hz as well as several resonances corresponding to the foundation’s modes are also located in the frequency range of this disturbance. Thereby, they are also excited. For the sake of easy comparison, the performance of different isolator’s configurations is compared using the root-mean-squared (RMS) value of the force transmissibility or the total acoustic power over selected frequency ranges. Since
the system resonances are not of interest in this study, the RMS metric is calculated for frequency ranges that exclude the system resonances, i.e. low frequencies are excluded.

### 3.2 IR Effects on Vibration Isolation

Substituting the parameters listed in section 3.1 into the reference model and using Equation (2.46), the force transmissibilities of the three isolators in the reference system can be calculated as a function of frequency. These transmissibilities are plotted in Figure 3.2. The amplitude of the transmissibility is denoted in decibel scale with reference value of 1. The solid lines are the transmissibilities calculated from the realistic isolator model in which the isolator’s distributed mass is taken into account. For comparison, the transmissibilities for the three isolators predicted from the massless model are also shown in the same figure as dotted lines.

Note that the transmissibility curves of isolators 1 and 2 are identical since the positions of isolators 1 and 2 are symmetric about the z-axis (refer to Figure 3.1). For the same reason, the system mode corresponding to the rotational DOF of the primary mass about z-axis cannot be seen on the transmissibility curves. A detailed explanation for this phenomenon can be obtained by recalling the transmissibility definition of Equation (2.44) or (2.45). At the frequency where the mode of the rotation about z-axis is excited, the displacements at the top ends of isolators 1 and 2 have the same amplitude while in the opposite direction; the displacement at the top end of isolator 3 is zero since isolator 3 is located on z-axis. Bear in mind that the properties of the three isolators are identical. Under these conditions, the transmitted forces through the isolators satisfy the following relationship: $f_{i1} = -f_{i2}$ where $i = 1, 2, 3$. Therefore, only the component $f_{i3}$ contributes to the transmissibility of the $i^{th}$ isolator $T_i$ (Equation (2.45)). Hence, only the system resonances corresponding to the translation and the rotation about x-axis can be seen in Figure 3.2. In addition to the system resonances, there are several other resonances that
correspond to the foundation’s modes and the IRs of the isolators. Some of the resonant frequencies of the system are listed in Table 3.1. It is observed that the foundation’s first mode at 73 Hz is very significant compared with other modes. This is because the three transmitted forces through the isolators are in phase at this mode. On the contrary, it is difficult to distinguish the foundation’s resonances of higher order. This indicates that the foundation’s resonances at high frequencies have little effect on the transmissibility of the system as long as the foundation is reasonably stiff. More detailed information on the effects of the foundation’s flexibility can be found in Section 3.4. The first two IRs of the isolators in the reference system are at frequencies above 1000 Hz.

Table 3.1: Resonant frequencies for the reference system

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>System resonances of primary mass</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>27.8</td>
<td>Translation in y-direction</td>
</tr>
<tr>
<td>2(^{a})</td>
<td>37.5</td>
<td>Rotation about z-axis</td>
</tr>
<tr>
<td>3</td>
<td>45.6</td>
<td>Rotation about x-axis</td>
</tr>
<tr>
<td>Foundation’s resonances</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>71.8</td>
<td>(1,1) mode</td>
</tr>
<tr>
<td>2</td>
<td>133.3</td>
<td>(2,1) mode</td>
</tr>
<tr>
<td>3</td>
<td>214.4</td>
<td>(1,2) mode</td>
</tr>
<tr>
<td>Isolators’ IRs</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1018.5</td>
<td>1(^{st}) IR</td>
</tr>
<tr>
<td>2</td>
<td>2037.0</td>
<td>2(^{nd}) IR</td>
</tr>
</tbody>
</table>

\(^a\) Not shown in Figure 3.2.
The results shown in Figure 3.2 give a quantitative measurement of the significance of the IR problem. For example, comparing the curves for isolator 1 predicted from two models that consider and neglect the isolator mass, respectively, it is seen that the force transmissibility of the realistic isolator can be 20 - 30 dB higher at IR frequencies than that of the ideal massless isolator. This conclusion is similar to what had been obtained by previous researchers [5, 6, 15-17, 19]. As for the asymptotic behavior, the transmissibility for the portion of the curves (the valleys) between the IR peaks decreases at about 6 dB per octave for the realistic model instead of 12 dB per octave for the massless case [5], which indicates that the traditional isolation model with massless isolators significantly over-estimates the performance of realistic isolators at high
frequencies where the IRs occur. It should be mentioned that the asymptotic decreasing of the transmissibility is approximately equal to 6 dB only for small values of the loss factor, e.g., for $\eta < 0.2$. In conclusion, the comparison made in Figure 3.2 implies that the massless isolator model does not work well at high frequencies where the IRs usually appear.

### 3.2.1 Comparison of Force Transmissibility Predictions of the 3DOF and SDOF Models

The model of the equivalent SDOF system was introduced in Section 2.2.6. To examine the difference of the predictions between the 3DOF and the SDOF models, the transmissibility calculated from the SDOF model (Equation 2.42) are plotted in Figure 3.3 together with the transmissibilities obtained from the 3DOF model. The only system resonance of the equivalent SDOF model is at 30.0 Hz, which corresponds to the translation of the primary mass.
It is interesting to observe that the transmissibility for one or more isolators in the reference system (3DOF) can be either higher or lower than the transmissibility for the SDOF system. For instance, the transmissibility of isolator 1 is 5 - 6 dB higher than that of the SDOF system at the IR frequencies. Part of the reason for this phenomenon lies in the fact that generally, the transmissibility curve starts decreasing at a rate that depends primarily on the isolator internal damping after the highest system resonance. Therefore, when the highest resonant frequency of the primary system increases, the value of the transmissibility at the IRs increases. Since the highest resonant frequency (45.6 Hz) of the reference 3DOF system is higher than the resonant frequency (30.0 Hz) of the equivalent SDOF system, the values of the transmissibility at the IRs of the 3DOF could
be larger than their counterparts of the SDOF system (Figure 3.2). Another reason for this behavior is related to the locations at which the isolators are connected to the primary mass. This will be discussed in section 3.2.3. The observation that the transmissibility in a multi-DOF system may be higher than that in its equivalent SDOF system at IR frequencies indicates that the IR problem in the multi-DOF system is more important than what the SDOF model reveals. The IRs can lead to significant deterioration of the performance of one or more isolators in a multi-DOF system but the SDOF model might fail to show this deterioration.

**3.2.2 Effects of Isolator Properties on Vibration Isolation – Parametric Study**

The effects of the IRs on the force transmissibility are shown in Figures 3.2 and 3.3 for a particular set of values of isolator and system parameters as described in the reference system. In order to evaluate the importance of the IRs in practical isolators and also gain in depth knowledge about IR problems, it is necessary to investigate how the isolator parameters affect the IRs and furthermore affect the isolator performance. It is found that the IRs, their frequencies and amplitudes, are significantly affected by three fundamental parameters: (1) the mass ratio ($\mu$), which is defined as the primary mass to the total mass of isolators, (2) the damping which may characterized as the loss factor ($\eta$) and (3) the Young’s modulus ($E$) of the isolator material.

It was mentioned in Chapter 1 that some previous researchers concluded that the IRs are not of the primary concern in the vibration isolation design [16, 17]. However, after the following parametrical studies in terms of the force transmissibility and considering the practical parameter ranges of isolator material, it will be shown that the IR problem is more severe than previously predicted.
Since at this point the characteristics of the IRs as a function of isolator properties are of interest, the number of DOF in the model does not play an important role. Therefore, for simplicity, the SDOF model is used in the following three sub-sections.

### 3.2.2.1 Mass Ratio

Derived from the model of the isolator with distributed mass, the force transmissibility of a SDOF system can be expressed directly as a function of the mass ratio as:

\[
T_{SDOF} = \left| \frac{1}{\cos(kL) - \mu k L \sin(kL)} \right|
\]  

(3.1)

There are two ways to change the mass ratio: (a) change the primary mass \(m\) and (b) change the density of the isolator, \(\rho\), while holding the other parameters constant. The former method leads to variation of the system resonant frequency; the latter one fixes the system resonance but changes the IR frequencies.

Figure 3.4 shows the theoretical curves of transmissibility for different mass ratios that were obtained by changing the primary mass, with the loss factor held constant at 0.1. For comparison purpose, the transmissibility of the massless isolator for \(\mu = 100\) is also shown. The resonances below 200 Hz correspond to the system resonance and the first mode of the foundation. It is observed that the system resonant frequency is lower for a high mass ratio and the level of transmissibility curve of high mass ratio is much smaller than that of low mass ratio. This may indicate that the IRs become less important as \(\mu\) increases because the transmissibility decreases with the mass ratio increasing. In other words, when the primary mass is very heavy compared to the isolator mass, the transmissibility may be already very low at frequencies where the IRs occur, so that the IRs might not be an issue. From this point of view, it is, therefore, desirable to have a mass ratio as large as possible, which is not always feasible in practical systems. In practice, most mounting systems have mass ratios ranging from 50 to 1000 [5, 17].
Figure 3.4: Effect of mass ratio on transmissibility – various mass ratios are obtained by changing the value of the primary mass.

Figure 3.5 shows transmissibility for different mass ratios that were obtained by varying the density of the isolator instead of the primary mass. The primary mass and the loss factor for each curve are fixed at 27.8 kg and 0.1, respectively. Since the primary mass and the stiffness of the isolator do not change, the system resonance appears at 30 Hz, i.e. it is not affected by the mass ratio. However, the IRs shift to lower frequencies and have narrower frequency intervals with decreasing mass ratio. For $\mu = 1000$, i.e. the density of the isolator is very small, there is only one IR barely occurring at 3000 Hz which is the highest frequency of the frequency range that is under investigation. In this
situation it is hard to say that high mass ratios are better than low mass ratios. The mass ratio that gives the best performance, i.e. the smallest transmissibility, in one frequency range may give poor performance in another frequency range. On the other hand, for a mounting system with a pre-specified or pre-designed system resonance, a lighter isolator with smaller density is always desirable, because it can move the IRs to higher frequencies. In the case where the IRs can be eventually shifted to a frequency range that is high enough so that the isolator performance in this frequency region isn’t of concern, the IRs can be neglected and the simple massless model may be appropriated. Unfortunately, this ideal situation is not found in practice because either extremely light isolator material is not always achievable or the loading capacity of such a light isolator may be very poor.

![Figure 3.5: Effect of mass ratio on transmissibility – various mass ratios are obtained by changing the density of the isolator.](image-url)
3.2.2.2 Damping

The isolator damping has significant influence on the vibration isolation. For example, the amplitudes at resonances are usually limited by material damping. In the traditional massless model, the isolator damping is generally characterized by the non-dimensional viscous damping ratio, $\zeta$. However, in the continuous isolator model used in this study, the internal damping of the isolator material is considered as structural damping represented by the loss factor $\eta$. To show the damping effect, Figure 3.6 plots the transmissibilities for a fixed mass ratio of 100 and for three loss factors, $\eta = 0.1, 0.2$ and 0.3.

![Figure 3.6: Effect of loss factor on transmissibility - showing at $\mu = 100$.](image-url)
It is seen from Figure 3.6 that the isolator damping is advantageous for attenuating the IRs. In detail, increasing damping (loss factor) decreases the transmissibility at all the resonant frequencies. This attenuation effect is more evident at high frequency range where the IRs occur. The asymptotic decrease of the valleys between the IR peaks becomes sharper with increasing damping. It is observed that, when the material loss factor reaches 0.3, the IRs are effectively damped. Furthermore, it is seen that the transmissibility at any frequency in the isolation range decreases to some extent instead of increasing with increasing loss factor. This behavior is opposite to the one predicted from the traditional massless model (Figure 1.3), which states that higher damping increases the transmissibility in the isolation range. This discrepancy is due to the different ways of representing the material damping in the models in Figures 1.3 and 3.6. The lumped parameter model presented in Chapter 1 uses a constant viscous damping ratio, while the model in this dissertation uses a constant loss factor to characterize the isolator damping (structural damping). In practical applications, the internal damping of isolator materials generally is better represented by a constant loss factor than by a constant viscous damping ratio. This is particularly true at high frequencies [19].

Mainly because of the advantage of high damping, some previous researchers concluded that the IRs problem is not important since the IRs can be effectively attenuated by simply increasing the damping of the isolator materials [16, 17]. However, this conclusion is only correct in theory. In practice, typical highly damped elastomers exhibit poor returnability and greater drift than elastomers with medium and low damping levels [7, 22]. These drawbacks limit the values of the loss factor in practical isolators. For rubberlike materials, most practical isolators have a loss factor between 0.03 and 0.2 [5, 7]. The loss factor is less than 0.05 for metal materials [16, 20]. Given these ranges for the loss factor, and the observations from Figure 3.6, it can be concluded that the IRs are significant as long as the isolator performance at the frequencies including the IRs are concerned.
In addition to the amplitude, the frequencies at which the IRs occur are also important in assessing the significance of the IRs in practical systems. From Figures 3.4 and 3.6, it can be observed that neither the mass ratio related with the primary mass nor the isolator loss factor has appreciable influence on the IR frequencies. The IR frequencies mainly depend on the density and Young’s modulus for a given isolator. The influence of the material density has already been taken into account and examined through the parameter of mass ratio; the effect of the modulus will be discussed in next section.

### 3.2.2.3 Young’s Modulus

Figure 3.7 shows the force transmissibility for values of Young’s modulus in the range from 10 MPa to 30 MPa. This is in the practical range for modulus of natural and neoprene rubber compounds, which is 2 to 50 MPa [7]. The mass ratio and loss factor used in Figure 3.7 are constant and equal to 100 and 0.1, respectively. It is observed that when the modulus decreases, the system resonant frequency as well as the IR frequencies decreases. Therefore, more IRs will appear below a given frequency (e.g. 3000 Hz) when the value of the modulus is reduced. The Young’s modulus affects the transmissibility levels too. Since the isolator stiffness is directly related to the Young’s modulus, all other parameters being equal, a smaller Young’s modulus indicates a softer isolator, and lower system resonance frequency. The traditional vibration model, which neglects IR, indicates that lowering the system natural frequency can improve the isolation efficiency by decreasing the transmissibility at high frequencies. Although this conclusion has been well adopted in the design theory when the massless isolator model is used, it could be misleading in real-life applications where the IRs exist. The following example (based on results presented in Figure 3.7) shows that, if the IRs are considered, lowering the system natural frequency by decreasing the Young’s modulus is not always an effective practice for improving the isolator performance at high frequencies.
Consider a system with an isolator whose Young’s modulus is 30 MPa. Suppose this system is subjected to a disturbance with energy in the range from 200 to 1500 Hz. If a designer finds that the isolation performance around 1250 Hz is unsatisfactory because of the first IR, and the designer is not aware of the isolator’s IRs, the designer is likely to opt for a softer isolator, for example one with modulus 20 MPa. However, although this new design will improve the isolation at around 1250 Hz as well as some other frequencies, the designer will find a new resonance occurring at 1000 Hz with almost the same amplitude as the resonance that was originally at 1250 Hz. The reason is that switching to a softer isolator does not attenuate the original IR - it shifts the IRs to lower frequencies. For instance, in the previous example, when the isolator modulus is reduced from 30 MPa to 20 MPa, the first IR frequency is consequently shifted from 1250 Hz to 1000 Hz, but the amplitudes at the first IR frequency for both isolators are almost the same at about –34 dB. This example reveals that, in this case, if the IRs are not recognized and properly attenuated, decreasing or increasing the modulus of the isolator can not improve the isolator performance.
Figure 3.7: Effect of Young’s modulus on transmissibility - showing at $\mu = 100$ and $\eta = 0.1$.

Among the three parameters – the mass ratio, loss factor and Young’s modulus, the last two completely depend on the properties of the isolator material. The influences of these two parameters on the IRs are summarized in Figure 3.8 in which the RMS transmissibility difference in the frequency rage of [200–3000] Hz, $\Delta RMS$, is plotted as a function of both the loss factor $\eta$ and Young’s modulus $E$. In the plot, $\Delta RMS$ is denoted in dB scale and is defined as $\Delta RMS = 20\log T_{RMS}(r) - 20\log T_{RMS}(i)$ where $T_{RMS}(r)$ and $T_{RMS}(i)$ represent the RMS transmissibility within [200–3000] Hz calculated for the
realistic model and ideal (massless) model, respectively. The [200–3000] frequency band was selected because it excludes all the system resonances of the reference system and includes the frequencies at which the first several IRs occur.

Figure 3.8: Effects of loss factor and Young’s modulus.

Figure 3.8 is illustrative in the sense that it implies how much improvement can be potentially obtained by suppressing the IRs. For example, if the IRs below 3000 Hz of an isolator with $E = 20$ MPa and $\eta = 0.1$ can be fully suppressed, the RMS transmissibility
can be approximately reduced by 4.5 dB. It is observed that the IRs are very significant when both the material damping and modulus are small. This is because the amplitudes of the IRs are higher for low damping and more IRs occur at frequencies below 3000 Hz for low modulus.

### 3.2.3 Effects of the Isolator Locations on the Force Transmissibility

It was mentioned earlier that the transmissibility for one or more isolators in a multi-DOF system are higher as compared to the isolator transmissibility predicted in a SDOF system. The isolator position has an important effect on the transmissibility curve at high frequencies. The effect of changing the isolator locations on the IRs can be examined by calculating the difference in the transmissibility level at an IR frequency (e.g. the first one) between the equivalent SDOF system and the 3DOF system. Since the isolation location does not affect the transmissibility in a SDOF system, the transmissibility of the equivalent SDOF system at the first IR frequency is always –34.8 dB (Figure 3.3). However, the transmissibilities for the three isolators in the 3DOF system change with the variation of isolation locations. Based on the reference 3DOF system described earlier, Figures 3.9(b) – 3.9(d) compares the transmissibility difference by moving the position of isolator 2 within a rectangular (shaded) area as shown in Figure 3.9(a) while keeping the positions of isolators 1 and 3 unchanged.

It is seen that the transmissibility levels at the first IR frequency of the isolators in the 3DOF system dramatically change as isolator 2 moves. Furthermore, it does not seem that there is a position for isolator 2 so that all the three isolators exhibit the best (or worst) performance simultaneously. For example, when isolation 2 is at the positions of \((\text{Dist}_x, \text{Dist}_z) = (0.4, 0.3)\) and \((0, 0)\), the transmissibility of isolator 1 at the first IR frequency will reach the lowest and highest values, respectively. The transmissibility difference calculated in Figures 3.9(b) – 3.9(d) can also acts as an indication of the
significance of the IRs with changes of isolator positions. For instance, when the isolator 2 is at \((\text{Dist}_x, \text{Dist}_z) = (0, 0.3)\) and \((\text{Dist}_x, \text{Dist}_z) = (0.4, 0.3)\), the first IR amplitude of isolator 1 is 9 dB higher and the first IR amplitude of isolator 2 is 11 dB higher than the one predicted by the equivalent SDOF system, respectively. These results reveal that the IR problem in a multi-DOF system could be much more important than that in a SDOF system.
Figure 3.9: Variation range of the position of isolator 2 (a) and transmissibility difference of isolator 1 (b), isolator 2 (c) and isolator 3 (d) as a function of the position of isolator 2.
3.3 \textbf{IR Effects on Structure Borne Noise}

In the prior section, force transmissibility has been employed as a metric for the isolator performance. The radiated sound power should be another important measure of the isolator performance. In this section, the importance of the IRs is investigated in terms of the radiated noise of the base structure, i.e. the foundation.

In Figure 3.10(a), the A-weighted sound power level radiated by the foundation of the reference system is presented assuming the external force to be $F_0 = 1$ N. The result calculated using the practical isolator model with inertia is compared to the sound power predicted using the ideal massless isolator model. The force transmissibility for isolator 1 of the reference system is also plotted in Figure 3.10(b). For completeness, the figure is generated by assuming that the external disturbance $F_0$ has constant spectral density from 1 to 3000 Hz. However, in practice, the disturbance in the low frequency range (e.g. <100 Hz) including the system resonant frequencies is very small. This is because the isolation systems are designed in such a way that the excitation energy mainly occurs in the isolation region, i.e. at frequencies higher than system resonances (Figure 1.3). Thus, for the reference system it is reasonable to assume that the excitation has little energy in frequencies below 200 Hz. Thus, the total sound power for the frequency band from 200 to 3000 Hz is used to quantify the isolator performance.

Figure 3.10(a) shows that the radiated sound power for the realistic isolator with mass is significantly higher than the radiated sound power for the ideal, massless isolator in the frequency range where the IRs occur. The total sound power in the [200–3000] Hz band for the cases of realistic and ideal massless isolators are 53.6 and 47.9 dBA, respectively. This result indicates that a total sound power reduction of up to 5.7 dB can be obtained if the IRs are fully suppressed. The potential sound power reduction at the specific IR frequencies can be as high as 30 dB. The total sound power reduction or the sound power
reduction at the specific frequency can also be interpreted as the detrimental effect of the IR on the radiated noise. Hence it can be concluded that these values reveal the significance of the IRs from the perspective of noise radiation.

Figure 3.10: (a) Radiated sound power by the foundation and (b) Force transmissibility of isolator 1 of the reference system.
3.3.1 Effects of Isolator Properties on Noise Radiation – Parametric Study

As shown earlier, the isolator properties affect the amplitudes and frequencies of the IRs and consequently the level of the radiated noise. Therefore, it is useful to investigate the effect of the isolator parameters on the radiated sound power as it has been done for the transmissibility in Figure 3.8. To quantify the effect induced by the IRs, the total sound power reduction $\Delta P$ is introduced. It is defined as the difference between the total sound power radiated by the foundation of a system with practical isolators and the sound power radiation for the same system but with ideal massless isolators.

The total sound power reduction, within the frequency range from 200 to 3000 Hz as a function of both the Young’s modulus and loss factor of the isolator material is plotted in Figure 3.11. The result in Figure 3.11 demonstrates the beneficial effect of high damping on the radiated noise as Figure 3.6 did for the force transmissibility. It is seen that when the loss factor increases, the sound power reduction decreases. This is because the IRs are effectively attenuated by the high damping. Figure 3.11 also shows that the noise reduction does not change monotonically with the Young’s modulus. Increasing the modulus will increase the overall response level (which tends to increase the radiated noise), but it will also shift the IR frequencies towards higher frequencies. The latter effect reduces the radiated noise because the excitation is assumed to have no energy above 3000 Hz. In general, increasing Young's modulus tends to reduce noise because the IR frequencies increase faster than the amplitudes do with increasing modulus.
Figure 3.11 demonstrates the potential in terms of sound power reduction that can be obtained by suppressing the IRs and making the practical isolators behave like the ideal massless isolators. It is observed that the detrimental effect of the IRs on the radiated noise indicated by $\Delta P$ is insignificant in the area where the loss factor is high and the Young’s modulus is large. However, this area is not preferred in practice because (a) high damping generally leads to large creep in viscoelastic materials and (b) large Young’s modulus results in high system resonance frequencies, which degrade the isolation.
performance. According to the parameters of most practical isolators made of viscoelastic materials [7], the total sound power reduction parameter would range from 3 to 22 dB.

It should be pointed out that Figure 3.11 is plotted by using values of Young’s modulus and loss factor mainly for elastomers. The observations and conclusions from Figure 3.12 are not valid for metal springs because, compared with elastomers, metals generally have very large modulus of elasticity and very small loss factor. Furthermore, a metal spring functions in a different way than an elastomer isolator. However, it is possible to show the effects of the IRs in the metal coil springs using the model presented in Chapter 2. In order to conduct the analysis, the coil springs are equated to “long-rods” and the equivalent “rod-lengths”, \( L_{rod} \), for the coil springs are calculated according to the method introduced in reference [9]. That is

\[
L_{rod} = \sqrt{\frac{2E}{G}} \left( \frac{D_{mean}}{d} \right) L_s
\]  

(3.4)

where \( E \) and \( G \) are the Young’s and shear modulus of the coil spring material, respectively; \( D_{mean} \) is mean coil diameter; \( d \) is the spring coil diameter and \( L_s \) is effective spring length which is proportional to the number of effective coils \( n \), and is given by \( L_s = \pi n D \). For steel, \( E/G \approx 2.5 \), hence Equation (3.4) becomes

\[
L_{rod} = \sqrt{5} \left( \frac{D}{d} \right) L_s = \sqrt{5} \frac{n \pi D^2}{d}
\]  

(3.5)

Figure 3.12 shows the force transmissibility and radiated sound power for the reference system while replacing the viscoelastic isolators with steel coil springs. The three coil springs are identical and have the same static stiffness as the previously used viscoelastic isolators. The loss factor used for the coil springs is 0.01, which is typical for metal material. Figure 3.12(a) compares the radiated sound powers at different frequencies for the coil springs with and without inertia. The total sound powers for the cases of realistic and ideal massless coil springs in the range from 200 to 3000 Hz are
65.8 and 42.1 dBA, respectively. This means that a total sound power reduction of 23.7 dB can be obtained by suppressing the IRs in these coil springs. This is a considerable amount comparing with the corresponding reduction in sound power for the viscoelastic isolators which is only 5.7 dB. The significance of IRs in metallic springs can be shown again by examining the transmissibility response. Figure 3.12(b) shows that at many frequencies the transmissibility curve for the coil spring with mass, i.e. with IRs, is more than 30 dB higher than that for the coil spring without mass. It is also noted that the first IR of the coil spring appears at a frequency as low as 161 Hz. In addition, the frequency density of the IRs shown in Figure 3.12(b) is very high (there are more than 10 IRs below 3000 Hz). To this end, a comparison can be made by recalling the transmissibility curves of the viscoelastic isolators shown in Figure 3.2 in which the first IR appears at 1018.5 Hz and there are only three IRs below 3000 Hz. It is seen that the IRs in metal springs appear at much lower frequencies and have much higher values compared to the IRs in viscoelastic isolators. This is primarily due to the mechanism that the metal spring functions and the low damping in metal material. Therefore, it is concluded that the IRs in metal springs are very significant; they greatly increase the sound power radiated by the foundation and the force transmissibility. Because of the demonstrated significance of the IRs in metal springs, a lot of research efforts on this issue can be found in the currently open literature. For example, references [9] and [11] present some detailed analysis about IRs in metal springs and their suppression. However, due to the lack of attention of IRs in isolators made of viscoelastic materials (rubber, elastomer, etc.), the analytical examples in this chapter, the experimental validations and also the proposed approaches in the next two chapters for attenuating the IRs are based on the rubber isolators.
Figure 3.12: (a) Radiated sound power by the foundation and (b) force transmissibility of isolator 1 of the reference system with metal coil springs.

3.3.2 Effects of Isolator Locations on Noise Radiation

In the reference 3DOF system, the excitation forces applying on the foundation by the isolators can be simply considered as three concentrated forces. When the foundation substructure is separated from the coupled vibration isolation system as shown in Figure 2.5, the problem of calculating the radiated acoustic power of the foundation is equivalent to the problem of calculating the sound power of a simply supported finite plate. The external excitation of this plate is three concentrated forces acting at three arbitrary
locations. Assuming that the input force vectors do not change, it is very reasonable to expect different values in sound power radiation when placing the three forces at different locations. For instance, if the three isolators together with the primary mass supported on their tops are shifted 0.175 m toward the negative $z$-direction (Figure 3.13(b)), the radiated acoustic power of the foundation will be 48.3 dBA. Although this number is only 0.7 dB higher than the original value (47.6 dBA), it indicates an approximate 17.5% increase in the radiated energy in linear scale.

In the above example, the transmitted forces through the $i^{th}$ isolator between cases 1 and 2 (Figure 3.13) are the same. It shows that when the isolators are connected to the foundation at different locations from the original positions, the radiated noise of the foundation may become higher. An even worse situation will occur when the connecting locations between the isolators and the primary mass are also changed. This situation is illustrated in case 3 of Figure 3.14(b). That is, the connecting point between isolator 2 and the primary mass is moved to the center of the bottom edge of the primary mass, and the positions of the three isolators on the foundation are more closer to the center of the foundation. According to the analysis in Figure 3.9, when isolator 2 is at the center of the bottom edge of the primary mass, the force transmissibility of isolator 1 and 2 are higher than before (i.e., when isolator 2 is at its original position). In this case, the total acoustic power within [200-3000] Hz is 49.1 dBA, which implies a 40% increase in radiated energy when comparing to case 1. Table 3.2 lists the isolator positions in the coordinates fixed on the primary mass and the foundation for the three cases mentioned above.
Figure 3.13: Changes of isolator positions on the foundation from the (a) original locations to the (b) new locations.
Figure 3.14: Changes of isolator positions on the foundation and the primary mass from the (a) original locations to the (b) new locations.
### Table 3.2: Positions of isolators used in Figures 3.13 and 3.14

<table>
<thead>
<tr>
<th>Case #</th>
<th>Position on the foundation coordinates ((x, y)) (m)</th>
<th>Position on the primary mass coordinates ((x', z')) (m)</th>
<th>Acoustic power in 200-3000 Hz (dBA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1 – original locations</td>
<td>Isolator 1 ((0.5, 0.325))</td>
<td>((-0.25, 0.175))</td>
<td>47.6</td>
</tr>
<tr>
<td></td>
<td>Isolator 2 ((1, 0.325))</td>
<td>((0.25, 0.175))</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Isolator 3 ((0.75, 0.675))</td>
<td>((0, -0.175))</td>
<td></td>
</tr>
<tr>
<td>Case 2</td>
<td>Isolator 1 ((0.5, 0.5))</td>
<td>((-0.25, 0))</td>
<td>48.3</td>
</tr>
<tr>
<td></td>
<td>Isolator 2 ((1, 0.5))</td>
<td>((0.25, 0))</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Isolator 3 ((0.75, 0.85))</td>
<td>((0, -0.35))</td>
<td></td>
</tr>
<tr>
<td>Case 3</td>
<td>Isolator 1 ((0.55, 0.49))</td>
<td>((-0.25, 0.175))</td>
<td>49.1</td>
</tr>
<tr>
<td></td>
<td>Isolator 2 ((0.75, 0.3))</td>
<td>((0, 0.3))</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Isolator 3 ((0.9, 0.75))</td>
<td>((0, -0.175))</td>
<td></td>
</tr>
</tbody>
</table>

In summary, the isolator locations affect both the transmissibility of individual isolator and the noise radiation of the foundation. When the isolation performance is unacceptable due to the IRs, suppressing the IRs would be necessary when changing isolator locations is not helpful or impractical.
3.4 Influence of the Foundation Flexibility on the IRs

In Section 3.2, it was already shown that the curve representing the force transmissibility as a function of the frequency has many resonance peaks corresponding to the foundation’s modes. The foundation’s flexibility, i.e. its resonances, affects the design of the isolation system in such a way that it significantly shifts resonant frequencies corresponding to the motion of the primary mass [44, 45]. However, it has little effect on the IR frequencies, because the IR frequencies solely depend on the boundary conditions and the properties of the isolator. As long as the dynamic stiffness of the foundation is high enough in the frequency range where the IRs occur, the boundary condition of the isolator can be considered as unchangeable in terms of the foundation flexibility, and so are the IR frequencies. Fortunately, it is normally the case in reality. Therefore, the effect of the foundation resonances on the force transmissibility is generally insignificant (Figures 3.3 – 3.7), especially at high frequencies where the IRs occur [46, 47]. Although the IR frequencies are nearly independent with the foundation’s modes, there is a situation where one IR frequency coincides with one of the foundation modes. At this mode, the foundation acts as a dynamic vibration absorber tuned to that IR frequency. In this case, the peak at that original IR frequency is suppressed, and two new resonance peaks appear on both sides of the original IR frequency. This phenomenon had been shown in Figure 3.7. Since the foundation flexibility does not significantly affect the characteristics of the transmissibility curve at high frequencies, this “DVA-like” effect is consequently slight.
Figure 3.15 compares the responses of the reference system with stiff and soft foundations. The foundation stiffness is adjusted by changing the thickness while keeping all other parameters constant. Figure 3.15(a) shows that the level of the radiated sound power for the soft foundation is higher than that for the stiff foundation at most frequencies. The reason is that a soft structure is more efficient for sound radiation than a stiff structure. The total sound power is 47.6 and 54.4 dBA for the stiff and soft foundations, respectively. The total sound power radiation by the soft foundation is 6.8 dB higher than that by the stiff foundation. From this perspective, increasing the
foundation stiffness is a good method for reducing the noise radiation, especially when the IRs occur. On the contrary, the foundation flexibility has little influence on the force transmissibility function. As shown in Figure 3.15(b), except for the obvious shifting of the resonant frequencies, the level of the transmissibility curves for soft and stiff foundations have little effect on the frequency range where the IRs occur.

### 3.5 Influence of the Property Variation of Isolator on IRs

The effects of changing the isolator material properties, i.e. Young’s modulus and loss factor, on the IRs have been investigated in Section 3.2.2. For simplicity, the isolator properties were employed to generate Figures 3.6 – 3.8 were assumed to be frequency and temperature independent. In practice, Snowdon [16, 17] and DeJong et al. [14] had shown that both the dynamic modulus and the loss factor of isolator material are functions of frequency and temperature. Specifically, for room temperatures and through the range of frequencies normally of concern in vibration problems, the dynamic modulus and loss factor generally slowly increase with increasing frequency and decreases with increasing temperature (Figure 1.7).

This temperature- and frequency-dependency information can be easily synthesized with results presented in Figures 3.6 and 3.7 to explore the significance of the IRs in practical isolators at different frequencies and temperatures. For example, it is observed from Figure 3.6 that high damping helps attenuate the IR amplitudes and from Figure 3.7 that increasing modulus moves IRs towards higher frequencies, which may be out of the range of interest. It is also known that practically, the modulus and damping will increase as the frequency increases. Thus, it turns out that the higher order IRs at very high frequencies can barely be observed physically. Therefore, the effects of the first several IRs are dominant within the range of practical interest. Consequently, if one wants to
improve the isolator performance by attenuating the IRs, one should primarily try to suppress the first several IRs.

3.6 Summary

Although several researchers have been aware of the IRs in isolators since the 1950s, they did not consider them to be of primary concern, particularly when viscoelastic isolators are used. This conclusion could be misleading because (a) previous researchers used a SDOF model, which may fail to reveal the importance of IRs, and (b) previous researchers did not investigate the effects of IRs on the radiated sound power.

In this chapter, the characteristics of the IRs, their frequencies and amplitudes, are investigated analytically. To evaluate the importance of the IRs, a system of a rigid primary mass connected to a flexible foundation through three isolators was considered as the reference 3DOF system whose parameters were selected based on practical information on rubber isolators. The 3DOF model has several advantages over the previously used simple SDOF model. For example, it can better represent a realistic vibration isolation system, and make it possible to examine the isolator location effect on the IRs. The significance of the IRs were demonstrated by comparing the force transmissibility of individual isolator and the radiated acoustic power of the foundation for the reference system with realistic and ideal massless isolators.

Comparison of the force transmissibility of the realistic isolator model with mass and the ideal massless isolator model shows that the effects of IRs are important. The transmissibility for the isolator with mass may be 20 – 30 dB higher than that for the massless isolator at the IR frequencies. This result is similar to that reported by previous researchers [4-5, 8-11]. Furthermore, this dissertation shows that the IRs can result in a more significant deterioration of the performance of one or more isolators in a multi-DOF system than previous researchers predicted using the SDOF model. Although this study
shows that the IRs in viscoelastic isolators can be attenuated using high damping of materials as previous studies stated, in real life it is impractical to build vibration isolators using materials with high damping. The loss factor in most practical viscoelastic isolators is too low to effectively suppress the IRs.

Besides evaluating the force transmissibility, one also needs to examine the radiated sound power to assess the importance of IRs. It was shown that neglecting the mass of isolators can lead to significant underestimation of the sound radiated by a foundation at frequencies around which the IRs occur. The analytical model shows that the total sound power, for the reference system, in the frequency band of [200–3000] Hz may increase 3 - 22 dB due to the IRs in the viscoelastic isolators.

The IRs in metal coil springs were briefly discussed in this paper. It is evident that the IR problem in metal springs is very significant from the perspective of either the force transmissibility or the radiated sound power.

The effects of the foundation’s flexibility were also examined. It was shown that the foundation flexibility does not play an important role in the force transmissibility function of the isolator. However, the noise radiation is greatly affected by the foundation’s flexibility.

Finally, the influence of the temperature- and frequency-dependency of the isolator properties on the IRs was briefly studied. It was concluded that higher order IRs may not be of practical interest because of their low amplitudes as the result of the isolator properties variation with the frequency. This implies that only the first two or three IRs may need to be considered and attenuated in practice.
Chapter 4

Passive Control of Internal Resonances

The characteristics of the IRs and their impacts on the isolator performance were investigated numerically in the previous chapter. It is concluded that if one wants to further improve the isolator performance, especially at high frequencies (e.g. > 500 Hz), one needs to consider the IR problem. In some situations, the IRs might be “avoided” by carefully selecting the isolator material and parameters for the isolation system. From the perspective of the attenuation of the IR problem, the process of selecting the isolator and system parameters can be considered as the control of IRs in the design stage. Another simple method to alleviate the adverse effects of IRs is to increase the isolator’s damping so that the IRs can be “internally” damped. However, as pointed out earlier, high damping isolators have other problems that limit their application in practical vibration systems. In most situations, the IRs may have to be suppressed or attenuated by way of
employing some additional devices that are attached to the isolators. In such cases, the
dynamic vibration absorber (DVA) is a good candidate to control the IRs since its nature
is to suppress the resonance of structures by “absorbing” the energy from the main
structure at a specific frequency. Depending on the actual requirements, the DVA can be
used either passively or actively. Each implementation has its own advantages and
disadvantages. This chapter concentrates on the control of IRs using passive DVAs
(PDVAs).

The first four sections in this chapter are theoretical and numerical analysis of using
PDVAs to suppress the IRs. The PDVA control concept is introduced in Section 4.1.
Based on the isolator model developed in Chapter 2, the model of the PDVA enhanced
isolator is derived in Section 4.2. It is followed by a brief discussion on the optimal
design of the DVA parameters. This optimization can be based on two objective
functions: one is the force transmissibility and the other is the radiated sound power from
the foundation. In Section 4.3, the PDVA enhanced isolator model is applied into the
3DOF reference system to examine the PDVA performance in attenuating the IRs. The
fourth section presents both the experimental validations of the IR phenomenon in
commercial isolators and some experimental results showing the performance of the
PDVA enhanced isolator.

4.1 Concept of Using DVA to Suppress IRs

Dynamic vibration absorbers are essentially mass-spring-damper subsystems which,
once connected to a main vibrating system, are capable of absorbing the vibration energy
at the attachment point. Usually, the DVA parameters (mass, spring stiffness and
damping) are chosen, i.e. tuned, so as to minimize vibrations at a specific frequency. In
other words, the motion of the DVA mass counters the motion of the main vibrating mass
most effectively at a single frequency. For this reason, the DVA is normally used to
control the vibration levels at resonant frequencies (Figure 1.9). The vibration amplitude of the main mass is reduced in a certain range of frequency which is a function of the DVA mass, stiffness and damping. For an example, Figure 4.1 illustrates the application of DVA in suppressing the IRs.

![Figure 4.1: Concept of using DVA to suppress the IRs.](image)

In Figure 4.1, a DVA is tuned to the first IR frequency of the isolator in the equivalent SDOF system. As a result, the amplitude of the force transmissibility at this frequency is greatly reduced and, due to the dynamics of the DVA, two new resonant peaks appear at
each side of the original IR peak. However, the amplitudes of these two new peaks could be much lower than that of the original IR peak by properly selecting the DVA parameters. This observation is the basis of using DVA to suppress the IRs, and thus to improve the isolator performance.

It has been shown that the mechanism of the IRs and the system resonance are different. The IRs are due to the distributed mass of the isolator itself, so that there are infinite number of IRs, while there are only limited number of system resonance corresponding to the number of DOF of the primary mass. Consequently, the application of DVAs in suppressing IRs is expected to be different from the traditional application of DVAs in suppressing the system resonance. This necessitates some discussion before the mathematical model of the DVA enhanced isolator can be developed. There are three basic questions: (a) which and how many IRs should be of the primary concern? This determines the frequency range in which the DVA should take effect. (b) Where should the DVA be attached so that it is most effective? And (c) is there any other practical factor that should be considered in the DVA application? These three questions are addressed from both analytical and practical perspectives in this section.

**Number and order of IRs**

The problem of determining the number of the IRs that are needed to be controlled is twofold in nature. Firstly, one needs to determine the frequency range in which the isolator performance is of the greatest interest for a practical vibration isolation problem. Secondly, one also needs to find out the most significant IRs among the infinite number of IRs for a given isolator working in a specific system.

The excitation input of practical isolation systems could be either harmonic, such as the unbalanced mass of rotational machinery (e.g. engine), or white noise, such as the
road surface excitation to vehicles. For either case, the highest frequency component has physical limit. For example, the highest disturbance frequency to the engine depends on the rotation speed of crankshaft while the highest excitation frequency due to the road surface is limited by the speed of vehicles. The normal frequency range, in which the practical excitation energy is significant for mechanical vibrations, is from several tens to several thousands Hertz. Consequently, the isolator performance in this frequency range is of the primary concern. Therefore, the IRs, if there is any, in this range should attract necessary attention. Below a certain frequency, for example 3000 Hz, there are probably only limited numbers of IRs starting with the very first one at a frequency higher than the system resonant frequency. In conclusion, based on the practical excitation range, only the first several IRs of an isolator need to be considered.

On the other hand, observed from Figure 3.6, the amplitudes of the higher order IRs decrease rapidly with the frequency, i.e. the higher order IRs are effectively damped out by the material damping of the isolator. It should be pointed out that Figure 3.6 was predicted under the assumption that the isolator can be modeled as a “long-rod”, i.e. the lateral deformation of the isolator under the longitudinal excitation is ignored. A more complex model based on the theory developed by A. E. Love [48] that accounts for the “lateral-inertia” shows that the magnitude of the higher order IRs decreases even more rapidly than that predicted in Figure 3.6 [15]. This leads to the same conclusion with the analysis based on the excitation. That is, the first several IRs have the most practical significance in terms of the isolator performance. In this dissertation, without loss of generality, the first two IRs of the isolator are chosen to be controlled and accordingly two DVAs are used.
DVA Attaching Positions

Since the DVA functions by absorbing the vibration energy, it is always desirable to attach the DVA at location of maximum vibration. The best attaching position can be obtained by examining the deformation of the isolator at different vibration modes of the primary mass. For instance, if a primary system whose mass is relatively much larger than the isolator’s mass (which is usually the case in practice) is connected to a rigid foundation through an isolator (refer to Figure 1.4(a)), the displacement fields of the isolator at the first three resonant frequencies can be calculated and are plotted in Figure 4.2.

It is seen that at the system resonance, the top of the isolator has the maximum displacement. Therefore, the DVA is classically attached to the primary structure when the system resonance is to be attenuated. However, this configuration is not desirable for suppressing the IRs because the ends of the isolator are motionless at the IRs. As a matter of fact, Figure 4.2 shows that in terms of the IRs, the mode shapes of the isolator can be approximated as if it undergoes longitudinal vibration with pinned-pinned boundary condition. The maximum displacement at IR modes takes place within the isolator, which indicates that the DVA should be attached or embedded directly into the isolator to suppress IRs. This conclusion is further validated in Figure 4.3.

Figure 4.3 shows the force transmissibility of the isolator in a SDOF system whose foundation is rigid. To suppress the first IR, a DVA is integrated into the isolator and is tuned to the first IR frequency. Three attaching positions are considered. They are a distance of 1/8, 1/4, and 1/2 of the isolator’s total length measured from the top of the isolator. In the three cases, the DVA parameters (stiffness, damping and tuning frequency) are the same. For comparison, the curve for the case without DVA is also plotted in the same figure. It is observed that when the DVA is placed at the center of the
isolator, the amplitude of the transmissibility curve at the first IR frequency reaches the lowest level. As expected, this result implies that the center of the isolator is the best position to attach DVA to suppress the first IR, i.e. anti-node of the mode. Referring to Figure 4.2, the maximum deformation of the isolator appears at the center cross-section of the isolator at the first IR mode. Likewise, the “one-quarter” position from both ends of the isolator is the best place to attach DVA to suppress the second IR because the maximum displacement appears at that position at the second IR mode (Figure 4.2).

As mentioned previously, this study develops and explores the technique of using DVAs to improve the isolator performance by suppressing the first two IRs. Hence, the idea of using two PDVAs is proposed. Based on the analysis in Figure 4.2, the two DVAs are placed at the position of one-quarter length from both ends of the isolator. This configuration is optimal for suppressing the second IR and is also acceptable for controlling the first IR.
Figure 4.2: Isolator deformation (a) and modal shape (b) at the system resonance and the first two IR frequencies of a SDOF vibration system (c).
Figure 4.3: Illustration of the best DVA attaching position for suppressing the first IR in a SDOF system with rigid foundation.

**Intermediate Mass**

An implementation issue will need to be considered when the DVA is directly embedded into an isolator. There is a need for a physical support through which the DVA can be connected to the isolator. This support could be in the form of a ring or thin plate in-between the isolator body as the one shown in Figure 4.4. The stiffness of this support is normally very high compared to the stiffness of the isolator so that its influence on the isolator stiffness is negligible. However, the mass of this support is not always small and
negligible compared to the mass of the isolator and the DVA. It acts as a concentrated mass within the isolator body and separates the isolator into two segments. Because of the way it is integrated in the isolator, it is called *intermediate mass* in this dissertation. The intermediate mass (IM) affects the dynamics of the isolator in both a desirable and an undesirable way. In the literature, an isolator incorporated with one or more IMs is referred as *compound isolator*.

![Schematic plot of the application of the intermediate mass](image)

**Figure 4.4:** Schematic plot of the application of the intermediate mass.

### 4.2 Mathematical Model

The mathematical model of the PDVA enhanced isolator is developed in this section. According to the analysis in Section 4.1, two PDVAs are attached at the one-quarter positions of the isolator. Between each DVA and the isolator body, an IM in the form of a thin plate or ring is needed to join the DVA and the isolator. Thus, it is important to investigate the effects of the two IMs on the isolator performance before adding the DVAs. To this end, the force transmissibility performance of a SDOF system (with IMs) is examined for both the ideal massless and realistic compound isolator.
4.2.1 Force Transmissibility of a System with Compound Isolator

The idealized compound system with one IM is illustrated in Figure 4.5. The spring elements, both with stiffness coefficient $k_s$, in this simplest compound system are considered massless so that the analysis can be focused on the influence of the IM, denoted as $m_i$. The force transmissibility of both the compound system and the simple system that does not have the IM are shown in Figure 4.6.

![Figure 4.5: The idealized compound system with massless isolator.](image)

It is seen from Figure 4.6 that the compound system has two resonant frequencies $f_1$ and $f_2$ in the order of $f_1 < f_0 < f_2$ where $f_0$ is the natural resonance frequency of the simple system. Although the compound system has the disadvantage of possessing an additional resonance, at high frequencies the transmissibility of the compound system decreases much more rapidly than the transmissibility of the simple system. The
decreasing rate of the transmissibility curve after the secondary resonance of the compound system is almost double than that of the simple system [15]. For the compound system to have the greatest effectiveness as an anti-vibration and noise device at high frequencies, it is desirable that the secondary resonance occurs, at the lowest possible frequency. This observation indicates that the compound isolator can be used to significantly improve the isolator performance at high frequencies. Therefore, it can also be of some interest in mitigating the performance loss in vibration and noise isolation due to the IRs since IRs usually occur at high frequencies.

Figure 4.6: Comparison of the force transmissibility of compound and simple system with massless isolator.
A SDOF compound system incorporating with a realistic isolator, i.e. an isolator with distributed mass, and two IMs is pictured in Figure 4.7. This structure is aimed to explore the influence of the IMs on the IRs and the isolator performance. Therefore, it will provide guidance to the design of the PDVA enhanced isolator.

According to the previous analysis on the DVA attaching positions in Section 4.1, the IMs are placed at one-quarter length from both ends of the isolator. The force transmissibility of this system as a function of frequency is plotted in Figure 4.8 together with the transmissibility of the same system but without IMs (i.e. simple system). Compared to the simple system, it is seen that, above 1100 Hz, the transmissibility of the compound system is reduced to a much lower level by inserting the IMs. That is to say, the amplitudes of the IRs above 1100 Hz are greatly reduced, and thus it is advantageous to incorporate IMs. However, there are two undesirable resonances between 500 and 1100 Hz. The resonant frequencies of the simple and compound systems shown in Figure 4.8 are listed in Table 4.1. It is observed that the system resonance corresponding to the primary mass for both systems is around 32 Hz. In additional to the system resonance, there are two IRs for the simple system and three IRs for the compound system below 3000 Hz. Here, the definition of the IR is extended to all resonances associated with the isolator except for the system resonance. In detail, the IR concept is extended to include the resonances due to not only the isolator’s distributed mass (original definition) but also the concentrated mass inserted into the isolator, i.e. the IM. However, strictly speaking, the resonances at 550 and 1050 Hz in Figure 4.8 are not IRs due to “wave like” effects because they correspond to the two concentrated IMs of \( m_{iu} \) and \( m_{il} \). The first IR due to the distributed mass of the isolator in the compound system is actually shifted to 2200 Hz, which is significantly higher than its counterpart (1017 Hz) in the simple system. This is because the IRs corresponding to the distributed mass occur at higher frequencies for smaller isolators and the original isolator is cut into three segments of shorter lengths by the two IMs.
Table 4.1: Resonant frequencies of the simple and compound systems

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resonant frequency (Hz)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simple system</td>
<td>32</td>
<td>1017</td>
<td>2034</td>
<td>N/A</td>
</tr>
<tr>
<td>Compound system</td>
<td>32</td>
<td>550</td>
<td>1050</td>
<td>2200</td>
</tr>
</tbody>
</table>

In summary, incorporating IMs is both advantageous and disadvantageous. For a given isolator, inserting IMs shifts its IRs to higher frequencies and increases its decreasing rate of the transmissibility curve at high frequencies. On the other hand, inserting IMs also induces new resonances (e.g. the resonances at 550 and 1050 Hz of the compound system in Figure 4.8). To some extent, these resonances can also be considered as “internal” resonances that may be able to be suppressed by DVAs. Therefore, from the standpoint of the IR control and improving the broadband isolator performance, both the DVA and the concomitant IM contribute to the performance improvement of the isolator.

Figure 4.7: The compound system of realistic isolator with two intermediate masses.
Figure 4.8: Comparison of the force transmissibility of compound and simple system with realistic isolator.

4.2.2 Model of the PDVA Enhanced Isolator

Figure 4.9 shows the configuration of the PDVA enhanced isolator which consists of a cylindrical isolator made of rubberlike material with two embedded passive DVAs. The DVAs are placed in the cylindrical cavity inside the isolator and each of them is connected to the isolator at the “one-quarter” position through a thin plate (IM). The isolator has complex modulus \( \tilde{E} = E(1 + j\eta) \), density \( \rho \) and length \( L \). Each DVA has mass \( m_{ai} \) and complex stiffness \( \tilde{k}_{ai} = k_{ai}(1 + j\eta_{ai}) \), where \( \eta_{ai} \) is the loss factor of the
DVA spring \((i = 1, 2)\). The upper and lower inserted plates are denoted as \(m_{iu}\) and \(m_{il}\), respectively. The isolator is separated into three segments marked as \(\mathbb{1}, \mathbb{2}\) and \(\mathbb{3}\) from top to bottom. Therefore, this PDVA enhanced isolator can be considered as a combination of three component isolators (isolators \(\mathbb{1}, \mathbb{2}\) and \(\mathbb{3}\)), two thin plates (IMs), and two DVAs.

![Figure 4.9: Sketch of the passive DVA enhanced isolator.](image)

The development of the dynamic model of the PDVA enhanced isolator involves finding the relationship between the two external forces \((F_T\) and \(F_B)\) and the resulting displacements \((X_T\) and \(X_B)\) at the two ends of the isolator. The EOMs of the DVA enhanced isolator is written by coupling the dynamics of the isolator components. That is
where $X_{iu}$, $X_{il}$, $X_{a1}$ and $X_{a2}$ represent the displacements of the upper IM, the lower IM, the first DVA and the second DVA, respectively. The matrix $[M]$ is given by

$$[M] = \begin{bmatrix} 0 & m_{iu} & m_{a1} & m_{a2} & 0 \\ m_{iu} & 0 & 0 & 0 & 0 \\ m_{a1} & 0 & 0 & 0 & 0 \\ m_{a2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.2}$$

and the stiffness matrix $[K_s]$ is given by

$$[K_s] = \begin{bmatrix} D_1^d & -D_1^o & 0 & 0 & 0 & 0 \\ -D_1^o & D_1^d + D_2^d + \tilde{k}_{a1} & -\tilde{k}_{a1} & 0 & -D_2^o & 0 \\ 0 & -\tilde{k}_{a1} & \tilde{k}_{a1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tilde{k}_{a2} & 0 \\ 0 & -D_2^o & 0 & -\tilde{k}_{a2} & D_2^d + D_3^d + \tilde{k}_{a2} & -D_3^o \\ 0 & 0 & 0 & 0 & -D_3^o & D_3^d \end{bmatrix} \tag{4.3}$$

where $D_i^d$ and $D_i^o$, for $i = 1, 2, 3$, are the terms of the dynamic stiffness matrix of component isolator 1, 2, and 3. They can be calculated from Equation (2.16). Replacing Equations (4.2) and (4.3) into Equation (4.1) yields the dynamic stiffness matrix of the DVA enhanced isolator:
\[
[D_{DVA}] = \begin{bmatrix}
D_1^d & -D_1^v & 0 & 0 & 0 & 0 \\
-D_1^v & T_{22} & -\tilde{k}_{a1} & 0 & -D_2^v & 0 \\
0 & -\tilde{k}_{a1} & T_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & T_{44} & -\tilde{k}_{a2} & 0 \\
0 & -D_2^v & 0 & -\tilde{k}_{a2} & T_{55} & -D_3^v \\
0 & 0 & 0 & 0 & -D_3^v & D_3^d 
\end{bmatrix}
\] (4.4)

where \( T_{22} = -\omega^2 m_{a1} + D_1^d + D_2^d + \tilde{k}_{a1} \), \( T_{33} = -\omega^2 m_{a1} + \tilde{k}_{a1} \), \( T_{44} = -\omega^2 m_{a2} + \tilde{k}_{a2} \) and \( T_{55} = -\omega^2 m_{d1} + D_2^d + D_3^d + \tilde{k}_{a2} \).

Equation (4.4) is the dynamic model of the PDVA enhanced isolator.

### 4.2.3 Model of Isolation System with PDVA Enhanced Isolator

The side view of the 3DOF reference system (Figure 2.5) introduced in Chapter 2 but with PDVA enhanced isolators is sketched in Figure 4.10. All the variables have the same meaning as those in Figure 2.5.

Comparing the structures showing in Figures 2.5 and 4.10, it is seen that they are basically the same system with the only difference in the isolator type. Therefore, the EOMs of the 3DOF reference system (Equation (2.40)) are still applicable at this point to calculate the displacements at the ends of the PDVA enhanced isolators. In order to derive the EOMs for the 3DOF system with enhanced isolators from Equation (2.40), the only modification required is using the model of the PDVA enhanced isolator shown in Equation (4.4). In detail, since the 6-by-6 matrix relates all the 6 displacements associated with one isolator (refer to Figure 4.9), there are 18 displacements in total for the 3DOF system with three enhanced isolators. Thus, all the matrices and vectors in Equation (2.40) should be extended properly to describe this nominal 18-DOF problem. This equation is then used to solve for the six displacements at each end of the three enhanced
isolators. After obtaining the motion of the isolators, the force transmissibility and the sound radiation from the foundation can be calculated according to the methods presented in Chapter 2.

Figure 4.10: 3DOF system with PDVA enhanced isolators.
4.3 Numerical Analysis

In this section, a parametric study on DVA properties is first conducted. Different from the parameter optimization which intends to seek the best value for each design variable, this parametric study aims to help understanding the effects of the DVA parameters’ variations on the isolator performance. Following the parametric study, the optimal design for the DVA parameters is briefly introduced. A performance comparison is then made between the original isolator and the PDVA enhanced isolator. This comparison shows the effectiveness of the concept of using DVAs to suppress the IRs. In the end, the robustness analysis of using one and two DVAs is presented.

4.3.1 Parametric Study

The proposed approach of suppressing the IRs in this study requires incorporating two DVAs into an isolator. Each DVA is characterized by the mass, damping, tuning frequency, and the IM to connect the DVA to the isolator. For simplicity, the two DVAs are assumed to be identical except for their stiffness, i.e. the DVAs have different tuning frequencies. Thus, there are a total of five parameters to be investigated: the DVA mass, the DVA damping, two tuning frequencies, and the IM.

The numerical model used here is the model of the reference system presented in Chapter 3 except that it is assumed that two DVAs are embedded into each isolator at the “one-quarter” length positions. Therefore, each isolator is separated by two IMs (or DVAs) into three segments of lengths 0.017, 0.033 and 0.017 m. All the parameters including the properties of the primary mass, the isolator and the foundation are kept unchanged as given in section 3.1. For the sake of clarity in the presentation, the isolator with and without DVAs is referred as enhanced and original isolator, respectively. From table 3.1, it is noticed that the system resonances of the reference system with the original
isolators are all below 50 Hz and the first two IRs appear at 1018.5 and 2037 Hz. Since a broadband performance of the isolator in the isolation region is concerned, the RMS value of the force transmissibility in the frequency range from 200 to 3000 Hz is used as the criteria for the parametric study. Without loss of generosity, the transmissibility is only calculated for isolator 1 (Figure 3.1) in the following parametric study.

4.3.1.1 Intermediate Mass

Although the IM is not a necessary component of the DVA, it is required to be used with the DVA. This is because of the following two reasons: (a) it acts as the physical connection between the DVA and the isolator and (b) it can help reduce the transmitted force at high frequencies, thus help mitigate the adverse effect of IRs. However, introducing the IM into the isolator also induces new problem - it has been shown previously that the IMs introduces new resonances. One way to avoid this disadvantage is to shift this new resonance to lower frequencies that are out of the range of interest. Usually, this is done by carefully selecting the inserting position and the mass value of the IM [15]. However, this method is not applicable here because in this study, the application of the IM is in conjunction with a DVA. Thus the insertion position of the IM is fixed at the “one-quarter” length position. Hence, it is necessary to investigate the effect of the IM in the enhanced isolator. For clarity, the influence of the IM will be examined along by considering the 3DOF reference system consisting of three compound isolators and each isolator with two IMs at the “one-quarter” positions, i.e., the DVAs are not included in the isolators.

Figure 4.11 shows the three-dimensional plot and the contour plot of the force transmissibility for isolator 1 as a function of both driving frequency and the IM. The two IMs are identical and changing in the range from 0 to 0.15 kg. The effect of the IM can be evaluated by comparing to the transmissibility for no IM, i.e. the baseline curve shown
as the blue solid line in Figure 3.2. It is seen that the transmissibility surface in Figure 4.11 has six resonant ridges. It is obvious that below the first IR, the IM does not have appreciable influence on the shape of the transmissibility curve, i.e. both the amplitudes and the resonant frequencies of the transmissibility curve are roughly independent to the IM. On the other hand, the IM greatly affects the characteristics of the IRs at high frequencies. The first two IRs are shifted to lower frequencies and the amplitudes also become higher with increasing IM. However, the transmissibility above the second IR frequency is significantly reduced by adding the IM. The transmissibility curve decreases significantly after the second IR with increasing IM. It is also noticed that the third IR frequency decreases slightly and the frequency span between the second and the third IRs increases when IM increases. These observations imply that a larger IM is preferred if the isolator performance at high frequencies is the main concern. However, a small IM is desirable if the isolator is required to perform well in the low and mid frequency ranges.
Figure 4.11: Force transmissibility of isolator 1 vs. frequency and intermediate mass.
4.3.1.2 DVA Mass

This and the following two sections will discuss the effects of the DVA parameters. For the sake of clarity, the IMs are assumed to be zero in these sections. The two DVAs have the same mass and damping and they are tuned to 1006 and 2300 Hz, respectively. These two frequencies are very close to the first and the second IRs of the original isolator.

In Figure 4.12, the transmissibility of isolator 1 is calculated for a DVA loss factor of 0.2. The DVA mass varies from 0.02 to 0.15 kg, which is about 22% to 165% of the isolator’s mass. Figure 4.12 shows that at low frequencies (e.g. below 500 Hz), the DVA mass have little effect on the transmissibility curve. At mid frequencies (e.g. 500 – 1000 Hz), there are two resonances due to the coupling effect of the DVAs and the original IRs. The amplitudes of these two resonances increase while their frequencies decrease with increasing DVA mass. At high frequencies (e.g. above 1000 Hz), the transmissibility decreases sharply as the DVA mass increases. However, when the frequency approaches 2500 Hz, the decreasing rate of the transmissibility becomes slow again for all DVA mass values.

Comparison of Figures 4.11 and 4.12 shows that the DVA has similar beneficial effect as the IM on the high frequency performance, i.e. significant improvement at high frequency with a detrimental effect at low and mid frequencies. This implies that it is not always necessary to use IM when implementing a DVA if it can be bonded to the isolator body directly. However, the synergistic integration of both IM and DVA can offer significant advantages as it will be demonstrated in later sections.
Figure 4.12: Force transmissibility of isolator 1 vs. frequency and DVA mass.
4.3.1.3 DVA Damping

In this study, the DVA damping is represented in the form of the structural damping, i.e. the loss factor. Among the DVA design parameters, the effect of the damping value on the DVA behavior is relatively straightforward. As it is well known, damping helps attenuate the resonant amplitudes. Therefore, a DVA with higher loss factor is generally desirable for improving the isolator performance.

4.3.1.4 Tuning Frequency

The effect of the two DVA tuning frequencies is investigated by calculating the RMS transmissibility. For simplicity, the DVA mass and loss factor are set to 0.05 kg and 0.2, respectively. The two tuning frequencies are allowed to vary in the range of 500 to 2500 Hz. Hence, the first two IR frequencies are included in this frequency range.

Figures 4.13 to 4.15 show the contour plots of the RMS transmissibility within [200–3000], [500–3000] and [700–3000] Hz, respectively. It is interesting to see that the curves in each figure are symmetric about the 45° line, i.e. the effect of the two DVAs on the isolator performance is basically the same. In these figures, the optimum DVA tuning frequencies at which the RMS transmissibility reaches the lowest value are indicated. It is observed that when both DVAs are tuned to a frequency around 500 Hz, the isolator has the best performance in the broadest frequency band of [200–3000] Hz. The best performance in the frequency range of [500–3000] Hz is obtained when one DVA is tuned to 550 Hz and the other DVA is tuned to 800 Hz. For tuning frequencies of 750 and 1700 Hz, the isolator has the lowest RMS transmissibility in the range of [700–3000] Hz. These results indicate that, as for other parameters, the tuning frequencies of the DVAs need to be optimized according to the specific frequency range of interest.
Figure 4.13: The RMS transmissibility of isolator 1 within [200–3000] Hz as a function of DVA tuning frequencies.
Figure 4.14: The RMS transmissibility of isolator 1 within [500–3000] Hz as a function of DVA tuning frequencies.
Figure 4.15: The RMS transmissibility of isolator 1 within [700–3000] Hz as a function of DVA tuning frequencies.
In the above contour plots, it is also interesting to further discuss the trends of each contour. In the three figures, the contours are almost either in vertical or horizontal direction depending on which DVA has the lower tuning frequency. This means that the isolator will keep roughly the same performance when the higher tuning frequency is changing while the lower tuning frequency is kept unchanged. In other words, the configuration of embedding two DVAs into one isolator offers good performance robustness with respect to the tuning frequency of one of the DVAs - the one tuned to the higher frequency. This conclusion is especially applicable for Figure 4.13 in which the isolator performance in the broadest frequency range is considered. In this case, the DVA that is tuned to the lower frequency has the most considerable effect on the isolator performance. This is because the transmissibility is higher at lower frequencies than at higher frequencies in nature, and the DVA with the lower tuning frequency is intended to improve the low frequency performance. Therefore, changes in low frequencies are dominant in the RMS transmissibility value.

4.3.2 Optimization of DVA Parameters

The previous parametric study provides some insight into the effect of each variable (DVA mass, loss factor, tuning frequency and the IM) on the isolator performance. Moreover, it leads to the conclusion that these parameters need to be optimized according to the frequency band within which the isolator performance is of the practical interest. It has been mentioned in section 4.1 that to obtain a good reduction in noise and vibration level at IR frequencies without significantly deteriorating the isolator performance at other frequencies, choosing the proper DVA parameters is very important. For a DVA with a given mass applied to a single primary structure supported by a massless spring, there is a classical procedure to calculate the optimum tuning frequency and damping [15]. However, reference [23] concluded that this procedure cannot simply be employed in designing the parameters for a DVA attached to a continuous rod such as the isolator
modeled in this dissertation. In this study, the design of the DVA parameters is formulated into an optimization problem with practical constrains. The optimization procedure used in this study was performed by Mr. Dhiraj Tiwari at the University of Toledo as his Master thesis. In this chapter, for the sake of completeness, only the selection of the design variables (DVA parameters) and the formulation of the objective functions will be briefly presented. For more information on this particular optimization problem, please refer to reference [49].

**Selection of the Design Variables**

The design variables used for each DVA are its mass, damping, and tuning frequency. Consequently, there are six design variables for the two DVAs embedded in an isolator and eighteen design variables for the 3DOF system shown in Figure 4.10. For simplicity, the properties of the three enhanced isolator, including the isolator material properties and the two DVAs’ parameters, are chosen to be identical. Therefore, only six parameters, i.e. two masses, two damping and two tuning frequencies for the two DVAs, are considered as the design variables in this study. It should be pointed out that, according to the previous parametric analysis, the values of IMs also affect the isolator performance. However, they are not considered as design variables in this study and are thereby given constant values in the optimization procedure. On the other hand, all the design variables are constrained to a range of practical values so that a PDVA enhanced isolator can be physically built and tested based on the optimization results.

**Formulation of the Objective Functions**

The performance of the isolator can be characterized by different criterion depending on the designer’s choice and therefore different objective functions can be selected. Both
the force transmissibility and the sound radiation of the foundation are used here as objective functions to be minimized. The optimization problem is formulated as

$$\text{Find } \mathbf{X} = [x_1, x_2, \ldots, x_N], \quad \mathbf{X}_l \leq \mathbf{X} \leq \mathbf{X}_u \quad (4.5)$$

$$\text{To minimize } f(\mathbf{X}) \quad (4.6)$$

where $\mathbf{X}$ represents the unknown vector which includes $N$ design variables, $\mathbf{X}_l$ and $\mathbf{X}_u$ are the lower and upper bounds for the design variables. Two objective functions were considered in the optimization: (a) the area between the transmissibility of the enhanced isolator and the ideal massless isolator and (b) the total radiated acoustic power. The first objective function is to improve the performance, in terms of the force transmissibility such that the realistic isolator to behave like the ideal massless isolator. The second objective function is intended to improve the isolator’s noise performance.

Lower and upper bound are given to each design variable in the optimization routine to yield a feasible configuration. Therefore the practical range for the DVA mass is assumed to be 0 – 0.15 kg. The loss factor is bounded between 0 and 0.4, because larger damping is difficult to achieve in practice. The range for the tuning frequency is 350 – 1200 Hz. The optimization was performed for two frequency ranges: [200–3000] Hz and [500–3000] Hz.

### 4.3.3 Performance of PDVA Enhanced Isolator – Numerical Simulation

The purpose of this section is to compare the performance of the PDVA enhanced isolator with optimized parameters to the original isolator. The performance comparison is based on the 3DOF reference system and is in terms of both the force transmissibility and the radiated acoustic power. As mentioned before, the IM is not a design variable and is assumed to be 0.05 kg for all cases.
Force Transmissibility Objective Function

The improvement on the vibration isolation by incorporating PDVAs into the original isolator is investigated using the force transmissibility as the objective function. The DVA parameters are tabulated in Table 4.2. Since the three isolators in the reference system are identical, the transmissibility of isolator 1 is again selected to investigate the system performance.

The transmissibility for the PDVA enhanced isolator 1 optimized for the force transmissibility in the [200–3000] Hz range is plotted in Figure 4.16. For comparison, the transmissibility curves for the system with original (practical) and ideal massless isolators are also shown in the same figure. It is seen that as expected the IRs of the original isolator are largely attenuated by the DVAs. Moreover, the transmissibility amplitude beyond 2000 Hz of the enhanced isolator is almost as low as the magnitude of the massless isolator. Since the DVAs introduce new resonances at low frequencies, it is observed that the transmissibility of the enhanced isolator between 200 and 500 Hz is a little higher than that of the practical isolator. In general, the decreasing rate of the transmissibility curve above 200 Hz for the PDVA enhanced is, however, much higher than that for the practical isolator (with inertia) and tend to approach the behavior of the massless isolator.
Figure 4.16: Force transmissibility of isolator 1 when the system has PDVA enhanced isolators and the DVA parameters are optimized for minimizing the transmissibility within [200–3000] Hz.

Figure 4.17 shows the performance of the enhanced isolator with DVA optimized for the transmissibility in the [500–3000] Hz range. The DVA parameters are again shown in Table 4.2. It is noticed that a considerable reduction in the transmissibility is obtained. In the frequency band of interest, the transmissibility of the enhanced isolator is 10 – 30 dB lower than that of the practical isolator. At some frequencies (e.g. 500 – 1000 Hz), the enhanced isolator performs even better than the massless isolator. However, the
transmissibility of the enhanced isolator around 450 Hz is significantly higher than that of the practical isolator.

Figure 4.17: Force transmissibility of isolator 1 when the system has PDVA enhanced isolators and the DVA parameters are optimized for minimizing the transmissibility within [500–3000] Hz.

A direct comparison can also be made between the two black curves corresponding to the enhanced isolators in Figures 4.16 and 4.17. It is noticed from Figure 4.16 that the enhanced isolator does not lead to a satisfying result within [200500] Hz, although its
performance within this frequency band was already taken into account in the optimization stage. On the contrary, a good performance is obtained in the [500–3000] Hz range. Furthermore, a better result is achieved for the same frequency range in Figure 17 in which the enhanced isolator was optimized with a focus on the high frequency performance. This comparison implies that the PDVA approach is more effective at high frequencies than at low frequencies. Since the IRs usually locate at high frequency range, it also can be concluded that the PDVA approach is effective for suppressing the IRs.

The DVA parameters used in Figures 4.16 and 4.17 summarized in Table 4.2 provide additional physical insight. It is seen that both DVA masses took the maximum value (upper constrain bound) of 0.15 kg for the [500–3000] Hz range case. This is consistent with the result shown in Figure 4.12, which indicates that larger DAV mass help reduce the transmissibility at high frequencies by shifting the IRs towards lower frequencies. On the other hand, the DVA masses for the case of [200–3000] Hz are relatively small so as to keep the response at the resonances as small as possible. Moreover, both high damping and low tuning frequencies were chosen for this case. This shows the effort by the optimization routine to attenuate the low frequency resonances.

<table>
<thead>
<tr>
<th>Target frequency range for the optimization (Hz)</th>
<th>200 – 3000</th>
<th>500 – 3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (kg)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DVA 1</td>
<td>0.097</td>
<td>0.15</td>
</tr>
<tr>
<td>DVA 2</td>
<td>0.06</td>
<td>0.15</td>
</tr>
<tr>
<td>Loss factor</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DVA 1</td>
<td>0.4</td>
<td>0.07</td>
</tr>
<tr>
<td>DVA 2</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Tuning frequency (Hz)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DVA 1</td>
<td>350</td>
<td>553</td>
</tr>
<tr>
<td>DVA 2</td>
<td>628</td>
<td>1032</td>
</tr>
</tbody>
</table>

To evaluate the broadband performance, Table 4.3 lists the overall RMS transmissibility for the results in Figures 4.16 and 4.17. It is seen that the RMS
transmissibility in the [500–3000] Hz range for the PDVA enhanced isolator is only one tenth of that of the practical isolator and is even smaller than the RMS value of the ideal massless isolator. This comparison again shows the effectiveness of using PDVA to suppress the IRs at high frequencies. In contrast, it is also noticed that the RMS transmissibility for the enhanced isolator from 200 to 3000 Hz is roughly the same with that of the practical isolator, which may seemingly implies no performance improvement. This result is easily explained since in Figure 4.20, it is clear that significant reduction was achieved above 500 Hz that is counter balanced by the increase at lower frequencies.

Table 4.3: Isolator performance in terms of RMS transmissibility value of isolator 1

<table>
<thead>
<tr>
<th>Target frequency range for the performance evaluation (Hz)</th>
<th>200 – 3000</th>
<th>500 – 3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>System with ideal massless isolators</td>
<td>0.356</td>
<td>0.087</td>
</tr>
<tr>
<td>System with practical isolators</td>
<td>0.59</td>
<td>0.444</td>
</tr>
<tr>
<td>System with DVA enhanced isolator optimized for the transmissibility within 200 – 3000 Hz</td>
<td>0.63</td>
<td>N/A</td>
</tr>
<tr>
<td>System with DVA enhanced isolator optimized for the transmissibility within 500 – 3000 Hz</td>
<td>N/A</td>
<td>0.044</td>
</tr>
</tbody>
</table>

**Acoustic Power Objective Function**

In this section, the acoustic power radiated by the foundation is used as the objective function in the optimum design of the PDVA enhanced isolators. Figures 4.18 and 4.19 show the A-weighted acoustic power spectra for PDVA enhanced isolators optimized for minimum noise radiation in the [200–3000] and [500–3000] Hz range, respectively. The optimum DVA parameters are shown in Table 4.4. From Figures 4.18 and 4.19, it is clear that the PDVA enhanced isolators significantly reduce the noise radiation in the frequency range of interest. At some frequencies (e.g. 2000 Hz in Figure 4.18, 1000 Hz in
Figure 4.19), the acoustic power with PDVA enhanced isolators can be attenuated by up to 30 dB.

Figure 4.18: Acoustic power radiated by the plate foundation when the system has PDVA enhanced isolators and the DVA parameters are optimized for minimizing the noise radiation within [200–3000] Hz.
Figure 4.19: Acoustic power radiated by the plate foundation when the system has PDVA enhanced isolators and the DVA parameters are optimized for minimizing the noise radiation within [500–3000] Hz.
To quantify the isolator’s performance, Table 4.5 compares the total acoustic power over the two frequency bands of interest for the various isolators. If the performance of the ideal massless isolator (row two) is considered as the baseline, a potential noise reduction of 5.7 and 11.2 dB can be obtained by fully suppressing the IRs in the two frequency ranges of interest, respectively. Using the PDVA enhanced isolator, the actual reduction is 2 and 20.3 dB, respectively. These results demonstrate that the passive DVAs are very effective for improving the isolator’s high frequency performance by suppressing the IRs. However, at low frequency the PDVA enhanced isolator leads to an increase in radiated noise.

Table 4.4: DVA parameters optimized for the acoustic power

<table>
<thead>
<tr>
<th>Target frequency range for the optimization (Hz)</th>
<th>200 – 3000</th>
<th>500 – 3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (kg)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DVA 1</td>
<td>0.1</td>
<td>0.15</td>
</tr>
<tr>
<td>DVA 2</td>
<td>0.06</td>
<td>0.15</td>
</tr>
<tr>
<td>Loss factor</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DVA 1</td>
<td>0.4</td>
<td>0.13</td>
</tr>
<tr>
<td>DVA 2</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Tuning frequency (Hz)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DVA 1</td>
<td>360</td>
<td>569</td>
</tr>
<tr>
<td>DVA 2</td>
<td>635</td>
<td>1016</td>
</tr>
</tbody>
</table>

Table 4.5: Isolator performance in terms of total acoustic power in dBA

<table>
<thead>
<tr>
<th>Target frequency range for the performance evaluation (Hz)</th>
<th>200 – 3000</th>
<th>500 – 3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>System with ideal massless isolators</td>
<td>47.9</td>
<td>41.2</td>
</tr>
<tr>
<td>System with practical isolators</td>
<td>53.6</td>
<td>52.4</td>
</tr>
<tr>
<td>System with DVA enhanced isolator optimized for the noise radiation within 200 – 3000 Hz</td>
<td>51.6</td>
<td>N/A</td>
</tr>
<tr>
<td>System with DVA enhanced isolator optimized for the noise radiation within 500 – 3000 Hz</td>
<td>N/A</td>
<td>32.1</td>
</tr>
</tbody>
</table>
Comparing the optimum parameters showing in Tables 4.2 and 4.4, it is interesting to notice that the values are very similar. This implies that minimizing the force transmissibility or the acoustic power basically results in the same DVA parameters.

### 4.3.4 Sensitivity Analysis

In this section, a sensitivity analysis is performed to quantify the robustness of the PDVA enhanced isolator to variations in the system parameters. Among the three DVA parameters (mass, damping and tuning frequency), the tuning frequency is the most critical factor affecting the DVA performance while damping has the least. Therefore, the tuning frequency is selected as the target parameter for the sensitivity study. Figure 4.20 shows the contour plot of the RMS transmissibility in the [500–3000] Hz range for isolator 1 as a function of the tuning frequencies for DVA 1 and 2. Note that the tuning frequencies vary by ±25% of their optimal values in Table 4.2. The closeness of the contour lines in the horizontal direction indicates that the isolator performance is sensitive to the tuning frequency of DVA 1, i.e. DVA tuned to 553 Hz. On the other hand, the performance is not sensitive to the tuning frequency of isolator 2. Therefore, the sensitivity analysis for the enhanced isolator incorporated with two passive DVAs can be further simplified to investigate the effect of single parameter, i.e. the tuning frequency of isolator 1.
Figure 4.20: The RMS transmissibility of isolator 1 within [500–3000] Hz vs. the variation of the tuning frequencies around their optimum values.

Figure 4.21 shows the RMS transmissibility of isolator 1 as a function of the normalized tuning frequency of DVA 1, i.e. normalized by the optimal tuning frequency. In this figure, the RMS transmissibility for the case of a single PDVA embedded at the center of the isolator (dotted line) is included. The single DVA has the same damping and tuning frequency as DVA 1 in the double-DVA approach. For a valid comparison, the mass of the single DVA is equal to the two DVA masses in the double-DVA approach (0.3 kg). It is clear that the double-DVA approach outperforms the single DVA case by a
factor of about 3. However, the performance of the double-DVA approach is degraded for tuning frequency ratios $\geq 1.2$. In fact, the single-DVA approach performs better than the double-DVA approach in this case.

![Graph showing RMS transmissibility](image)

**Figure 4.21:** RMS transmissibility of isolator 1 calculated for the frequency band of [500–3000] Hz and for enhanced isolators incorporated with 1 or 2 DVAs

The above analysis indicates that the double-DVA enhanced isolator is capable of maintaining stable performance to a large degree as the DVA parameters change. Furthermore, it is also demonstrated that the double-DVA approach results in better
isolator performance than the single-DVA approach as long as the shift in the tuning frequency is no more than 1.2 times the optimum value.

4.4 Experimental Validation

In this section, the performance of the proposed PDVA enhanced isolator is investigated experimentally. For the sake of simplicity, all the experiments were based on a SDOF system consisting of a primary mass connected to a foundation through a single isolator. A rigid foundation is used when the experimental goal is to show the IRs in practical isolators.

The first part of this section describes the experimental setup. The second part compares the experimental results for a commercial rubber isolator and the theoretical predictions for the same isolator based on the “long-rod” model developed in Chapter 2. This comparison is also used as the model validation. The validated model is then employed in the optimization routine to find the best DVA parameters. A prototype PDVA enhanced isolator was built. The last part of this section presents the experimental results of the PDVA enhanced isolator.

4.4.1 Experimental Setup And Test Specimen

**SDOF System with Rigid Foundation**

Figure 4.22 shows the schematic plot of the test setup consisting of a primary mass mounted on a rigid foundation through the isolator. The primary mass is excited by an electromagnetic shaker along the center axis of the isolator. The shaker is driven with white noise in two frequency ranges, e.g. [0–1600] or [0–3200] Hz. Two force transducers are used. One is inserted between the shaker and the primary mass. The other
is inserted between the isolator and the foundation. Thus, the force transmissibility, i.e. the transfer function between the two force transducers, was directly measured. Figure 4.23 is a picture taken for the actual setup in which the massive steel flange acts as the rigid foundation.

Figure 4.22: Schematic plot of the test setup consisting of a primary mass mounted on a rigid foundation through an isolator.
Figure 4.23: Actual test setup of the SDOF system with rigid foundation.
**SDOF System with Flexible Foundation**

Figure 4.24 illustrates the diagram of the test setup for the SDOF system mounted on a flexible foundation. The primary mass is again driven by a shaker with band limit white noise. The force transmissibility is still measured by the two force transducers. The foundation is a simply supported aluminum plate with dimensions of 711.2 x 508 mm and thickness of 6 mm. The isolator is mounted at the center of the plate. The plate properties are: density of 2700 kg/m³, Young’s modulus of 7.31 x 10¹⁰ Pa, Poisson’s ratio of 0.33 and the loss factor of 0.01. A picture of the actual setup is shown in Figure 4.25.

The sound radiation is then experimentally estimated by measuring the velocity distribution of the plate. To this end, the surface of the plate is partitioned into a 3-by-5 array of elementary radiators. The geometrical center of each element is instrumented with an accelerometer to measure the acceleration and thus the velocity. The total acoustic power of the foundation is estimated as

$$P(\omega) = V^H RV$$

(4.7)

where $V$ is the velocity vector of the elementary radiators, the superscript $H$ denotes the complex conjugate transpose, and $R$ is the radiation resistance matrix of the plate. The entries in matrix $R$ is calculated from the physical parameters for the fluid (air) and plate. A detailed discussion on Equation (4.7) can be found in reference [50].
Figure 4.24: Schematic plot of the test setup consisting of a primary mass mounted on a flexible foundation through an isolator.
Figure 4.25: Actual test setup of the SDOF system with flexible foundation; (a) surface mounting of accelerometers; (b) a shaker driven primary mass is mounted on the flexible foundation through an isolator.
Test Specimen and Prototype of PDVA Enhanced Isolator

Three isolators were tested. They are referred as the original, hollow, and PDVA enhanced isolator. Figure 4.26 shows photographs of the original and hollow isolators. The original isolator is a commercial rubber mount manufactured by Lord Corporation. It has a diameter of 79.2 mm and a length of 76.2 mm. The density of the isolator material is 1103 kg/m³. Two thin steel plates with 79.2 mm diameter are glued at both ends of the isolator; hence it is known as a sandwich mount. To incorporate DVAs into the isolator, another isolator was built using the same type of sandwich mount. This is the hollow isolator inside which there is a cylindrical cavity with a diameter of 45 mm. This cavity is designed to house the two DVAs. Referring to Figure 4.26(b), the hollow isolator consists of three pieces with length \( L_1 = L_3 = 20 \text{ mm} \) and \( L_2 = 36 \text{ mm} \). The length of \( L_1 \) or \( L_3 \) (20 mm) is roughly one fourth of the isolator’s total length (76 mm). This is in accordance to the suggestion from the previous analysis which states that the DVAs should be attached at the “one-quarter” position. A steel washer (0.078 kg) is attached at the end of each part of \( L_1 - L_3 \) (refer to Figure 4.30 (a)). The hollow isolator components are assembled using bolts (see four threaded holes). These steel washers play the roll of the IMs.

This result easily explained since in Figure 4.20 it is clear that significant reduction was achieved above 500 Hz that is counter balanced by the increase at lower frequencies.
Figure 4.26: Photographs of (a) the original and the hollow isolators (b) the inner structure of the hollow isolator.
The PDVA enhanced isolator is shown in Figure 4.27. The two DVAs are clamped between $L_1$, $L_2$ and $L_3$. Each DVA mass is coupled to the isolator body through a multi-layer plate spring which consists of several layers of metallic and viscoelastic materials. The plate spring has the same diameter as the isolator. The DVA damping is obtained mainly by means of the surface friction between the adjacent layers of the plate spring. The material, thickness and shape of each layer can be adjusted to achieve the desired damping and stiffness. This adjustment usually has to be accomplished by trial and error.
4.4.2 Experimental Demonstration of the IRs and Model Validation

To actually demonstrate the IRs in a practical isolator and show their adverse effect on the isolator performance, the original isolator was tested using the setup shown in Figure 4.22. The primary mass was 0.278 kg. The red curve in Figure 4.28 is the measured force transmissibility of the SDOF system with the original isolator. It is seen that the system resonance is at 424 Hz and the first two IRs are at 1400 and 2230 Hz, respectively. The force transmissibility is about 8 dB at the first IR and –5 dB at the second IR.

In the numerical simulation, the isolator properties were assumed to be constants. However, the analytical prediction using constant isolator properties does not match the experimental result very well. In Figures 1.6 and 1.7, it was already seen that the dynamic modulus and damping of rubber isolators are functions of frequency and temperature. However, since isolators usually work in the constant-temperature condition in numerous practical cases, the temperature-dependency of the isolator properties is not considered in this study. Therefore, to better predict the behavior of the tested isolator, it is assumed that the modulus of elasticity and loss factor of the isolator material are linear functions of frequency. Using the experimental transmissibility results at the first two IRs, the damping at the IRs is estimated by means of the half power method and the elasticity modulus is calculated by matching the IR frequencies. After some mathematical manipulations, the frequency-dependent modulus and loss factor for the isolator material are found to be

\[ E = 4.256 \times 10^7 - 9105f \]  \hspace{1cm} (4.8)

and

\[ \eta = 0.05765 + 3.446 \times 10^{-5}f \]  \hspace{1cm} (4.9)

The complex modulus is actually a quadratic function of frequency, which is given by

\[ \bar{E} = E(1 + j\eta) = 2.4536 \times 10^6 + 941.714f - 0.3138f^2 \]  \hspace{1cm} (4.10)
Since the three types of isolators are made of the same rubber material, the dynamic parameters estimated from the test on the original isolator are assumed to be also valid for the hollow and enhanced isolators.

Substituting the above expressions for the modulus and loss factor into the “long-rod” isolator model, the analytically predicted transmissibility for the original isolator is shown as the blue curve in Figure 4.28. It is seen that the analytical model matches the experimental data very well. On the other hand, the model predicts the positions of the first two IRs accurately while there is an approximate 3 dB difference in the IR amplitudes. Using this model, the transmissibility function can be also predicted for the case where the isolator is assumed to be massless (black curve). Comparing the red curve and the black curve, it clearly shows that the true transmissibility of the practical isolator at the IR frequencies is at least 20 dB higher than that of the ideal one.

![Figure 4.28: Experimental result of the original isolator- Force transmissibility.](image-url)
To further validate the model, the theoretical and experimental transmissibility curves of the SDOF system with the hollow isolator are compared in Figure 4.29. Because the hollow isolator is longer and has a smaller cross-sectional area than the original isolator, the system resonance occurs at a lower frequency of 310 Hz. The first two IRs of the hollow isolator are at 950 and 1310 Hz, respectively. After incorporating the IMs of the hollow isolator and the dynamic parameters estimated in Equations (4.8 - 10) into the isolator model, the transmissibility for the hollow isolator is predicted (blue curve). It can be seen the analytical prediction agrees well with the experimental result up to 1600 Hz (an analysis on the coherence function shows that the experimental result is not reliable beyond 1600 Hz). This comparison validates the feasibility of the isolator model.
developed on the basis of the “long-rod” assumption and the concept of IMs. Therefore, this model is used for the design of the DVA parameters.

When comparing the experimental results of the hollow isolator with the original one, it is seen, as expected, that the first two IRs of the hollow isolator appear at lower frequencies with higher amplitudes. In addition, the transmissibility curve of the hollow isolator beyond the second IR decreases more rapidly than that of the original isolator. This behavior is due to the influence of the IMs (inserted steel washers) in the hollow isolator.

4.4.3 Performance of PDVA Enhanced Isolator - Experimental Validation

The performance of the PDVA enhanced isolator is experimentally measured using both the force transmissibility and the radiated acoustic power by the foundation. The test setup is the SDOF system mounted on the flexible foundation as shown in Figure 4.24. Using the analytical model, the desired DVA parameters were optimized to achieve the best isolator performance. As shown in Table 4.6, it is found again that the parameters of the two DVAs optimized for the transmissibility and the acoustic power are very similar. The table also lists the parameter of the actual prototype PDVA enhanced isolator. In the fabrication of the PDVA enhanced isolator, the key priority was to achieve the correct tuning frequencies. Although parameters of the PDVA enhanced isolator prototype does not match perfectly with the “optimally designed system”, it is still very useful to demonstrate the feasibility of the concept of incorporating PDVAs into the isolator to suppress the IRs and hence improve the isolation performance.
Table 4.6: Optimal and practical DVA parameters for the SDOF system

<table>
<thead>
<tr>
<th></th>
<th>Mass (kg)</th>
<th>Loss factor</th>
<th>Tuning frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical parameters</td>
<td>DVA 1</td>
<td>0.15</td>
<td>0.4</td>
</tr>
<tr>
<td>(optimized for the</td>
<td>DVA 2</td>
<td>0.15</td>
<td>0.17</td>
</tr>
<tr>
<td>transmissibility)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theoretical parameters</td>
<td>DVA 1</td>
<td>0.15</td>
<td>0.4</td>
</tr>
<tr>
<td>(optimized for the</td>
<td>DVA 2</td>
<td>0.15</td>
<td>0.19</td>
</tr>
<tr>
<td>acoustic power)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Practical parameters</td>
<td>DVA 1</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>used in the experiment</td>
<td>DVA 2</td>
<td>0.144</td>
<td>0.25</td>
</tr>
</tbody>
</table>

After constructing the prototype of the PDVA enhanced isolator, its performance was then tested. In the experiment, the primary mass was 1 kg and the excitation input to the shaker was a white noise in [0–1600] Hz range. For comparison, the original and the hollow isolators were also tested on the same setup. It is important to note that the results of the original isolator are not directly comparable to the results obtained for the other two isolators. This is because the dimensions of the original isolator are different from the dimensions of the hollow and the enhanced isolators. However, the performance of the original isolator can be used as a reference. By comparing to this reference, the advantage and effectiveness of the PDVA enhanced isolator can be demonstrated.

Figure 4.30 shows the experimental results on the transmissibility measurements. It is seen that, from an overall point of view, the transmissibility amplitude of the PDVA enhanced isolator is significantly lower than that of either the hollow or the original isolator. At some frequencies, a reduction of 30 dB is observed. In other words, the PDVAs effectively reduce the transmitted force through the isolator, especially at high frequencies where the IRs occur.
The PDVA effects can also be quantified by calculating the RMS transmissibility in the designated working range, i.e. the isolation range of the isolator. To determine the isolation range, it is required to know the system resonance frequency. Because the stiffness of the aluminum plate foundation is relatively low, the shape of the transmissibility curve is significantly affected by the foundation’s modes. Therefore, it is difficult to recognize the system resonance and IRs from the experimental data. However, a numerical simulation shows that the system resonance is around 200 Hz and the first two IRs are at 850 and 1200 Hz respectively for the hollow isolator. It means that the
lowest isolation frequency can be estimated at 283 \( (\sqrt{2} \times 200) \) Hz. Hence the RMS transmissibility is calculated in Table 4.7 for the frequency range of [300–1600] and [500–1600] Hz. It is noticed that the RMS transmissibility of the hollow isolator is reduced by 15% and 29% by using passive DVAs for the frequency range of [300–1600] and [500–1600] Hz, respectively. Among the three isolators, the enhanced isolator results in the smallest transmitted force at most frequencies of the isolation range.

<table>
<thead>
<tr>
<th>Target frequency range for the performance evaluation (Hz)</th>
<th>300 – 1600</th>
<th>500 – 1600</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMS transmissibility</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Original isolator</td>
<td>7.02</td>
<td>3.41</td>
</tr>
<tr>
<td>Hollow isolator</td>
<td>6.38</td>
<td>5.25</td>
</tr>
<tr>
<td>DVA enhanced isolator</td>
<td>5.42</td>
<td>3.73</td>
</tr>
</tbody>
</table>

Figure 4.31 shows the A-weighted acoustic power radiated by the foundation as a function of frequency. The three curves are estimated from the experimental data and are normalized by the input force applied on the primary mass by the shaker, i.e. sound radiation per unit input force. Table 4.8 lists the total acoustic power calculated for different frequency bands. It is seen that the hollow isolator results in lower noise radiation at frequencies above 1300 Hz than the original isolator. This is because of the dynamics of the IMs. However, the IMs are also responsible for the noise increase around 900 Hz. From an overall point of view, the foundation radiates approximately the same level of noise when the original or the hollow isolator is used. On the other hand, the PDVA enhanced isolator significantly reduces the radiated acoustic power. Comparing to the hollow isolator case, a 25 dB noise reduction is obtained around 900 Hz by using the enhanced isolator. Moreover, the PDVA enhanced isolator also results in a 2.7 and 8.8 dB
reductions in the total acoustic power in the frequency range of [300–1600] and [500–1600] Hz, respectively.

The experimental results on both the transmissibility and the radiated acoustic power have shown that the approach of using PDVAs to attenuate the IRs is very effective. The isolator performance at high frequencies is significantly improved by adding PDVAs into the isolator. However, the performance of the PDVA enhanced isolator at low frequency region is not always acceptable because of the dynamics (i.e. new resonant peaks) introduced by the DVAs and IMs.

![Figure 4.31: Experimental acoustic power for the original, hollow and PDVA enhanced isolators when the foundation is flexible - the results were normalized by the input disturbance.](image-url)
Table 4.8: Total acoustic power calculated from the experimental data

<table>
<thead>
<tr>
<th>Target frequency range for the performance evaluation (Hz)</th>
<th>300 – 1600</th>
<th>500 – 1600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total acoustic power (dBA)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Original isolator</td>
<td>94</td>
<td>93.5</td>
</tr>
<tr>
<td>Hollow isolator</td>
<td>94</td>
<td>93.6</td>
</tr>
<tr>
<td>DVA enhanced isolator</td>
<td>91.3</td>
<td>84.8</td>
</tr>
</tbody>
</table>

4.5 Summary

The concept of using passive DVAs to attenuate the IRs has been examined and discussed in detail. The novelty of the DVA application in this study is that the PDVA is embedded into the isolator body directly instead of the primary mass to which it is traditionally attached. To suppress the first two IRs, two DVAs were employed. Each of them is attached at the “one-quarter” position. Once the structure of the PDVA enhanced isolator is determined, its mathematical model is derived. The DVA parameters are then optimized by minimizing the force transmissibility or the radiated acoustic power. It is observed that these two objective functions generally lead to similar results, which indicates that if an isolator is good for vibration isolation, it is also good for noise isolation. This is because that, in this study, the force transmitted through the isolator is assumed to be the only excitation for the noise radiation. Therefore, reducing the transmitted force, i.e. vibration, also means reducing the excitation energy to the base structure and consequently reducing the noise radiation. The minor differences between the optimum results obtained from the two objective functions are due to the influence of the A-weighting function applied on the calculation of the acoustic power.

The performance of the PDVA approach was simulated using the 3DOF reference model. An at least 10 dB reduction in the force transmissibility was observed around the IR frequencies. For the acoustic power, by using the passive DVAs, a 2 and 20.3 dB
reduction were obtained in the [200–3000] and [500–3000] Hz frequency bands, respectively.

To verify the importance of the IRs in practical isolators, a commercial rubber mount was tested. The experimental transmissibility curve shows similar phenomenon as predicted by the analytical models. This result clearly demonstrates that the IR is a practical problem that degrades the isolator performance. To experimentally validate the effectiveness of the proposed approach of using DVAs to suppress the IRs, a PDVA enhanced isolator was built and tested. It has been seen that the enhanced isolator leads to significant reduction in both the force transmissibility and the radiated acoustic power. When comparing the experimental data for the isolator with and without DVAs, a reduction of up to 30 dB is achieved for the force transmissibility and 8.8 (or 2.7) dB for the total acoustic power within [500–1600] (or 300–1600) Hz.

Both the analytical and the experimental results in this chapter have shown that the PDVA approach is very effective to improve the isolator performance by attenuating the IRs, especially at high frequencies. However, those results also show that the PDVA enhanced isolator does not result in satisfying performance at low frequencies. This is because that the DVA as well as the IMs induces new resonances at the low frequency region. Therefore, if a broadband, including both the low and the high frequency, isolator performance is of interest, the passive approach alone is not good enough to achieve the goal. In this case, the active control technique is proposed to work together with the passive DVAs. This is the “hybrid controlled” DVA enhanced isolator which will be discussed in detail in Chapter 5.
Chapter 5

Hybrid Control of Internal Resonances

It has been shown that the passive DVA approach is very effective for suppressing the IRs and improving the isolator performance at high frequencies. However, the isolator performance at low frequencies (e.g., between 200 and 500 Hz) is adversely affected by the newly introduced dynamics of the passive elements. An effective approach to overcome this drawback is the addition of an active component to the PDVA approach. This active/passive or hybrid DVA (HDVA) approach is expected to (a) effectively attenuate the transmitted force through the isolator at low frequencies and (b) further improve the isolator performance at high frequencies.

This chapter investigates the approach of using hybrid DVAs to suppress the IRs and therefore improve the isolator performance over the whole isolation range. In Section 5.1,
some fundamentals of the hybrid DVA enhanced isolator are addressed. First, different configurations of active force are discussed. Second, based on the passive configuration, the structure of the HDVA enhanced isolator is illustrated. An analytical model for the HDVA enhanced isolator is then developed. According to the model, the ideal control force that can lead to a zero transmitted force is calculated. Finally, the control algorithm used to regulate the active force is briefly discussed. Since the effectiveness of the hybrid DVAs largely depends on the capacity of the actual control actuator, the performance of the HDVA enhanced isolator is mainly studied experimentally. The experimental setup and results are presented in Section 5.2. It is found that, as expected, the HDVAs significantly improve the isolator performance at low frequencies. Moreover, both the force transmissibility and the radiated sound power at high frequencies are further attenuated compared to the case where the passive technique is applied alone.

5.1 Theoretical Developments

5.1.1. Configurations of the HDVA Enhanced Isolator

On the basis of the configuration of the PDVA enhanced isolator, three possible HDVA configurations are illustrated in Fig. 5.1. They are (a) single DVA with the active force acting between the DVA mass and the isolator body, (b) two DVAs connected to the isolator body at the same position with the active force acting between the two DVA masses, and (c) two DVAs connected to the isolator body at different positions with the active force acting between the two DVA masses.
Figure 5.1: Schematic plots of (a) single-mass HDVA, (b) dual-mass HDVA and (c) two DVAs connected to the primary structure at different positions with active force acting in between the two DVA masses.
Figure 5.1(a) represents the conventional implementation of the active force. In this implementation, the active force, $F_c$, is inserted between the DVA reaction mass and the primary structure to which the DVA is attached, i.e., the isolator body. In Figure 5.1(b), two DVA masses are connected to the primary structure through two sets of spring and damping elements. Indeed, this implementation is equivalent to the configuration of two separate DVAs attaching in parallel at the same position of the primary structure. The two DVA masses are then coupled by the active force. This configuration is called dual reaction masses HDVA or simply dual-mass HDVA [37, 51]. In Figure 5.1(c), the active force is also applied between the two DVA masses. However, in contrast to the configuration of Figure 5.1(b), the two DVAs in Figure 5.1(c) are connected to the isolator body at different positions - each is 1/4 of the isolator’s total length away from its nearest end of the isolator. The difference between the configurations shown in Figures 5.1(b) and (c) may not lead to any difference in performance if the primary structure to which the DVAs are attached is rigid body. But it will be explained later that this structural difference results in distinct behavior of the active force when the primary structure is a continuous system, such as a rod which is the case modeled for the isolator in this dissertation.

In Figure 5.1, the primary mass, $m$, is excited by an external force $F_0$. The purpose of adding the active force is to, ideally, minimize the transmitted force through the isolator to the base structure. Since the base structure is rigid (for the case shown in Figure 5.1), according to the isolator model (Equation (2.6)) a zero transmitted force requires a zero displacement at point $\circ$. Point $\circ$ is actually the isolator’s cross-section at which DVA 1 ($m_{a1}$) is connected to the isolator as indicated in Figure 5.1. The effectiveness of the active control can then be evaluated by calculating the amplitudes of the “ideal” control forces that are required to drive the motion at point $\circ$ to zero. Hence, the control configuration that requires the smallest control force has the best performance. In other words, the best implementation of the active force requires less energy to achieve the
same level of attenuation comparing to other implementations. Heilmann and Burdisso [37] discussed the performance of the conventional single-mass HDVA and the dual-mass HDVA in detail. They concluded that the dual-mass HDVA only needs half of the control energy as required by the single-mass HDVA to obtain the same level of reduction in the response of the primary structure. According to their conclusion, configuration (a) is excluded as a candidate for suppressing the IRs.

In frequency domain, the equation-of-motions (EOMs) of the system shown in Figure 5.1(b) can be written as

\[
\begin{bmatrix}
-\omega^2 m_{a1} + \tilde{k}_{a1} & 0 & -\tilde{k}_{a1} & 0 \\
0 & -\omega^2 m_{a2} + \tilde{k}_{a2} & -\tilde{k}_{a2} & 0 \\
-\tilde{k}_{a1} & -\tilde{k}_{a2} & D_{dL/4}^d + D_{dL/4}^o + \tilde{k}_{a1} + \tilde{k}_{a2} & -D_{dL/4}^o \\
0 & 0 & -D_{dL/4}^o & -\omega^2 m + D_{dL/4}^d \\
\end{bmatrix}
\begin{bmatrix}
X_{a1} \\
X_{a2} \\
X_1 \\
X_0 \\
\end{bmatrix}
= \begin{bmatrix}
F_c \\
-F_c \\
0 \\
F_0 \\
\end{bmatrix}
\]

(5.1)

where \( \omega \) is the radian frequency, \( F_0 \) is the external driving force, \( F_c \) is the control force and \( X_0, X_1 \) are the displacement of the primary mass, \( m \), and the displacement of the isolator’s cross section indicated by \( \odot \), respectively. The DVA mass, complex stiffness and displacement are denoted by \( m_{ai}, \tilde{k}_{ai}, \) and \( X_{ai} \), for \( i = 1, 2 \). The complex stiffness is a combination of stiffness \( k_{ai} \) and loss factor \( \eta_{ai} \), which is expressed as \( \tilde{k}_{ai} = k_{ai}(1 + j\eta_{ai}) \).

The isolator properties are described by a 2-by-2 dynamic matrix. The diagonal and off-diagonal terms in this matrix is denoted as \( D_{dL/4}^d \) and \( -D_{dL/4}^o \), respectively. Where the coefficient \( \alpha \) can be 1/4, 2/4 and 3/4 and \( L \) is the total length of the isolator. Detailed derivation and calculation of the dynamic stiffness matrix of an isolator can be found in Chapter 2.
Letting $X_i$ in Equation (5.1) be zero, the ideal control force $F_c$ is solved by re-writing Equation (5.1) in the following form:

\[
\begin{bmatrix}
-\omega^2 m_{a1} + \tilde{k}_{a1} & 0 & 0 & -1 \\
0 & -\omega^2 m_{a2} + \tilde{k}_{a2} & 0 & 1 \\
-\tilde{k}_{a1} & 0 & -D^o_{3L/4} & 0 \\
0 & 0 & -\omega^2 m + D^d_{3L/4} & 0 \\
\end{bmatrix}
\begin{bmatrix}
X_{a1} \\
X_{a2} \\
X_0 \\
F_c \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
F_0 \\
\end{bmatrix}
\]

After some mathematical manipulations, the ideal control force is obtained as

\[
F_c = \frac{(-\omega^2 m_{a1} + \tilde{k}_{a1})(-\omega^2 m_{a2} + \tilde{k}_{a2})D^o_{3L/4}}{(-\omega^2 m + D^d_{3L/4})(\omega^2 m_{a2}\tilde{k}_{a1} - \omega^2 m_{a1}\tilde{k}_{a2})} F_0 
\]

For the system shown in Figure 5.1(c), the EOMs are given as

\[
\begin{bmatrix}
-\omega^2 m_{a1} + \tilde{k}_{a1} & 0 & 0 & 0 \\
0 & -\omega^2 m_{a2} + \tilde{k}_{a2} & 0 & 0 \\
-\tilde{k}_{a1} & 0 & D^d_{3L/4} & -D^d_{3L/4} \\
0 & -\tilde{k}_{a2} & D^d_{3L/4} & D^d_{3L/4} \\
\end{bmatrix}
\begin{bmatrix}
X_{a1} \\
X_{a2} \\
X_1 \\
X_0 \\
\end{bmatrix} =
\begin{bmatrix}
-F_c \\
F_c \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\omega^2 m_{a1} + \tilde{k}_{a1} & 0 & 0 & 0 \\
0 & -\omega^2 m_{a2} + \tilde{k}_{a2} & 0 & 0 \\
-\tilde{k}_{a1} & 0 & D^d_{3L/4} & 0 \\
0 & -\tilde{k}_{a2} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
X_{a1} \\
X_{a2} \\
X_1 \\
X_0 \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
F_0 \\
\end{bmatrix}
\]

where $X_2$ is the displacement of the isolator’s cross section indicated by $\Box$, and all other variables are the same as used in Equation (5.1).

The ideal control force is then calculated by, again, letting $X_i = 0$:
Furthermore, $F_c$ can be written in an explicit form:

$$F_c = \frac{-(-\omega^2 m_{a1} - \tilde{k}_{a1})D_{L/4}^o}{(-\omega^2 m + D_{L/4}^o)} F_0,$$

where

$$\Omega = \frac{(-\omega^2 m_{a1} + \tilde{k}_{a1} + \tilde{k}_{a1}\tilde{k}_{a2})}{-\omega^2 m_{a2} + 1} + \frac{\tilde{k}_{a1}}{D_{2L/4}^o (D_{2L/4}^d + D_{L/4}^d + \tilde{k}_{a2})} + \frac{(D_{L/4}^o)^2 \tilde{k}_{a1}}{(-\omega^2 m + D_{L/4}^d)D_{2L/4}^o}$$

In Figure 5.2, the ratios of the control force to the disturbance force for configurations (b) and (c) are compared. The two DVAs are tuned to 1165 and 2329 Hz, which are the first and second resonant frequencies of the isolator, respectively. According to Equation (5.3), the ideal control force is zero at the resonances of the passive DVAs. Therefore, the DVA tuning frequencies are clearly identified by the two anti-resonances on the plots. At low frequencies below first DVA tuning frequency, the ideal control force required by configuration (c) is much smaller than that required by configuration (b). Recalling the expected performance of the active control approach mentioned earlier, high control authority at low frequencies would be preferred. Therefore, configuration (c) is chosen in this study to construct the HDVA enhanced isolator.
5.1.2 Proposed HDVA Enhanced Isolator

From the previous analysis, the HDVA enhanced isolator can be obtained by simply adding an active control force between the two DVA masses of the PDAV enhanced isolator. This is illustrated in Figure 5.3. It is noticed that no damping or spring element is coupled between the two DVAs. This is because that the presence of the passive coupling (e.g., spring and damping elements) between the two DVA masses raise the requirement for the control energy [37].
Figure 5.3: Schematic plot of the hybrid DVA enhanced isolator.
5.1.3 Introduction of the Control Algorithms for the Active Force

There are two fundamentally different approaches which can be used for implementing the active force in the HDVA enhanced isolator: feedback and feedforward control. They both have their own advantages and disadvantages.

Feedback control involves a “feedback” signal derived or measured from the system response, which could be displacement, velocity, acceleration, transmitted force or the radiated acoustic power. This signal is then amplified, passes through a compensator (controller) and used to drive the control actuator (active force) to cancel the residual system response. Feedback control is more practical to implement; it can be used in most systems. However, an inherent disadvantage of feedback control systems is their conflict between the feedback gain and the system stability. That is, if the gain is set too high, the system is prone to go unstable; but a high feedback gain is necessary for good performance of active vibration and noise control systems.

On the other hand, feedforward control involves feeding a reference signal related to the disturbance into the controller which then generates a signal to drive the control actuator (active force) in such a way as to cancel the system response to the disturbance. Usually, feedforward control provides better results than feedback control. Moreover, feedforward control is inherently more stable than feedback control. Therefore, feedforward control is preferred whenever it is available. However, a reference signal related with the disturbance is needed to implement the feedforward algorithm and a good reference signal cannot always be obtained in practice. Hence, there are situations where the feedforward control is not applicable. Another problem associated with the feedforward control is the system causality. To obtain best results for the feedforward control, a system has to be causal, which means that the reference signal fed into the controller has to be obtained far enough in advance of the disturbance reaching the error
sensor so as to allow time for the processing and propagation of the control signal to meet and cancel the disturbance signal at the error sensor.

In this study, the broadband filtered-x least-mean-square (LMS) adaptive feedforward algorithm is used to control the active force. This algorithm is implemented using a digital filter and is briefly introduced in this chapter. However, a complete description and discussion of the filtered-X LMS control algorithm is beyond the scope of this dissertation and can be found in many references [52 - 56].

A typical block diagram of the single-input, single-output (SISO) filtered-x LMS control algorithm is illustrated in Figure 5.4. The primary and the secondary paths of the plant, i.e. the isolation system, are characterized by the z-domain transfer functions $G_p(z)$ and $G_s(z)$, respectively. The input to the primary path of the plant is the external disturbance $x_k$, and the input to the secondary path of the plant is the control signal $u_k$. The plant output $e_k$ is the combination of the outputs of the primary and the secondary paths $d_k$ and $y_k$, which is written as

$$e_k = d_k + y_k$$  \hspace{1cm} \text{(5.8)}

where the subscript $k$ denotes the signal sample at time $t_k$. Here, the input $x_k$ could be the previously mentioned disturbance force $F_0$ and the output $e_k$ could be either the transmitted force or the radiated acoustic power from the foundation. It is noticed that the secondary path output $y_k$ in Equation (5.8) can be replaced in terms of the control input. That is:

$$e_k = d_k + u_k G_s(z)$$  \hspace{1cm} \text{(5.9)}
The control signal $u_k$ is obtained by filtering a reference signal which is coherent to the disturbance signal for the case shown in Figure 5.4 through an adaptive finite impulse response (FIR) filter, $W(z)$. The FIR filter has the form:

$$W(z) = W_0 + W_1z^{-1} + W_2z^{-2} + \cdots + W_Nz^{-N} = \sum_{i=0}^{N} W_i z^{-i} \quad (5.10a, b)$$

where $z^{-i}$ represents a delay of $i$ samples such that $z^{-i}x_k = x_{k-i}$. Replacing $d_k$ and $u_k$ in Equation (5.9) in terms of the input $x_k$ yields

$$e_k = [G_p(z) + W(z)G_s(z)]x_k \quad (5.11)$$

From Equation (5.11), it is observed that the system output can be controlled by designing the FIR filter. This is done by using the LMS algorithm. In detail, the LMS algorithm adapts the filter coefficients ($W_i$) to minimize a quadratic cost function of the
plant response. For the system represented by Equation (5.11), the cost function is easily defined as the mean square value of the output (error) signal. That is
\[ C(W_l) = \mathbf{E}[e_k^2] \] (5.12)
where the operator \( \mathbf{E}[\cdot] \) denotes the operation of calculating the expected value. The weights of the FIR filter is then updated as
\[ W_i(k + 1) = W_i(k) - \mu_s \frac{\partial C}{\partial W_i} \] (5.13)
where \( \mu_s \) is the step size used in the LMS method when minimizing the cost function [55]. According to Equation (5.12), the partial differential of the cost function with respect to the filter coefficient can be written as
\[ \frac{\partial C}{\partial W_i} = 2 \mathbf{E}\left[e_k \frac{\partial e_k}{\partial W_i}\right] = 2 \mathbf{E}\left[e_k G_s(z)x_{k-i}\right] = 2 \mathbf{E}\left[e_k \hat{x}_{k-i}\right] \] (5.14a-c)
in which
\[ \hat{x}_{k-i} = G_s(z)x_{k-i} \] (5.15)
The sequence \( \hat{x}_{k-i} \) is obtained by filtering the disturbance input through the transfer function of the secondary (control) path. Hence, it is termed filtered-x signal. Therefore, the secondary path filter representing the dynamics between the control-error signals must be estimated before the cost function can be minimized. The estimation of this secondary path filter is denoted as \( \hat{G}_s(z) \) in Figure 5.4.

5.2 Performance of HDVA Enhanced Isolator - Experimental Validation

In this section, an experimental evaluation of the proposed HDVA enhanced isolator is presented. The test bench is the SDOF system with the simply supported aluminum plate foundation as described in Chapter 4. A prototype HDVA enhanced isolator was built. Due to the practical limitations, the DVA parameters used in the prototype were not able
to comply with the optimal results obtained from the numerical analysis. Because of this imperfection, the prototype HDVA enhanced isolator might not lead to the potentially maximum improvement in the isolator performance. However, this prototype isolator can still demonstrate the effectiveness of the hybrid control approach.

5.2.1. Prototype of HDVA Enhanced Isolator and Experimental Setup

Prototype of HDVA Enhanced Isolator

As mentioned earlier, the difference between the PDVA enhanced isolator and the HDVA enhanced isolator is the presence of the active force. In the HDVA enhanced isolator, a pair of active force acts between the two DVA masses. In the prototype, the active force is implemented using a voice coil – magnet system.

As shown in Figure 5.5, the magnet is fixed to one DVA mass and the coil to the other DVA mass. Both the masses comprise several smaller parts while the major part for each mass is a steel cylinder of 44 mm in diameter. When a current passes through the coil, there is an electromagnetic force acting between the coil and the magnet, and hence the two DVA masses. This is the active control force which causes the masses to attract or repel each other. Its amplitude and direction are controlled by the amplitude and the polarity of the current. The first DVA mass in Figure 5.5(a) has a hollow cavity aligned to its center. A rare-earth magnet slug is mounted in this cavity and a circular steel cap is attached on the top of the magnet. The inner wall of the mass and the steel cap formed a gap to house the voice coil. This gap also acts as a path for the magnetic flux to flow. In addition, a steel alignment shaft extends from the top of the cap along the axis of the mass. The second DVA mass in Figure 5.5(b) consists of a plastic piece and a steel piece that are mounted together axially. A cylindrical voice coil is attached on the top of the plastic piece and is aligned with the axis of the mass. The mass also contains a linear
bearing which is mounted in the center of the plastic piece. The alignment shaft on the first mass will fit into the bearing when the two masses are assembled together so that the coil is guaranteed to be in the gap in the magnetic field path. The sensitivity of the coil – magnet pair was empirically determined to be 0.72 N/V [51].

The two DVA masses are attached to circular plate springs and then embedded into the hollow isolator by means of studs and the threaded holes on the intermediate plate as shown in Figure 5.5(c). A photograph of the actual HDVA enhanced isolator is also shown in Figure 5.6. As it can be seen, each DVA mass is mounted axially to the center of a plate spring. The thickness and shapes of the plates are adjusted to achieve the desired stiffness. Similar as used before, some viscoelastic material layers are inserted in the plate spring so as to achieve the proper DVA damping. Due to the size of the coil and the magnet and some other practical limitations, there is no too much room to adjust the DVA parameters towards the optimal values. The actual DVA parameters used in the prototype are listed in Table 5.1. After assembling the hybrid DVAs and the hollow isolator together, it is seen that the hybrid DVAs are positioned in the cavity of the hollow isolator, and the bearing/shaft pair of the two DVAs allows the relative motion between the DVA masses while maintaining the axial alignment which keeps the copper coil centered within the flux gap.

<table>
<thead>
<tr>
<th>DVA</th>
<th>Mass (kg)</th>
<th>Loss factor</th>
<th>Tuning frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.12</td>
<td>0.08</td>
<td>1060</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.11</td>
<td>650</td>
</tr>
</tbody>
</table>
Figure 5.5: Illustration of (a) first DVA mass, (b) second DVA mass, and (c) assembly of HDVA enhanced isolator.
Figure 5.6: Picture of the HDVA enhanced isolator.
Control System Setup

The test setup for the HDVA enhanced isolator is shown in Figure 5.7 in which a primary mass of 1 kg is mounted on the flexible foundation through the isolator. The foundation and the isolator material properties are the same as described in section 4.5.

The filtered-$x$ LMS algorithm is run from a personal computer which is used as the controller. The built-in reference signal that is coherent with the disturbance signal is filtered through a FIR control filter, i.e. the compensator, to generate the control signal. The weights of the FIR filter are updated by the LMS algorithm which seeks to minimize the signal from the error sensor. Depending on the criterion of interest, the error sensor could be the force transducer that measures the transmitted force or one of the fifteen accelerometers that measures the response of the plate foundation. The FIR filter has 256 coefficients. The sampling frequency of the controller is 5000 Hz. To assure a causal system, the disturbance signal generated by the controller and sent to the shaker is delayed by 64 time steps. The disturbance and the control signals are low-pass filtered with a cutoff frequency of 1600 Hz. Since the isolator is expected to take effect at frequencies beyond the system resonance, its performance above this resonant frequency is of practical concern. Therefore, the error signal is band-pass filtered in a frequency band. Two bands are used in the experiments. They are 200 – 1600 and 500 – 1600 Hz. The first frequency band approximately begins with the system resonant frequency of the SDOF system shown in Figure 5.7 and includes all frequencies in the pre-assumed isolation range (300 – 1600 Hz). The second frequency band emphasizes the isolator performance in the frequency range where the IRs appear. The force transmissibility and the radiated acoustic power are again measured by the two force transducers and estimated by the fifteen accelerometers, respectively.
Figure 5.7: Test setup for the HDVA enhanced isolator.
5.2.2 Experimental Results

As mentioned earlier, two types of error signal can be used in the experiments. When the control goal is to minimize the transmitted force, the output from force transducer 2 (Figure 5.7) is used as the error signal. On the other hand, since the radiated acoustic power of the foundation is directly related to the surface vibration, the accelerometer attached at the center of the plate was also selected as the error sensor. The error signal is then fed into the LMS algorithm through a band-pass filter. According to the error signal and the range of the band-pass filter, four test cases are suggested as shown in Table 5.2.

Table 5.2: Test cases for the experiments on the HDVA enhanced isolator

<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error signal</td>
<td>Force</td>
<td>Acceleration</td>
<td>Force</td>
<td>Acceleration</td>
</tr>
<tr>
<td>Band-pass filter range (Hz)</td>
<td>200 - 1600</td>
<td>200 - 1600</td>
<td>500 - 1600</td>
<td>500 - 1600</td>
</tr>
</tbody>
</table>

Figure 5.8 shows the force transmissibility of the SDOF system with the HDVA enhanced isolator for test case 1. For comparison, the transmissibilities for the cases of hollow isolator and HDVA enhanced isolator without control are also plotted in the same figure. When the controller is turned off, i.e. no active force, the HDVA enhanced isolator turns into a PDVA enhanced isolator. It is seen that the PDVAs only lower the transmissibility level from [600–1600] Hz. This result is consistent with the conclusion obtained in Chapter 4. However, when the feedforward controller is turned on, the transmissibility level is significantly reduced in the whole isolation range from [300–1600] Hz. At frequencies between 200 to 600 Hz, the HDVA is able to result in an up to 20 dB reduction relative to the hollow and the PDVA isolator. The RMS transmissibility values in the [300–1600] frequency band for the cases of hollow isolator (no DVA), HDVA-control off (PDVA) and HDVA-control on are 6.38, 5.76 and 0.55, respectively.
It is noticed that the RMS transmissibility value for the case when the active control is turned on is only 8.6% of that for the case of the hollow isolator. This clearly demonstrates the effectiveness of the HDVA approach. The filtered-\(x\) LMS feedforward control algorithm not only seeks to suppress the IRs but also tries to reduce the response level at any other frequency included in the error signal. Thus, it improves the isolator performance in a broadband range.

Figure 5.8: Experimental force transmissibility for the HDVA enhanced isolator when the error signal is the transmitted force and is band-pass filtered between [200-1600] Hz.
The experimentally estimated acoustic power for test case 1 is presented in Figure 5.9. To make the results comparable with each other, the radiated acoustic power for each case was normalized by the amplitude of the input disturbance from the shaker. The total acoustic power in the [300–1600] frequency range for the case when the HDVA (control on) is 84.9 dBA. In the same frequency range, the total acoustic power for the cases of hollow isolator (no DVA) and HDVA-control off are 94 and 89.7 dBA, respectively. It is seen that with the help of the active force, the HDVA effectively reduce the noise level at low frequencies. Some improvements in the high frequency range above 1000 Hz is also observed but not as significant as those at low frequencies.

Figure 5.9: Experimental acoustic power for the HDVA enhanced isolator when the error signal is the transmitted force and is band-pass filtered between [200-1600] Hz.
Since the filtered-\(x\) LMS controller cancels the content of the error signal which is correlated with the reference, the coherence between the error and the disturbance signals can also be considered as a gauge for the potential effectiveness of the controller. Figure 5.10 shows the coherence function between the error and the disturbance signals with and without control for test case 1. The average coherence in the 200 – 800 and 1000 – 1400 frequency ranges was reduced from about 1 to about 0.3. Referring to Figures 5.8 and 5.9, these two ranges are the regions where the significant attenuation is observed. However, the coherence does not change below 200 Hz, between 800 and 1000 Hz, and above 1400 Hz. There is no control below 200 Hz because the band-pass filter filtered out any component in the error signal below 200 Hz. It is noticed that the 800 – 1000 Hz band is in between the two tuning frequencies of the DVAs. It is not clear the reason for the poor performance of the controller in the last two frequency bands.

![Figure 5.10: Coherence function between error and disturbance signals when the error signal is the transmitted force and is band-pass filtered between [200-1600] Hz.](image)
Figure 5.11: Time history of the (a) error signal – control off, (b) error signal – control on, and (c) control signal when the error signal is the transmitted force and is band-pass filtered between [200-1600] Hz.

The effectiveness of the HDVA enhanced isolator can also be demonstrated by comparing the time domain error signals with and without control as shown in Figures 5.11(a) and 5.11(b), respectively. The root-mean-square values of the error signals for these two cases are 0.5763 and 0.1115 volt, respectively. This indicates that the controller successfully reduces the error signal by 80%. For reference, the control signal is plotted in Figure 5.11(c).
Figure 5.12: Comparison of the experimental (a) force transmissibility and (b) acoustic power when different error sensor is used and when the error signal is band-pass filtered between [200-1600] Hz.

Figure 5.12 compares the results of test cases 1 and 2, i.e. the error signal is the transmitted force and acceleration at the center of the plate, respectively. The RMS transmissibility and total acoustic power in the frequency range of [300–1600] Hz for test case 2 are 0.36 and 83.5 dBA, respectively. In a broadband sense, the accelerometer as
the error sensor yielded better results than the force sensor used in test 1. Below 300 Hz, the transmitted force as the error signal clearly results in better performance than the acceleration.

In test cases 3 and 4, the band-pass filter was set for a narrower frequency range. That is from 500 to 1600 Hz, which is also the range where the IRs occur. Figure 5.13 shows the experimental force transmissibility for test case 3, i.e. using the force as error signal. Again for comparison, the transmissibilities for the cases of hollow isolator and HDVA enhanced isolator without control are also plotted in the same figure. It is seen that the attenuation due to the HDVAs appears at frequencies above 400 Hz. The RMS transmissibility values in the [500–1600] Hz frequency band for the cases of hollow isolator (no DVA), HDVA-control off and HDVA-control on are 5.25, 2.99 and 0.12, respectively. A reduction of nearly 98% was obtained by using the HDVA approach. The transmissibility curve for the case of HDVA with control is relatively flat in the [500–1600] Hz frequency as compared to the same curve in Figure 5.8. This phenomenon implies that the controller performs better in a narrower frequency range than it does in a large frequency range as in test case 1. A check on the coherence function verifies this conclusion. As shown in Figure 5.14, the coherence for test case 3 in the frequency range of [800–1000] Hz and [1400–1600] Hz is much smaller than that for test case 1 (Figure 5.10).
Figure 5.13: Experimental force transmissibility for the HDVA enhanced isolator when the error signal is the transmitted force and is band-pass filtered between [500-1600] Hz.

Figure 5.14: Coherence function between error and disturbance signals when the error signal is the transmitted force and is band-pass filtered between [500-1600] Hz.
Figure 5.15 presents the normalized experimental acoustic power for test case 3. Similar attenuation pattern is observed as for the transmissibility. It is seen that the radiated noise is significantly reduced at high frequencies (e.g. above 1400 Hz). The total acoustic power in the [500–1600] Hz frequency range for the cases of hollow isolator (no DVA), HDVA-control off and HDVA-control on are 93.6, 85.8 and 84.6 dBA, respectively. It is interesting to note that a significant reduction in the transmitted force spectrum does not translate in the same type of reduction in the acoustic power spectrum.
As before, a comparison between the experimental results of test cases 3 (force as error signal) and 4 (accelerometer as error signal) is shown in Figure 5.16. It is again found that for the high frequency performance, the acceleration as the error signal is better. However, when the transmitted force is used as the error signal, the controller has slightly better performance at low frequencies.

![Figure 5.16: Comparison of the experimental (a) force transmissibility and (b) acoustic power when different error sensor is used and when the error signal is band-pass filtered between [500-1600] Hz.](image-url)
The experimental results for the HDVA enhanced isolator are summarized in Tables 5.3 and 5.4. It is seen that the PDVA enhanced isolator has better performance than the hollow isolator. In addition, the HDVA enhanced isolator performs the best. Compared with the hollow isolator, more than 90% reduction in the RMS transmissibility and more than 10 dB attenuation in the radiated noise were obtained by using the HDVA approach.

Table 5.3: RMS transmissibility value calculated from the experimental data

<table>
<thead>
<tr>
<th>Target frequency range for the performance evaluation (Hz)</th>
<th>300 – 1600</th>
<th>500 – 1600</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMS transmissibility</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hollow isolator</td>
<td>6.38</td>
<td>5.25</td>
</tr>
<tr>
<td>PDVA enhanced isolator</td>
<td>5.42</td>
<td>3.73</td>
</tr>
<tr>
<td>HDVA enhanced isolator – control off</td>
<td>5.76</td>
<td>2.99</td>
</tr>
<tr>
<td>HDVA enhanced isolator – control on</td>
<td>Error signal – force</td>
<td>0.55 (tc1)</td>
</tr>
<tr>
<td></td>
<td>Error signal – acceleration</td>
<td>0.36(tc2)</td>
</tr>
</tbody>
</table>

Table 5.4: Total acoustic power calculated from the experimental data

<table>
<thead>
<tr>
<th>Target frequency range for the performance evaluation (Hz)</th>
<th>300 – 1600</th>
<th>500 – 1600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total acoustic power (dBA)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hollow isolator</td>
<td>94</td>
<td>93.6</td>
</tr>
<tr>
<td>PDVA enhanced isolator</td>
<td>91.3</td>
<td>84.8</td>
</tr>
<tr>
<td>HDVA enhanced isolator – control off</td>
<td>89.7</td>
<td>85.8</td>
</tr>
<tr>
<td>HDVA enhanced isolator – control on</td>
<td>Error signal – force</td>
<td>84.9(tc1)</td>
</tr>
<tr>
<td></td>
<td>Error signal – acceleration</td>
<td>83.5(tc2)</td>
</tr>
</tbody>
</table>

① These are the experimental results for the passive DVA approach as presented in Table 4.7 or 4.8.
When control is off, the HDVA approach becomes the PDVA approach. However, the DVA parameters were different from those in Chapter 4.

The test case (tc) number is indicated in the parenthesis.

5.3 Summary

In this chapter, a passive/active or hybrid DVA (HDVA) enhanced isolator is proposed for suppressing the IRs. Based on the PDAV enhanced isolator, a prototype of HDVA enhanced isolator was built. The active force was implemented by means of a voice coil – magnet pair. The adaptive filtered-$x$ LMS feedforward control algorithm was chosen to control the active force in the experiments. This algorithm seeks to minimize the error signal by adapting the coefficients of a FIR control filter. Two types of error signals were used in the feedforward control experiments. One is the transmitted force and the other is the acceleration at the center of the plate foundation. It is observed that the HDVA enhanced isolator performs relatively better at low frequencies when the error signal is the transmitted force and at high frequencies when the error signal is acceleration. However, from the overall point of view, both cases result in substantial broadband reduction in both force transmissibility and radiated acoustic power. Comparing to the performance of the hollow isolator, the amplitude of the transmitted force is reduced by more than 90% by using the HDVA approach. For the acoustic power, the reduction is more than 10 dB in the experiments.

It should be noticed that when the control goal is to reduce the force transmissibility, the transmitted force should be selected as the error signal because it directly measure the function it is sought to be reduced. However, either error signal used in the experiments (force or acceleration) is not a direct metric for the radiated noise. Hence, better noise reduction would be expected if a signal from an acoustic transducer (e.g. a microphone) is directly used as the error signal.
Chapter 6

Conclusions and Future Work

In this chapter, the study on the IRs in isolators and their control is briefly reviewed and the main conclusions summarized. Recommendations for future work are also discussed.

6.1 Conclusions

It was mentioned that many researchers have been aware of the IRs in vibration isolators and their adverse effects to the isolator performance since the 1950s [1, 5-9, 15-17]. However, they did not considered that the IRs were of primary concern, particularly when viscoelastic isolators are used. This conclusion is not accurate because (a) previous researchers used a SDOF model, which might fail to reveal the importance of IRs, and (b)
previous researchers did not investigate the effects of IRs on the radiated sound power. Furthermore, to the best of the author’s knowledge, no attempt was made to improve the isolator performance by suppressing IRs.

To investigate the importance of IRs as well as their characteristics, an analytical model was developed in Chapter 2. This model consists of a rigid primary mass connected to a flexible foundation through multiple isolators. As used by previous researchers, the isolator was simply modeled as a one-dimensional “long-rod”. The IRs are associated to the “standing-wave” frequencies of this long-rod. The main difference between the model used in this dissertation and previously employed SDOF model is that the primary mass was considered to have three DOFs and the foundation to be flexible. This 3DOF model is considered to be more practical since most realistic vibration problems deal with multi-DOF systems and flexible foundations. Hence it makes it possible to examine the IRs in multi-DOF systems and their effects on the noise radiation. To demonstrate the significance of the IRs, the force transmissibility and the sound power radiated by the foundation were used as the two metrics to measure the isolator performance.

Using the 3DOF analytical model, the IRs’ characteristics are investigated analytically in Chapter 3. Comparison of the force transmissibility of the individual isolator and the radiated acoustic power of the foundation for the system with realistic and ideal massless isolators shows that the effects of IRs are important. The transmissibility for the isolator with inertial effects is 20 – 30 dB higher than that for the massless isolator at the IR frequencies. Furthermore, this dissertation shows that the IRs can result in a more significant deterioration of the performance of one or more isolators in a multi-DOF system than previous researchers predicted using the SDOF model. Evaluation on the radiated sound power shows that neglecting the mass of isolators can lead to significant underestimation of the sound radiated by the foundation of the vibration system at
frequencies around the IRs. The analytical model indicates that the total sound power in the frequency band of [200–3000] Hz is 3 - 22 dB higher (due to the IRs) as compared to the predicted power using the massless isolator model.

In Chapter 4, the IR phenomenon and the “long-rod” isolator model was verified experimentally. It was shown that after adjusting some of the isolator parameters, the model successfully predicted the IRs in a commercial rubber isolator. As a first attempt to suppress the IRs, the innovative concept of inserting passive DVAs into the isolator was proposed and discussed in detail (referred as PDVA enhanced isolator). Different from the conventional DVA application, in this study two DVAs are directly embedded into the isolator body instead of the primary mass to which they are traditionally attached. A parameter study of the PDVA enhanced isolator was carried out to gain insight into the influence of each DVA parameter on the isolator performance. It is found that the DVA mass and the tuning frequency greatly affect the isolator performance while the DVA damping is relatively less important. To obtain the best performance, the DVA parameters were optimized based on minimizing of either the transmissibility or the radiated acoustic power over a frequency band. The performance of the PDVA enhanced isolator was then simulated using the 3DOF reference model. At least 10 dB reductions in the force transmissibility were demonstrated around IR frequencies. For the total acoustic power, the PDVA enhanced isolator resulted in a 2 and 20.3 dB reduction in the [200–3000] and [500–3000] frequency bands, respectively. The passive DVA effect was also tested experimentally on a prototype PDVA enhanced isolator. It was demonstrated that the PDVA enhanced isolator leads to significant reductions in both the transmissibility and the radiated acoustic power. The experimental results show that the PDVA enhanced isolator yield a reduction of up to 30 dB for the force transmissibility and 8.8 (or 2.7) dB for the total acoustic power within [500–1600] (or [300–1600] Hz) frequency band.
Both the analytical and the experimental results in Chapter 4 have shown that the PDVA approach is effective to improve the isolator performance by attenuating the IRs, especially at high frequencies (usually above the first IR). However, it was also shown that the PDVA enhanced isolator does not always result in good performance at low frequencies ranging from the highest system resonance to the first IR. This is because that new resonances induced by the DVA masses at the low frequency region are not avoidable if only passive elements are used. Therefore, the active control technique was suggested to be employed along with the passive DVAs to further improve the isolator performance.

In Chapter 5, on the basis on the PDVA enhanced isolator, an active force was applied between the two DVA masses to form the HDVA enhanced isolator. This hybrid approach for suppressing the IRs was mainly investigated experimentally. In the experiments, the active force was implemented by means of a voice coil – magnet pair. The adaptive filtered-x LMS feedforward control algorithm was selected to control the active force. This algorithm seeks to minimize the error signal by adapting the coefficients of a FIR control filter. Corresponding to the two metrics of the isolator performance, two types of error signals were used. One is the transmitted force which is related with the force transmissibility and the other is the acceleration at the center of the plate foundation which is related with the sound radiation of the foundation. It was observed that the HDVA enhanced isolator performs relatively better at low frequencies when the error signal is the transmitted force and at high frequencies when the error signal is the acceleration. In an overall point of view, the HDVA enhanced isolator results in considerably better broadband reduction in both force transmissibility and radiated acoustic power than the PDVA enhanced isolator does. Compared to the case of hollow isolator without the DVA treatment, the amplitude of the transmitted force is reduced by more than 90% by using the HDVA approach. For the noise radiation, the reduction in the total acoustic power is more than 10 dB in the experiments.
In brief, the main conclusions of this dissertation are:

1. Traditional massless isolation model tends to over-estimate the isolation efficiency because it ignores the isolator’s mass. For a practical isolator, IR can increase the force transmissibility by as much as 20 ~ 30 dB at some frequencies.

2. IR is more significant when multiple DOFs exist in the system. Due to the inherent limitation, the SDOF model cannot reveal this phenomenon.

3. From the point of view of noise, the IR is important because it can significantly increase the total sound radiation by the foundation in the most audible frequency range.

4. The method of adding DVAs into the isolator shows much potential to attenuate the IR problem. The passive DVA approach is very effective to improve isolator performance at high frequencies. In order to obtain better performance at low frequencies and further improvement at other frequencies, active control technique may be required. It was demonstrated that, in the designated broadband isolation range, the performance of the Hybrid controlled DVA enhanced isolator can be as good as the performance of the ideal massless isolator.

The major contributions of this dissertation are:

1. On the basis of the achievements of some previous researchers, this dissertation is a continued study on the IRs in vibration isolators and their effects on the isolator performance. Different from other researchers, this dissertation uses a more practical model. This is a 3DOF vibration model which includes a primary mass connected to a flexible foundation through three isolators. The advantages of this model are that it
predicts the IRs in practical systems more actually, helps investigate the IRs in multi-DOF systems, is able to examine the isolator location effects on the IRs and makes it possible to study the IR problem from the point of view of noise radiation.

2. Besides the vibration transmissibility, this study also investigates the IR effects on the radiated acoustic power of the foundation, which were not included in the current available literatures authored by other researchers. Because the IRs usually appear in the most audible frequency range, it is very important to look at the IR problem from the perspective of the noise radiation.

3. To the best of the author’s knowledge, this study is the first research effort on the possible approaches for attenuating the IR problem and improving the isolator performance. A novel configuration of embedding DVA into the conventional isolator body to form the DVA enhanced isolator was proposed and validated. Both a passive and a hybrid DVA enhanced isolator prototypes were built and tested. The analytical and experimental results demonstrated that the approach of using DVA to suppress the IRs is very effective.

6.2 Future Work

The current available models on IRs including the isolator model used in this dissertation assume that the isolator can be represented as a “long-rod”, i.e. the length of the isolator is much larger than its lateral dimension so that the lateral deformation of the isolator under the longitudinal excitation is ignored. Although it was demonstrated that this approximation is feasible for isolators primarily in long cylindrical shapes and at relatively low frequencies (e.g. the frequency range in which the first several IRs appear), its application is fairly limited. Moreover, since the lateral effect is ignored, the predictions from the “long-rod” model always have some discrepancies from the
experimental results - the higher the frequency, the larger the discrepancy. Therefore, it is suggested that the isolator model should be modified according to the more comprehensive theory developed by A. E. Love [48]. In Love’s theory, the “lateral-inertia” of the isolator is accounted for. Thus, a more accurate prediction on the frequencies and amplitudes of the IRs can be expected.

It was experimentally demonstrated that the performance of the PDVA enhanced isolator highly depends on the DVA parameters which should be decided according to the preferred isolation region and the system characteristics. Therefore, in order to implement this technique in practical systems, a design methodology need to be developed. This methodology should include the selection of the configuration for the PDVA enhanced isolator (e.g. number and location of DVAs), deciding the frequency range in which the DVA parameters should be optimized and the objective function for the optimization routine. The latter two parts are primarily related with the design requirements of the isolation system.

For the hybrid control approach, this study demonstrated its effectiveness using the filtered-x LMS feedforward control algorithm. This algorithm seeks to minimize the error signal. Hence, it is desirable to have an error signal which is also the target signal or directly related with the target signal to be controlled. In experiments performed Chapter 5, when the control goal is to minimize the force transmissibility, the signal from the transducer measuring the transmitted force through the isolator was used as the error signal; when the noise energy radiated from the foundation is the primary target to be reduced, the signal of the accelerometer at the center of the foundation was used as the error signal. In the former case, the control goal is coincident with the error signal. Therefore, the best performance was expected. However, in the latter case, the error signal is not the best one for reducing the noise radiation. Therefore, more experimental investigations on the control of radiated noise from the foundation are suggested for
future work. Similar with the case for reducing the transmissibility, the ideal error signal for reducing the noise radiation should be the A-weighted total sound power radiated by the foundation. The non-weighted sound power may be obtained in two distinct ways. One is to measure the velocity distribution of the foundation using accelerometers; the radiated sound power can be estimated using the method presented in Equation (4.7). The other method is to measure the sound pressure level at several points on a hemispherical surface surrounding the foundation using microphones; the sound power can then be determined according to the measurement standards [57]. This method may need to use an anechoic room. The estimated non-weighted sound power is subsequently passed through the A-weighting network (electric circuits) to generate the A-weighted signal. Better results are expected if this ideal error signal can be practically obtained and fed back to the controller.

Although the feedforward control resulted in very promising results, its application is limited. This is because the reference signal is difficult to obtain in a timely manner (causality problem) in practice, especially for the case of broadband control. Therefore, to practically implement the HDVA enhanced isolator, further investigations and analyses are needed. Moreover, the performance of the HDVA enhanced isolator should be investigated using feedback control algorithms which do not have causality problem.

As mentioned before, this dissertation mainly discussed the IRs in viscoelastic isolators, such as rubber mounts, etc. However, the IRs or the wave effects should be considered as a general phenomenon existing in any kind of practical isolator. Therefore, the concept of using DVA to improve the isolator performance can also be extended to other types of isolators. One example is the hydraulic isolator. It is well known that many engine mounts and transmission mounts of vehicles are hydraulic systems, which is the practical motivation to explore the technique of embedding DVA into hydraulic mounts. The primary difference between the hydraulic mount and the mount made of solid
material is that the functional medium of the hydraulic mount is fluid. Thus, the two fundamental questions needed to be answered for this future research direction are (a) the characteristics of the IRs in hydraulic mounts and (b) how to integrate DVAs into hydraulic mounts.
Bibliography


Appendices

A: Development of the Force Vector

In Section 2.2.4, the three derived excitation forces are computed from the original external force using the static equilibrium condition of force and moment. This section demonstrates that the previously used approach is still valid for the case when dynamic forces are considered.

Consider the system shown in Figure 2.5. The EOMs of the primary mass can be directly written with respect to the motion of its CG, i.e., the translation $Y_{cg}$ and rotations $\Theta_x$ and $\Theta_z$ about x- and z- axes respectively. This is shown in Equation (A.1).

$$
\begin{align*}
F_0 &= m\ddot{Y}_{cg} - r_1 \\
-F_0L_{Fz} &= J_{xx} \ddot{\Theta}_x - r_2 \\
F_0L_{Fx} &= J_{zz} \ddot{\Theta}_z - r_3
\end{align*}
$$

(A.1)
where \( r_i \) (where \( i = 1, 2, 3 \)) is the reaction force or moment acting on the primary mass by the isolators. Equation (A.1) can be rewritten in matrix form as

\[
F_0 \begin{bmatrix} 1 \\ -L_{Fz} \\ L_{Fx} \end{bmatrix} = \begin{bmatrix} m \\ J_{xx} \\ J_{zz} \end{bmatrix} \begin{bmatrix} \ddot{Y}_{cg} \\ \dot{\Theta}_x \\ \dot{\Theta}_z \end{bmatrix} - \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (A.2)
\]

Notice from Equation (2.25), the motion of the CG of the primary mass can be expressed in terms of the three translations, \( Y_{Ti} \) (where \( i = 1, 2, 3 \)), at the mounting points between the isolators and the primary mass. Therefore, the accelerations of the CG can also be expressed in terms of the accelerations of the three translations as shown in Equation (A.3).

\[
\begin{bmatrix} \ddot{Y}_{cg} \\ \dot{\Theta}_x \\ \dot{\Theta}_z \end{bmatrix} = [T_m]^{-1} \begin{bmatrix} \ddot{Y}_{T1} \\ \ddot{Y}_{T2} \\ \ddot{Y}_{T3} \end{bmatrix} \quad (A.3)
\]

Substituting Equation (A.3) into Equation (A.2) yields:

\[
F_0 \begin{bmatrix} 1 \\ -L_{Fz} \\ L_{Fx} \end{bmatrix} = \begin{bmatrix} m \\ J_{xx} \\ J_{zz} \end{bmatrix} [T_m]^{-1} \begin{bmatrix} \ddot{Y}_{T1} \\ \ddot{Y}_{T2} \\ \ddot{Y}_{T3} \end{bmatrix} - \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (A.4)
\]

Multiplying Equation (A.4) by \([T_m]^T\) gives:

\[
[T_m]^T F_0 = [T_m]^T \begin{bmatrix} m \\ J_{xx} \\ J_{zz} \end{bmatrix} [T_m]^{-1} \begin{bmatrix} \ddot{Y}_{T1} \\ \ddot{Y}_{T2} \\ \ddot{Y}_{T3} \end{bmatrix} - [T_m]^T \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (A.5)
\]

Using Equation (2.29), Equation (A.5) is further simplified as:

\[
[T_m]^T \begin{bmatrix} 1 \\ -L_{Fz} \\ L_{Fx} \end{bmatrix} F_0 = [M] \begin{bmatrix} \ddot{Y}_{T1} \\ \ddot{Y}_{T2} \\ \ddot{Y}_{T3} \end{bmatrix} - [T_m]^T \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (A.6)
\]
Equation (A.6) is the EOMs of the primary mass with respect to the three translations at the mounting points between the isolators and the primary mass; thus it is equivalent to Equation (2.33). Comparing with Equation (2.33), the second term on the right hand side of Equation (A.6) is the reaction force vector, i.e. \( \{ F_{e1}, F_{e2}, F_{e3} \}^T \); the term on the left hand side of Equation (A.6) is the external force vector consisting of the three sub-forces acting at the mounting points between the isolators and the primary mass. That is

\[
\begin{bmatrix}
-T_{m} \quad 1 \\
\end{bmatrix} \begin{bmatrix}
L_{Fz} \\
L_{Fx}
\end{bmatrix} F_0 = \begin{bmatrix}
F_{T1} \\
F_{T2} \\
F_{T3}
\end{bmatrix} \tag{A.7}
\]

Notice that Equation (A.7) is the same as Equation (2.32), which shows that, for the case discussed in this study, the three dynamic component forces \( F_{Ti} \) can be computed from the single dynamic force \( F_0 \) according to the static equilibrium condition.
B: Vibration Model of the Simply Supported Rectangular Plate under a Concentrated Force Excitation

The displacement response of a simply supported rectangular plate under the excitation of a concentrated force is given by [39, 40]:

\[ W(x, z, \omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Phi_{mn}(x, z) q_{mn}(\omega) \]  

(B.1)

where \( \Phi_{mn}(x, z) \) is the modal shape and \( q_{mn}(\omega) \) is the modal amplitude of the \((m, n)\)th mode, which are given as:

\[ \Phi_{mn}(x, z) = \frac{2}{\sqrt{m} \sqrt{a b}} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi z}{b} \right) \]  

(B.2)

and
\begin{equation}
q_{mn}(\omega) = (\omega_{mn}^2 - \omega^2)^{-1} \frac{2}{\sqrt{m_d ab}} \sin\left(\frac{m\pi x_f}{a}\right) \sin\left(\frac{n\pi z_f}{b}\right) F(\omega)
\end{equation}  \tag{B.3}

in which

\( \omega_{mn} \) : the \((m,n)\) natural frequency of the simply supported plate;

\( a, b \) : length and width of the plate;

\( m_d \) : mass per unit area of the plate;

\( (x_f, z_f) \) : position of the external force on the plate;

\( F(\omega) \) : Fourier Transform of the time-dependent external force.

The velocity response of the plate is given by:

\begin{equation}
V(x, z, \omega) = j\omega W(x, z, \omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Phi_{mn}(x, z) q_{mn}(\omega) \cdot j\omega
\end{equation}  \tag{B.4}
C: Noise Radiation of Planar Structures

The mechanism of sound generation by surface vibration is the acceleration of fluid in contact with the surface. It is informative to consider the vibration characteristics of a plate before proceeding to the question of its noise radiation.

1. Infinite Plate

Considering an infinite, uniform thin plate lying in the $x$-$z$ plane as shown in Figure (C.1), the bending wave (or the flexural wave) equation in rectangular Cartesian coordinates is given as [42]

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial z^2} + \frac{\partial^4 w}{\partial z^4}\right) + m_s \frac{\partial^2 w}{\partial t^2} = 0 \tag{C.1}$$

where $t$ denotes the time, $w$ is the displacement response of the plate, $m_s$ is the mass per unit area of the plate and $D$ is termed the bending stiffness of the plate, which is given by
\[ D = \frac{Eh^3}{12(1-\nu^2)} \]  

in which \( E \) is the Young’s modulus, \( \nu \) is the Poisson’s ratio of the plate material and \( h \) is the plate thickness.

Equation (C.1) is also the EOM for the transverse vibration of the plate. The solution for the transverse displacement has the form of

\[ w(x, z, t) = A_d e^{j(\omega t - k_x x - k_z z)} \]

where \( \omega \) is the radian frequency of the transverse wave, \( A_d \) is the displacement amplitude and \( k_x \) and \( k_z \) are the wavenumbers of the bending waves traveling in the \( x \) and \( z \) directions, respectively. Substituting Equation (C.3) into Equation (C.1) yields

\[ [D(k_x^4 + 2k_x^2k_z^2 + k_z^4) - m_z\omega^2]A_d = 0 \]

or

\[ D(k_x^2 + k_z^2)^2 - m_z\omega^2 = 0 \]

Let \( k_b = \sqrt{k_x^2 + k_z^2} \), which is normally called the flexural wavenumber of the plate. Substitution \( k_b \) into Equation (C.4b) gives

\[ Dk_b^4 - m_z\omega^2 = 0 \]  

(C.5)

The flexural wavenumber can then be calculated as

\[ k_b = (m_z\omega^2 / D)^{\frac{1}{4}} \]

(C.6)

The acoustic field generated by the bending wave in the plate is a plane wave traveling at an angle from the plate (refer to Figure C.1(b)). That is

\[ p(x, y, z, t) = A_p e^{i(\omega t - k_x x - k_z z - k_y y)} \]

where \( A_p \) is the amplitude of the acoustic pressure and the wavenumber components are related to the acoustic wavenumber \( k = \omega / c \), here \( c \) is the speed of sound in the fluid, as

\[ k^2 = k_x^2 + k_z^2 + k_y^2 \]

(C.8)
At the interface between the structure and the fluid, the particle velocity in the normal direction must match the structural surface velocity. Thus, according to the Euler’s equation [38]

\[
- \frac{1}{j\omega \rho_a} \frac{\partial p}{\partial y} \bigg|_{y=0} = \hat{w}(x,0,z,t) = j\omega A_p e^{j(\omega t-k_x x-k_z z)}
\]

(C.9)

where \( \rho_a \) is the fluid density. Substituting Equation (C.7) into Equation (C.9) yields

\[
- \frac{j k_y}{j\omega \rho_a} A_p e^{j(\omega t-k_x x-k_z z)} = j\omega A_f e^{j(\omega t-k_x x-k_z z)}
\]

(C.10)

which indicates that the \( x \)- and \( z \)- components of the acoustic wavenumbers have to be the same as the corresponding bending wavenumbers, i.e. \( k_{xf} = k_x \) and \( k_{zf} = k_z \). Replacing \( k_y^2 = k_x^2 + k_y^2 \) into Equation (C.8), it is found that the \( y \)-component of the acoustic wavenumber, \( k_y \), is obtained as

\[
k_y = \pm \sqrt{k_x^2 - k_y^2}
\]

(C.11)
Figure C.1: (a) Transverse vibration of an infinite thin plate; (b) A plane bending wave propagating in the plate.
The appropriate sign of the square root in Equation (C.11) is determined by the characteristics of the acoustic and the bending wavenumbers. Figure C.2 shows the two wavenumbers inside the square root sign, i.e., the acoustic wavenumber and the flexural wavenumber, as a function of frequency, which is known as the dispersion curves.

From Figure C.2, three distinctive conditions are possible to determine the \( y \)-component of the acoustic number. They are:

(a) \( k_b < k \): the phase speed of the plate bending waves is greater than the speed of sound in the fluid. Thus, these bending waves are referred as supersonic waves. In this case, plane sound waves travel away from the surface of the plate at an angle to the
normal given by $\cos \alpha = k / k_y$, and the positive sign in Equation (C.11) is selected. In literatures, the condition of $k_b < k$ is usually referred as the radiation condition.

(b) $k_b > k$: the phase speed of the plate bending waves is less than the speed of sound in the fluid. These bending waves are referred as subsonic waves. In this case, $k_y$ is imaginary; the disturbance in the fluid decays exponentially with the distance $y$. The negative sign of the square root must be selected so that the acoustic pressure at the infinite distance converges to zero.

(c) $k_b = k$: the phase speed of the plate bending waves is equal to the speed of sound in the fluid. In this case, $k_y = 0$, which implies infinite impedance. The boundary condition can not be satisfied in practice because finite vibration produces infinite pressure. The frequency at which $k_b = k$ is called the critical frequency. At frequencies lower than the critical frequency free bending waves do not radiate sound. On the other hand, at frequencies higher than the critical frequency free bending waves radiate sound into the surrounding fluid.

The amplitude of the acoustic pressure can also be calculated from Equation (C.10). That is

$$A_p = j \omega A_d \frac{\omega \rho_a}{k_y}$$  \hspace{1cm} (C.12)

After determining the value of $k_y$ from Equation (C.11) and substituting it together with Equation (C.12) into Equation (C.7) yields the expression of the sound pressure in the half space above the plate:

$$p(x, y, z, t) = j \omega A_d \frac{\omega \rho_a}{k_y} e^{i(\omega - k, x - k, z - k, y)}$$  \hspace{1cm} (C.13)
As pointed out earlier, when the wavelength of the plate vibrations is greater than the wavelength in the ambient medium, i.e. $k_b < k$, the plate radiates a plane wave into the medium. In the far field, the sound intensity can be calculated as

$$I = \frac{|p(x, y, z)|^2}{2\rho_a c}$$

(C.14)

**2. Baffled Finite Plate**

The radiation from bending waves on infinite plates has been discussed in the previous section. It was found that for infinite plates, there is a critical frequency. Only above this frequency, the bending waves on the plates can radiate sound into the ambient medium. However, any practical problem involves in a finite structure whose radiation behavior cannot be predicted directly using the infinite plate theory.

In order to explore the sound radiation by a finite plate, it is instructive to consider a plate strip with finite length and infinite width that is cut from an infinite plate. As shown in Figure C.3, the plate strip (2-dimensional) is embedded in an infinite baffle. It has length $l$, a given velocity distribution $v(x)$ in the region $0 < x < l$. Since the velocity distribution along the width direction is assumed to be uniform, the resulting sound pressure can be treated as a two-dimensional problem in the $x$-$y$ plane.

![Figure C.3: Sound radiations from a 2-dimensional finite plate.](image-url)
Since the plate extends only in a finite length, its sound field does not have the simple form as presented in Equation (C.7). For the finite plate, one approach is to use Raleigh integral to calculate the sound pressure, and from that, the radiated power. However, it is significantly simpler to investigate the sound radiation by the structure sketched in Figure C.3 as if it were an infinite plate. The velocity of the 2-dimensional plate (including the baffle) can be represented as a linear contribution of bending waves in an infinite plate. This is accomplished by taking the wavenumber transform.

In detail, along the imaginary infinite plate there travel back and forth a number of waves of various lengths (and thus various wavenumbers) and of amplitudes and phases. These waves add up to the velocity distribution $v(x)$ in the finite plate. That is, one needs to find out the distribution $\hat{v}(k_x)$, in the imaginary infinite plate, of plane waves with different wavenumber $k_x$, the sum or integral of which gives the velocity $v(x)$ in the region $0 < x < l$, and zero outside of this region. Thus, the distribution $\hat{v}(k_x)$ must satisfy the relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{v}(k_x) e^{-i k_x x} dk_x = v(x)$$

(C.15)

The form of Equation (C.15) is similar to the inverse Fourier transform of $v(x)$. Thus, the velocity distribution of $\hat{v}(k_x)$ may be calculated as

$$\hat{v}(k_x) = \frac{1}{l} \int_{0}^{l} v(x) e^{i k_x x} dx$$

(C.16)

Equation (C.16) is termed the wavenumber transform of the velocity distribution.

Since now the wave component $\hat{v}(k_x)$ is propagating in an infinite plate, the corresponding surface pressure field can be written in a form analogous to Equation (C.13):
\[
\hat{p}(k_y) \bigg|_{y=0} = \hat{v}(k_y) \frac{\omega \rho_o}{k_y} \tag{C.17}
\]

where, again \( k_y = \sqrt{k^2 - k_x^2} \) whose sign depends on the values of \( k \) and \( k_x \).

The total sound pressure resulted from the bending waves on the finite plate strip thus is given by adding up all the pressure of each wave component. That is

\[
p(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\omega \rho_o \hat{v}(k_x)}{\sqrt{k^2 - k_x^2}} e^{-jk_x x} e^{-\sqrt{k^2 - k_x^2} y} \, dk_x \tag{C.18}
\]

The power radiated per unit width of the plate is then found by evaluating the expression [42]

\[
P(\omega) = \frac{1}{2} \text{Re} \left\{ \int_0^1 p(x,0) v^*(x) \, dx \right\} \tag{C.19}
\]

where \( \text{Re} \) is the operator of retaining the real part of the complex expression and \( v^*(x) \) is the complex conjugate of \( v(x) \). The integral in Equation (C.19) may be evaluated by replacing the corresponding parts using Equations (C.15) and (C.18) as

\[
P(\omega) = \frac{\omega \rho_o}{4\pi} \int_{-\sqrt{k^2 - k_x^2}}^{\sqrt{k^2 - k_x^2}} \frac{|\hat{v}(k_x)|^2}{\sqrt{k^2 - k_x^2}} \, dk_x \tag{C.20}
\]

Note that the integration range in Equation (C.20) is decided by the fact that \( \sqrt{k^2 - k_x^2} \) is real only for \(-k < k_x < +k\).

It is also worthwhile to mention that Equation (C.20) usually has a non-zero value for the bending wave propagating at an arbitrary frequency \( \omega \) in a finite plate. Thus, there is no “cut-on” frequency as for the finite plate, i.e. the finite plate embedded in a baffle can radiate sound into the ambient fluid at any frequency. However, the capability of sound radiation of the finite plate does change a lot with frequencies. In the traditional vibration analysis, the response of a finite structure is usually expressed as a linear combination of
the modes. Thus, it is instructive to consider the radiation characteristics of different modes.

Assuming that the finite plate in Figure C.3 is simply supported by the baffle. Its velocity distribution at the mode, i.e., the $n^{th}$ model shape, can be expressed as

$$v_n(x) = V_n \sin \left( \frac{n\pi x}{l} \right) ; \quad 0 < x < l \quad \text{(C.21)}$$

where $V_n$ is the mode amplitude. The wavenumber transform of $v_n(x)$ is

$$\hat{v}_n(k_x) = V_n \int_0^l \sin \left( \frac{n\pi x}{l} \right) e^{jk_xx} \, dx \quad \text{(C.22)}$$

It is observed from Equation (C.20) that the spectrum of the modulus squared of $\hat{v}(k_x)$ may be a good indicator of the capability of the sound radiation. Hence, the spectrum of $\hat{v}_n(k_x)$ is shown in Figure C.4. Two cases are plotted, one is the fundamental mode and the other one represents a higher mode. It is seen that the spectrum is dominated by the wavenumber components around the free wavenumber, $k_n = n\pi/l$, of the $n^{th}$ mode, except for the fundamental mode $n = 1$. Therefore, the radiation increases rapidly as $k_x$ approaches $k_n$. On the other hand, based on the radiation condition, only the components (shaded area) whose wavenumbers, $k_x$, are less than the acoustic wavenumber, $k$, can radiate sound. It is then concluded that for frequencies such that (a) $k < k_n$, the mode is a poor radiator and (b) $k > k_n$, the mode is an efficient radiator. In Figure A.4, the fundamental model of case (a) is more efficient in radiating sound than case (b). As the vibration frequency for a given mode is increased, the corresponding acoustic wavenumber increases. As shown in Figure C.4, the position of $k$ moves further toward $+k_x$ direction. Therefore, more radiating wavenumber components are included in the range of $-k < k_x < +k$ and the radiation efficiency increases.
The radiation efficiency of a simply supported square plate was calculated and plotted in literature [41] for low-numbered modes. For reference, the figure is copied here as Figure C.5 in which the radiation efficiency is plotted as a function of the ratio of acoustic wavenumber to the plate flexural wavenumber. The mode number in the form of \((m, n)\) for each curve is marked in the figure.
Figure C.5: Radiation efficiency for low-numbered modes of a square plate (Frank Fahy, Copyright 1987, Academic Press).

It is observed from Figure C.5 that, in general, for \( k/k_b < 1 \), the radiation efficiencies of the modes for which both \( m \) and \( n \) are odd are the highest, among which the fundamental mode \((1, 1)\) dominates. Those of the even-even modes are the lowest. The radiation efficiency of all modes becomes asymptotic to unity at high frequencies when the acoustic wavenumber is larger than the structural wave number. In this sense, the point
where \( k/k_b = 1 \) is the critical frequency of the finite plate. Recall that this point is also the critical frequency for the infinite plate. However, it implies different physical meanings for these two cases. That is, below the critical frequency, an infinite plate can not radiate sound while a finite plate can radiate sound but inefficiently; above the critical frequency, both the infinite and the finite plates can radiate sound into the ambient fluid efficiently.

To this end, the characteristics of the finite plate in sound radiation have been thoroughly discussed. Generally speaking, the plate is a good radiator at high frequencies that may coincide with several IR frequencies. Therefore, the IRs in the isolators might result in serious noise problem in practice. It is also noted from Figure C.5 that the plate flexural wavenumber, \( k_b \), plays an important role in the radiation efficiency. In the range of \( k/k_b < 1 \) and for a given wave with frequency \( \omega \), a smaller \( k_b \) indicates a larger radiation efficiency. Since the flexural wavenumber largely depends on the plate parameters such as the thickness, density, Young’s modulus and so on, the plate parameters also affect the sound energy radiation. On the other hand, it has been shown that some modes of the plate vibration are efficient in radiating sound while some others are not. In the cases where the isolators are placed at such locations that some of the “efficient” modes are excited, the plate radiates more sound energy than other cases.
D: A Simplified Numerical Method for Calculating the Acoustic Power Radiated by a Planar Structure Embedded in a Baffle

Equation (2.47) gives the expression for calculating the acoustic power radiated by a planar structure embedded in an infinite baffle. It is noticed that the integral area is specified by \( k \geq \sqrt{k_x^2 + k_z^2} \), which stems from the radiation condition. On the boundary when \( k = \sqrt{k_x^2 + k_z^2} \), the denominator of the integrand is zero. Therefore, a singularity problem occurs if one wants to calculate the acoustic power directly according to Equation (2.47). On the other hand, it is observed that the two-dimensional integral in Equation (2.47) is actually defined within a circle. This circle is centered at the origin of Cartesian coordinates. Its radius is \( k \). Since a two-dimensional integral defined within a circular area usually has a much simpler form in polar coordinates than that in Cartesian coordinates, Equation (2.47) can be rewrite in the polar coordinates as
\[ W(\omega) = \frac{\omega \rho}{8\pi^2} \int_0^{2\pi} \frac{r^2}{\sqrt{k^2 - r^2}} \int_0^k |V(r, \theta)|^2 \, r \, dr \, d\theta \]  

(D.1)

where distance \( r \) and angle \( \theta \) are the two components in polar coordinates. The circular integral area is then defined by the new bounds of \( 0 \leq r \leq k \) and \( 0 \leq \theta \leq 2\pi \). To transform Equation (2.47) into Equation (D.1), the following relationships are used

\[ k_x = r \cos \theta \, , \, k_y = r \sin \theta \, , \, \text{and} \, \, d_{k_x}d_{k_y} = rd_r d_\theta \]  

(D.2-4)

Although Equation (D.1) is more concise than Equation (2.47), the singularity problem still occurs when \( k = r \). To avoid the singularity problem, the zeros of the denominator of the integrand have to be gotten rid of. Inspired by the form of the denominator and the range of \( r \) in Eq. (D.1), the formula is further simplified by using some trigonometric function. In order to do this, a new variable \( \varphi \) is introduced so that

\[ r = k \sin \varphi \, (0 \leq \varphi \leq \pi / 2) \]  

(D.5)

\[ \sqrt{k^2 - r^2} = \sqrt{k^2 - (k \sin \varphi)^2} = k \cos \varphi \]  

(D.6)

and

\[ rd_r d_\theta = k \sin \varphi d_{k \sin \varphi} d_\theta = k^2 \sin \varphi \cos \varphi d_\varphi d_\theta \]  

(D.7)

Submitting Equations (D.5-7) into Equation (D.1) yields

\[ W(\omega) = \frac{\omega \rho}{8\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{|V(\varphi, \theta)|^2}{k \cos \varphi} \cos \varphi \sin \varphi d_\varphi d_\theta \]  

(D.8)

Fortunately, the term in the denominator of Equation (D.8) can be canceled out by the same term in the nominator. This leads to the final simplified form of the formula for calculating the acoustic power without the singularity problem as shown in Equation (D.9).

\[ W(\omega) = \frac{\omega \rho}{8\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} (k \sin \varphi) \sin \varphi d_\varphi d_\theta \]  

(D.9)
Vita

Yu Du was born on October 22, 1973 in Beijing, China. He received his Bachelor of Science degree and Master of Science degree in Vehicle Engineering in 1997 and 1999, respectively, from Department of Automobile Engineering of Tsinghua University, Beijing, China. Upon his graduation from Tsinghua University, Yu Du immediately began the Ph.D. studies at Virginia Polytechnic Institute and State University in the fall of 1999. He worked in the Vibration and Acoustics Laboratories (VAL) as a Graduate Research Assistant under the guidance of Dr. Ricardo A. Burdisso and Dr. Efstratios Nikolaidis (The University of Toledo). His main research interests are acoustics, controls, dynamics and mechanical vibrations. Yu Du completed his Ph.D. degree in spring, 2003, and accepted a research and development engineer position at Adaptive Technologies, Inc. (ATI) located in Blacksburg, VA.