Interference Avoidance based Underlay Techniques for Dynamic Spectrum Sharing

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Dynamic spectrum sharing (DSS) is a new paradigm for spectrum allocation that is expected to lead to more efficient spectrum usage and alleviate the spectrum-scarcity that has been perceived in recent years. DSS refers to the opportunistic, dynamic, and uncoordinated use of the spectrum by multiple, possibly non-cooperating, systems. It allows bands which may be underutilized by incumbent or legacy systems to be shared by agile or cognitive radios on a “do no harm” basis.

An ideal DSS technique is one which efficiently uses the allocated spectrum and maximizes the performance of the DSS network while causing no interference to the legacy radio system with which it coexists. We address this issue in our work by investigating desirable features for DSS with respect to the impact on a legacy radio system as well as the performance of a DSS network. It is found that “ideal” DSS techniques with respect to both objectives are characterized by the removal of the strongest interferers in the system and averaging of the remaining interference. This motivates the use of an interference avoidance (IA) based underlay technique for DSS. The performance benefit provided by this technique, over an IA-based overlay technique, is shown to increase with the transmission bandwidth available to the DSS system. It is also shown that this technique is more robust to inaccuracies in the system knowledge required for implementing IA.

An example of an IA-based underlay technique is a spreading-sequence-based transmission scheme that employs sequence adaptation to avoid interference. We use game-theoretic tools to design such schemes for distributed or ad hoc networks. The designed schemes can also be used to avoid interfering with other agile or static radios. We then extend this work to Ultra Wideband systems which can maximally exploit the gains from the proposed scheme due to the large transmission bandwidths.
Dedicated to the memory of the victims of 16 April 2007.
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Chapter 1

Interference Avoidance based Underlay Techniques for Dynamic Spectrum Sharing

1.1 Introduction

In recent years, rapid advances in wireless technology coupled with advances in chip technology have fueled a dramatic increase in the number of possible wireless applications, devices, and services. However, the actual deployment of new services has been hindered by the fact that FCC spectrum allocations are based on reserving portions of the spectrum for specific applications. Studies have shown that a large percentage of the allocated spectrum, including spectrum with highly desirable propagation characteristics like the TV spectrum, is unused over time and space. For instance, researchers at Kansas University found that the average national spectrum occupancy is $5.2\%$ with a maximum occupancy of $13.2\%$ in New York City. Hence, the perceived spectrum scarcity is more a result of regulatory policy than an actual scarcity of spectrum. It is also believed that recent technical advances can be used to build wireless devices, such as software radios and cognitive radios, which do not require rigid bandwidth allocations to limit interference between systems [1]. This has led to a renewed interest in spectrum access and prompted research in the direction of more flexible spectrum policy where devices share spectrum and use it on a need to basis.

Dynamic spectrum sharing (DSS) refers to the opportunistic, dynamic, and uncoordinated use of the spectrum by possibly non-cooperating systems. It allows bands which may be underutilized by incumbent or legacy systems to be shared by agile or cognitive radios on a “do no harm” basis. It is expected that DSS could lead to a more efficient usage of the spectrum and could also spawn innovative technologies for shared usage of spectrum. The expectations are, in part, due to the success story of the ISM band, which was allocated for unlicensed usage and which led to a plethora of new wireless applications and created many important technologies and wireless standards. The first step in the direction of DSS is the IEEE 802.22 standard for Wireless Regional Area Networks (WRAN) [2], which is currently being drafted. This standard oversees the utilization of geographically unused TV spectrum on a strictly non-interfering basis by license-exempt multi-vendor devices for wireless broadband access.

The two main aspects of the design of a DSS system are: coexistence with legacy or incumbent devices and the multiple access control (MAC) design for distributed DSS systems. DSS devices are expected to not cause any interference to the incumbent or legacy system. However, DSS systems might not always have access to perfect information about the legacy system. In addition, since DSS allows multiple uncoordinated systems to share the spectrum, spectrum access strategies that alleviate spectrum contention between DSS-enabled devices need to be developed. These issues are expanded on in later sections.
1.2 Approaches to Dynamic Spectrum Sharing

Two of the main approaches that have been proposed for a DSS radio system that intends to co-exist with a static radio system (the term “static radio” refers to an existing legacy user or incumbent user of the spectrum and is so used since the legacy radios are not expected to dynamically alter their transmission characteristics based on the interference caused by a DSS system) are the interference avoidance (IA) based intelligent spectrum overlay technique and the interference averaging (or spreading) based spectrum underlay technique. A third possibility is a hybrid IA-based spectrum underlay which combines the two approaches in an attempt to combine the benefits provided by each.

1.2.1 Spectrum Overlay with Interference Avoidance

IA-based spectrum overlay is a cognition-based narrowband (NB) technique where the DSS radio dynamically chooses a frequency band for transmission. The choice is made such that the interference caused to the static radio system is minimized or is below some predefined maximum. This technique requires knowledge of the locations and transmissions of the static radios which ideally can be estimated by sensing or obtained through the use of a local spectrum usage database (Note that it might be impossible in some situations, for example TV receivers, to obtain perfect information about the location of the static radio receivers.). Some example schemes are described in [3], [4], [5] and [6].

1.2.2 Spreading-based Spectrum Underlay

Interference averaging refers to transmission techniques where radios spread their signals across the entire bandwidth available to the DSS radio system. Ideally, no single source dominates the interference caused to the static radio system in this scheme. Code Division Multiple Access (CDMA) [7] [8] and Ultra Wide Band (UWB) [9] are two technologies that utilize this approach. Spreading based systems have, in the past, been studied for spectrum sharing with the GSM standard in cellular systems ([10], [11] and many others). This was the original ISM-band approach.

1.2.3 Interference Avoidance based Spectrum Underlay

As mentioned, a third possibility would be to combine the two approaches to harness the benefits of each. To this end, we motivate the use of a spreading-based underlay technique that also employs IA for DSS. In this transmission technique, the DSS radios spread their signal over the entire available bandwidth. However, they also avoid frequencies in which they can sense static radio transmissions. IA techniques such as notching ([12] and [13]) and waveform adaptation ([14], [15] and references within) can be used to this end. The adaptive frequency hopping technique implemented in Bluetooth version 1.2 [16] is also an example of this hybrid DSS technique. As will be shown in this dissertation, this technique can, to some extent, alleviate the burden of gathering perfect knowledge about the static system for the IA scheme. This technique is also referred to in this dissertation as “spreading-based underlay with IA”.

1.3 Research Challenges for Dynamic Spectrum Sharing

An ideal DSS scheme is one which efficiently uses the allocated spectrum and maximizes the performance of the DSS network while causing no interference to the static radio system with which it coexists. However, the design of an ideal DSS scheme entails a variety of technical challenges ranging from the design of
wide-band sensing mechanisms, interference metrics and knowledge of the static system to the design of distributed DSS approaches and security issues. We delineate a few of these, which will be the focus of our research, in this section.

1.3.1 Interference to Legacy Systems

The DSS network does not necessarily have access to the complete, perfect and instantaneous interference profile at the legacy receivers that are influenced by the actions of the dynamically adapting DSS radio. This arises due to fact that perfect knowledge of the interference profile at any receiver requires accurate information about the location of the receiver, the characteristics of the transmission to the receiver and the sum interference power seen at the receiver due to transmissions from other interferers in the network. However, practical deployment issues like imperfect sensing and the hidden node problem preclude the former two conditions while the third condition requires considerable coordination and feedback in the network. In some scenarios, access to such information might even be impossible to obtain since the legacy system is not usually required to cooperate with the DSS network. For instance, consider the case of the co-existence of wireless broadband access with television broadcasts in the TV spectrum. The legacy receiver (which is the television set in this case) does not transmit any signal that indicates the TV transmission to which it is tuned or even its existence. Consequently, it is not possible to completely predict the interference profile at the TV receiver.

Given that perfect information about the interference profile at affected legacy receivers is almost impossible to obtain, all DSS techniques result in some amount of aggregate interference at the legacy receivers in the system. Hence, it becomes important to study DSS approaches and identify desirable features for DSS techniques with respect to their resulting interference distributions at the legacy or static receiver. This is also relevant with respect to the new “interference temperature” model proposed by the FCC that places a maximum threshold for the aggregate interference-plus-noise level at a receiver as opposed to the current approach that manages interference by limiting the transmit power of individual devices [17].

Comparisons of different transmission schemes with respect to interference distributions have been previously studied in the literature. For example, there exists some previous work ([18] and references within) on the comparison of DS-CDMA and FH-CDMA schemes with respect to the system capacity of ad hoc networks implementing these schemes. There also has been some work on analyzing the mutual interference between specific DSS schemes (for instance, [19] and references within which specifically analyze the interference between 802.11 and Bluetooth systems). However, a general framework that compares basic approaches to DSS based on their impact to a static radio system, especially when including the effect of imperfect knowledge, does not exist. Further, such analysis is absolutely vital for determining the impact of DSS on legacy systems.

1.3.2 Performance of Distributed Networks

Since DSS techniques are also ideally required to lead to an efficient usage of the available spectrum, it is imperative that desirable features for spectrum access are also investigated from the perspective of optimizing the performance of the DSS network. DSS networks may have centralized or de-centralized architectures. Centralized networks have been well-studied in the context of cellular networks and it has been shown that the sum-capacity (sum-rate) of a network is maximized by avoiding strong interferers and averaging over weak interferers [20]. Note that such an allocation is possible by the use of an spreading-based waveform adaptation scheme [21], an example of an IA-based spectrum underlay scheme.

Determining the actual capacity of networks with distributed receivers, on the other hand, has proved an onerous task since the interference profiles at different receive-nodes are different and there does not need to
be any symmetry in user-pair interactions. However, considerable work has been carried out in establishing capacity order-bounds and scaling laws (as a function of node-density) for these networks ([22], [23], [24], [18] and references within). In [23], it was shown that for power-limited systems, a CDMA-based (or spreading based) multiple access scheme is optimal in the limit of infinite bandwidth since in this limit the interference between spectrum-sharing devices becomes negligible. The relevance of this result to practical networks which are bandwidth-limited has yet to be established. However, when only a smaller amount of bandwidth is available to the network, it was shown in the paper that a hybrid FDMA-CDMA technique, where FDMA is used in a local region and CDMA is used for handling interference from outside the local region, achieves the lower-bound on the optimal capacity. In [25] and [26], it was shown that in under-loaded systems when the signal attenuation in the network is large, water-filling techniques or spreading the signal over all available dimensions are better than IA-based techniques. The opposite is found to be true in networks where the mutual interference between spectrum sharing devices can be large due to small attenuation factors. However, this work only considers a small subset of possible network scenarios (under-loaded networks with two users or networks where all channels have equal gains).

Though the analysis scenarios are restrictive, the above results seem to indicate that allocations similar to the optimal allocations in a centralized network, where the strongest interferers at a receiver are avoided and the power from the weaker interferers are averaged, could be optimal from the perspective of a distributed network as well. A more comprehensive study which confirms these possible inferences and explicitly compares different approaches to DSS in distributed networks needs to be undertaken.

1.3.3 Design of DSS schemes for Distributed Networks

Once the desired characteristics for DSS techniques are known, specific DSS schemes that incorporate these characteristics and network algorithms for dynamically adapting transmission waveforms according to the interference environment and the legacy radio transmissions need to be designed. DSS allows multiple “uncoordinated systems” to share the spectrum. Also, as discussed earlier, access to perfect information about the interference profile at affected legacy receivers might lead to considerable over-head in the network or might even be impossible to obtain. As a consequence, the adaptation algorithms need to be amenable to a distributed implementation and should preferably adapt according to the interference environment perceived at its own receiver and with no or limited feedback from other spectrum users. However, in this scenario, taking into consideration the fact that transmit waveform adaptations themselves change the interference environment, it is necessary to investigate the over-all convergence and stability of the distributed adaptation algorithm in the network. It is also important to ensure that the adaptations by individual radios result in a desirable state for the network and improve (or maintain) some network performance measure.

Previous design of transmit waveform adaptation techniques has mostly focused on centralized systems. Due to the inherent structure of these networks, the interference profiles for different users are symmetric. This property leads to the convergence of simple iterative adaptation schemes where each user greedily adapts to maximize or improve the signal to interference and noise ratio (SINR) at its receiver ([21], [14], [27] and references within). However, in networks with distributed receivers, direct application of greedy adaptation algorithms might not lead to convergence [28]. This is caused by the asymmetry of the mutual interference between users at different receivers, leading the users to adapt their sequences in conflicting ways and resulting in resource allocation cycles. Since, each adaptation, in general, requires considerable feedback from the receiver to the transmitter, these allocation cycles are expensive with respect to the network overhead and are undesirable from a network performance perspective. Hence it is essential to investigate adaptation algorithms which are specifically tailored for distributed networks. Game theory, which models contention between players for a common resource, could be useful in the design of algorithms for these networks [29].
1.4 Dissertation Structure and Scope

In this dissertation, we address some of the research issues delineated in the previous section. Specifically, the goal of this work is to develop dynamic spectrum sharing schemes which are efficient with respect to the performance of a spectrum sharing network, minimize the impact to existing static radio systems and are amenable to a distributed and uncoordinated implementation in the network. To achieve this goal, we do the following:

- Develop a framework for analyzing the impact of DSS techniques on existing legacy static radio systems
- Develop a framework for analyzing the performance of distributed networks that implement DSS techniques
- Identify desirable features for “ideal” DSS techniques based on the above analyses
- Develop specific DSS schemes that incorporate the desirable characteristics identified above
- Design radio resource management algorithms for these DSS schemes that can be implemented in uncoordinated and distributed networks

More specifically, we initially develop a framework to determine the impact of different DSS techniques on the performance of legacy radio systems with which they coexist. The framework is based on the interference profile that results at a legacy receiver due to different DSS techniques. The framework also incorporates imperfections in system knowledge including the hidden node problem and imperfect sensing. Desirable features of DSS techniques with respect to the impact on legacy systems are identified based on the framework. It is found that the least interference is caused to a legacy system by a DSS technique that removes strong interferers and averages the power of the remaining interferers to the legacy system. It is also found that the benefit provided by this technique increases with an increase in the available bandwidth. In addition, this scheme is more robust to imperfect information about the legacy system. The inclusion of log normal shadowing is shown to further accentuate these performance trends.

The desirable characteristics of DSS techniques with respect to the performance of a distributed network that implements these techniques are then investigated. The sum of the outage-capacities of users in the network is used as the performance metric. Results indicate that, similar to the previous scenario and as is hinted in previous research, DSS techniques that mitigate interference from the strongest interferers in the systems while averaging the remaining interference are optimal with respect to maximizing network performance. The inclusion of log normal shadowing and an increase in the number of available transmission dimensions are shown to accentuate these performance trends.

Taking into account these observations, spreading-based techniques which implement some form of IA to reject the strongest interferers in the network are found to be the beneficial with respect to both the impact on the legacy system as well as the performance of a DSS network. This motivates the use of the IA-based spectrum underlay technique, introduced earlier, for DSS.

A spreading-sequence-based scheme that implements waveform (sequence) adaptation [14] for IA is a specific example of the IA-based spectrum underlay technique. Therefore, waveform adaptation algorithms for DSS amenable to a distributed implementation in networks with distributed receivers are developed. The designed algorithms allow users to dynamically adapt their transmit waveforms according to the interference environment and legacy radio transmissions. Two different objectives are considered for the adaptations: reduction of sum-interference in the network and target-performance realization. The development is based on a potential game model [30] that allows individual adaptation to improve a network performance measure.
Properties of potential games are then used to investigate and establish the convergence of the designed algorithms to desirable steady states for the network.

As mentioned, an increase in transmission bandwidth available to DSS radio systems is shown to accentuate the performance benefit provided by IA-based underlay techniques. This motivates the investigation of these techniques for UWB systems. Towards this end, we develop a novel spreading-sequence-based transmission scheme which results in increased energy capture and increased tolerance to interference. The proposed scheme also allows a reduction in receiver complexity. Finally, adaptation algorithms for this scheme in centralized and distributed multiuser networks are investigated.

1.5 Outline of Document

The rest of the document is organized as follows: Chapter 2 reviews and develops approaches to modeling a communication network and deriving interference power distributions at a receiver. Specifically, Sections 2.3.3 and 2.5 develop some new approaches while the remaining sections review existing work. The analysis in this chapter is employed in subsequent chapters to construct the frameworks used for the investigation of DSS techniques. In Chapter 3, a framework to investigate the impact of DSS on legacy radio systems is constructed and desirable features for DSS schemes with respect to this framework are investigated. In Chapter 4, a framework to investigate the performance of distributed DSS networks is developed and desirable DSS schemes with respect to this framework are investigated. It is found that an IA-based spectrum underlay scheme is beneficial with respect to both objectives. Spreading-sequence-based waveform adaptation techniques are an example of an IA-based spectrum underlay scheme. Distributed implementations of this scheme are developed in the remaining chapters. Specifically, Chapter 5 reviews and develops game-theory concepts which are utilized to construct distributed waveform adaptation schemes. In this chapter, existing work is reviewed in Sections 5.2-5.4 while Section 5.5 develops a new convergence result. Chapters 6 and 7, develop distributed waveform adaptation schemes for networks with centralized and distributed receivers, respectively. Chapter 8 develops a joint power control and waveform adaptation framework which enables users in the network to achieve their target-performances. Waveform adaptation strategies for UWB systems are investigated in Chapter 9. Finally, Chapter 10 summarizes and delineates the contributions of our research.
Chapter 2

Interference Modeling

2.1 Introduction

To investigate the performance of dynamic spectrum sharing schemes, a framework to model the network is one of the basic tasks to be accomplished. There are two aspects to the design of such a framework: the first is a model for the spatial distribution of nodes in the network and the second is a model for the interference power distributions that result from a particular choice of spatial distribution. In this chapter, we review and develop some new approaches to modeling the spatial distribution of a communication network and deriving the interference power statistic. We start in Section 2.2 with an introduction to spatial Poisson processes, which has been extensively used in the existing literature to model the spatial distribution of nodes in a network. The distribution of the interference power statistic while using a Poisson distributed network is investigated in Section 2.3 and in Section 2.4. In Section 2.5, Cox Poisson processes, which can be used to model non-homogeneous distributions of nodes in a network, are discussed. Finally, Section 2.6 concludes the chapter and delineates its contributions.

2.2 Spatial Poisson Point Processes

A homogeneous communication network can be represented by a uniform distribution of radios in a given area. The number of radios in a subset (of positive measure) of the given area is then given by a binomial distribution. In the limit of a large sample size, the binomial distribution tends to a Poisson distribution with parameter \( \lambda \) equal to the mean of the binomial distribution ([31] and Appendix A). Therefore, the number of radios in a subset of the given area, when the area under consideration is large, follows a Poisson distribution. The probability that there are \( k \) SS radios in a region unit area is then given by

\[
P(k) = \frac{e^{-\lambda}(\lambda)^k}{k!}.
\]

(2.1)

This Poisson distribution of radios in a given area is an example of a spatial Poisson point process. Such a point process has been extensively used in the literature to model radio networks. Some relevant papers are [32], [33], [34], [35], [36], and [37]. It is also mentioned in [32] that networks generated according to the Longley-Rice model [38], which incorporates effects of irregular terrain, show a close structural resemblance to networks generated using a Poisson distribution of nodes.
2.2.1 Definition and Properties

The definition and some properties of spatial Poisson point processes are delineated here. We closely follow the notations and text from [39]. A Poisson point pattern or process $\Pi$ is a random set or random counting measure such that when applied to a measurable space $E$, the value of $\Pi(E)$ is the number of points of $\Pi$ that lie in the set $E$. If the set $E$ is bounded, then $\Pi(E)$ is finite point pattern. A Poisson point process is characterized by the following properties:

- **Simplicity**: The Poisson point process is simple, i.e., there is never more than one point on a single location.

- **Poisson distribution of point counts**: The number of points of $\Pi$ in a bounded set $E$ with measure $\nu(E)$ is a Poisson variable with mean parameter $\lambda \nu(E)$, for some parameter $\lambda$.

- **Independent scattering**: For all disjoint subsets $A$ and $B$, $\Pi(A)$ and $\Pi(B)$ are independent random variables.

The Poisson point process is stationary if the statistical properties of the point process does not depend upon the location of the observation.

A marked Poisson point process is a point process that has a value or mark associated with each point. Thus, a marked point process on $\mathbb{R}^d$ is a random set $\Pi = \{[x_n; m_n]\}$, where $x_n$ is the location of a point and $m_n$ is the mark associated with the point. The marks might be dependent or independent of the location of the spatial point. For example, in a radio network modeled by a point process, the mark $m_n$ might be the transmit power of a radio. The transmit power of a radio may be based on the location of the radio or based on some other arbitrary criteria independent of the location.

The following theorem can be used to derive distributions and statistical properties of a spatial Poisson point process.

**Theorem 1 (Campbell's Theorem [39])** For any non-negative measurable function $f(.)$ and a Poisson point process over measurable space $A$,

$$
E \sum_{[x, m(x)] \in \Pi} f(x, m(x)) = \int_A \int f(x, m(x)) M_x(dm) \Lambda(dx)
$$

(2.2)

Here, $\Lambda(x)$ is the intensity measure over some measurable space $x$ with mean value $\lambda$ and $M_x$ is the distribution of the mark of a point at $x$. If $\Lambda$ is independent of the location,

$$
E \sum_{[x, m(x)] \in \Pi} f(x, m(x)) = \lambda \int_A f(x, m(x)) M_x(dm) dx
$$

(2.3)

2.2.2 Operations on Point Process

Three fundamental operations on point processes are thinning, clustering and superposition.

**Thinning**

A thinning operation uses a definite rule to delete points of a point process $\Pi$ to yield thinned point process $\Pi_{th}$. Note that $\Pi_{th} \subset \Pi$. An independent thinning process is one in which the deletion of a point is independent of the location as well as the deletions of other points of the point process. An example is $p$—thinning,
where a point is deleted with probability $p$. A generalization of $p$—thinning, is $p(x)$—thinning, where the probability of deletion is dependent on the location of the point. The characteristics of an independently thinned point process are easy to calculate if the characteristics of the original point process are known. For example the intensity measure of a $p(x)$—thinned point process is given by

$$\Lambda_{th} = \int_A p(x)\Lambda(dx).$$

(2.4)

The point process resulting from an independent thinning of a Poisson point process is also Poisson. The analysis of dependent thinning, on the other hand, is more involved. An example of a dependent thinning process is the Matern hard-core process which will be discussed in Chapter 4. An example of a scenario that can be modeled using the thinning process is a network where radios select a transmission band from the $N$ available transmission bands. The network of radios transmitting on the same transmission-band will be a $\frac{1}{N}$-thinned process of the original network.

Clustering

In clustering, every point of a point process is replaced by a cluster of $N^x$ points. The replacement clusters are also point processes. The resultant point process is called the cluster point process $\Pi_{cl}$ and $\Pi_{cl} = \bigcup_{x \in \Pi} N^x$. An example of a cluster process is the Neyman-Scott process which is formed when a homogeneous independent clustering is applied to a stationary Poisson point process. A cluster process could be used to model a network of small mesh networks.

Superposition

These point processes are generated by the union of two independent point processes. Since the sum of two Poisson random variables is Poisson, if the superposed point processes are Poisson, the resultant point process is also Poisson with intensity equivalent to the sum of the intensities of the original point processes.

2.3 Distribution of Interference Power

Performance analysis of communication networks generally requires knowledge of the interference power statistic at a radio in the network. Note that the interference power term is different from the interference term used in the decision statistic at the output of a receiver (the former term is the sum of powers of the interferers while the latter term is the sum of amplitudes of the interferers). The interference power term is more pertinent for the study of the performance of a user (and consequently the network) and is considered in our work. This section illustrates how the distribution of the interference power statistic can be calculated when using a Poisson point process to model the communication network.

2.3.1 System Model

We assume that the radio at which the interference statistic is to be determined is situated at the center of a large region in which other radios are distributed according to a marked spatial Poisson process. Interference is caused at the central radio by transmissions from the other radios in the network. This is modeled by letting the mark associated with a radio (i.e., each point of the Poisson process) to be the interference power caused by the radio at the central receiver. We assume that the interference power is only a function of the
distance between the radio and the central receiver and the path-loss factor. The effect of other factors, such as fading or shadowing, can easily be incorporated into the analysis. Such an extension is used in Chapter 3 to incorporate log normal shadowing. Interference power at the central receiver (assuming a matched filter) is equivalent to the sum of powers of signals received from all the interfering radios in the large region. The normalized interference at the central radio (assuming that the power received from an interfering radio at a distance of 1m is 1watt) is given by

\[ X^n = \sum_{i \in J^n} g(r_i). \]  \hspace{1cm} (2.5)

Here, \( n \) is intensity of the spatial Poisson point process of interfering radios per unit area, \( J^n \) denotes the set of interfering radios in the region, \( r_i \) is the distance of the \( i^{th} \) interfering radio from the central radio and \( \frac{1}{g(r_i)} \) is the path-loss suffered by a signal at distance of \( r_i \)m relative to the path-loss at a distance of 1m. \( g(r) \) is assumed to satisfy the following properties:

1. \( g(r) \) is monotonically decreasing or in other words, \( \lim_{r \to 0} g(r) = \infty \) and \( \lim_{r \to \infty} g(r) = 0 \).
2. \( \lim_{r \to \infty} r^2 g(r) = 0 \).

Note that, if the latter condition is not satisfied, the interference power at a given receiver would be a function of the network size and would be infinite for an infinite sized network ([34] and [40]). Also, as we shall see in the next subsection, the characteristic function of the interference power would not exist [37]. Further note that for realistic systems, while \( \alpha \leq 2 \) is possible for short distances, it is not possible for sufficiently long distances especially in the limit \( r \to \infty \). \( g(r) \) is henceforth specified as

\[ g(r) = \frac{1}{r^\alpha}; \ \ \ \alpha > 2. \]  \hspace{1cm} (2.6)

The distribution of the the random variable \( X^n \) is the basic analysis to be accomplished. This statistic can be used to evaluate performance metrics such as the probability of outage or the average performance of networks. Subsection 2.3.2 investigates the scenario where interfering radios are distributed in a circular region. Subsection 2.3.3 investigates the scenario where interfering radios are distributed in an annular region around the central receiver.

### 2.3.2 Circular Region

We first investigate the scenario, where interfering radios are distributed in a large circular region with radius extending to infinity. Let \( X^n_{0,a} \) denote the sum of received power at the static receiver from interfering radios Poisson distributed with intensity parameter \( n \) in a circular region with outer-radius \( a \). Then,

\[ X^n_{0,a} = \sum_{J^n_{0,a}} g(r_i). \]  \hspace{1cm} (2.7)

Here, \( J^n_{0,a} \) denotes the set of interfering radios at distance \( r_i \) from the static receiver such that \( 0 \leq r_i \leq a \). The distribution of \( X^n_{0,a} \) is derived from its characteristic function. This analysis was initially developed in [33] and [41]. It is shown in [34] that closed form expressions for the PDF and CDF of \( X^n_{0,a} \) are possible when \( \alpha = 4 \). In [42], it is shown that closed form the PDFs form alpha-stable distributions for all values of \( \alpha > 2 \). The above analysis is extended to correlated transmissions in [43]. Note that expression (2.6) for path-loss holds only when the receiver is in the far-field of the transmitter. In general, as the distance between the transmitter and receiver tends to zero, the received power does not tend to infinity but to some finite maximum value. However, expression (2.6) for \( 0 \leq r_i \leq a \) is used in these papers since it leads to stable distributions and makes the analysis more tractable [35].
The cumulative distribution function (CDF) of \( X_{0,a} \) for a given parameter \( n \) is given by
\[
\phi_{X_{0,a}}(\omega, n) = \mathbb{E} \left( e^{i\omega X_{0,a}^n} \right) .
\]
(This may be evaluated by conditioning on the number of radios (Poisson distribution as mentioned before) in the given area.
\[
\mathbb{E} \left( e^{i\omega X_{0,a}^n} \right) = \mathbb{E} \left( e^{i\omega X_{0,a}^n} \mid k \text{ in } D_{\alpha} \right)
= \sum_{k=0}^{\infty} \frac{e^{-n\pi a^2} (n\pi a^2)^k}{k!} \mathbb{E} \left( e^{i\omega X_{0,a}^n} \mid k \text{ in } D_{\alpha} \right).
\]
Here, \( D_{\alpha} \) denotes a disc of radius \( a \). If an SS radio is placed at a point on \( D_{\alpha} \) with uniform probability, the probability that the radio is at a distance \( r \) from the center of the disc is given by
\[
p_r(r) = \frac{2r}{a^2}; \quad 0 \leq r \leq a.
\]
Random variable \( X_{0,a}^n \) is the sum of \( k \) random variables \( (X_{0,a}^n = \sum_{j=0}^{j_0} g(r_j)) \) and the underlying probability density function of each random variable (i.e., the distribution of \( \pi \)) is given by (2.10). Also, the characteristic function of a sum of random variables is the product of the individual characteristic functions. The expectation term in (2.9) can therefore be replaced as follows:
\[
\phi_{X_{0,a}}(\omega, n) = \sum_{k=0}^{\infty} \frac{e^{-n\pi a^2} (n\pi a^2)^k}{k!} \left( \int_0^a \frac{2r}{a^2} e^{i\omega g(r)} dr \right)^k.
\]
Note that \( \sum_{k=0}^{\infty} e^{-n\pi a^2} (n\pi a^2)^k \left( \int_0^a \frac{2r}{a^2} e^{i\omega g(r)} dr \right)^k = 1 \), since this is a sum over a Poisson distribution with parameter \( n\pi a^2 \left( \int_0^a \frac{2r}{a^2} e^{i\omega g(r)} dr \right) \). Using this fact, we get
\[
\phi_{X_{0,a}}(\omega) = \exp \left( n\pi a^2 \left( \int_0^a \frac{2r}{a^2} e^{i\omega g(r)} dr - 1 \right) \right).
\]
Using integration by parts to evaluate the inner integral and letting \( a \to \infty \), we get
\[
\phi_{X_{0,\infty}}(\omega) = \exp \left( -in\pi \omega \int_0^\infty r^2 e^{i\omega g(r)} g'(r) dr \right).
\]
Making change of variable \( t = g(r) \),
\[
\phi_{X_{0,\infty}}(\omega) = \exp \left( in\pi \omega \int_0^\infty \left[ g^{-1}(t) \right]^2 e^{i\omega t} dt \right).
\]
Using (2.6), the characteristic function can be written as
\[
\phi_{X_{0,\infty}}(\omega) = \exp \left( in\pi \omega \int_0^\infty t^{\beta} e^{i\omega t} dt \right).
\]
Here, \( \beta = \frac{2}{a} \). The characteristic function can now be represented in terms of the Gamma function, \( \Gamma(\cdot) \), as
\[
\phi_{X_{0,\infty}}(\omega) = \exp \left( -n\pi \Gamma(1 - \beta) e^{-\pi\beta/2} \omega^\beta \right); \quad \omega \geq 0.
\]
The cumulative distribution function (CDF) of \( X_{0,\infty} \) can then be expressed as [44]
\[
F_{X_{0,\infty}}(y) = \Pr \left( X_{0,\infty} < y \right) = \frac{2}{\pi} \int_0^\infty \text{Re} \left( \phi_{X_{0,\infty}}(\omega) \right) \frac{\sin \omega y}{\omega} d\omega.
\]
The above expression can be numerically evaluated using series expansions to the desired accuracy [44]. Alternatively, probability of outage can be calculated from the following infinite series expression for the CDF of \( X_{0,\infty}^n \) [35]

\[
F_{X_{0,\infty}^n}(y, \alpha) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{2k}{\alpha}\right)}{k!} \left(\frac{\rho}{y^{2/\alpha}}\right)^k \sin k\pi \left(1 - \frac{2}{\alpha}\right).
\]

Here, \( \rho = \pi n \Gamma(1 - \frac{2}{\alpha}) \). When \( \alpha = 4 \), the CDF of \( X_{0,\infty} \) has a closed form expression [35] given by

\[
F_{X_{0,\infty}}(y) = \text{erfc}\left(\frac{\pi^{3/2} n}{2\sqrt{y}}\right).
\]

### 2.3.3 Annular Region

Interferers are distributed in an annular region with inner-radius \( \epsilon \) and an outer-radius that extends to infinity. The inner radius constraint can arise in a variety of network scenarios. For instance, it can be the result of the incorporating the minimum possible physical distance between two radios and hence a minimum distance between the central radio and an interferer. Another example scenario is the modeling of the out-of-cell interference in a cellular network. Again note that expression (2.6) for path-loss holds only when the receiver is in the far-field of the transmitter. However, the minimum possible physical distance (\( \epsilon > 0 \)) between two interfering radios places a natural constraint on the power received from the interferer and we assume that the path loss expression holds for all the other scenarios (\( r \geq \epsilon \)).

The interference statistic for the annular region is given by

\[
X_{\epsilon,a}^n = \sum_{J_{\epsilon,a}^n} g(r_i).
\]

Here, \( J_{\epsilon,a}^n \) denotes the set of interfering radios at distance \( r_i \) from the central receiver such that \( \epsilon \leq r_i \leq a \).

The characteristic function of \( X_{\epsilon,a}^n \) can be calculated in a manner similar to the derivation in the previous subsection. The characteristic function of \( X_{\epsilon,a}^n \) for a given parameter \( n \) is given by

\[
\Phi_{X_{\epsilon,a}^n}(\omega) = E\left(e^{i\omega X_{\epsilon,a}^n}\right).
\]

This may be evaluated by conditioning on the number of radios in the given area.

\[
E\left(e^{i\omega X_{\epsilon,a}^n}\right) = E\left(E\left(e^{i\omega X_{\epsilon,a}^n} | k \text{ in } D_{\epsilon,a}\right)\right) = \sum_{k=0}^{\infty} \frac{e^{-n\pi(a^2-\epsilon^2)}(n\pi(a^2-\epsilon^2))^{k}}{k!} E\left(e^{i\omega X_{\epsilon,a}^n} | k \text{ in } D_{\epsilon,a}\right).
\]

Here, \( D_{\epsilon,a} \) denotes an annular ring with inner radius \( \epsilon \) and outer radius \( a \). If an SS radio is placed at a point on \( D_{\epsilon,a} \) with uniform probability, the probability that the radio is at a distance \( r \) from the center is given by

\[
p_r(r) = \frac{2r}{(a^2-\epsilon^2)}; \quad \epsilon \leq r \leq a.
\]

Random variable \( X_{\epsilon,a}^n \) is the sum of \( k \) random variables \( (X_{\epsilon,a}^n = \sum_{J_{\epsilon,a}^n=0}^{k} g(r_i)) \) and the underlying probability density function of each random variable (i.e., the distribution of \( r \)) is given by (2.23). Also, the characteristic function of a sum of random variables is the product of the individual characteristic functions. The expectation term in (2.22) can therefore be replaced as follows:

\[
\Phi_{X_{\epsilon,a}^n}(\omega) = \left(\int_{\epsilon}^{a} \frac{2r}{(a^2-\epsilon^2)} e^{i\omega g(r) dr}\right)^k.
\]
As in the previous subsection, note that \( \sum_{k=0}^{\infty} e^{-n\pi(a^2-e^2)} \frac{1}{k!} \left( n\pi \left( a^2 - e^2 \right) \int_{e}^{a} \frac{2r}{(a^2-e^2)} e^{i\omega g(r)}dr \right)^k = 1 \), since this is a sum over a Poisson distribution with parameter \( n\pi \left( a^2 - e^2 \right) \int_{e}^{a} \frac{2r}{(a^2-e^2)} e^{i\omega g(r)}dr \). Using this fact, we get

\[
\Phi_{\mathcal{X}_{0,a}^n}(\omega) = \exp \left( n\pi \left( a^2 - e^2 \right) \left( \int_{e}^{a} \frac{2r}{(a^2-e^2)} e^{i\omega g(r)}dr - 1 \right) \right). \tag{2.25}
\]

Using integration by parts and letting, \( a \to \infty \), the inner integral can be evaluated to get

\[
\phi_{\mathcal{X}_{0,\infty}^n}(\omega) = \exp \left( n\pi e^2 \left( 1 - e^{\frac{\omega}{\pi}} \right) - in\pi\omega \int_{0}^{1} t^{-\beta} e^{i\omega t}dt \right). \tag{2.26}
\]

The characteristic function can now be expressed in terms of the incomplete Gamma function, \( \Gamma_{inc}(\cdot) \), as

\[
\phi_{\mathcal{X}_{0,\infty}^n}(\omega) = \exp \left( n\pi e^2 \left( 1 - e^{\frac{\omega}{\pi}} \right) - n\pi(-i\omega)^\beta \Gamma_{inc} \left( \frac{\omega}{\epsilon a}, 1 - \beta \right) \right). \tag{2.27}
\]

The CDF of \( \mathcal{X}_{0,\infty}^n \) can be calculated in a manner similar to (2.17) in the previous subsection and is given by

\[
F_{\mathcal{X}_{0,\infty}^n}(y) = \Pr \left( \mathcal{X}_{0,\infty}^n < y \right) = \frac{2}{\pi} \int_{0}^{\infty} \text{Re} \left( \phi_{\mathcal{X}_{0,\infty}^n}(\omega) \right) \frac{\sin \omega y}{\omega} d\omega \tag{2.28}
\]

The above expression can be numerically evaluated using series expansions to the desired accuracy [44]. Note that, as opposed to the CDF of \( \mathcal{X}_{0,\infty}^n \) considered in the previous subsection, closed-form expressions for the CDF of \( \mathcal{X}_{0,\infty}^n \) are not available. Hence the evaluation of distribution of \( \mathcal{X}_{0,\infty}^n \) involves considerable computational complexity. We therefore analyze alternate ways to model the distribution of the interference statistic \( \mathcal{X}_{0,\infty}^n \) using well-known distributions and bounds in the next section.

## 2.4 Modeling Interference Power in an Annular Region

Three methods to model the interference power statistic at the central receiver are discussed in this section. The first an Edgeworth Gaussian approximation with correction terms based on the higher order cumulants to model the interference. This method has been traditionally used to calculate the out-of-cell interference statistic for CDMA systems ([37] and [40]) and is computationally less intensive than the characteristic function approach. However, it has been observed via simulations that that the modified Gaussian distribution using Edgeworth expansion is a good approximation only when the interfering radios are not too close to the central receiver (The minimum distance between an interfering radio and the central receiver should be \( > 10m \)). The second method models the interference power statistic using a heavy-tailed log normal distribution. This method is found to be sufficiently accurate for our purposes and for most practical values (\( > 0.1m=10cm \)) of the minimum distance between an interfering radio and the central receiver and is used for most of the interference modeling in our work. Finally, a modified-Chernoff bound on the interference power statistic proposed in [37] is also discussed.

### 2.4.1 Gaussian Approximation

An alternate way to approximate the sum interference power at the static receiver, \( \mathcal{X}_{0,a}^n \), is to use a Gaussian approximation [7]. However, simulation results show that the distribution of \( \mathcal{X}_{0,a}^n \) is skewed to the left. This is due to the fact that interferers very close to the receiver terminal contribute a disproportionately large amount of interference. An Edgeworth expansion of the characteristic function (2.25) using higher
order cumulants of $X^n_{e,a}$ is used to approximate the distribution of $X^n_{e,a}$ in [37] and [40]. The Edgeworth approximation yields a Gaussian distribution together with a skewness correction factor for $X^n_{e,a}$.

Using this approach, the probability density function for $X^n_{e,a}$ can be approximated by

$$p_{X^n_{e,a}}(x) \approx q(\hat{x}) \left(1 + t_1 + t_2 + t_3\right).$$

(2.29)

Here, $\hat{x} = (x - \text{mean}(x) ) / \sqrt{\text{var}(x)}$, $q(\hat{x})$ is the standard Normal density function with mean zero and variance one, and

$$t_1 = \frac{k_3}{6} h_3(\hat{x}),$$

$$t_2 = \frac{k_4}{24} h_4(\hat{x}) + \frac{k_3^2}{72} h_6(\hat{x}),$$

$$t_3 = \frac{k_5}{120} h_5(\hat{x}) + \frac{k_3 k_4}{144} h_7(\hat{x}) + \frac{k_3^3}{1296} h_9(\hat{x}).$$

(2.30)

In the above expression, $k_r = m_r m_2^{-r/2}$ for $r = 3, 4, \ldots$, where $m_k$ is the $k^{th}$ cumulant of $X^n_{e,a}$ and $h_k$ is the $k^{th}$ Hermite polynomial. The first few Hermite polynomials are as follows [45]:

$$h_0(y) = 1$$
$$h_1(y) = y$$
$$h_2(y) = y^2 - 1$$
$$h_3(y) = y^3 - 3y$$
$$h_4(y) = y^4 - 6y^2 + 3$$
$$h_5(y) = y^5 - 10y^3 + 15y$$
$$h_6(y) = y^6 - 15y^4 + 45y^2 - 15$$
$$h_8(y) = y^8 - 28y^6 + 210y^4 - 420y^2 + 105$$

The cumulative distribution of $X^n_{e,a}$ can be approximated by

$$F_{X^n_{e,a}}(y) = \Pr(X^n_{e,a} < y) \approx 1 - 0.5\text{erfc}(\hat{y}) - q(\hat{y}) \left(1 + f_1(y) + f_2(y) + f_3(y)\right).$$

(2.31)

Here, $\text{erfc}(\cdot)$ is the complementary error function and

$$f_1(x) = \frac{k_3}{6} h_2(\hat{x}),$$

$$f_2(x) = \frac{k_4}{24} h_3(\hat{x}) + \frac{k_3^2}{72} h_5(\hat{x}),$$

$$f_3(x) = \frac{k_5}{120} h_4(\hat{x}) + \frac{k_3 k_4}{144} h_6(\hat{x}) + \frac{k_3^3}{1296} h_8(\hat{x}).$$

(2.32)

This technique gives us an easier method to evaluate the distribution of the interference statistic than the previous method which uses numerical analysis. The cumulants of $X^n_{e,a}$, required for the computation of $p_{\text{out}}(\gamma_i)$, are calculated in the next sub-section.

**Cumulants of the distribution**

The moment generating function of $X^n_{e,a}$ is given by

$$\Phi_{X^n_{e,a}}(s) = E \left(e^{sX^n_{e,a}}\right).$$

(2.33)
From (2.25), \( \Phi(s) \) is found to be
\[
\Phi_{X_{\epsilon,a}^n}(s) = \exp \left( n\pi \left( a^2 - \epsilon^2 \right) \left( \int_\epsilon^a \frac{2r}{(a^2 - \epsilon^2)} e^{sg(r)} \, dr - 1 \right) \right). \tag{2.34}
\]
The \( k \)-th cumulant of \( X_{\epsilon,a}^n \), \( m_k \), is given by [46] (See Appendix B for a discussion of moments and cumulants)
\[
m_k = d^k \left( \frac{\ln \Phi_{X_{\epsilon,a}^n}(s)}{ds^k} \right)_{s=0} \tag{2.35}
\]
This can be expanded to
\[
m_k = d^k \left( n\pi \left( a^2 - \epsilon^2 \right) \left( \int_\epsilon^a \frac{2r}{(a^2 - \epsilon^2)} e^{sg(r)} \, dr - 1 \right) \right)_{s=0}. \tag{2.36}
\]
The series expansion for an exponential function can be used to yield
\[
m_k = d^k \left( n\pi \left( a^2 - \epsilon^2 \right) \left( \int_\epsilon^a \frac{2r}{(a^2 - \epsilon^2)} \sum_{n=1}^\infty \frac{(sg(r))^n}{n!} \, dr \right) \right)_{s=0}. \tag{2.37}
\]
Simplifying, we get
\[
m_k = n\pi \left( a^2 - \epsilon^2 \right) \int_\epsilon^a \frac{2r}{(a^2 - \epsilon^2)} (g(r))^k \, dr
= \frac{2n\pi}{(k\alpha - 2)} \left( \frac{1}{\epsilon^k - 2} - \frac{1}{a^k - 2} \right). \tag{2.38}
\]
Note that the above expression for the moment of \( X_{\epsilon,a}^n \) can also be directly computed using Theorem 1 and Equation (1) for the Poisson point process. With respect to Equation (1), \( r \) is the integration space, \( f(x, m(x)) = g(r)^k \), \( M_x(dm) = \frac{2r}{(a^2 - \epsilon^2)} \) and \( \lambda = n\pi(a^2 - \epsilon^2) \).

It is observed via simulations that the modified Gaussian distribution using Edgeworth expansion is a good approximation for the distribution of \( X_{\epsilon,a}^n \) when \( \epsilon > 10 \). However, the approximation is not sufficiently accurate for small values of \( \epsilon \).

### 2.4.2 Log Normal Approximation

Simulation results show that the distribution of \( X_{\epsilon,a}^n \) is positively skewed and heavy-tailed. Also, large values for \( X_{\epsilon,a}^n \) are caused by interfering radios very close to the static receiver. However, the probability of radios being very close to the static receiver is small. The log normal distribution is positively skewed, heavy-tailed and is suitable to model random variables that are constrained by zero but have a few very large values [47]. Taking into account these observations, we attempt to fit the parameters of \( X_{\epsilon,a}^n \) to the log normal distribution using a simple moment matching approach.

The probability density function of a log normal variable is given by
\[
p(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp \left( -\frac{(\ln (x/m))^2}{2\sigma^2} \right). \tag{2.39}
\]

The parameters \( m \) and \( \sigma \) can be found by using the following relations:
\[
m_1 = m \exp \left( \frac{1}{2} \sigma^2 \right), \quad m_2 = m^2 \exp \left( \sigma^2 \right) \left( \exp (\sigma^2) - 1 \right) \tag{2.40}
\]
Here, $m_k$ is the $k^{th}$ cumulant of the $X^n_{\epsilon,a}$ and can be calculated as shown in the following sub-section.

The CDF of the interference statistic is as follows:

$$F_{X^n_{\epsilon,a}}(y) = \Pr(X^n_{\epsilon,a} < y) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{\ln(y/m)}{\sigma \sqrt{2}}\right).$$

(2.41)

A Kolmogorov-Smirnov (KS) statistical goodness of fit test ([48]) was performed to test the log normal distribution hypothesis for $X^n_{\epsilon,a}$ and the hypothesis passed the test for $\epsilon > 0.1m$. An example fit of the simulated interference statistic to the log normal distribution is shown in Figure 2.1. The log normal distribution can thus be used to model the interference statistic for a larger range of values than the Gaussian method (Recall that the Gaussian approximation method using the Edgeworth expansion is only valid for $\epsilon > 10m$). It is also a more convenient method to evaluate the distribution of the interference statistic than the characteristic-function-based method which requires numerical analysis. It is therefore used for all subsequent interference statistic analysis in the dissertation.

![Figure 2.1: Comparison of simulated and theoretical outage probability distributions w.r.t. the threshold interference power at the static receiver. SS radio interferers are assumed to distributed in a disc with radius extending from $\epsilon = 0.5m$ to 100m. The density of interferers, $N = 1$. $P_a = -60$dBw.](image)

### 2.4.3 Chernoff Bound

The Chernoff bound for the probability of outage for a given threshold $\gamma$ is given by

$$P_{out}(\gamma_i) = \Pr\{X^n_{\epsilon,a} \geq \gamma_i\} \leq \mathbb{E}\left(e^{\hat{v}(X^n_{\epsilon,a} - \gamma_i)}\right)$$

(2.42)

where $\hat{v}$ is the solution to

$$\hat{v} = \arg\min_{v>0} \mathbb{E}\left(e^{v(X^n_{\epsilon,a} - \gamma_i)}\right) = e^{-\gamma_i} \mathbb{E}\left(e^{v(X^n_{\epsilon,a})}\right).$$

(2.43)

The moment generating function, $\Phi_{X^n_{\epsilon,a}}(v) = \mathbb{E}\left(e^{v(X^n_{\epsilon,a})}\right)$, for random variable $X^n_{\epsilon,a}$ is given by (2.25). The Chernoff bound is approximated in the following way [37]: The Taylor series is used to expand the
expression of $\Phi_{X_{\epsilon a}}$ to give

\[
\ln P \{ X_{\epsilon a}^n \geq \gamma_i \} \leq -v\gamma_i + vm_1 + \frac{v^2}{2!} m_2 + \ldots + \frac{v^i}{i!} m_i + R_i(v) \tag{2.44}
\]

where $m_i$ is the $i^{th}$ cumulant of the distribution and

\[
R_i(v) = \int_{\epsilon}^{a} n\pi \left( a^2 - \epsilon^2 \right) \frac{2r}{a^2 - \epsilon^2} e^{2r g(r)} (z g(r))^{i+1} dr.
\tag{2.45}
\]

When $g(r) \leq K; \forall r$, where $K \geq 1$ is a constant,

\[
R_i(v) < \frac{e^{vK}}{(i+1)!} \int_{\epsilon}^{a} n\pi \left( a^2 - \epsilon^2 \right) \frac{2r}{a^2 - \epsilon^2} (z g(r))^{i+1} dr = \frac{e^{vK}}{(i+1)!} m_{i+1}.
\tag{2.46}
\]

Substituting, the bound can now be written as

\[
\ln P \{ X_{\epsilon a}^n \geq \gamma_i \} \leq -\hat{v}\gamma_i + \hat{v} m_1 + \frac{\hat{v}^2}{2!} m_2 + \ldots + \frac{\hat{v}^i}{i!} m_i + \frac{e^{vK}}{(i+1)!} m_{i+1}. \tag{2.47}
\]

Here, $\hat{v}$ is the solution to

\[
\hat{v} = \arg \min_{v>0} f(v) = -v\gamma_i + vm_1 + \frac{v^2}{2!} m_2 + \ldots + \frac{v^i}{i!} m_i + \frac{e^{vK}}{(i+1)!} m_{i+1}. \tag{2.48}
\]

It is shown that $v$ is a convex function and has a unique minimum. Hence $\hat{v}$ can be calculated using any optimization routine.

When $\epsilon << 1, K = g(\epsilon) >> 1$. Then the exponential term, $e^{\hat{v}K}$, becomes the dominating term in the modified Chernoff bound and the bound becomes very loose. A tighter upper bound can be obtained by only considering the interference from radios present at a distance greater than 1m from the center of the disc and assuming that an outage is automatically caused if one or more radios are present inside a 1m radius. The upper bound for the probability of outage for this scenario is then given by

\[
p_{out}(\gamma_i) = P \{ X_{\epsilon a}^n \geq \gamma_i \} + \left( 1 - P \{ X_{\epsilon a}^n \geq \gamma_i \} \right) \left( 1 - e^{-N_1 N_1^0} 0! \right). \tag{2.49}
\]

Here, $\gamma_i$ is the interference threshold for outage and $N_1$ is the average number of radios inside the 1m radius from the center of the disc.

It is also to be noted that the bound can be used only if $g(r) \leq K$. If log normal shadowing is included in the calculation of the interference statistic, i.e., if, $g(r) = \frac{10z_{10}}{10^z}$, where $z$ is a log normal random variable, then there is no positive and finite $K$ such that $g(r) \leq K$ and the bound cannot be used.

An alternate method to bound the interference and hence the outage probability, based on Chebyshev’s inequality, is suggested in [18]. However, it is expected that the Chernoff bound will be tighter that the bound derived in this paper.

### 2.5 Non-homogeneous Networks

A Cox process or a doubly stochastic Poisson process is a point process with a random intensity measure such that the process is Poisson conditional on the realization of the intensity measure. It is used to generate nonhomogeneous networks. *Clustering* can also be used to model non-homogeneity in networks. An example is the Neyman-Scott process [39]. In the following subsection we calculate the MGF and derive the first two moments of a marked Cox process. We then discuss an example of a Cox process, log Gaussian Cox process.
2.5.1 Derivation of Generating Function and Moments for Marked Cox Process

The generating function of the Cox process is given by [39]

\[ G(s) = \int_{\mathcal{M}} G_{\Lambda}(s) \, Q(d\Lambda) \]  

(2.50)

Here, \( G_{\Lambda} \) is the generating function of the Poisson process of intensity measure \( \Lambda \). \( x \in \mathbb{R}^d \) and \( A \) is the volume over which the Poisson process exists in the \( d \)-dimensioned space. \( Q(.) \) is the distribution function for the intensity over \( \mathcal{M} \), the space of non-negative locally finite measures on \( \mathbb{R}^d \).

Now consider a simple marked Poisson process with mark \( m_x \), corresponding mark distribution \( p(x, m_x) \) and intensity \( \Lambda \). Consider a function \( f(x, m_x) \) defined on each point of the Poisson process. If a path-loss model is considered as in the previous section, \( f(x, m_x) = g(x) = \frac{1}{x^\alpha} \). The moment generating function of the sum of \( f(x, m_x) \) is given by [49]

\[ G_{m,\Lambda}(s) = \exp \left( - \int_{A} \int_{\mathcal{M}} (1 - e^{-sf(x,m_x)}) \, p(x, dm) \, \Lambda(dx) \right) \]  

(2.51)

If the marked process is only dependent upon the position \( x \) in \( \mathbb{R}^d \), we get

\[ G_{m,\Lambda}(s) = \exp \left( - \int_{A} (1 - e^{-sf(x,m_x)}) \, p(x) \, \Lambda(dx) \right) \]  

(2.52)

The MGF of a Cox process with an underlying marked Poisson process is then given by

\[ G(s) = \int_{\mathcal{M}} \exp \left( - \int_{A} (1 - e^{-sf(x,m_x)}) \, p(x) \, \Lambda(dx) \right) \, Q(d\Lambda) \]  

(2.53)

The moments of the Cox process can be now be calculated from the MGF as follows: The \( k^{th} \) moment is given by

\[ m_k = \left. \frac{d^k G(s)}{ds^k} \right|_{s=0} \]  

(2.54)

Since the outer integral is not a function of \( s \),

\[ \frac{d^k G(s)}{ds^k} = \int_{\mathcal{M}} \frac{d^k}{ds^k} \left( \exp \left( - \int_{A} (1 - e^{-sf(x,m)}) \, p(x) \, \Lambda(dx) \right) \right) \, Q(d\Lambda) \]  

(2.55)

The series expansion of an exponential function could be used to yield

\[ \frac{d^k G(s)}{ds^k} = \int_{\mathcal{M}} \frac{d^k}{ds^k} \left( \exp \left( \sum_{n=1}^{\infty} \frac{sf(x,m_x)^n}{n!} \, p(x) \, \Lambda(dx) \right) \right) \, Q(d\Lambda) \]  

(2.56)
The first moment is then given by

\[ \mu_{c,1} = \left[ \frac{dG(s)}{ds} \right]_{s=0} \]

\[ = \int_{M} \left[ \exp \left( \int_{A} \sum_{n=1}^{\infty} \frac{sf(x,m_{x})^{n}}{n!} p(x) \Lambda(dx) \right) \right] \left( \int_{A} \frac{f(x,m_{x})}{(n-1)!} \sum_{n=2}^{\infty} s^{n-1} f(x,m_{x})^{n} p(x) \Lambda(dx) \right) \] \[ Q(d\Lambda) \] \[ = \int_{M} \int_{A} f(x,m_{x}) p(x) \Lambda(dx) Q(d\Lambda) \]

\[ = \int_{A} f(x,m_{x}) p(x) \int_{M} \Lambda(dx) Q(d\Lambda) \]

(2.57)

From [39], Section - 7.2.2 which discusses the moment measures of a random distribution over \( \mathbb{R}^{d} \) (as mentioned before \( \Lambda \) is a random measure over \( \mathbb{R}^{d} \)), \( \int \Lambda(dx)Q(d\Lambda) \) is the first moment of \( \Lambda \) over \( dx \). This is given by \( m_{\Lambda,1}dx \) where \( m_{\Lambda,1} \) is the mean density of \( \Lambda \) over unit area. Therefore,

\[ \mu_{c,1} = \int_{A} f(x,m_{x}) p(x) m_{\Lambda,1}dx \]

(2.58)

Here, \( m_{f,1} = \int_{A} f(x,m_{x}) p(x) dx \).

The second moment is given by

\[ \mu_{c,2} = \left[ \frac{d^{2}G(s)}{ds^{2}} \right]_{s=0} \]

\[ = \int_{M} \left[ C \left( \int_{A} f(x,m_{x})^{2} + \sum_{n=3}^{\infty} \frac{s^{(n-2)} f(x,m_{x})^{n}}{(n-2)!} p(x) \Lambda(dx) \right) + CD^{2} \right] \left( \int_{A} \frac{f(x,m_{x})}{(n-1)!} \sum_{n=2}^{\infty} s^{n-1} f(x,m_{x})^{n} p(x) \Lambda(dx) \right) \] \[ Q(d\Lambda) \]

\[ = \int_{M} \int_{A} f(x,m_{x})^{2} p(x) \Lambda(dx) Q(d\Lambda) + \int_{M} \left( \int_{A} f(x,m_{x}) p(x) \Lambda(dx) \int_{A} f(y,m_{y}) p(y) \Lambda(dy) \right) Q(\Lambda) \]

(2.59)

\[ = m_{\Lambda,1}m_{f,2} + \int_{M} \left( \int_{A} f(x,m_{x}) f(y,m_{y}) p(x) p(y) \Lambda(dx) \Lambda(dy) \right) Q(\Lambda) \]

\[ = m_{\Lambda,1}m_{f,2} + \int_{A} \int_{A} f(x,m_{x}) f(y,m_{y}) p(x) p(y) \left( \int_{M} \Lambda(dx) \Lambda(dy) Q(\Lambda) \right) \]

Here, \( m_{f,2} = \int_{A} f(x,m_{x})^{2} p(x) dx \). From [39], Section - 7.2.2, \( \left( \int \Lambda(dx) \Lambda(dy) Q(\Lambda) \right) = E[\Lambda(dx)\Lambda(dy)] \). Let the pair correlation function of \( \Lambda \) be given by \( h(x,y) = \frac{E[\Lambda(dx)\Lambda(dy)]}{E[\Lambda(dx)]E[\Lambda(dy)]} \). Substituting, we get,

\[ \mu_{c,2} = m_{\Lambda,1}m_{f,2} + m_{\Lambda,1}^{2} \int_{A} \int_{A} f(x,m_{x}) f(y,m_{y}) p(x) p(y) h(x,y) dx dy \]

(2.60)
The variance of the marked cox process is therefore given by
\[
\sigma_c^2 = \mu_{c,2} - \mu_{c,1}^2
= m_{\Lambda,1} m_{f,2} + m_{\Lambda,1}^2 \left( \int_A \int_A f(x,m_x) f(y,m_y) p(x)p(y) h(x,y) \, dx \, dy - m_{f,1}^2 \right) .
\] (2.61)

### 2.5.2 Log Gaussian Cox Process

A log Gaussian Cox process [50] is an example of a Cox process in which the density measure \( \Lambda \) follows a Log Gaussian distribution. The density can be represented as
\[
\Lambda(s) = \exp\{Y(s)\} \quad (2.62)
\]
Here, \( Y(s) \) is a real-valued Gaussian process with mean \( \mu \), variance \( \sigma^2 \) and correlation function \( r(s_1, s_2) = \frac{\text{Cov}(Y(s_1), Y(s_2))}{\sigma^2} \). The homogeneous Poisson process is the limit of a Cox process as \( \sigma^2 \) tends to zero. When \( r(.) = 1 \), a mixed Poisson process with randomized intensity which is log Gaussian distributed is obtained. From [50], the parameters of the intensity measure \( \Lambda \) are as follows: The mean
\[
m_{\Lambda,1} = \exp\left(\mu + \frac{\sigma^2}{2}\right) ,
\] (2.63)
the variance
\[
\sigma_{\Lambda}^2 = \left(\exp(\sigma^2) - 1\right) \exp\left(2\mu + \sigma^2\right)
\] (2.64)
and the pair correlation function
\[
h(x, y) = \exp\left(\sigma^2 r(x, y)\right) .
\] (2.65)

The moments of a marked log Gaussian Cox process can be obtained by substituting the above values in Equations (2.58) and (2.61).

### 2.6 Conclusions and Contributions

In this chapter, the use of a spatial Poisson process to model a communication network was investigated. The distribution of the interference power at a node in the network was then evaluated using a characteristic-function-based approach and by using moment-matching methods to fit it to some well-known distributions. It was found that the log normal distribution adequately modeled the distribution of the interference statistic. Finally, Cox processes, which could be used to model non-homogeneous networks were discussed.

The original contributions in this chapter are as follows:

- The derivation of the characteristic function for the interference power statistic in an annular region.
- The use of a log normal distribution to model the interference power statistic in an annular region.
- The derivation of the MGF and the first two moments of the Cox process.
Chapter 3

Analysis of Spectrum Sharing Techniques with respect to Legacy Static Radio Systems

3.1 Introduction and Problem Statement

One of the most important challenges faced by a dynamic spectrum sharing (SS) system which coexists with a legacy system is the limitation in obtaining perfect information about the actual interference profile seen at the legacy receiver. This almost precludes the possibility of practical SS techniques that are guaranteed to cause no interference to a legacy system. Hence it becomes important to study SS schemes from the perspective of the aggregate interference caused at a legacy receiver.

We address the above issue in this chapter by developing a framework to determine the interference profile due to dynamic SS at a legacy receiver under different system scenarios which include the hidden-node and imperfect-sensing problems. By analyzing the distribution of this interference and the resultant outage probability at the legacy receiver and by comparing two common approaches to SS - interference-avoidance-based overlay techniques and interference-averaging-based underlay techniques - we identify desirable characteristics for SS radio systems. We then leverage this knowledge to motivate the use of a hybrid approach for SS that combines the benefits of interference-averaging and interference-avoidance, resulting in a substantial reduction of the outage probabilities at the legacy receiver.

3.2 Motivation and Previous Work

As mentioned earlier, two of the main approaches that have been proposed for a SS radio system that intends to co-exist with a static radio system are the interference averaging (or spreading) based spectrum underlay technique and the interference avoidance (IA) based intelligent spectrum overlay technique. Interference averaging refers to wideband (WB) transmission techniques where radios spread their signals across the entire bandwidth available to the SS radio system. Ideally, no single source dominates the interference caused to the static radio system in this scheme. Code Division Multiple Access (CDMA) [7] [8] and Ultra Wide Band (UWB) [9] are two technologies that utilize this approach. IA-based spectrum overlay is a cognition-based narrowband (NB) technique where the SS radio dynamically chooses a frequency band for transmission. The choice is made such that the interference caused to the static radio system is minimized or is below some predefined maximum. This technique requires knowledge of the locations and transmissions of the static radios which can be ideally estimated by sensing or obtained through the use of a local spectrum usage database (Note that it might be impossible in some situations, for example TV receivers, to obtain perfect information about the location of the static radio receivers.). Some example schemes discussed in
A third possibility would be to combine the two approaches to harness the benefits of each. To this end, we motivate the use of a hybrid IA-based spectrum underlay technique (also referred to as the spreading-based underlay technique that employs IA). In this transmission technique, the SS radios spread their signal over the entire available bandwidth. However, they also avoid frequencies in which they can sense static radio transmissions. IA techniques such as notching ([12] and [13]) and waveform adaptation ([14] and [15]) can be used to this end. The adaptive frequency hopping technique implemented in Bluetooth version 1.2 [16] is an example of this hybrid SS technique.

An ideal SS scheme for a cognitive radio network is one which maximizes the capacity of the network while causing no interference to the static radio system with which it coexists. However, the cognitive radio system does not necessarily have access to perfect information about the actual interference profile seen at the static receivers (due to imperfect knowledge of the static system including imperfect knowledge of the location of static receivers or due to the lack of knowledge of the interference power at the static receiver from other interferers in the system) and this results in some amount of aggregate interference at a static receiver. Hence, it becomes important to study SS approaches with respect to their resulting interference distributions at the static receiver. This is also relevant with respect to the new “interference temperature” model proposed by the FCC that places a maximum threshold for the aggregate interference-plus-noise level at a receiver as opposed to the current approach that manages interference by limiting the transmit power of individual devices [17].

We examine the interference caused to static radio systems in this paper. Specifically, we formulate a framework based on the distribution of interference power and the resulting outage probability at a static receiver placed in a field of SS radios. We then investigate the impact of different SS techniques on the static system with respect to this framework. We incorporate the effects of imperfect static radio system information, specifically the hidden-node problem, imperfect sensing and out-of-band interference at the static receiver, as well as the effect of log normal shadowing on the comparative performance of different SS schemes.

Comparison of different transmission schemes with respect to interference distributions is a subject that has been well-studied in the literature. For example, there exists some previous work ([18] and references within) on the comparison of DS-CDMA and FH-CDMA schemes with respect to the system capacity of ad hoc networks implementing these schemes. There also has been some work on analyzing the mutual interference between specific SS schemes (for instance, [19] and references within which specifically analyze the interference between 802.11 and Bluetooth systems). However, to the best of our knowledge, no analysis similar to that in this paper currently exists that develops a general framework to compare different SS approaches based on their impact on a static radio system, especially when including the effect of imperfect knowledge. Further, such analysis is absolutely vital for determining the impact of dynamic spectrum sharing on legacy systems.

The distribution of the interference power at a receiver situated in a field of interferers is one of the basic analysis to be accomplished in this chapter. As mentioned in Chapter 2, three different approaches have been used in the literature to model this interference statistic. The first is a numerical approach based on the characteristic function of the interference variable ([34], [35], [33] and references within). This method is extended to the current scenario in Chapter 2. However, it is found to be computationally intensive. The second uses an Edgeworth Gaussian approximation with correction terms based on the higher order cumulants to model the interference. This method has been traditionally used to calculate the out-of-cell interference statistic for CDMA cellular systems ([37] and [40]) and is easier to compute than the first approach. However, it has been observed via simulations that the modified Gaussian distribution using Edgeworth expansion is a good approximation only when the interfering radios are not too close to the central receiver (The minimum distance between an interfering radio and the central receiver should be greater than 10m). This requirement is too strict a limit since, in our analysis, the minimum distance between two radios is assumed to be only restricted by the minimum possible physical distance between two radios. The
third approach uses bounds on the interference and hence the probability of outage at a central receiver [18]. The bounds are found to be too loose for our purposes.

In Chapter 2, we proposed the use of a log normal distribution to model the interference statistic. This method was found to be sufficiently accurate for all practical values (> 0.1m = 10cm) of the minimum distance between an interfering radio and the central receiver. We therefore use this technique for all the interference-analysis in this chapter.

The rest of the chapter is organized as follows. The system model is discussed in Section 3.3. In Section 3.4, the scenario with perfect static system knowledge is considered. Outage probability expressions and interference statistics are formulated for the two basic SS approaches. These are then used to motivate a hybrid underlay approach that also incorporates IA and the performance of the three SS schemes is analyzed. The impact of imperfect knowledge of the static radio system is incorporated into the framework in Section 3.5. It is shown that the lack of complete and perfect knowledge accentuates the benefit of using the hybrid scheme over the other SS approaches. The effect of non-homogeneity in the distribution of SS radios in the network is investigated in Section 3.5. The influence of Log normal shadowing is incorporated in Section 3.7. Finally, Section 3.8 concludes the chapter and delineates original contributions.

### 3.3 System Model

A static receiver is assumed to be at the center of a circular region with radius extending to infinity. SS transmit-radios are assumed to be uniformly distributed in this circular region and the mean number of SS transmit-radios per unit area is assumed to be \( N \). Using the fact that the Poisson distribution is the limit of a binomial distribution ([31] and Appendix A), it can be shown that the number of SS transmit-radios in an area of measure \( A \) follows a Poisson process with parameter \( NA \). The probability that there are \( k \) SS radios in a region with this area is then given by

\[
P(k) = \frac{e^{-NA}(NA)^k}{k!}.
\]

Such a distribution for radios has been previously considered in [34], [35], [33], and [37]. A detailed discussion of spatial Poisson processes is given in Chapter 2.

Interference is assumed to be caused to the static receiver by SS radios that transmit in the same frequency band as the static radio transmission. Note that the interference between static radios in a static radio network is peripheral to the analysis and is not taken into consideration. The interference power at the static receiver is modeled as a random variable. The distribution of the random variable and the resulting outage probability at the static receiver are the basic analyses to be accomplished. The following are the minimal set of assumptions used in this analysis:

1. The static radio system is a NB system with transmission bandwidth \( B \).
2. The distance between a static transmitter and the corresponding static receiver is denoted by \( r_s \). The power received from the static transmitter, at a distance of 1m, is denoted by \( P_s \). (\( P_s = KP_t \), where \( P_t \) is the transmit power of static transmitter and \( K \) is constant that depends upon antenna gain, carrier wavelength and system loss factors.)
3. The data transmission bandwidth of an SS radio is \( B \) without loss of generality (The outage probabilities can be appropriately scaled if this is not the case).
4. The total transmission bandwidth available to the SS system is \( N_B B \), where \( N_B \) is a positive integer.
5. All SS radios transmit with the same power. The power received from an SS radio transmitter at a distance of 1m is denoted by $P_a$. Power control can be incorporated into the analysis by assuming a distribution for the transmit powers of the SS radios. It is expected that this inclusion will not change the relative performance trends presented in this paper. However, it will be examined in future work.

In addition, the signals received from the SS radios are assumed to suffer a loss in power that follows an exponential propagation law. Let $\frac{1}{g(r)}$ be the path-loss suffered by a signal at distance $r$ from the transmitter relative to the path-loss at distance 1m from the transmitter. $g(r)$ is assumed to satisfy the following properties:

1. $g(r)$ is monotonically decreasing i.e. $\lim_{r \to 0} g(r) = \infty$ and $\lim_{r \to \infty} g(r) = 0$.

2. Path loss exponent $\alpha$ is greater than 2 i.e. $\lim_{r \to \infty} r^2 g(r) = 0$.

Note that, if the latter condition is not satisfied, the sum interference power at a receiver would be a function of the network size and would be infinite for an infinite sized network ([34] and [40]). Further note that for realistic systems, while $\alpha \leq 2$ is possible for short distances, it is not possible for sufficiently long distances especially in the limit $r \to \infty$. $g(r)$ is henceforth specified as

$$g(r) = \frac{1}{r^\alpha}; \quad \alpha > 2. \quad (3.2)$$

Note that the above expression for path-loss holds only when the receiver is in the far-field of the transmitter. In general, as the distance between the transmitter and receiver tends to zero, the received power does not tend to infinity but to some finite maximum value. However, the minimum possible physical distance ($r_{phy} > 0$) between two interfering radios places a natural constraint on the power received from the interferer and we assume that the path loss expression holds for all the other scenarios ($r \geq r_{phy}$).

In the analysis presented here, an outage is assumed to be caused at the static receiver if

$$\frac{P_s/r_s^\alpha}{\sum_J P_a g(r_i)} \leq \gamma. \quad (3.3)$$

Here, $J$ is the set of all SS radios in the system which transmit in the same frequency band as the static transmitter, $\gamma$ is the minimum SIR required at a static receiver for successful reception and $r_s$ is the distance of the $i^{th}$ SS radio in set $J$ from the static receiver in meters. This implies that for a fixed $r_s$, an outage is caused at the static receiver if the normalized interference power is above some threshold $P_i$:

$$\sum_J g(r_i) \geq P_i = \frac{P_s/r_s^\alpha}{P_a \gamma}. \quad (3.4)$$

As mentioned, the static receiver is assumed to be at the center of a circular region with radius $a$ (that extends to infinity) in which SS radios are Poisson distributed. It is also assumed that the minimum distance between the static receiver and an SS radio is greater than some positive value $\epsilon > 0$ (due to the minimum physical distance constraint or the implementation of an interference avoidance scheme). Hence the circular region has an inner-radius $\epsilon$. The sum of the normalized received power at the static receiver from interfering radios Poisson distributed with intensity $n$ per unit area in this annular region is modeled using a random variable $X_{\epsilon,a}^n$.

$$X_{\epsilon,a}^n = \sum_{J_{\epsilon,a}^n} g(r_i). \quad (3.5)$$

Here, $J_{\epsilon,a}^n$ denotes the set of interfering radios at distance $r_i$ from the static receiver such that $\epsilon \leq r_i \leq a$. 

We use a log normal distribution to model the interference statistic, \( X_{n,a} \). From the analysis in Chapter 2, Section 2.4.2, the probability of outage for threshold \( P_i \) (calculated using the inverse CDF function for a log normal distribution) for \( X_{n,a} \) is as follows:

\[
p_{\text{out}}(P_i) = \Pr(X_{n,a} > P_i) = \frac{1}{2} - \frac{1}{2}\erf\left(\frac{\ln(P_i/m)}{\sigma \sqrt{2}}\right).
\] (3.6)

The parameters \( m \) and \( \sigma \) can be found by using the following relations:

\[
m_1 = m \exp\left(\frac{1}{2}\sigma^2\right), \quad m_2 = m^2 \exp\left(\sigma^2\right) (\exp(\sigma^2) - 1).
\] (3.7)

Here, \( m_k \) is the \( k^{th} \) cumulant of the \( X_{n,a} \) and is given by (Equation (2.38) in Chapter 2)

\[
m_k = \frac{2n\pi}{(k\alpha - 2)} \left( \frac{1}{e^{k\alpha - 2}} - \frac{1}{a^{k\alpha - 2}} \right)
\] (3.8)

### 3.4 Scenario with Perfect Static System Knowledge

In this section, SS radios are assumed to have access to perfect information about the locations and transmission frequency bands of the static radios. Also, the effect of out-of-band transmissions by SS radios (transmissions by SS radios in frequency bands outside the transmission band used by the static radio system) on the static radio system is assumed to be negligible. The effect of imperfect knowledge about the static radio system including the effect of out-of-band interference on the static receiver will be analyzed in the next section.

#### 3.4.1 Spectrum Sharing Schemes

Since SS radios are assumed to have complete information about the location and transmissions of the static radio receiver and transmitter pair, an SS radio implementing IA is assumed to not transmit in a frequency band used by the static radio if its distance from the static receiver is such that its transmission could cause an outage at the static receiver. Let \( r_{\min} \) denote this distance from the static receiver together with some safety margin. Thus no SS radio employing IA that exists within a radius of \( r_{\min} \) from the static receiver transmits in the same frequency as the static receiver. This scheme thus precludes the possibility of outage being caused at the static receiver due to an individual SS radio transmission when IA is used. However an outage could still be caused at the static receiver due the fact that the sum of interference powers from SS radios, that are distributed in the annular region with inner radius \( r_{\min} \) and outer radius extending to infinity around the static receiver, could be larger than the interference threshold \( P_i \). The interference statistic at the static receiver for a SS scheme employing IA is therefore given by \( X_{r_{\min},\infty}^{n} \). The outage probability expressions for the two main SS approaches in this scenario are as follows:

1. **Scheme-1: Spectrum Overlay with IA**

   In this scheme, the SS radio system is a NB system. A radio transmits in a given frequency band (of width \( B \), from the \( N_B \) available bands, only if there are no static radio transmissions with which it could interfere in this frequency band. Therefore an SS radio within a radius of \( r_{\min} \) from the static receiver does not transmit in the same frequency as the static receiver and the outage probability at the static receiver is given by

\[
p_{\text{out}}^{\text{NB-IA}}(P_i) = \Pr(X_{r_{\min},\infty}^{N_B} > P_i) = \Pr(X_{r_{\min},\infty}^{N_B} > P_i).
\] (3.9)
Here, $N_n$ is the density of radios transmitting in the frequency band used by the static receiver. It is assumed that SS radios are uniformly distributed among the set of available frequency bands. Hence, $N_n = N/N_B$.

2. Scheme-2: Spreading-based Spectrum Underlay

In this scheme, the SS radio system is a wideband (WB) system. All SS radios spread their transmission power over the entire available bandwidth $N_B B$. Therefore, the power received from an SS radio transmitter in any frequency band of width $B$ at a distance of 1m is given by $P_i/(N_B)$ and the number of radios that transmit in any frequency band is equal to the total number of radios in the system. Also, since this scheme does not implement IA, interfering radios are distributed in the annular region around the static receiver with radius extending from $r_{phy}$ to infinity. The outage probability at the static receiver is therefore,

$$p_{out}^{WB}(P_i) = Pr\left(\frac{X_{r_{phy},\infty}}{N_B} > P_i\right). \quad (3.10)$$

3.4.2 Discussion of Interference Statistics

In this subsection, general characteristics of the interference statistics are investigated to derive trends and desired properties for SS schemes. The $k^{th}$ cumulant of the interference distribution at the static receiver when interfering radios are distributed in an annular region with inner radius $\epsilon$ and outer radius extending to $\infty$ is (from (3.8))

$$m_k = 2\frac{n\pi}{(k\alpha - 2)} P_x^k \frac{1}{\epsilon^{k\alpha - 2}}.$$

Here, $n$ is the density and $P_x$ is a scaling factor for the transmit power of interfering SS radios. The $k^{th}$ cumulant decreases linearly with a reduction in $n$, exponentially with a reduction in $P_x$ and exponentially (with a factor that depends upon the path loss exponent $\alpha$) with an increase in $\epsilon$. It can thus be seen that, in general, the value of a cumulant is impacted most by the minimum distance between the static receiver and an interfering SS radio, $\epsilon$, followed by the power scaling factor $P_x$ and then finally by the density of interferers $n$. Clearly, to reduce the mean or variance, we wish to first increase $\epsilon$ and then decrease $P_x$.

A reduction in the mean of the interference power distribution (cumulant $m_1$) shifts the distribution to the left resulting in smaller outage probability at the static receiver for a given tolerable (or threshold) interference power. A reduction in the variance (cumulant $m_2$) narrows the density distribution of the interference power and translates to a faster decay of the outage probabilities with increasing tolerable interference powers at the static receiver. It is thus seen that the removal of contributions to the interference statistic from the strongest interferers in the system i.e. interferers which are closest to the static receiver (in other words, increasing $\epsilon$) has the potential to dramatically decrease the outage probabilities at the static receiver. However, when the contribution from the $\epsilon$ term is ignored, reduction in the transmit power of interferers benefits the system more than a reduction in their density.

Consider the $k^{th}$ cumulants of the IA-based overlay scheme and the spreading-based underlay scheme,

$$m_k^{NB} = 2\frac{(N/N_B)\pi}{(k\alpha - 2)} \frac{1}{\epsilon_{NB}^{k\alpha - 2}} \quad \text{and} \quad m_k^{WB} = 2\frac{N\pi}{(k\alpha - 2)} \frac{1}{N_B} \frac{1}{(\epsilon_{WB})^{k\alpha - 2}}. \quad (3.12)$$

Here, $\epsilon_{NB}$ and $\epsilon_{WB}$ are the minimum distances between an SS radio interferer and the static receiver for the overlay and the underlay schemes respectively. Note that the overlay scheme achieves a reduction in the density of SS interferers (by a factor of $1/N_B$ since each SS radio only transmits in one of the $N_B$ available transmission bands). On the other hand, the underlay scheme achieves a corresponding reduction in transmit power level (by a factor of $P_x = 1/N_B$ due to spreading), though the density of interferers remains the same. As noted above, the reduction in power is more beneficial than the corresponding reduction in
node density. However, the overlay scheme is capable of implementing IA w.r.t. the static radio system thereby increasing the minimum distance between an overlay interferer and the static receiver resulting in $\epsilon^{NB} >> \epsilon^{WB} = r_{phy}$. Since the contribution from the $\epsilon$ term is the largest, for reasonable values of $\epsilon$, the term is able to scale the cumulants of the interference power distribution such that IA-based overlay scheme provides substantial benefits over the spreading-based underlay scheme. This is shown by noting that $m_1^{NB} < m_1^{WB}$ when $\epsilon^{NB} > \epsilon^{WB}$ and $m_2^{NB} < m_2^{WB}$ when $\epsilon^{NB} > \epsilon^{WB} N_k^{\alpha - 2}$. The latter condition is satisfied for reasonable values of the parameters. For example, when $\epsilon^{WB} = 0.1m$, $N_B = 256$ and $\alpha = 3$, $m_2^{NB} < m_2^{WB}$ if $\epsilon^{NB} > 0.4m$.

Now consider the scenario where $\epsilon^{WB} = \epsilon^{NB}$. This results in $m_1^{NB} = m_1^{WB}$ and $m_2^{NB} > m_2^{WB}$ since, as mentioned before, when the contribution from the $\epsilon$ term is negated, the reduction in variance achieved by reducing $P_x$ is larger than the reduction achieved by reducing $N$. Also note that the reduction increases with an increase in the spreading factor $N_B$.

This preliminary analysis thus motivates hybrid SS schemes that can decrease the average interference power of the SS interferers in addition to employing some kind of IA to increase the minimum distance $\epsilon$ between an interfering SS radio and the static receiver. The hybrid IA-based spectrum underlay scheme, described in the next section, is an example of such a scheme. The inferences in this section are further investigated in the subsequent sub-sections using the actual outage probability distributions. It should be noted that these same conclusions provide the motivation for many CDMA schemes which use orthogonal waveforms in-cell but allow random spread interference from out-of-cell.

### 3.4.3 Interference Avoidance based Spectrum Underlay

This scheme is referred to as Scheme-3. In this scheme, the SS radio system is a WB system. In addition, the radios are assumed to be able to null or notch frequencies in which there exists a static radio transmission with which they could interfere and spread their transmission power over the remaining available portion of bandwidth. The power received from an SS transmit-radio at a distance of 1m in a frequency band of width $B$, if there exists no static radio transmission that can be interfered with, is hence given by $P_a/N_B$. Note that, similar to Scheme-2, the number of radios that transmit in any frequency band is equal to the total number of radios in the system. However, since an SS radio within a radius of $r_{min}$ from the static receiver does not transmit in the same frequency as the static receiver, unlike in Scheme-2, the annular region around the static receiver in which interfering radios are distributed has an inner-radius of $r_{min}$. The outage probability at the static receiver is therefore given by

$$P_{out}^{WB-IA}(P_i) = Pr \left( \frac{X_{r_{min},\infty}^N}{N_B} > P_i \right).$$

(3.13)

### 3.4.4 Outage Probability Performance Analysis

Example cases are used here to study the outage probability based performance at the static radio receiver in the presence of different SS schemes. In all the discussed examples, the transmit power of the SS radio is taken to be 10mW, the transmit power of the static radio is taken to be 100mW and the power loss at a distance of 1m is assumed to be 40dB (hence, $P_a = -60$dBw and $P_s = -50$dBw). This attenuation is typical in frequency bands of interest. The interference statistics can be appropriately scaled if this is not the case. This scaling does not influence the comparative performance results of different SS schemes. The path loss exponent, $\alpha$, is assumed to be 3 and $r_s = 5m$. It is also assumed that the minimum physical distance between two radios, $r_{phy} = 0.1m = 10cm$. Unless otherwise mentioned, the average number of SS radios per unit area (in m$^2$), $N$, is taken to be 1.
In the presence of perfect static system knowledge, an SS radio implementing IA is assumed to not transmit in the same frequency as the static transmitter if its distance from the static receiver is less than 20m ($r_{min} = 20$). This ensures that up to an SIR requirement of 30dB, no outage is caused at the static receiver due to a NB transmission from a single SS radio. To ensure a fair comparison, WB SS radios implementing IA within the same distance from a static receiver are assumed to not transmit in the same frequency as the static transmitter. Outage probabilities for the overlay and underlay schemes with IA (Scheme-1 and Scheme-3) are plotted in Figure 3.1. As a reference, outage probabilities for the overlay and underlay schemes without the incorporation of IA are plotted in Figure 3.2.

It can be observed by comparing Figure 3.1 with Figure 3.2 that IA drastically reduces the outage probabilities. SS radios implementing IA exploit the available perfect information about the static radio system to maintain an increased minimum distance $r_{min} >> r_{phy}$ between an interfering SS radio and the static receiver leading to a much smaller mean and variance than SS radios that do not implement IA (Scheme-2). This shifts the probability density function (PDF) of the interference power at the static receiver to the left and makes it narrower (Figure 3.3) which in turn reduces the outage probabilities for a given threshold interference power.

Now consider the hybrid WB underlay technique that implements IA (Scheme-3) and thus also exploits the information about the static radio system to maintain an increased minimum distance $r_{min} > r_{phy}$. As shown before in Sub-section 3.4.2, $m_{1}^{NB-IA} = m_{1}^{WB-IA}$ and $m_{2}^{WB-IA} = \frac{2N\pi}{(2\alpha-2)N_{B}r_{min}^{2\alpha-2}} < m_{2}^{NB-IA} = \frac{2N\pi}{N_{B}(2\alpha-2)r_{min}^{2\alpha-2}}$, resulting in a narrower PDF for Scheme-3 than the PDF for Scheme-1 (Figure 3.3). Thus, as can be observed from Figure 3.1, Scheme-3 guarantees smaller outage probabilities than Scheme-1 for a given interference threshold except for a small cross-over region.

Outage probability curves are also plotted in Figure 3.1 for a hybrid WB SS scheme, where the transmit signal is only spread over a fraction of the available spectrum $N_{frac} < N_{B}$. Hence the power received from an SS transmitter in a frequency band of width $B$, if there exists no static radio transmission that can be interfered with, is given by $P_{s}/N_{frac}$ and the number of interferers in a frequency band assuming SS radios are uniformly distributed over the frequency bands is given by $N/(N_{B}/N_{frac})$. This scheme is thus
Figure 3.2: Outage probability vs interference threshold when the SS radios have no information about the static radio system. Interferers are distributed in a disc with radius extending from 0.1 m to $\infty$.

Figure 3.3: Probability density function of the interference power at the static receiver when interferers are distributed in a disc with radius extending from $\epsilon$ m to $\infty$. Transmission bandwidth available to SS radio system, $N_B = 64$. Note that the PDFs are not normalized.
an intermediate between Scheme-1 and Scheme-3. It can be noticed by comparing the performance curves that the interference reduction at the static receiver when using a hybrid scheme is more pronounced when a larger spreading factor is available to and employed by the SS radio system. These trends thus re-iterate the observations in Subsection 3.4.2.

Note that this outage probability analysis can also be used to determine the optimum range \( r_{\text{min}} \) for a static radio system within which SS radios should not transmit in the same transmission frequency as the static radio system.

### 3.5 Scenario with Imperfect Static System Knowledge

Practical deployment issues like imperfect sensing, the hidden-node problem, limited memory and limited access to information about the static radio system etc, might preclude the availability of accurate knowledge at the SS radios of the static radio system including the locations of the static radio receivers. This section incorporates this lack of complete and perfect knowledge of the static radio system and evaluates its influence on the performance of different SS schemes with respect to their impact on the static receiver.

#### 3.5.1 Hidden Node Problem with Perfect Sensing

In most network scenarios (for example, in a television network), the static receiver (for example, the TV receiver) does not usually transmit any signal that indicates a static radio transmission in a particular frequency band. In this case, perfect knowledge of the location of the static receiver, ideally required for IA, requires considerable coordination (to implement some form of distributed sensing) and feedback in the SS network and in some cases (for example, in a television network), might even be impossible to obtain. Therefore, a more practical approach is one in which the SS radios scan their environment for transmissions from the static radio transmitter and do not transmit in the frequency band in which they sense some transmission. In this scenario, interference is caused at the static receiver by transmissions from SS radios which are hidden (i.e. are outside the range in which the SS radios can sense transmissions) from the static radio transmitter. Hence, unlike the perfect system knowledge scenario, an outage can be caused at the static receiver by an individual SS radio interferer in addition to an outage caused by the sum of received powers from SS radio interferers. Note that the analysis in this sub-section assumes SS radios are able to perfectly sense the static radio transmission when inside the sensing range. The effect of imperfect sensing is discussed in the next sub-section.

Let \( r_{\text{as}} \) be the range from the static transmitter within which the SS radio can identify static radio transmissions and let \( \theta \) be the angle of the line joining the SS radio transmitter and the static receiver w.r.t. the line joining the transmitter and receiver. These variables are illustrated in Figure 3.4. Consider a disc of radius \( a \). Then the area outside the sensing region is given by \( \pi(a^2 - r_{\text{as}}^2) \). The probability density function of the radio distribution with respect radius \( r \) and \( \theta \) is given by

\[
 f_{r,\theta}(r, \theta) = \frac{r}{\pi(a^2 - r_{\text{as}}^2)}; \quad \max(r_{\text{min}}(\theta), r_{\text{phy}}) \leq r \leq a \quad 0 \leq \theta \leq 2\pi.
\]  

Here, \( r_{\text{min}}(\theta) \) is the minimum distance of the agile transmitter from the static receiver such that the static transmitter is hidden from it. It is a function of \( \theta \) and is given by (Figure 3.4) \( r_{\text{min}}(\theta) = r_s \cos \theta + r_{\text{as}} \sin \left( \cos^{-1}\left( \frac{r_{\text{as}} \sin \theta}{r_{\text{as}}} \right) \right) \). Let \( \Phi(\theta) = \max \{r_{\text{min}}(\theta), r_{\text{phy}}\} \). The interference statistic at the static receiver for a spectrum sharing scheme employing IA in the absence of perfect system knowledge is given by

\[
 X_{\Phi(\theta),\infty}^n = \sum_{j\in\Phi(\theta),\infty} g(r_j).
\]
Here, \( J_{\tau(\theta),\infty} \) is the set of SS radios that are outside the sensing region of the static transmitter, i.e., \( r \) satisfies \( r(\theta) \leq r_i \leq \infty \) and \( \theta \) satisfies \( 0 \leq \theta \leq 2\pi \). Following the analysis in Chapter 2, Section 2.3.3, the MGF of \( X_{\tau(\theta),\infty} \) can be written as

\[
\Phi_{X_{\tau(\theta),a}}(s) = \exp \left( n\pi \left( a^2 - r_{as}^2 \right) \int_0^{2\pi} \int_{r(\theta)}^{\infty} r(\theta) \frac{r}{\pi (a^2 - r_{as}^2)} e^{sg(r)} drd\theta - 1 \right). \tag{3.16}
\]

The \( k \)th cumulant of \( X_{\tau(\theta),\infty} \) (again following the analysis for the derivation of moments from the MGF which is described in Section 2.4) is then given by

\[
m_k = n \int_0^{2\pi} \int_{r(\theta)}^{\infty} r(\theta)^k drd\theta, \tag{3.17}
\]

which can be determined using numerical integration. The outage probabilities for the three schemes can be determined as in Sub-section 3.4 (Equations (3.9) and (3.13)) by substituting \( X_{\tau_{\min},\infty} \) with \( X_{\tau(\theta),\infty} \).

**Outage Probability Performance Analysis**

The radius of the sensing region around the static transmitter is assumed to be 15m \( (r_{as} = 15) \). The received power at this distance is approximately 75dB less than the transmit power due to path loss. The outage probability distributions at the static receiver for the underlay and overlay techniques implementing IA are plotted in Figure 3.5. As expected, larger outage probabilities are seen at the static receiver in this scenario as opposed to the scenario with perfect static system knowledge. This is due to the additional interference.
Figure 3.5: Outage probability vs interference threshold when the SS radios have perfect and imperfect information (due to the hidden node problem) about the static radio system. For the imperfect information scenario, the distance between the static transmitter and receiver, $r_s = 5$ m and the sensing radius, $r_{as} = 15$ m. For the perfect information scenario, the radius around the static receiver in which no interferers are present is 20m. $N_B = 512$.

Figure 3.6: Outage probability vs the density of SS radios for the perfect and imperfect information (due to the hidden node problem) scenarios. SIR threshold $\gamma = 10$ dB. For the imperfect information scenario, the distance between the static transmitter and receiver, $r_s = 5$ m and the sensing radius, $r_{as} = 15$ m. For the perfect information scenario, the radius around the static receiver in which no interferers are present is 20m. $N_B = 512$. 

caused by SS radios which are hidden from the static transmitter and whose distance from the static receiver is less than \( r_{\text{min}} \).

Comparing Scheme-1 and Scheme-3, it is seen that except for a small crossover region at very high outage probabilities the IA-based overlay scheme causes more interference at the static receiver. As before, this is due to the fact that the means for both schemes are the same (\( m_1^{\text{NB-IA}} = m_1^{\text{WB-IA}} \)) while the variance for Scheme-3 (\( m_2^{\text{WB-IA}} = \frac{N}{N_B} \int_0^{2\pi} \int_0^{\infty} r(g(r)) dr d\theta \)) is less than the variance for Scheme-1 (\( m_2^{\text{NB-IA}} = \frac{N}{N_B} \int_0^{2\pi} \int_0^{\infty} r(g(r))^2 dr d\theta \)). It is also seen that the increase in interference caused at the static receiver due to the absence of perfect information is greater for the IA-based overlay scheme as compared to the hybrid spreading-based underlay scheme with IA. This is intuitive since in the absence of perfect information due to the hidden node problem, some radios that are close to the static receiver interfere with the static radio transmission. In this scenario, it is beneficial if the transmit power of these SS radios is distributed or averaged over all frequencies (as in an underlay scheme) instead of being concentrated in the transmission frequency of the static radio (as in an overlay scheme). These results are re-iterated in Figure 3.6 which show the outage probabilities at the static receiver versus SS radio density for a fixed tolerable interference power threshold at the static receiver. Also, as expected, the density of SS radios that can be supported by both schemes when perfect information is available at the SS radios is greater than the density that can be supported when perfect information is unavailable.

### 3.5.2 Imperfect Sensing

In the absence of a local database, SS radios attempt to locate static radio receivers and static radio transmissions by either sensing signals from the static receivers (if the static receiver transmits a signal indicating a static radio transmission in a particular frequency band) or by sensing transmissions from static transmitters. In either scenario, sensing errors might occur and interference could be caused at the static receiver by an individual SS radio transmission. Hence, unlike the perfect system knowledge scenario, an outage can be caused at the static receiver by an individual SS radio interferer in addition to an outage caused by the sum of received powers from SS radio interferers. The probability of sensing error (\( p_{\text{se}} \)), in general, increases as the distance between the radio transmitting the signal and the SS radio increases and can be modeled as

\[
 p_{\text{se}}(r) = \begin{cases} 
 Cr^q; & r_{\text{phy}} \leq r \leq r_{\text{se}} \\
 1; & r > r_{\text{se}} 
\end{cases} 
\]  

(3.18)

Here, \( C \) is some weighting constant such that at some distance \( r_{\text{se}} \) between the SS radio and the radio transmitting the signal to be sensed, \( P_{\text{se}}(r_{\text{se}}) = 1 \) (\( C = \frac{1}{r_{\text{se}}} \)). Note that the analysis in this paper assumes that the minimum distance between two radius is lower-bounded due to physical constraints (\( r_{\text{phy}} > 0 \)). Therefore, \( p_{\text{se}} \) is bounded away from zero and the constant \( C \) can also be chosen to take into account the SNR wall that arises during sensing due to noise-uncertainty. \( q \) is a factor which can be used to shape the distribution of \( p_{\text{se}} \). We assume that sensing error is directly proportional to the path-loss, i.e., \( q = \alpha \). We consider the following two scenarios: the scenario where the static signal (to be sensed) is transmitted from the static receiver and the scenario where it is transmitted from the static transmitter. Note that the latter scenario incorporates the hidden node problem as well.

**Sensing signal from static receiver**

The static receiver is assumed to transmit a signal indicating a static radio transmission in a particular frequency band. The probability of a SS radio making an error while sensing this signal is assumed to be given by \( p_{\text{se}}(r) \), where \( r \) is the distance between the SS radio and the static receiver. Thus in addition to the interference caused by radios outside the circular region with radius \( r_{\text{min}} \), interference is caused at the static receiver by radios inside the circular region that make an error in sensing the signal from the static receiver.
The interference statistic at the static receiver is therefore,

\[ X_{se,sky,a}^n = \int_{r_{sky}}^{R} \sum_{J_{r,r}^{p}(pse(r))} g(r) dr + \sum_{J_{r,a}^{p}} g(r). \]  \hspace{1cm} (3.19)

Here, \[ \hat{r} = \min \{ r_{min}, r_{se} \} \] and \[ J_{r,r}^{p}(pse(r)) \] denotes the set of interfering radios at distance \( r \) from the static receiver. Define

\[ p_{res}(r) = \begin{cases} p_{se}(r) ; & r \leq \hat{r} \\ 1 ; & r > \hat{r} \end{cases} \]  \hspace{1cm} (3.20)

The interference statistic can now be re-written as

\[ X_{se,sky,a}^n = \int_{r_{sky}}^{a} \sum_{J_{r,r}^{p}(p_{res}(r))} g(r) dr. \]  \hspace{1cm} (3.21)

Note that a radio in the annular region with inner-radius \( r_{sky} \) and outer-radius \( a \) (the number of radios is Poisson distributed with intensity measure \( n \)) interferes with the static radio with a probability of \( p_{res}(r) \). This new distribution of interfering radios can be modeled by a thinning operation on the original Poisson distribution where a point of the original Poisson process is retained with probability \( p \). The MGF of the thinned Poisson process, \( X_{se,sky,a}^n \) is given by (using results for thinned Poisson process given in [39] and the derivation of the MGF in Chapter 2, Section 2.3.3)

\[ \Phi_{X_{se,sky,a}^n}(s) = \exp \left( n\pi \left( a^2 - r_{sky}^2 \right) \left( \int_{r_{sky}}^{a} \frac{2r}{(a^2 - r^2_{sky})} p_{res}(r) e^{sg(r)} + 1 - p_{res}(r) \right) dr - 1 \right) \]  \hspace{1cm} (3.22)

The \( k \)-th cumulant of \( X_{se,sky,a}^n \) can be evaluated as before (Section 2.4) to yield

\[ m_k = \frac{d^k \left( \ln \Phi_{X_{se,sky,a}^n}(s) \right)}{ds^k} \bigg|_{s=0} = \frac{d^k \left( n\pi \left( a^2 - r_{sky}^2 \right) \int_{r_{sky}}^{a} \frac{2r_{res}(r)}{(a^2 - r_{sky}^2)} \sum_{k=1}^{\infty} \frac{(sg(r))^k}{k!} dr \right)}{ds^k} \bigg|_{s=0} \]  \hspace{1cm} (3.23)

The \( k \)-th cumulant of \( X_{sky,\infty}^n \) is therefore given by

\[ m_k = 2n\pi \int_{r_{sky}}^{\hat{r}} r_{res}(r) (g(r))^k dr = 2n\pi \int_{r_{sky}}^{\hat{r}} r_{sky}(r) (g(r))^k dr + 2n\pi \int_{\hat{r}}^{\infty} r (g(r))^k dr \]

\[ = \frac{2n\pi C}{(k\alpha - q - 2)} - \frac{1}{r_{sky}^{k\alpha - q - 2}} + \frac{2n\pi}{(k\alpha - 2)} \frac{1}{\hat{r}^{k\alpha - 2}}. \]  \hspace{1cm} (3.24)

The outage probabilities for the three schemes can be determined in a manner similar to that given in Subsection 3.4 by substituting \( X_{sk,min,\infty}^n \) with \( X_{se,sky,\infty}^n \).

**Sensing signal from static transmitter**

The SS radios are assumed to sense transmissions from the static transmitter and the probability of a SS radio making a sensing error is assumed to be \( p_{se}(r_t) \), where \( r_t \) is the distance between the SS radio and the
static transmitter. Let $\theta$ be the angle between the line joining the SS radio to the static receiver and the line joining the static transmitter to the static receiver. Let $r$ be the distance between the SS radio and the static receiver. Distance $r_i$ can be expressed in terms of $\theta$ and $r$ as $r_i(r, \theta) = \sqrt{(r^2\sin^2\theta + (r_n - r\cos\theta)^2)}$. The interference statistic at the static receiver for a spectrum sharing scheme employing IA with the inclusion of the hidden-node problem and sensing errors is then given by

$$X^n_{tse, \text{phy}, a} = \int_0^{2\pi} \left( \int_{r_{\text{phy}}} \left( \sum_{r_{\text{se}}(r, \theta)} r \cdot p_{\text{se}}(r) \cdot e^{sg(r)} dr \right) d\theta \right) g(r) dr + \sum_{J^n_{\text{phy}, a}} g(r) d\theta. \quad (3.25)$$

Here, $\hat{r}(\theta) = \max \{ r_{\text{min}}(\theta), r_{\text{phy}} \}$ and $J^n_{\text{phy}, a}$ is the set of SS radios that are outside the sensing region of the static transmitter, i.e., $r_i \leq r_i \leq a$ and $\theta$ satisfies $0 \leq \theta \leq 2\pi$. The MGF of $X^n_{\text{phy}, \infty}$, a thinned Poisson process similar to the process in the previous section, can be written as

$$\Phi^{X^n_{tse, \text{phy}, a}}(s) = \exp \left( n \left( r^2 - r^2_{\text{phy}} \right) \left( \int_0^{\hat{r}(\theta)} \int_{r_{\text{phy}}} \frac{r}{r^2_{\text{se}} - r^2_{\text{phy}}} p_{\text{se}}(r) e^{sg(r)} dr d\theta \right) \right) \exp \left( n\pi \left( a^2 - r^2_{\text{se}} \right) \left( \int_0^{\hat{r}(\theta)} \frac{r}{(a^2 - r^2_{\text{se}})} e^{sg(r)} dr d\theta - 1 \right) \right). \quad (3.26)$$

The $k^{th}$ cumulant of $X^n_{tse, \text{phy}, a}$ can be derived as in the previous subsection and is given by

$$m_k = n \int_0^{2\pi} \int_{r_{\text{phy}}} r p_{\text{se}}(r, \theta) \left( g(r) \right)^k dr d\theta + n \int_0^{2\pi} \int_{\hat{r}(\theta)}^\infty \left( g(r) \right)^k dr d\theta, \quad (3.27)$$

The value of $m_k$ can be evaluated using numerical integration. As before, the outage probabilities for the SS schemes can be determined by substituting $X^n_{tse, \infty}$ with $X^n_{tse, \text{phy}, \infty}$ in Equations (3.9) and (3.13).

**Outage Probability Performance Analysis**

It is observed via simulations that the log normal distribution does not accurately model the interference statistic with imperfect sensing when $r_{\text{phy}} < 5$ m. This can be attributed to the fact that the probability of an interferer being present inside the 5 m radius is very small, the interference that could be contributed by it, if present, is disproportionately large, considerably biasing the mean and variance of the distribution. Since the probability of finding an SS interferer at a distance less than 5 m around the static receiver is very small, we attempt to model the actual interference power distribution with imperfect sensing (and $r_{\text{phy}} = 0.1$ m) by computing the cumulants of the log normal distribution using a value of 5 m for $r_{\text{phy}}$. Figure 3.7 shows the outage probabilities w.r.t. different SS radio densities for a fixed tolerable interference power threshold at the static receiver. It is seen that the log normal distribution with $r_{\text{phy}} = 5$ m accurately models the actual distribution for relatively high and medium outage probabilities. However, it underestimates the outage probabilities at smaller values since the interference power contributed by SS radio interferers which might be present at a distance less than 5 m from the static receiver is ignored in the log normal modeling. Nevertheless, the figures indicate that the log normal modeling can provide valuable comparisons between different SS schemes though it is restricted outage probabilities greater than 0.5%.

Outage probabilities of the underlay and overlay SS schemes implementing IA with imperfect sensing for different SS radio densities are plotted in Figure 3.8 and Figure 3.9 for $N_B = 64$ and $N_B = 512$ respectively. It is seen that, in general, imperfect sensing causes a degradation in performance when compared to the scenario with perfect static system knowledge. It can also be observed that the performance loss is larger when the sensing is based on a signal from the static transmitter than when based
Figure 3.7: Comparison of simulated and theoretical outage probability distributions w.r.t. the density of SS radios for the scenario with imperfect static system information due to imperfect sensing. Simulated curves assume SS radios to be distributed in a disc with inner radius $r_{phy} = 0.1m$ while theoretical curves are generated using $r_{phy} = 5m$. The outer radius of the disc is assumed to be to $100m$. SIR threshold $\gamma = 10dB$ and $r_{se} = 60m$. $N_B = 64$.

Figure 3.8: Outage probability vs the density of SS radios for the perfect and imperfect information (due to imperfect sensing) scenarios. SIR threshold $\gamma = 10dB$ and $r_{se} = 60m$. $N_B = 64$. 
3.5.3 Out-of-band Interference to the Static Radio System

In this subsection, we consider the effect of out-of-band transmission by SS radios (transmissions by SS radios in frequency bands outside the transmission band used by the static transmitter) on the static receiver which might arise due to poor front-end selectivity of the static receivers (e.g. TV receivers). Since the SS radios may not have any knowledge of the out-of-band interference they might cause at the static receiver, SS interferers are distributed in the entire annular region with radius extending from \( r_{phy} \) to infinity in these
bands. The interference statistic at the static receiver is then given by
\[ X^n_{\text{oob}, r_{\text{phy}}, \infty} = X^n_{r_{\text{min}}, \infty} + \sum_{l \in \mathcal{L}} \beta_l X^n_{r_{\text{phy}}, \infty} \]  
(3.28)
Here, \( \mathcal{L} \) are the frequency bands in which out-of-band interference is experienced by the static receiver and \( \beta_l \) models the sensitivity of the static radio to the interference power in these bands. Note that the first term represents the in-band interference at the static receiver. It can be seen that the distribution of interfering SS radios is the sum of \(|\mathcal{L}| + 1\) independent spatial Poisson processes. Since the number of radios transmitting in any frequency band is assumed to be independent, the MGF of \( X^n_{\text{oob}, r_{\text{phy}}, \infty} \) is the product of the MGFs of the individual Poisson processes [39] and is given by (derivation of an individual MGF is given in Chapter 2, Section 2.3.3)
\[ \Phi_{X^n_{\text{oob}, r_{\text{phy}}, \infty}} = \exp \left( n\pi \left( \int_{r_{\text{min}}}^{\infty} 2re^{sg(r)} - 1 \right) + \sum_{l \in \mathcal{L}} \beta_l n\pi \left( \int_{r_{\text{phy}}}^{\infty} 2re^{sg(r)} - 1 \right) \right). \]
(3.29)
Following the analysis in the previous sections, if \( n \) is the density of SS interferers in a frequency band and if \( P_x \) is a factor used to scale the transmit power of interfering SS radios, the \( k^{th} \) cumulant is given by
\[ m_k = \frac{2n\pi}{(k\alpha - 2)} P^k_x \frac{1}{r_{\text{min}}^{k\alpha - 2}} + \sum_{l \in \mathcal{L}} \frac{\beta_l 2n\pi}{(k\alpha - 2)} P^k_x \frac{1}{r_{\text{phy}}^{k\alpha - 2}}. \]
(3.30)
From the above expression it can be seen that the means for Scheme-1 and Scheme-3 are the same. However, the variance of Scheme-3, given by \( m^2_{WB-IA} \), is less than the variance of Scheme-1, given by \( m^2_{NB-IA} \). It is also seen in comparison to the perfect static system knowledge scenario (3.4) that the inclusion of the out-of-band interference exacerbates the difference in the variance of the two schemes. From this discussion it can be concluded that the performance improvement of the spreading-based underlay scheme that incorporates IA (Scheme-3) over the overlay scheme (Scheme-3) is accentuated by the inclusion of the effect of out-of-band interference at the static receiver.

This section thus showed that the characteristics of the interference distribution at the static receiver and the resulting comparative performance trends of SS schemes are maintained and even accentuated when the influence of imperfect static system knowledge is included in the analysis.

### 3.6 Performance in a Non-homogeneous Network

In this section, we compare the performances of Scheme-1 and Scheme-3 in a non-homogeneous network. Non-homogeneity is incorporated by modeling the network using the Cox process discussed in Section 2.5, Chapter 2. We again use an exponential path-loss model. With reference to the discussion in Section 2.5, the function \( f(x, m) \) is therefore specified as \( f(x, m) = P_x \cdot r \). Here, \( r \) is the distance of a node from the central receiver and \( P_x \) is the power scaling factor. The \( k^{th} \) moment of the marked process is then given by
\[ \mu_{f,k} = \frac{2P^k_x}{(k\alpha - 2)} \left( \frac{1}{R_{\text{min}}^{k\alpha - 2}} - \frac{1}{R_{\text{max}}^{k\alpha - 2}} \right) \]
(3.31)
Note that \( P_x = 1 \) for Scheme-1 and \( P_x = \frac{1}{N_B} \) for Scheme-3. The first and second moments for Scheme-1 and Scheme-3 are therefore given by
\[ \mu^1_{NB-IA} = \frac{2}{(\alpha - 2)} \left( \frac{1}{R_{\text{min}}^{\alpha - 2}} - \frac{1}{R_{\text{max}}^{\alpha - 2}} \right); \quad \mu^1_{NB-IA} = \frac{2}{N_B(\alpha - 2)} \left( \frac{1}{R_{\text{min}}^{\alpha - 2}} - \frac{1}{R_{\text{max}}^{\alpha - 2}} \right) \]
(3.32)
and
\[ \mu^{WB-IA}_1 = \frac{2}{(2\alpha - 2)} \left( \frac{1}{R_{min}^{2\alpha} - 2} - \frac{1}{R_{max}^{2\alpha} - 2} \right) ; \mu^{WB-IA}_2 = \frac{2}{N_B (2\alpha - 2)} \left( \frac{1}{R_{min}^{2\alpha} - 2} - \frac{1}{R_{max}^{2\alpha} - 2} \right). \]

For Scheme-3, the intensity of nodes is \( N_B \Lambda \). Therefore, \( \mu^WBA = N_B \mu_{\Lambda,1} = N_B \mu_{\Lambda,2} \) and \( \mu^WBA = N_B \mu_{\Lambda,2} = N_B \mu_{\Lambda,1} \).

From the expressions, in Section 2.5, the mean of the cox process for both schemes are as follows:
\[ m_{c,1}^{WB-IA} = \mu_{\Lambda,1}^{NB-IA} \mu_1 \] (3.34)
and
\[ m_{c,1}^{WB-IA} = N_B \mu_{\Lambda,1} \mu_1^{WB-IA} = N_B \mu_{\Lambda,1} \frac{\mu_1^{WB-IA}}{N_B} = \mu_{\Lambda,1} \mu_1^{WB-IA} \] (3.35)
respectively. The variances are given by
\[ m_{c,2}^{NB-IA} = \mu_{\Lambda,1} \mu_2^{NB-IA} + \mu_{\Lambda,1}^2 \left( \int_A \int_A f(x,m_x) f(y,m_y) p(x) p(y) g(x,y) dxdy - m_{f,1}^2 \right) \] (3.36)
and
\[ m_{c,2}^{WB-IA} = N_B \mu_{\Lambda,1} \mu_2^{WB-IA} + \frac{N_B^2 \mu_{\Lambda,1}^2}{N_B^2} \left( \int_A \int_A f(x,m_x) f(y,m_y) p(x) p(y) g(x,y) dxdy - m_{f,1}^2 \right) \]
\[ = \frac{N_B \mu_{\Lambda,1} \mu_2^{WB-IA}}{N_B} + \mu_{\Lambda,1}^2 \left( \int_A \int_A f(x,m_x) f(y,m_y) p(x) p(y) g(x,y) dxdy - m_{f,1}^2 \right) \]
\[ = \frac{\mu_{\Lambda,1} \mu_2^{WB-IA}}{N_B} + \mu_{\Lambda,1}^2 \left( \int_A \int_A f(x,m_x) f(y,m_y) p(x) p(y) g(x,y) dxdy - m_{f,1}^2 \right) \]
\[ < m_{c,2}^{WB-IA} \] (3.37)

From, the above expressions it can be seen that the means for Scheme-1 and Scheme-3 are the same. However, the variance of Scheme-3 is less than the variance of Scheme-1. It is also seen the incorporation of non-homogeneity in the network exacerbates the difference in the variance of the two schemes (as compared to a homogeneous network as considered in Section 3.4). It is thus seen that the trends seen in homogeneous networks are magnified by incorporating non-homogeneity in the network.

### 3.7 Impact of Fading

In this section, we include the effect of log normal shadowing on the interference statistic. The effect of small-scale fading in addition to log normal shadowing was also investigated via simulations and analysis. However, it was found that the sum-interference power at the central static receiver is dominated by log normal shadowing and the inclusion of small-scale fading does not significantly alter the comparative results in the below discussion.

The received power at distance \( r \) from the \( i \)th SS radio with the inclusion of log normal shadowing is given by
\[ P(r_i) = P_a \left( \frac{1}{r_i} \right)^\alpha 10^{z_i / 10}. \] (3.38)
Here, $z_{i,\sigma_l}$ is a zero-mean Gaussian random variable (in dB) with standard deviation $\sigma$ (also in dB) used to represent log normal shadowing. The log normal distribution describes the random shadowing effects due to different levels of clutter on the propagation path. The value for $\sigma$ is computed using measurements and is typically between 4dB and 12dB. The normalized sum interference power at the static receiver from SS radios Poisson distributed in an annular region with inner radius $\epsilon$ and outer radius $a$ is given by

$$X_{\text{shad},\epsilon,a}^n = \sum_{i=1}^{n} (s_i).$$

(3.39)

Here, $s_i = \frac{10^{z_{i,\sigma_l}/10}}{r_i^\alpha}$, is the power of the $i^{th}$ interfering SS radio at the static receiver. Let the distribution of $s_i$ be given by $p_s(s)$. Then the characteristic function of $X_{\text{shad},\epsilon,a}^n$ is given by (following the analysis in Section 2.3.3)

$$\phi_{X_{\text{shad},\epsilon,a}^n} (\omega) = \exp \left( n\pi \left( a^2 - \epsilon^2 \right) \left( \int_{s} p_s(s) e^{i\omega s} ds - 1 \right) \right).$$

(3.40)

The $k^{th}$ cumulant of $X_{\text{shad},\epsilon,a}^n$ is therefore given by

$$m_k \left( X_{\text{shad},\epsilon,a}^n \right) = n\pi \left( a^2 - \epsilon^2 \right) \int_{s} p_s(s) s^k ds.$$

(3.41)

Since the joint distribution of two independent distributions is the product of the two distributions,

$$n\pi \left( a^2 - \epsilon^2 \right) \int_{s} p_s(s) s^k ds = n\pi \left( a^2 - \epsilon^2 \right) \int_{\epsilon}^{a} \frac{2r}{(a^2 - \epsilon^2)} (g(r))^k dr \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sigma_l}} e^{\frac{-z^2}{2\sigma_l^2}} dz.$$

(3.42)

Here, $C = \frac{1}{10 \log_{10} e}$. The $k^{th}$ cumulant of $X_{\text{shad},\epsilon,\infty}^n$ is therefore given by

$$m_k \left( X_{\text{shad},\epsilon,\infty}^n \right) = \frac{2n\pi e^{\sigma^2 c^2 k^2/2}}{(k\alpha - 2) (e^{k\alpha} - 2)}.$$

(3.43)

We again attempt to fit the distribution of $X_{\text{shad},\epsilon,\infty}^n$ to a log normal distribution using a simple moment matching technique. Note that this method is very similar to the technique illustrated in [51] where a log normal variable is used to model a linear combination of a set of log normal variables with one-sided random weights (the one-sided random weights in the current scenario are the path-loss factors). The simulated and theoretical outage probability curves w.r.t. different node densities are shown in Figure 3.10. The value of $\sigma_l = 6$dB and $\epsilon = 20$m. It is seen that the log normal distribution accurately models the distribution outage probabilities greater than 1%.

Figure 3.11 compares the outage probabilities of the underlay and overlay SS schemes implementing IA with the inclusion of shadowing. Perfect static system knowledge is assumed for IA and the minimum distance between an interfering SS radio and the static receiver is assumed to be 20m ($r_{\text{min}}$). It is observed that excepting for a small crossover region, the hybrid underlay scheme with IA gives lower outage probabilities than the IA-based overlay scheme. The margin of performance improvement can be noticed to increase with an increase in the spreading factor. When compared with the outage probability curves without the inclusion of log normal shadowing, it is seen that log normal shadowing leads to a loss in performance in general. However, the margin of performance loss is greater for the overlay scheme than the underlay scheme. This is due to the additional factor $(e^{\sigma^2 c^2 k^2/2} > 1)$ by which the cumulants of the distributions are weighted leading to a magnification of the relative interference statistics. It is thus seen that shadowing accentuates the performance improvement that can be achieved using the hybrid spreading-based underlay approach with IA for SS.
Figure 3.10: Comparison of simulated and theoretical outage probability distributions w.r.t. the density of SS radios with the inclusion of the effect of log normal shadowing. SIR threshold $\gamma = 10\text{dB}$ and $\sigma_l = 6\text{dB}$. SS radios are assumed to be distributed in a disc with radius extending from $r_{\text{min}} = 20\text{m}$ to $100\text{m}$. $N_B = 64$.

Figure 3.11: Outage probability vs the density of SS radios when SS radios have perfect information about the static radio system and with the inclusion of the effect of log normal shadowing. Interferers are distributed in a disc with radius extending from $20\text{m}$ to $\infty$. SIR threshold $\gamma = 10\text{dB}$ and $\sigma_l = 6\text{dB}$. 
3.8 Conclusions, Design Trends and Contributions

In this chapter, we developed a framework to investigate the impact of dynamic spectrum sharing radio systems on legacy static radio systems with which they must coexist. The framework allows for inclusion of the effect of imperfect information about the actual interference profile seen at the legacy receiver. The problem was analyzed by considering the distribution of interference power (modeled by a log normal distribution) and consequent outage probability at a single static radio receiver situated at the center of a circular field of SS radios. The outage probability statistics and example scenarios were then used to derive desirable features for SS techniques.

It was shown that the minimum distance $\epsilon$ between an interfering SS radio and the static receiver has the largest impact on the outage probability. Hence IA techniques, which increase the minimum distance $\epsilon$ and thereby remove the strongest interferers, are able to dramatically reduce the interference seen at a static receiver. However, averaging the interference power over all available transmission dimensions (as in an underlay scheme) is more advantageous than a corresponding reduction in the density of interferers (as in an overlay scheme) for interference from users beyond $\epsilon$. Hence interference averaging is more effective in protecting the legacy system from far-away interference. These results motivate the use of hybrid IA-based spectrum underlay technique, that removes the strongest interferers to a legacy system and averages the power of the remaining interferers. Examples of this scheme include adaptive frequency hopping, sequence-adaptation and notched-transmission techniques. The hybrid scheme decreases the interference power of the SS interferers through averaging in addition to employing IA to increase the minimum distance $\epsilon$. The hybrid scheme thus performs better (causes less outage at the static receiver except for a small cross-over region that occurs at high outage probabilities) than the IA-based spectrum overlay scheme. The results are analogous to lessons learned in cellular systems where in-cell interference should ideally be avoided while out-of-cell interference is best averaged. Also, the benefit provided by the hybrid scheme over the IA-based spectral overlay scheme is more pronounced as the transmission bandwidth available to the SS radio system is increased. This motivates the use of Ultra-wideband (UWB) techniques with band-notching or UWB OFDMA techniques, which avoid specific bands, for spectrum sharing. In addition, the hybrid scheme is less affected by the absence of perfect information about the static radio system required for IA. This provides a larger flexibility in the practical deployment of this technique. The inclusion of log normal shadowing in the analysis further accentuates the above performance trends.

The original contributions in this chapter are as follows:

- A framework to investigate the impact of DSS radio systems on legacy static radio systems was developed. The framework incorporates practical deployment issues such as, the hidden-node problem, imperfect sensing and out-of-band interference, that result in imperfect static system knowledge.

- Based on the framework, desirable features for spectrum sharing techniques with respect to the performance of distributed networks were identified.

The publications that resulted from this chapter are as follows:


Chapter 4

Analysis of Dynamic Spectrum Sharing Network Performance

4.1 Introduction and Problem Statement

An ideal spectrum sharing (SS) scheme is one that minimizes interference to the legacy radio system while maximizing the performance of the network that implements the scheme. In the previous chapter, we studied desirable characteristics of SS schemes with respect to the legacy system. In this chapter, we investigate the second aspect, the performance of a network that implements spectrum sharing. Towards this end, we develop a framework to compare the performance of SS schemes in a distributed network. The sum of outage-capacities of users in the network is used as the performance metric. The relative performance trends of different schemes are then used for identifying features of transmission techniques which make them desirable from the perspective of improving the performance of the distributed network. We start with a literature review which summarizes the research in this field and motivates the analysis approach used in our work.

4.2 Related Literature and Motivation

SS networks may have centralized or de-centralized architectures. Centralized networks have been well-studied in the context of cellular networks and it has been shown that the sum-capacity (sum-rate) of a network is maximized by avoiding strong interferers and averaging over weak interferers [20]. Determining the capacity of networks with distributed receivers, on the other hand, has proved an onerous task since the interference profiles at different receive-nodes are different and there does not need to be any symmetry in user-pair interactions. However, considerable work has been carried out in establishing capacity order-bounds and scaling laws (as a function of node-density) for these networks ( [22], [23], [24], [18] and references within). In [23], it was shown that for power-limited systems, a CDMA-based (or spreading based) multiple access scheme is optimal in the limit of infinite bandwidth since in this limit the interference between spectrum-sharing devices becomes negligible. The relevance of this result to practical networks which are bandwidth-limited has yet to be established. However, when only a smaller amount of bandwidth is available to the network, it was shown in the paper that a hybrid FDMA-CDMA technique, where FDMA is used in a local region and CDMA is used for handling interference from outside the local region, achieves the lower-bound on the optimal capacity. In [25] and [26], it was shown that in under-loaded systems when the signal attenuation in the network is large, water-filling techniques (which spread the signal over all available dimensions) are better than IA-based techniques. The opposite is found to be true in networks where
the mutual interference between spectrum sharing devices can be large due to small attenuation factors. However, this work only considers a small subset of possible network scenarios (under-loaded networks with two users or networks where all channels have equal gains).

Though the analysis scenarios are restrictive, the above results seem to indicate that allocations similar to the optimal allocations in a centralized network, where the strongest interferers at a receiver are avoided and the power from the weaker interferers are averaged, could be optimal from the perspective of a distributed network as well. A more comprehensive study which confirms these possible inferences and explicitly compares different approaches to SS in distributed networks needs to be undertaken and is the focus of this chapter.

A comparison of direct-sequence CDMA (DS-CDMA) and frequency-hopping CDMA (FH-CDMA) schemes for ad hoc networks was presented in [18]. The paper only considers slow frequency-hopping, which results in the capacity expressions for FH-CDMA being similar to expressions for a narrowband (NB) transmission scheme. Note that with fast frequency-hopping, the expressions for FH-CDMA are similar to that of DS-CDMA. The comparison is based on bounds on the outage-capacity statistics at a receiver placed in the center of a circular network in which radios that implement the above two techniques are Poisson distributed. An ALOHA-based random access Multiple-Access-Control (MAC) protocol is assumed for the transmission schemes. Transmission capacity (defined as the product of the maximum transmission-density, the average rate of a successful transmission and the probability of a successful transmission) is used as the performance metric. It was shown in the paper that, in general, FH-CDMA performs better than DS-CDMA. In addition the following performance trends are established:

1. Transmission-capacity of the network decreases with path-loss exponent for DS-CDMA and increases with path-loss exponent for FH-CDMA, resulting in FH-CDMA being increasingly superior to DS-CDMA with path-loss exponent.
2. Transmission capacity of the network is almost constant with the spreading factor for FH-CDMA, whereas the transmission capacity of the network decreases with the spreading factor in DS-CDMA.

Hence, it was shown that FH-CDMA (with characteristics similar to a NB transmission scheme) is preferable to DS-CDMA (an interference-averaging-based scheme) in ad hoc networks. However, it can be seen that the above trends are dominated by the poor performance of interference-averaging schemes in the presence of the near-far interference problem. This limitation was addressed in [52], by comparing FH-CDMA with a DS-CDMA scheme that implements successive interference cancellation (SIC). It was shown that with perfect SIC which can completely cancel interference from a finite number (K) of strongest interferers, DS-CDMA performs better than FH-CDMA. However, an SIC technique that does not completely cancel the strongest interferers might perform worse than FH-CDMA. In addition, it was shown that even with the inclusion SIC, the transmission capacity of the network decreases with spreading factor for the DS-CDMA schemes. Hence, the paper recommended that though interference-averaging with SIC should be implemented, the spreading factor should be made as small as possible. However, we feel that this result is an artifact of the assumption used in the paper that the number of interferers that can be rejected (K) stays constant with an increase in the spreading factor. The number of interferers that can be rejected (K), in general, increases with the spreading factor used. Also, as mentioned earlier, the two described papers ([18] and [52]), use an ALOHA-based MAC. The inclusion of a more sophisticated MAC might influence the relative performance trends of the schemes.

In [22], a seminal paper on the capacity of wireless ad hoc networks, upper and lower capacity order-bounds were derived by using the the concept of an exclusion region. An exclusion region is a region around each receiver such that no interferers exist inside this region. The exclusion region concept brings an aspect of TDMA to the MAC. In the paper, the exclusion region for a receiver was assumed to be larger than the distance between the receiver and its intended transmitter. It was shown that the capacity of the
network decreases with an increase in the size of this exclusion region. In [53], optimal flow-rates for an
ad hoc network are investigated by jointly optimizing over routing, scheduling and power-control. It was
shown that the optimal solution is characterized by an exclusion region around the destination. These results
motivate the use of an exclusion region based MAC protocol for the framework developed in this chapter.

In [54], the size of the exclusion region (referred to in the paper as critical radius), was investigated for
CDMA networks. It was shown that large performance improvements are possible in a CDMA system by
employing an exclusion region and it was suggested (based on performance bounds derived in [22]) that
there exists an optimal size for the exclusion region that maximizes transmission-capacity.

In this chapter, we develop a framework to investigate desirable features for (spectrum-sharing) trans-
mision techniques in general and to compare the relative merits of NB and Wideband (WB) interference-
averaging-based techniques in particular, with respect to the performance of a distributed (or equivalently of
an ad hoc) network. Note that the NB technique is equivalent to the interference-avoidance (IA) based over-
lay technique used for SS with legacy systems and considered in the previous chapter (Chapter 3). Likewise,
the WB technique is equivalent to the interference-averaging-based spectrum underlay technique used for
SS with legacy systems. As mentioned, WB schemes are severely limited by their performance in near-far
interference scenarios. Hence, the performance of a WB scheme which implements interference-mitigation
techniques, specifically the MMSE receiver, is also investigated. The analysis utilizes the distribution of
SINR at the output of an MMSE receiver, which is a function of the spreading factor among other param-
eters. Hence, unlike the approach in [52], we incorporate the effect of spreading factor on the rejection
capabilities of the receiver. The framework is based on the interference statistics at a central receiver placed
in a circular region in which other transmit-receive node-pairs are uniformly distributed (as mentioned,
a similar system model is used in [18]) and employs an exclusion-region-based MAC. The concept of a
Matern hard-core process [39] is used to model the exclusion-region. Transmission-density with respect
to an outage-constraint (which is a scaled version of transmission-capacity used in [18]) and sum-outage-
capacity (defined as the product of the transmission-density in the network and the average-outage-capacity
for a node-pair) are used as performance metrics.

The rest of the chapter is organized as follows: The system model used for the framework is described
in Section 4.3. Section 4.4 discusses the exclusion-based MAC. The evaluation of interference statistics
for different network scenarios is discussed in Section 4.5. The relative performance of NB and WB are
investigated in Section 4.6. Performance evaluation with the inclusion of an MMSE receiver for the WB
scheme is carried out in Section 4.7. Finally, Section 4.8 summarizes the chapter.

### 4.3 System Model

We analyze the performance of a transmit-receive link in which the receiver is situated at the center of
a circular region with radius $r_{\text{max}}$. Other transmit-receive node-pairs that share the spectrum with this
central transmit-receive node-pair are assumed to be uniformly distributed in this circular region. The central
receiver-node is assumed to be at a distance $r_s$ from its transmit-node. The system model is illustrated
in Figure 4.1. We also restrict ourselves to a single-hop transmission-link between the transmit-receive
nodes. The total bandwidth available to the network is denoted by $N B$ where $N$ is a positive integer and
$B$ is the data rate. The spreading factor of a transmission scheme is denoted by $N_B \leq N$. Note that
$N_B = 1$ corresponds to a narrowband transmission scheme. The density of transmit-receive node-pairs in
a transmission bandwidth of $N_B B$ is assumed to be $\lambda$. Hence, assuming a uniform distribution of nodes in
the transmission bands, the total density of nodes is $\lambda_{\text{BB}}$. Without loss of generality, the data transmission
bandwidth $B$ is assumed to be 1Hz. The results can easily be scaled for other values. The performance of
the central transmit-receive node is assumed to be representative of the performance of a single-link in a
single-hop ad hoc network in which multiple node-pairs are uniformly distributed in a circular region.
Interference is caused at the central receiver by radios in the circular region that are transmitting in the same frequency band (or signal dimension) as the central link. In the analysis presented here, an outage is assumed to be caused at the central receiver if the signal to interference noise ratio (SINR) at the central receiver is less that some SINR threshold $\gamma$. The interference power at the central receiver (after despreading if spreading is used) under consideration is modeled as a random variable. The distribution of the random variable and the resulting outage-capacity at the central receiver are the basic analyses to be accomplished. The following are other assumptions and notations used in this analysis: All transmit-nodes are assumed to transmit at the same power-level. The efficacy of power-control will be addressed in future work. The power received from a transmit-node at a distance of 1m is denoted by $P_t$. In addition, transmitted signals are assumed to suffer a loss in power that follows an exponential propagation law. The path-loss exponent is denoted by $\alpha$ and it is assumed that $\alpha > 2$. Note that if this condition is not satisfied, as mentioned in Chapter 2, the sum-interference power at the receiver would be a function of the network size and would tend to infinity as $r_{\text{max}} \rightarrow \infty$ ([34] and [40]).

### 4.4 Exclusion-region-based MAC

We assume that there exists an exclusion region around a receiver in which no interfering transmitter is present. We denote the radius of this exclusion region by $r_{\text{min}}$. This minimum distance constraint can be incorporated into the network by implementing a **Matern hard-core process** [39], a spatial point process where constituent points are forbidden to lie closer together than a certain minimum distance ($r_{\text{min}}$). This process thus describes patterns produced by the locations of centers of non-overlapping circles of radius $r_{\text{min}}/2$ and is generated as follows: Consider the initial stationary distribution of nodes with density $\lambda$, denoted by $\Phi_0$. The points of $\Phi_0$ are marked independently by random numbers uniformly distributed over $(0, 1)$. A dependent thinning process is then applied, where a point $x$ of $\Phi_0$ with mark $m(x)$ is retained if
Figure 4.2: Transmission node-density vs exclusion region for $\lambda$ and upper-bound $\lambda_{\text{max}}$. $P_o = 0.01$, $SINR = 3$ (corresponding to 1 bit/ Hz/ sec), $r_s = 5m$ and $\alpha = 3$.

the circle $b(x, r_{\text{min}})$ contains no point of $\Phi_b$ with marks smaller than $m(x)$. The thinned process is thus given by

$$\Phi = \{ x \in \Phi_b : m(x) < m(y) \ \forall y \in \Phi_b \cap b(x, r_{\text{min}}) \backslash \{x\} \}$$

(4.1)

The density of the thinned process is given by [39]

$$\lambda_{\text{m}}(r_{\text{min}}) = \frac{1 - \exp\left(-\lambda_b \pi r_{\text{min}}^2\right)}{\pi r_{\text{min}}^2}.$$  

(4.2)

The distribution of nodes that transmit in the same frequency band is assumed to follow a Matern hard-core process with density $\frac{\lambda}{2}$. (The density of the total number of nodes, sum of transmit and receive nodes, is assumed to be equal to $\lambda_{\text{m}}$). Hence, $\lambda = \frac{\lambda}{2}$. It is to be noted that this density does not represent the closest possible packing of circles. An upper bound on the packing density of circles of radius $r_{\text{min}}$ is given by

$$\lambda_{\text{max}} = \frac{1}{\pi r_{\text{min}}^2}.$$  

(4.3)

However, it is observed that the performance results obtained using the Matern-hard core process are almost identical to the performance results obtained by using the upper-bound on the density of nodes. An example scenario is illustrated in Figure 4.2 which plots the density of SS nodes that can be accommodated, for different values of $r_{\text{min}}$, such that an outage probability of 0.01 and Signal-to-Interference-Noise-Ratio (SINR) of at least 3 is guaranteed at the central receiver. It is seen that the plots generated using a Matern hard-core process with density $\lambda_{\text{m}}$ and those generated with density $\lambda_{\text{max}}$ match closely (The trends in the plot are discussed in detail in later sections). Therefore, the Matern hardcore process yields densities which are very close to the maximum possible packing densities. It is hence used for most of our subsequent analyses.

Note that, while using a Matern hard core process, the location of an individual transmit node is dependent on the location of other nodes in its neighborhood. Therefore, it is not possible to obtain a closed form

1 $b(x, r_{\text{min}})$ refers to a circle with $x$ as the center and $r_{\text{min}}$ as the radius.
expression for the distribution of the location of a single transmit node in the circular region (or more specifically the distribution of the interference power received at the center of the circular region from a transmit node). It will be seen in the subsequent sections that an expression for this distribution is essential to derivation of the SINR at the central receiver. Hence we will show via simulations that the interference profile at the central receiver due to a Matern hardcore process can be approximated by a distribution obtained by uniformly and independently distributing nodes with density $\lambda_m$ (the reduced density obtained from the Matern hardcore process) in an annular region, with inner-radius $r_{\min}$ and outer-radius $r_{\max}$, around the central receiver. Figure 4.3 compares the inverse-cdfs obtained from the Matern hardcore process and the reduced density process. It is seen that the interference profile yielded by both processes are very close. Therefore a uniform reduced-density process, which is more amenable to generating interference distributions will be used in all subsequent sections.

4.5 Modeling the Interference Statistic

As mentioned before, the distribution of the interference power at the central receiver is the basic analysis to be accomplished. Here, we investigate this distribution under two scenarios: first, when a matched filter is used at the central receiver (for $N_B \geq 1$) and second, when an interference-mitigator in the form of a Minimum Mean Square Error (MMSE) filter is employed at the central receiver (for $N_B > 1$). In both scenarios, the transmit node-pairs in the network are assumed to use random spreading codes with spreading factor $N_B$ (when $N_B > 1$).

4.5.1 Interference Statistic with Random Spreading and a Matched Filter

Let $P_x$ be a scaling factor for the power received at the detector from interfering transmit-nodes that accounts for the effect of spreading while using random transmit sequences and a matched filter. For a spreading factor
of $N_B$, it is assumed that $P_x = \frac{1}{N_B}$ ([18] and [7]). The sum-interference power at the centralreceiver when $P_t = 1 \text{watt}$ is then given by

$$X^\lambda_{r_{\min}, r_{\max}} = \sum_{J_{r_{\min}, r_{\max}}} \frac{P_x}{r_i^{\alpha}},$$

(4.4)

Here, $r_i$ is the random variable that denotes the distance of the $i^{th}$ transmitter from the central receiver and $J_{r_{\min}, r_{\max}}$ denotes the set of interfering radios such that $r_{\min} \leq r_i \leq r_{\max}$. The $k^{th}$ moment of $X^\lambda_{r_{\min}, r_{\max}}$ can be derived as follows:

Let $r$ represent the distance of a transmit-node from the central receive-node. The distribution of $r$ for a node in the network is

$$f_r (r) = \frac{2r}{(r_{\max}^2 - r_{\min}^2)}, \quad r_{\min} \leq r \leq r_{\max}.$$

(4.5)

The power received from a transmit-node at distance $r$ assuming unit-received power at a distance of 1m is given by $\frac{1}{r^\alpha}$. The distribution of the received power can be derived to be

$$f_p (p) = \left| \frac{f_r (r)}{\frac{d}{dr}(\frac{1}{r^\alpha})} \right| = \frac{2}{\alpha (r_{\max}^2 - r_{\min}^2) p^{\frac{\alpha + 2}{\alpha}}}, \quad \frac{1}{r_{\max}^\alpha} \leq p \leq \frac{1}{r_{\min}^\alpha}.$$

(4.6)

The $k^{th}$ moment of the received power from a transmit-node is given by

$$\kappa_k = \frac{2P_x^k}{\alpha (r_{\max}^2 - r_{\min}^2)} \int_{r_{\max}^\alpha}^{r_{\min}^\alpha} p^k p^{\frac{\alpha + 2}{\alpha}} dp = \frac{2P_x^k}{(k\alpha - 2) (r_{\max}^2 - r_{\min}^2)} \left( \frac{1}{r_{\min}^{\alpha - 2}} - \frac{1}{r_{\max}^{\alpha - 2}} \right).$$

(4.7)

The first two cumulants of the distribution of the power from a single transmit node are then $m_1^1 = \kappa_1$ and $m_2^1 = \kappa_2 - \kappa_1^2$ (The relationship between moments and cumulants are stated in Appendix B). Using the assumption that the transmit-nodes are uniformly and independently distributed in the circular region, the $k^{th}$ cumulant of $X^\lambda_{r_{\min}, r_{\max}}$ can be expressed as

$$m_k = \lambda \pi \left( r_{\max}^2 - r_{\min}^2 \right) m_1^1.$$

(4.8)

Here, $\lambda \pi \left( r_{\max}^2 - r_{\min}^2 \right)$ is the number of transmit-nodes in the circular region. The mean and variance of the distribution are thus given by

$$m_1 = \frac{2\lambda \pi P_x}{(\alpha - 2)} \left( \frac{1}{r_{\min}^{\alpha - 2}} - \frac{1}{r_{\max}^{\alpha - 2}} \right).$$

(4.9)

and

$$m_2 = 2\lambda \pi P_x^2 \left( \frac{1}{(2\alpha - 2)} \left( \frac{1}{r_{\min}^{2\alpha - 2}} - \frac{1}{r_{\max}^{2\alpha - 2}} \right) - \frac{2}{(\alpha - 2) (r_{\max}^2 - r_{\min}^2)} \left( \frac{1}{r_{\min}^{\alpha - 2}} - \frac{1}{r_{\max}^{\alpha - 2}} \right)^2 \right).$$

(4.10)

Note that as $r_{\max} \to \infty$, the mean and variance are

$$m_1 = \frac{2\lambda \pi P_x}{(\alpha - 2) r_{\min}^{\alpha - 2}}$$

(4.11)

and

$$m_2 = \frac{2\lambda \pi P_x^2}{(2\alpha - 2) r_{\min}^{2\alpha - 2}}$$

(4.12)

respectively.
An outage is assumed to be caused at the central receiver if the following condition is satisfied:

\[
\frac{P_t/r_s^\alpha}{\sigma^2 + P_tX_{r_{\min},r_{\max}}^\lambda} \leq \gamma
\]  

(4.13)

Here, \(\sigma^2\) is the noise power and \(\gamma\) is the minimum SINR required at a central receiver for successful reception. This implies that an outage is caused at the central receiver if the normalized interference power is above some threshold \(P_i\), i.e., if the following condition is satisfied:

\[
X_{r_{\min},r_{\max}}^\lambda \geq P_i = \frac{P_t/r_s^\alpha}{P_t\gamma} - \frac{\sigma^2}{P_t}.
\]  

(4.14)

A log normal distribution is used to model the interference statistic. The parameters of \(X_{r_{\min},r_{\max}}^\lambda\) are fit to the log normal distribution using a simple moment matching approach. The probability density function of a log normal variable is given by

\[
p(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(x/m_l))^2}{2\sigma_l^2}\right),
\]  

(4.15)

where parameters \(m_l\) and \(\sigma_l\) represent the mean and standard-deviation of the underlying Gaussian process. The parameters \(m_l\) and \(\sigma_l\) for fitting the random variable \(X_{r_{\min},r_{\max}}^\lambda\) to a log normal distribution can be found by using the following relations:

\[
m_1 = m_l \exp\left(\frac{1}{2}\sigma_l^2\right), \quad m_2 = m_l^2 \exp\left(\sigma_l^2\right) \left(\exp\left(\sigma_l^2\right) - 1\right).
\]  

(4.16)

Here, \(m_k\) is the \(k^{th}\) cumulant of the \(X_{r_{\min},r_{\max}}^\lambda\). The probability of outage can then be expressed as

\[
P_o = p_{out}(P_i) = \Pr\left(X_{r_{\min},r_{\max}}^\lambda > P_i\right) = \frac{1}{2} - \frac{1}{2} \text{erf}\left(\frac{\ln(P_i/m_l)}{\sigma_l\sqrt{2}}\right) = Q\left(\frac{\ln(P_i/m_l)}{\sigma_l}\right).
\]  

(4.17)

Similar to the modeling of the interference statistic in Chapter 2 (which is a slightly different from the interference-statistic used here), it is seen that the heavy-tailed log normal distribution closely approximates the heavy-tailed interference statistic. Note that the log normal approximation gives us a closed form expression for the interference statistic which is useful in deriving comparative performance trends.

The outage-capacity for a node-pair whose receiver is at the center of the circular region in the scenario discussed in this sub-section is given by

\[
C_{out} = \frac{1}{2} \log_2 \left(1 + \frac{P_t/r_s^\alpha}{N_0 + P_i}\right).
\]  

(4.18)

Note that the expression assumes that the multiple access interference statistic is treated as Gaussian noise. Such an expression for capacity has been previously used in [22], [23], [18] etc.

4.5.2 Interference Statistic with Random Spreading and a MMSE Receiver

Let the network under consideration have \(K\) transmit-nodes (and hence \(K - 1\) interfering-nodes to the central receiver) and let the transmit-node corresponding to the central receive-node be indexed by 1 in the following analysis. Let the power received from the \(k^{th}\) transmit-node at the central receiver be denoted
by $P_k$ and the random signature sequence used by the $\mu$th transmit-node be denoted by $s_k \in \mathbb{R}^{N_B}$. Note that $N_B$ is the spreading factor or, in other words, the number of available signal dimensions. The random sequence $s_k$ is specified as follows ([55]):

$$s_k = \frac{1}{\sqrt{N_B}}(V_{k1}, \ldots, V_{kN})^T, \ k = 1, \ldots, K \tag{4.19}$$

The random variables $V_{kn}$ are independent and identically distributed with zero mean and unit variance. The normalization ensures that $\mathbf{E}[[|s_k|^2]] = 1$. The assumption that $\mathbf{E}[V_{kn}^4] < \infty$ is also imposed on the sequences.

Theorem 3.1 in [55] shows that the random SINR, $\beta_{N_B}$, at the output of a MMSE receiver, when the number of users per signal dimension $\zeta = \frac{K}{N}$ is fixed, converges in probability to $\beta^*$ as $N_B \to \infty$. Here, $\beta^*$ is the unique solution to the following equation:

$$\beta^* = \frac{P_1}{\sigma^2 + \zeta \int_0^\infty \frac{P_1}{P_1+p\beta^*} dF(p)} \tag{4.20}$$

Here, $p$ is the power received from an interfering transmit-node and $F(p)$ limiting empirical distribution of the power received from an interfering transmit node. Note that $P_1 = P_T / r_s^\alpha$.

In [56], it is shown that in networks where the received powers at a receive-node from all the transmit-nodes in the network are equal, the distribution of $\beta_{N_B}$ is asymptotically Gaussian as $N_B \to \infty$. The mean of the Gaussian distribution is given by $\beta^*$ and the variance of the distribution is given by the following expression:

$$\sigma_{\text{MMSE}}^2 = \frac{2}{N_B} \int \left[ \frac{P_1}{(\lambda_e + \sigma^2)} \right]^2 dG^*(\lambda_e) + \frac{(\mathbf{E}[v_{11}^4] - 3)}{N_B} \left[ \int \frac{P_1}{(\lambda_e + \sigma^2)} dG^*(\lambda_e) \right]^2 \tag{4.21}$$

Here, $G^*(\cdot)$ is the limiting distribution of the eigenvalues of the interference cross-correlation matrix give by $S_{-1}T_{-1}S_{-1}^T$, where $S_{-1} = [s_2, \ldots, s_K]$ and $T_{-1} = \text{diag}[P_2, \ldots, P_K]$. The existence of a limiting distribution to which the empirical distribution of $S_{-1}T_{-1}S_{-1}^T$ converges is shown in the references within [56]. Equation 4.21 can be solved for the variance by noting that [56]

$$\int \frac{P_1}{(\lambda_e + \sigma^2)} dG^*(\lambda_e) = \beta^* \tag{4.22}$$

and

$$\int \frac{P_1}{(\lambda_e + \sigma^2)^2} dG^*(\lambda_e) = -\frac{d\beta^*}{d\sigma^2} \tag{4.23}$$

Expressions for the mean and variance in the equal received-power scenario are derived in [56] by solving the above equations.

In our analysis here, we assume that the SINR distribution at the output of the MMSE receiver when the received powers at a receive-node from all other transmit-nodes in the network are not equal also asymptotically converges to the Gaussian distribution $N(\beta^*, \sigma_{\text{MMSE}}^2)$ (Refer to the appendix for a more detailed discussion). Expressions for the mean and variance in the scenario where transmit-nodes are uniformly distributed with reduced density (according to (4.2)) in an annular region with inner-radius $r_{\text{min}}$ and outer-radius $r_{\text{max}}$ and the receive-node is situated at the center of the annular region can be calculated as follows:

From the discussion in the previous sub-section, the distribution of the received power is given by

$$f_p(p) = \frac{2}{\alpha (r_{\text{max}}^2 - r_{\text{min}}^2)} p^{\frac{2-d}{\alpha} - 1}, \frac{r_{\text{max}}^2}{r_{\text{min}}^2} \leq p \leq 1 \tag{4.24}$$
The mean of the Gaussian distribution \( \beta^* \) is then given by
\[
\beta^* = \frac{P_1}{\sigma^2 + \zeta P_1 A(\beta^*)},
\] (4.25)
where
\[
A(\beta^*) = \frac{2}{\alpha (r_{\text{max}}^2 - r_{\text{min}}^2)} \int_{r_{\text{min}}^1}^{r_{\text{max}}^1} \frac{p}{P_1 + p\beta^*} p^{\frac{\alpha - 2}{2}} dp
\]
\[
= \frac{2}{\alpha (r_{\text{max}}^2 - r_{\text{min}}^2)} \int_{r_{\text{min}}^1}^{r_{\text{max}}^1} \frac{p^{\frac{\alpha - 2}{2}}}{P_1 + p\beta^*} dp
\]
\[
= \frac{2}{(\alpha - 2) P_1 (r_{\text{max}}^2 - r_{\text{min}}^2)} \left[ 2 F_1 \left[ \frac{\alpha - 2}{\alpha}, 1, \frac{\alpha - 2}{\alpha} + 1, \frac{-\beta^*}{P_1 r_{\text{min}}^1} \right] \right] - 2 F_1 \left[ \frac{\alpha - 2}{\alpha}, 1, \frac{\alpha - 2}{\alpha} + 1, \frac{-\beta^*}{P_1 r_{\text{max}}^1} \right].
\] (4.26)

Here, \( 2F_1 \) is the Gauss hypergeometric function defined as follows:
\[
2F_1 (\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n) \Gamma(\beta + n)}{\Gamma(\gamma + n)} z^n,
\] (4.27)
where \( \Gamma \) is the Gamma function.

To calculate the variance of the Gaussian distribution, the differential of \( \beta^* \) with respect to \( \sigma^2 \) is to be calculated. This is accomplished as follows: Equation (4.20) can be rewritten as
\[
\beta^* \sigma^2 + \zeta \beta^* \int_{r_{\text{min}}^1}^{r_{\text{max}}^1} \frac{P_1 p}{P_1 + \beta^* p} dF_p(p) = P_1.
\] (4.28)

Taking the differential w.r.t. \( \sigma^2 \) on both sides we get
\[
\beta^* + \sigma^2 \frac{d\beta^*}{d\sigma^2} + \zeta \frac{d\beta^*}{d\sigma^2} \int_{r_{\text{min}}^1}^{r_{\text{max}}^1} \frac{P_1 p}{P_1 + \beta^* p} dF_p(p) + \zeta \frac{d\beta^*}{d\sigma^2} \int_{r_{\text{min}}^1}^{r_{\text{max}}^1} -\frac{P_1 p^2}{(P_1 + \beta^* p)^2} dF_p(p) = 0
\] (4.29)

The terms can be re-arranged to get
\[
-\frac{d\beta^*}{d\sigma^2} = \beta^* \left( \sigma^2 + \zeta \int_{r_{\text{min}}^1}^{r_{\text{max}}^1} \frac{P_1 p}{P_1 + \beta^* p} dF_p(p) - \zeta \beta^* \int_{r_{\text{min}}^1}^{r_{\text{max}}^1} -\frac{P_1 p^2}{(P_1 + \beta^* p)^2} dF_p(p) \right)^{-1}
\] (4.30)
The last term in the inverse term can be evaluated as follows:

$$\frac{1}{r_{\min}^2} \int_{r_{\max}^2}^{1} \frac{-p^2}{(P_1 + p\beta \gamma)^2} dF_p(p)$$

$$= \frac{2}{\alpha (r_{\max}^2 - r_{\min}^2)} \int_{r_{\max}^2}^{1} \frac{-p^\alpha}{(P_1 + p\beta \gamma)^2} dp$$

$$= \frac{1}{(\alpha - 1) P_1^2 (r_{\max}^2 - r_{\min}^2)} \left[ 2 F_1 \left[ \frac{2\alpha - 2}{\alpha}, 2, \frac{2\alpha - 2}{\alpha} \right] + 1, \frac{-\beta \gamma}{r_{\min}^2} \right] - 2 F_1 \left[ \frac{2\alpha - 2}{\alpha}, 1, \frac{2\alpha - 2}{\alpha} + 1, \frac{-\beta \gamma}{r_{\max}^2} \right]$$

(4.31)

Using the expressions (4.21), (4.23), (4.25), (4.26), (4.30) and (4.31), the variance of the limiting SINR distribution at the output of the MMSE receiver can be determined.

An outage is assumed to be caused at the central receive-node if the SINR at the output of the MMSE receiver is less than some threshold $\gamma$. The probability of outage is thus given by the following expression:

$$P_o = p_{out}(\gamma) = Pr(\beta N_B < \gamma) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\gamma - \beta \gamma}{\sqrt{2\sigma_{MMSE}}} \right)$$

(4.32)

The fit of the moments of the distribution is shown in Figure 4.4. It is seen that the Gaussian distribution does not perfectly model the actual distribution. However, the difference in performance is negligible for our analyses where we compare the relative performance of the scheme with that of a narrowband transmission scheme ($N_B = 1$). This is illustrated in Figure 4.5 which plots the outage-capacity versus node-density for both schemes (The trends in the plot are discussed in detail in later sections.).

The outage-capacity for a node-pair whose receiver is at the center of the circular region in the scenario discussed in this subsection is given by

$$C_{out} = \frac{1}{2} \log_2 (1 + \beta N_B)$$

(4.33)

### 4.6 Comparison of Narrowband and Wideband Transmission Schemes

In this section, we investigate the influence of the exclusion region and the relative performance of a narrowband (NB) transmission scheme (corresponding to $N_B = 1$) and a wideband (WB) interference-averaging-based transmission scheme (corresponding to $N_B > 1$) with a matched-filter (interference-mitigation is not employed). Again note that the NB technique is equivalent to the IA-based overlay technique and the WB technique is equivalent to the interference-averaging-based spectrum underlay technique, used for SS with legacy systems. For ease of presentation, the arguments in this section use analytical expressions evaluated in the limit $r_{\max} \to \infty$. The parameters used for the numerical analysis in this section, unless otherwise mentioned, are as follows: The path-loss exponent $\alpha$ is assumed to be 3. The minimum physical distance between two nodes, $r_{\text{phy}} = 0.1 \text{m}$. The maximum radius of the circular region, $r_{\max} = 1000 \text{m}$. The outage probability is $1e - 2$.

#### 4.6.1 Discussion of Interference Statistics

As discussed in the previous section, the sum-interference power at the central receiver while using random spreading codes and a matched filter is modeled using a log normal random variable. The parameters of
Figure 4.4: The actual and approximate (using the Gaussian approximation) mean and “mean - a single standard deviation” of the SINR distribution at the output of a MMSE receiver. $r_{\text{min}} = 1\text{m}$, $r_{\text{max}} = 50\text{m}$ and $N_B = 128$.

Figure 4.5: Simulated and theoretical outage probability curves that show the variation of outage capacity with the density of nodes in the circular region. Outage probability $= 1e^{-4}$. 
the corresponding log normal distribution can be found by using the relationships in (4.16) and substituting from (4.11) and (4.12) to get,

$$\sigma^2 = \ln \left( \frac{m_2}{m_1} + 1 \right) = \ln \left( \frac{1 + h(\alpha) 2\lambda \pi r_{\text{min}}^2}{h(\alpha) 2\lambda \pi r_{\text{min}}^2} \right)$$

(4.34)

and

$$m = \frac{m_1}{\exp(\sigma^2/2)} = \frac{2\lambda \pi P_x}{h(\alpha) 2\lambda \pi r_{\text{min}}^2} \sqrt{\frac{h(\alpha) 2\lambda \pi r_{\text{min}}^2}{1 + h(\alpha) 2\lambda \pi r_{\text{min}}^2}}.$$  

(4.35)

Here, $h(\alpha) = \frac{(2\alpha-2)}{(\alpha-2)}$. The outage probability given in Equation (4.17) can be expressed as follows:

$$\frac{1}{2} \text{erf} \left( \frac{\ln(P_i/m)}{\sigma \sqrt{2}} \right) = \frac{1}{2} - P_o$$

(4.36)

Let $A = \sqrt{2} \text{erf}^{-1}(1 - 2P_o)$. Then,

$$A\sigma + \ln(m) = \ln(P_i)$$

(4.37)

Using the notation, $x = h(\alpha) 2\lambda \pi r_{\text{min}}^2$, the above expression can be written as

$$A \sqrt{\ln \left( \frac{1 + x}{x} \right)} + \ln \left( \frac{x \sqrt{x}}{\sqrt{1 + x} (\alpha - 2) h(\alpha) r_{\text{min}}^\alpha} \right) = \ln(P_i).$$

(4.38)

From Equation (4.18), it can be seen that for a fixed outage probability (and hence for fixed $A$), $P_i$ is inversely proportional to the outage capacity and can be used as a measure of the interference-power in the system. Therefore the above expression (Equation (4.38)), can be used to analyze the impact of various parameters on the outage-capacity of a node in the system.

We first analyze the effect of density $\lambda$. Note that the first term of the above equation is a monotonically decreasing function of $x$ (which is in turn directly proportional to $\lambda$) for positive $x$ (as is the case here) and the second term is a monotonically increasing function of $x$ for positive $x$. Also note that the second term dominates the first term. Hence, as can be expected, the sum-interference power increases and consequently outage-capacity decreases with an increase in node-density $\lambda$ in the circular region.

Next, we consider the radius of the exclusion region $r_{\text{min}}$. Note that $x$ is directly proportional to $r_{\text{min}}^2$. However, as compared to the previous scenario, the second term has an additional factor of $\frac{1}{r_{\text{min}}^\alpha}$ with $\alpha > 2$, making the second term also a monotonically decreasing function of $r_{\text{min}}$. Hence both terms decrease with an increase in $r_{\text{min}}$, resulting in a decrease in sum interference power and consequently an increase in outage capacity with an increase in the size of the exclusion region.

Now, consider the power scaling factor $P_x$. Recall that for a spreading factor, $N_B$, the power scaling factor is $\frac{1}{N_B}$. The factor only appears in the numerator of the second term. Hence, when the powers of the interfering nodes in the network gets scaled up, as is intuitive, the interference power at the central receiver also increases, resulting in a decrease in outage-capacity of the central receiver. It can also be inferred from the above discussion that outage-capacity of a node is impacted most by a change in the exclusion radius, followed by a change in the power scaling factor and finally, by a change in the density of nodes (As also shown in Chapter 3).

Now consider the effect of the path-loss exponent on the outage-capacity of a node. Equation (4.38) can be re-written in terms of the SINR threshold $\gamma$ corresponding to the interference threshold $P_i$ as follows:

$$A \sqrt{\ln \left( \frac{1 + x}{x} \right)} + \ln \left( \frac{x \sqrt{x}}{\sqrt{1 + x} (\alpha - 2) h(\alpha) r_{\text{min}}^\alpha} \right) = \ln \left( \frac{P_i}{\gamma} \right)$$

(4.39)
The terms \((\alpha - 2)\) and \(h(\alpha)\) are monotonically increasing functions of \(\alpha\). Hence when the radius of the exclusion-region, \(r_{\text{min}} = r_s\), SINR increases with an increase in the path-loss exponent, in turn resulting in an increased node-density (when the target outage-capacity and hence SINR is fixed). However, when \(r_{\text{min}} \neq r_s\), SINR depends upon the relative values of \(r_s\) and \(r_{\text{min}}\). This trend is illustrated in Figure 4.6 which plots the variation of transmission-density that can be accommodated vs the exclusion radius for a given outage-probability and outage-capacity (and hence fixed SINR) and different path-loss exponents.

It is seen that when the value of \(r_s\) is less than \(r_{\text{min}}\), transmission node-density increases with \(\alpha\) since the interference power is reduced for greater path-loss exponents. However when the value of \(r_s\) is much larger than the value of \(r_{\text{min}}\), node density decreases with increasing path-loss exponent. This effect can be attributed to the fact that as the path-loss exponent increases the power received from the intended transmitter also reduces. Since \(r_s >> r_{\text{min}}\), the decay of the power received from the intended transmitter occurs at a faster rate than the decay of the interference power.

### 4.6.2 Existence of an Optimal Exclusion Radius

Consider expression (4.38). It can be seen that the sum interference power decreases with an increase in the exclusion radius. Alternatively, if the SINR (and therefore \(P_t\)) required for a node-pair transmission is kept constant, it can be seen that an increase in exclusion radius increases the density of nodes \(\lambda\) that can be accommodated in the system for a specific outage-probability. However from expression (4.2), it can be seen that the density of nodes decreases with an increase in exclusion radius due to the physical constraint in the number of circles of a given radius that can be accommodated in a given area. Hence for a given outage-probability and outage-capacity for a node-pair, there exists an optimum exclusion radius that maximizes the transmission-density of the network. Note that the optimum exclusion-radius can be numerically solved for by optimizing Expression (4.38) over \(r_{\text{min}}\) subject to the condition in Expression (4.2).

The results in the above discussion are re-iterated in Figure 4.7 which plots the transmission-density
Figure 4.7: Transmission-density vs normalized radius of exclusion region for varying spreading factor $N_B$. $P_o = 10^{-2}$, $SINR = 3$ (corresponding to 1 bit/ Hz/ sec) and $\alpha = 3$.

(density per unit bandwidth) vs the exclusion radius for a fixed outage-probability, a fixed outage-capacity and for different spreading factors. It is seen that there exists an optimal value of the exclusion radius that maximizes transmission-density. Note that there are two regions in the performance plots: the first, where node-density increases with increasing radius of the exclusion region ($r_{\text{min}}$) and the second, where node-density decreases with $r_{\text{min}}$. In the first region, the decrease in sum-interference-power, due to increase in exclusion-radius, allows a larger number of interferers to be accommodated in the same region for fixed values of outage-probability and SINR threshold. In the second region, the increase in SINR, due to the increase in exclusion-radius, cannot be compensated for by an increase in transmission-density since transmission-density is limited by the number of nodes that can be physically accommodated in a given area ($\frac{\lambda_m(r_{\text{min}})}{2N_B}$).

Now, consider the relationship between exclusion-region and spreading factor. It can be seen from (4.38) that an increase in the spreading factor, for a given exclusion-radius, reduces the sum-interference power at the receiver and hence correspondingly increases the outage-capacity. If the outage-capacity is fixed, the increase in SINR due to an increase in the spreading factor can be compensated for by increasing the transmission-density. However, this is subject to the condition that the increased transmission-density can be physically accommodated in unit area. Hence, in the “first region”, transmission-density can be expected to increase with an increase in spreading-factor. In the “second region”, transmission density for a given spreading factor is limited by $\frac{\lambda_m(r_{\text{min}})}{2N_B}$. Hence, in this region, transmission-density can be expected to decrease with an increase in spreading-factor. This results in the optimum transmission-density decreasing with increase in spreading-factor. These trends are illustrated in Figure 4.7.

In the second region, as mentioned before, an increase in SINR cannot be compensated for by an increase in transmission-density. This results in the actual achievable SINR, in this region, being much greater than the threshold value. This is illustrated in Figure 4.8 which shows that actual achieved SINR when the system is designed for a target SINR of 3 (4.77dB). Therefore, in the “second region”, transmission-density might not be a useful measure of performance. An alternate approach is to analyze the maximum sum-outage-
capacity of the network (defined as the outage-capacity of a node multiplied by the transmission-density). This is discussed in the next section.

4.6.3 Sum-outage-capacity for Fixed Exclusion Radius

Using the expression for the sum-interference power $P_i$ (4.38), the outage-capacity of the central node-pair, when the spreading factor of the transmission scheme is $N_B$ (i.e., with $P_x = \frac{1}{N_B}$), can be rewritten as follows:

$$C_{out} = \frac{1}{2} \log_2 \left( 1 + \frac{P_T/r_s^\alpha}{N_0 + \left( \frac{\pi x^2}{N_B \sqrt{1+x(\alpha-2)\kappa(\alpha)r_{min}}^\alpha} \right) \exp \left( A \sqrt{\ln \left( \frac{1+x}{x} \right)} \right) } \right). \quad (4.40)$$

Assuming, $\lambda$ (the density of nodes in a transmission bandwidth $N_B$) is constant, it can be observed that the interference power decreases and consequently the outage-capacity increases with an increase in the spreading factor. The total density of nodes that can be accommodated in the network for a spreading factor of $N_B$ and available bandwidth $N$ is given by $\lambda \left( \frac{N}{N_B} \right)$. Hence if $\lambda$ is kept constant, the total density of nodes that can be accommodated in the network decreases with the spreading factor. As mentioned before, the outage-capacity of the central node-pair is assumed to be representative of the performance of a single-link in an ad-hoc network with multiple node-pairs distributed in a circular region. Hence, the sum of the outage-capacities of node-pairs in a unit area of this circular region is given by

$$C_{sum, out} = \frac{\lambda N}{N_B} C_{out}. \quad (4.41)$$

It can be seen from the above discussion that increasing the spreading factor results in an increase in the SINR and hence outage-capacity of a node-pair. However, it also results in a decrease in the total density of the nodes that can be accommodated in the network. Since the decrease in density linearly decreases
the sum-outage-capacity and the increase in SINR only logarithmically increases the sum-outage-capacity, an increase in the spreading factor of the transmission scheme decreases the sum-outage-capacity of the network.

The above trends are illustrated in Figure 4.9 and Figure 4.10. The plots are generated by optimizing over the transmission density to maximize the sum-outage-capacity of a network for a fixed exclusion radius. The optimization is performed by using Equation (4.41) as the objective function and a gradient-based line search over the transmission density. Note that it is possible that the gradient-based routine finds solutions which are “locally” optimum instead of “globally” optimum. However multiple starting points are used to alleviate this problem and it is found that the same solution is obtained in all scenarios. The obtained solutions also agree with trends expected from the analytical expressions. It is seen that maximum sum-outage-capacity of the network decreases while the outage-capacity of an individual node-pair increases with an increase in the spreading factor. The optimization usually leads to a solution where the density of nodes that can transmit in the same transmission dimension (\(\lambda\)) is equal to the maximum possible density for a given exclusion-radius, i.e. is given by \(\frac{\lambda_m}{N}\). Hence transmission-density (given by \(\frac{\lambda_m}{N}\)) decreases with spreading factor. This is illustrated in Figure 4.11. Since, as discussed above, variation in density has a larger impact than a corresponding variation in outage-capacity, sum-outage-capacity decreases with increase in \(N\).

Note that Figure 4.9 illustrates the existence of an optimal-exclusion-region with respect to maximizing the sum-outage-capacity in the network as well.

### 4.6.4 Sum-outage-capacity for Fixed Transmission Density

We now analyze the performance trend with varying spreading factors when the total density of nodes (denoted by \(\lambda_{total}\)) in the network is kept constant for all spreading factors. Therefore, if the spreading factor is \(N_B\), the density of nodes in the same transmission dimension (of size \(N_B\)) is given by \(\lambda = \frac{\lambda_{total}}{N_B}\). To maximize SINR, the radius of the exclusion radius is set to the maximum allowable value for a given density.
Figure 4.10: Outage-capacity per node corresponding to the optimized sum-outage-capacity in Figure 4.9 vs normalized radius of exclusion region. $P_0 = 10^{-2}$ and $\alpha = 3$.

Figure 4.11: Node-density corresponding to the optimized sum-outage-capacity in Figure 4.9 vs normalized radius of exclusion region. $P_0 = 0.01$ and $\alpha = 3$. 
of nodes in a transmission bandwidth. Therefore, if the spreading factor is $N_B$, the radius of the exclusion region is given by $r_{N_B} = \sqrt{\frac{1}{\pi 2\lambda}} = \sqrt{\frac{N}{\pi 2\lambda_{total} N_B}}$ (assuming $\lambda_m \approx \lambda_{max}$). It is thus seen that for a fixed total density of nodes, the possible size of the exclusion region decreases with an increase in the spreading factor. On the other hand, the interference power at the central receiver also gets scaled down by a larger term, $P_x = \frac{1}{N_B}$, with an increase in the spreading factor. Now, consider expression (4.38). It can be seen that the interference power at the central receiver is affected more by the scaling of the exclusion-radius by $\sqrt{\frac{1}{N_B}}$ than the scaling of transmit-power by $\frac{1}{N_B}$. Therefore, it can be concluded that the interference power at the central receiver, for a fixed total density of nodes, increases with the spreading factor. This results in a decrease in the outage-capacity of individual node-pairs with an increase in the spreading factor. This is illustrated in Figure 4.12, Figure 4.13 and Figure 4.14. Since the total density of nodes in the network is fixed, the sum-outage-capacity of the network (product of total density of nodes and outage-capacity) also decreases with an increase in spreading factor.

4.6.5 Summary of Performance Trends

The discussion in the previous subsections show that outage-capacity for a single node is influenced most by the exclusion radius, followed by the scaling of the transmit power of interfering radios and finally, by the the density of interfering radios. On the other hand, the sum-outage-capacity of a network is influenced most by the density of nodes that can be accommodated in the network followed by the outage-capacity of individual node-pairs (or links). This leads to the following two performance trends:

1. When a fixed total density of nodes is assumed in the network, sum-outage-capacity decreases with an increase in spreading factor. This is due to the fact that though the increase in spreading factor increases the factor by which the transmit power of interfering radios is scaled down, it also decreases the possible size of the exclusion region in the network, resulting in a decrease in the outage-capacity.

2. When a fixed exclusion-radius is assumed, sum-outage-capacity of the network decreases with in-
Figure 4.13: Variation of outage capacity with the density of nodes in the circular region. Outage probability $= 1e^{-2}$.

Figure 4.14: Variation of outage capacity with the spreading factor for the WB-MF scheme. Note that $N_B = 1$ corresponds to the NB interference avoidance scheme. $r_{min} = 1m$ and density is $\frac{1}{2\pi}$ per unit bandwidth.
crease in spreading factor. This is due to the fact that though an increase in spreading factor increases
the outage-capacity of a node-pair (due to the scaling down of interference power), it leads to a de-
crease in the total density of nodes that can be accommodated in the network and hence leads to a
decrease in the sum-outage-capacity of the network.

It is thus seen that narrowband ($N_B = 1$) SS (in other words, IA-based overlay) schemes are preferable
with respect to improving the performance of a network as well as improving the performance of individual
nodes (when the network is of fixed size). The performance trends are primarily a result of the size of the
exclusion-region: smaller exclusion-radius increases interference while a large exclusion region limits the
number of nodes that can be accommodated. However, interference-mitigation techniques for spreading-
based systems have the ability to avoid the strongest interferers in the system and thus increase effective
distance between an interferer and the receiver (in other words, virtually increase the size of the exclusion
region). The impact of these techniques on the performance trends are analyzed in the next section.

4.7 Performance Comparison of NB Scheme and WB Scheme with MMSE
Receiver

In this section we include the effect of interference-mitigation techniques, specifically the effect of using a
MMSE filter, on the performance of the spreading-based SS techniques. The relative performance of the
following three SS techniques are therefore analyzed in this section: NB SS technique (referred to as the
NB scheme), WB spreading-based SS technique with a matched filter (referred to as the WB-MF scheme)
and WB spreading-based SS technique with a MMSE receiver (referred to as the WB-MMSE scheme). The
system parameters used for numerical analysis are the same as those used in the previous section. For the
WB-MMSE scheme, we use expressions from Section 4.5.2.

4.7.1 Effect of Exclusion Radius

The variation of the performance of the three SS schemes with respect to the radius of the exclusion-region
is investigated here. The total density of nodes ($\lambda_{total}$) to be accommodated in the network is kept constant
across the considered values of the exclusion radius and is fixed at some value which is less than the al-
lowable density for the largest value of the exclusion region and the considered spreading factor. In other
words, $\lambda_{total} < \frac{1}{2\pi r_{min}} \forall r_{min}$. Figure 4.15 plots the outage-capacity achieved by the three SS techniques.
It is seen that the performance of the NB scheme and the WB-MF scheme deteriorate with a decrease in
the exclusion region, as is expected. However, the performance of the WB-MMSE scheme is relatively
unaffected by a variation in the exclusion region. This result can be ascribed to the fact that the MMSE
receiver rejects the strongest interference components (which are dependent on the radius of the exclusion
region) and the performance of the receiver is mostly dependent on the smaller interference components
from far-away transmitters.

4.7.2 Sum-outage-capacity for Fixed Transmission Density

The total density of nodes in the network ($\lambda_{total}$) is kept constant for all spreading factors. As before, to
maximize the SINR for a given density of nodes, the radius of the exclusion radius is set to the maximum
allowable value. Therefore, in the case of the NB scheme, $r_{min,NB} = \sqrt{\frac{N}{\pi 2\lambda_{total}}}$, and in the case of the
WB scheme, $r_{min,WB} = \sqrt{\frac{N}{\pi 2\lambda_{total} N_B}}$. However, for the WB interference-averaging scheme with the
MMSE receiver, since (as seen in the previous subsection) the performance is relatively unaffected by the
value of \( r_{\text{min}} \), we include the case where the inner radius is set to \( r_{\text{phy}} \) (the minimum possible physical distance between two nodes) in addition to the case where the inner radius is equal to \( r_{\text{min},WB} \). Note that, \( r_{\text{phy}} \leq r_{\text{min},WB} < r_{\text{min},NB} \).

Figure 4.15 shows the outage probability curves for the three different schemes. It is seen that at high outage probabilities, the NB scheme outperforms the other two schemes. However, for smaller outage probabilities, the WB-MMSE scheme outperforms the other schemes. This trend is re-iterated in Figure 4.12 and 4.13 which show the outage-capacities for the users over different densities of the nodes. This trend can be explained from the fact that the MMSE receiver rejects the strongest components and averages over the remaining interference. Hence the mean of the interference power might be larger than that of the NB scheme (due to the fact that interference from all the nearest radios are completely rejected in the NB scheme) but the variance of the interference power might be smaller due to the averaging of the interference power from interfering radios which are further away. It can also be observed from the figures that, as also seen in the previous section, the performance of the WB-MMSE scheme is relatively insensitive to the value of the exclusion radius.

Also note that the performance of the WB-MF scheme is substantially worse than that of the other two schemes. As discussed earlier, this is due to the fact that since \( r_{\text{min},WB} < r_{\text{min},NB} \) and since no interference-mitigation technique is used, stronger un-rejected nearby interferers are present in the WB system as compared to in the NB or WB-MMSE schemes. The interference power from these interferers negates the improvement provided by averaging interference and significantly deteriorates the performance of the scheme compared to other two schemes.

Now, consider the variation of the relative performances of the three schemes with an increase in spreading factor. As discussed earlier (and illustrated in Figure 4.14), the performance of the WB-MF scheme deteriorates with an increase in spreading factor. Note that a spreading factor of one corresponds to the NB scheme. We now consider the performance of the WB-MMSE scheme. Figure 4.17 shows the variation of the outage-capacity of the central node with the spreading factor for different outage probabilities at the central receiver. It is seen that the WB-MMSE scheme performs substantially better than the WB-
Figure 4.16: The complement of the CDF function vs the interference power at the central receiver (the outage probability in other words). $r_{\text{min}} = 1$m and density is $\frac{1}{2\pi}$ per unit bandwidth.

Figure 4.17: Variation of outage capacity with the spreading factor for the WB-MMSE scheme. Note that $N_B = 1$ corresponds to the NB interference avoidance scheme. $r_{\text{min}} = 1$m and density is $\frac{1}{2\pi}$ per unit bandwidth.
MF scheme. In addition, except at very high outage-probabilities, the WB-MMSE scheme (with sufficient spreading) performs better than the NB scheme. These results can be attributed to the fact, as mentioned before, the WB-MMSE scheme is unaffected by variations in $\lambda_{\text{min}, W B}$ (since the MMSE filter mitigates the interference from the strongest interferers unlike the MF filter in the WB-MF scheme) and that the increase in spreading factor increases the averaging of the interference from far-away interferers. The latter effect reduces the variance of the interference-power distribution. It can also be seen from the figure that increasing the spreading factor increases the range of outage-probabilities over which the WB-MMSE scheme out-performs the NB scheme.

### 4.7.3 Sum-outage-capacity with Fixed Exclusion Radius

As noted earlier, for a fixed exclusion-region, though an increase in spreading factor increases the outage-capacity of a node-pair (due to the scaling down of interference power), it leads to a decrease in the total density of nodes that can be accommodated in the network and hence leads to a decrease in the sum-outage-capacity of the network. However, additional gain is provided due to the use of interference-mitigation techniques in the WB-MMSE scheme, which could offset the loss due to decrease in the total density of nodes in the network with an increase in spreading factor. Figures 4.18 and 4.19 plot the transmission density and the outage-capacity respectively for different exclusion-radii. For the NB and the WB schemes, the plots are generated by optimizing over transmission-density to maximize the sum-outage-capacity of a network for a fixed exclusion radius using a gradient-based line search as before. For the WB-MMSE scheme, the plots are generated by assuming $\lambda_{W B} = \frac{\lambda}{2N_B}$ instead of optimizing for the sum-outage-capacity over the density of nodes. Hence the generated sum-outage-capacities and outage-capacities are less than or equal to the maximum achievable values. From Figure 4.19, it can be seen that the improvement in outage-capacity provided by the WB-MMSE scheme is large compared to the improvement provided by the WB-MF scheme.

Figure 4.20 illustrates the maximum achievable sum-outage-capacity. It can be observed that the increase in outage-capacity for the WB-MF scheme is not large enough to compensate for the loss in sum-capacity due to the reduction in the density of nodes with respect to the NB scheme. This results in lower sum-capacity for the former scheme as compared to the latter scheme. However, when an MMSE receiver is
Figure 4.19: Outage-capacity at the central-receiver corresponding to the maximization of sum-capacity. Outage probability = $1e - 2$.

Figure 4.20: Optimized sum-outage-capacity vs normalized exclusion radius. Outage probability = $1e - 2$. 
incorporated with the WB scheme, it is seen that the increase in outage-capacity effectively compensates for the loss in node-density and provides an increase in sum-outage-capacity as compared to the NB scheme.

4.7.4 Impact of Fading

The analysis in the previous sections only considered the effect of the exponential path-loss on the interference-power statistic. In this section, we incorporate the effect of log normal shadowing. Performance trends are generated using simulations of the network since it was found that the log normal approximation does not adequately model distribution of interference power at the central receiver when the effect of log normal shadowing is included in the analysis. The effect of small-scale shadowing was also investigated. However, it was found that sum-interference power at the central receiver is dominated by log normal shadowing and the inclusion of small-scale fading does not significantly alter the relative trends discussed here.

Figure 4.21 plots the outage-capacity of the NB scheme and the WB-MMSE scheme when the total density of nodes in the network is kept constant and the radius of the exclusion radius is fixed in a manner similar to that used in Section 4.7.2. It is seen that the WB-MMSE scheme performs better than the NB scheme for outage-probabilities of $1e^{-1}$ and $1e^{-2}$. Recall that when log normal shadowing was not included in the analysis the WB-MMSE scheme outperformed the NB scheme only at the lower outage-probability. Therefore the inclusion of log normal shadowing accentuates the performance improvement of the WB-MMSE scheme over the NB scheme. This can be explained by the fact that the inclusion of log normal shadowing increases the variance of the interference power at the receiver for both schemes. An increase in variance, in general, decreases the outage-capacity for a given outage-probability. However, the interference-averaging performed at the receiver of a WB system is able to offset the increase in variance due to log normal shadowing.
4.7.5 Summary of Performance Trends

A WB scheme which implements interference-mitigation has the ability to remove the strongest interferers in the system and thus effectively combat the near-far interference scenario that plagues WB schemes. In addition, as opposed to NB schemes, this scheme also averages the power from the unrejected interferers. Also, as discussed earlier, the poor performance of pure WB schemes are primarily the result of the exclusion region: smaller exclusion regions increase interference while larger exclusion regions reduce the density of nodes that can be accommodated in the network. The use of the MMSE receiver has the ability to remove the strongest (and hence) closest interferers, thus virtually increasing the size of the exclusion region. These factors result in the following trends, established in this section:

1. When a fixed total density of nodes is assumed in the network, the use of a WB-scheme with interference-mitigation increases the sum-outage-capacity of the network with an increase in spreading factor (as opposed to the result in [52]).
2. When a fixed total density of nodes is assumed in the network, and when a sufficient spreading factor is used in the network, WB schemes with interference-mitigation result in larger sum-outage-capacity for the network that NB transmissions schemes. The performance gap is accentuated by increase in spreading factor and decrease in the outage-probability (and hence increase in the performance of nodes) required in the network.
3. The inclusion of shadowing in the analysis, accentuates the above performance trends.
4. For a fixed exclusion-radius, the use of a WB-scheme with interference-mitigation results in a larger sum-outage capacity for the network than a NB scheme.

These results thus illustrate that a transmission scheme which removes the strongest interferers at a receiver while averaging the remaining interference is preferable with respect to improving the performance of the network as well as that of individual nodes. It is also illustrated that if such a scheme is implemented, the performance of the network (and of individual nodes) improves with increase in spreading factor. However, note that as (as shown in the previous section), if the strongest interferers at a receiver cannot be rejected, then increasing the spreading factor decreases performance. Hence the choice of spreading factor for a network is contingent on the choice of the interference-avoidance scheme to be implemented in the network.

Also note that the inclusion of interference-mitigation for WB schemes, make them relatively insensitive to the size of the exclusion radius. This fact simplifies the design of the MAC layer for this scheme since strict exclusion regions do not have to be established.

4.8 Conclusions and Contributions

In this chapter, a framework was formulated to identify desirable features for transmission schemes w.r.t. to the performance of a distributed network. The framework incorporated an exclusion-region-based MAC. Sum-outage-capacity of the network was used as the performance metric. It was found that similar to the optimal allocations in centralized networks, transmission schemes where the strongest interferers at a receiver are avoided and the power from the weaker interferers are averaged, are preferable from the perspective of a distributed network as well. The performance benefit provided by the scheme is accentuated when the effect of shadowing is included in the analysis. It was also shown that if such a scheme is implemented, the performance of the network improves with an increase in spreading factor (as opposed to the result in [52]). However, if the strongest interferers at a receiver cannot be rejected, then increasing the spreading factor
decreases performance (Similar results are shown in [52], though for an ALOHA-based MAC). Hence the choice of spreading factor for a network is contingent on the choice of the interference-avoidance scheme implemented in the network.

The existence of an optimal exclusion-radius for narrowband schemes and for wideband schemes (that do not implement interference-mitigation) with respect to maximizing the transmission-density of a network and with respect to maximizing the sum-outage-capacity of the network was also established in the chapter (It was previously suggested in [54], that an optimal exclusion radius might exist, though this was not analytically confirmed, as is done in this chapter). Larger exclusion-regions result in a decrease in the number of nodes that can be accommodated in the network while smaller exclusion-regions result in increased interference at a receiver. Using the log normal approximation for the interference statistic, expressions were formulated which could be used to numerically calculate the optimal exclusion-radius for a given spreading factor and a required outage-capacity for the nodes in the network or a required sum-outage-capacity for the network.

In addition, it was shown that the use of interference-mitigation techniques make wideband schemes relatively insensitive to the size of the exclusion region. This fact simplifies the design of the MAC layer for these schemes since strict exclusion regions do not have to be established.

Note that there is considerable synergy between the performance statistics of an individual node in the SS network and that of a legacy static radio placed in a network of SS radios (Chapter 3). However, the sum-performance of a network of SS nodes, analyzed in this chapter, is additionally complicated by the fact that the transmission strategy of a node is dependent on the transmissions of the other nodes in the network (as opposed to the legacy static radio which does not adapt to the interference from the SS system and whose transmission strategy is thus independent of the nodes in the SS system). The role of the exclusion-based MAC in modeling the interaction between nodes and the resultant impact on the sum-performance of the network is thus vital to the analysis and is what separates the analysis in this chapter from that in Chapter 3.

The original contributions in this chapter are as follows:

- A framework to investigate the relative benefits of different spectrum sharing schemes with respect to the performance of a distributed network that implements the schemes was developed. The framework incorporates an exclusion-region-based MAC, which is suggested to be the optimal MAC in distributed networks.

- Based on the framework, desirable features for spectrum sharing techniques with respect to the performance of distributed networks were identified.

- Expressions that characterize the optimal exclusion-radius for an exclusion-region-based MAC protocol were derived.

The publications that resulted from this chapter are as follows:


Chapter 5

Potential Game Model

5.1 Introduction

The waveform adaptation algorithms developed in our work are based on a potential game model. In this chapter, we briefly introduce the potential game model and discuss some relevant convergence results. We also develop a new convergence result for potential games, which is useful in constructing reduced-feedback implementations of the developed waveform adaptation algorithms.

The chapter is organized as follows: A brief introduction to game-theory is provided in Section 5.2. The potential game model is introduced in Section 5.3. Section 5.4 presents some convergence results for potential games which are useful in establishing the convergence of potential-game-based algorithms to desirable solutions. A new convergence result is developed in Section 5.5. Finally Section 5.6 concludes the chapter.

5.2 Normal Form Games

Game theory is a branch of applied mathematics that models interactions between rational decision makers with formalized incentive (preference) structures and provides tools to predict and analyze the outcome of these interactions. Game theoretic models can hence be used for the design and analysis of distributed algorithms in which individual nodes adapt their actions and contend for a common resource. A normal form game \[57\] is the most frequently used game model, where a game is represented as the following tuple:

\[
\Gamma = \langle \mathcal{K}, \{A_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}} \rangle,
\]

(5.1)

Here, \( \mathcal{K} = \{1, 2, \ldots, |\mathcal{K}|\} \) is the set of players of the game. The set of actions available for player \( k \) is denoted by \( A_k \) and the utility function associated with each player \( k \) by \( u_k \). If the set of all available actions for all players is represented by \( A = \times_{k \in \mathcal{K}} A_k \), then \( u_k : A \rightarrow \mathbb{R} \). The utility function for each player is thus a function of the actions in the game. An element \( a \in A \) is called an action profile. Player \( k \) prefers an action profile \( a \) over an action profile \( a' \) if \( u_k (a) \geq u_k (a') \). The notation \( A_{-i} \) is used to denote the product space \( \times_{k \neq i} A_k \). Let \( a_{-i} \) be an element of \( A_{-i} \).

A Nash Equilibrium (NE) for a game is an action profile from which no player can increase its utility by unilateral deviations. An action profile, \( a \in A \), is a NE if and only if

\[
u_k (a) \geq u_k (b_k, a_{-k}) \quad \forall k \in \mathcal{K}, b_k \in A_k.
\]

(5.2)
Here, \((b_k, a_{-k}) = (a_1, \ldots, a_{k-1}, b_k, a_{k+1}, \ldots, a_{|K|})\) refers to the action profile in which the action of user \(k\) is changed from \(a_k\) to \(b_k\), while the actions of all other players in the game remain the same. Nash equilibria form the steady states of the game.

Suppose that a normal form game is played repeatedly. At each stage of the game, players choose actions that improve their utility functions in a round-robin fashion. The criteria for a particular choice of action gives rise to the best and better response dynamics defined below:

1. **Best response dynamic:** At each stage, a player \(k\) deviates from \(a_k \in A_k\) to some action \(b_k \in A_k\) if \(u_k(b_k, a_{-k}) \geq u_k(c_k, a_{-k})\), \(\forall c_k \in A_k\). Note that a NE is an action profile, \(a \in A\), such that \(a_k\) is a best response for every player \(k \in K\).

2. **Better response dynamic:** At each stage, a player \(k\) deviates from \(a_k \in A_k\) if there exists an action \(b_k \in A_k\) such that \(u_k(b_k, a_{-k}) > u_k(a_k)\).

Two games with the same sets \(K\) and \(A\) are said to be best-response-equivalent if \(\forall x \in A\) and every player \(k \in K\), the best responses of both games coincide. Better-response-equivalence has a corresponding definition. The convergence properties of best and better response dynamics are identical for games which are better-response-equivalent. The properties of only the best response dynamic are identical for games which are best-response-equivalent. A more extensive overview of game theory is provided in [58].

In addition to the notations used above, the following notations are also used in this chapter. Given \(x_0 \in A\), \(N_\epsilon(x_0) = \{x \in A : d(x, x_0) < \epsilon\}\) is the neighborhood of \(x_0\), where \(d(x, x_0)\) is the Euclidean distance metric on \(A\). \(N^*_\epsilon(x_0) = N_\epsilon(x_0) - \{x_0\}\) is the deleted neighborhood of \(x_0\). Given a set, \(B \subseteq A\), \(\overline{B}\) denotes its closure and \(P(B)\) denotes its power set.

### 5.3 Potential Games

A potential game ([30], [59] and [58]) is a normal form game such that any changes in the utility function of a player in the game due to a unilateral deviation by the player is reflected in a global function referred to as the potential function. Potential games can be grouped into different types based on the relationship between the potential function and the utility functions of the players in the game. A function \(V : A \rightarrow \mathbb{R}\) is called

1. an **exact potential function** if \(\forall k \in K, a \in A\) and \(b_k \in A_k\)
   \[u_k(a) - u_k(b_k, a_{-k}) = V(a) - V(b_k, a_{-k}).\] \hspace{1cm} (5.3)

2. an **ordinal potential function** if \(\forall k \in K, a \in A\) and \(b_k \in A_k\)
   \[u_k(a) \geq u_k(b_k, a_{-k}) \Leftrightarrow V(a) \geq V(b_k, a_{-k}).\] \hspace{1cm} (5.4)

3. a **best response (BR) potential function** if \(\forall k \in K\) and \(a_{-k} \in A_{-k}\)
   \[\arg \max_{a_k \in A_k} u_k(a_k, a_{-k}) = \arg \max_{a_k \in A_k} V(a_k, a_{-k}).\] \hspace{1cm} (5.5)

A game is an exact, ordinal or BR potential game if there exists an exact, ordinal or BR potential function respectively for the game. In addition, a game is a transformable ordinal potential game if there exists an ordinal transformation \(f_k : u_k(A) \rightarrow \mathbb{R}, \forall k \in K\) such that the game \(\Gamma = \langle K, \{A_k\}_{k \in K}, \{\hat{u}_k\}_{k \in K}\rangle\) with \(\hat{u}_k(a) = f_k(u_k(a))\) is an exact potential game.
Consider a coordination game \( \Gamma = \left< K, \{ A_\parallel \| \in K, \{ V_\parallel \| \in K \} \right> \), where each player’s utility function is replaced by the potential function. Exact and ordinal potential games are best and better-response-equivalent to this coordination game. The BR potential game is best-response-equivalent to this coordination game. Hence, in general, improving each user’s utility also increases the value of a global potential function making these games easy to analyze. In addition, if the potential function is also a global network performance measure, these games give a framework where users can serve the greater good by following their own best interest, i.e., can maximize a global utility by only trying to maximize their own utilities. Note that since a potential game is best-response-equivalent to its corresponding coordination game any result for the best response dynamic of a coordination game also applies to a potential game. A similar inference holds for better-response-equivalent games. Also, the maximizers of the potential function are NE for the potential game.

5.4 Convergence Properties for Potential Games

Some convergence properties for potential games, relevant to the waveform adaptation algorithms that will developed in later chapters, are delineated in this section. A more exhaustive discussion of convergence results for Potential games can be found in [58]. A special-case of Zangwill’s convergence theorem-C [60], stated below, is used to aid the convergence analysis.

**Theorem 2 (Special Case of Zangwill’s Convergence Theorem)** Let the correspondence \( \Phi : X \rightarrow P (X) \), determine an algorithm that given a point \( x^0 \) generates a sequence \( \{ x^t \} \) through the iteration \( x^{t+1} \in \Phi (x^t) \). Let a solution set, \( S^* \subset X \) be given. Suppose

1. All points \( \{ x^t \} \) are in a compact set \( S \subset X \).
2. There is a continuous function \( V : X \rightarrow \mathbb{R} \) such that
   \begin{itemize}
   \item[(a)] If \( x \in X \) is not a solution, then \( V (x') > V (x) \) for any \( x' \in \Phi (x) \).
   \item[(b)] If \( x \in S^* \), then either the algorithm terminates or for any \( x' \in \Phi (x) \), \( V (x') \geq V (x) \).
   \end{itemize}
3. Given any convergent subsequence \( x^t \rightarrow x' \), \( t \in T \), if \( x' \) is not a solution, then there is a convergent subsequence \( \{ V (x^t') \} \) \( T' \), (Note \( T' \) need not be contained in \( T \), such that
   \[ \lim_{t \in T'} V (x^t') > V (x') \].

Then either the recursion \( x^{t+1} \in \Phi (x^t) \) arrives at a solution, or the limit of any convergent subsequence of \( \{ x^t \} \) is a solution (is in \( S^* \)).

**Proof** Given in Appendix D.

5.4.1 Best Response Convergence

Theorem 1 can be directly used to show the convergence of a potential game, with a compact action space, to the NE under a best response dynamic. In the context of this game, let \( \Phi \) represent the map from the action set to the set of all best responses after one round-robin iteration (Refer to Appendix E for a more detailed definition). Since the action space is a compact set, the points generated by the adaptation process are in a compact set and condition 1 of theorem 1 is satisfied. Let \( S^* \) be formed by the Nash equilibria of
the game. By the definition of a NE, the potential function satisfies the properties required of function $V(.)$ (conditions 2 and 3) when best response iterations are used. Condition 4 is satisfied as follows:

$$\Phi$$ is an upper-semi-continuous (u.s.c) correspondence (Shown in Appendix E). An u.s.c correspondence in a compact space has a closed graph (Proposition 9.8 in [61]). Hence the algorithm map, $\Phi$, is closed and therefore, if there exists a sequence $x' \rightarrow x'$, $t \in T$, then there exists a subsequence $x^{t+1} \rightarrow x''$, $t \in T$ such that $x'' \in \Phi(x')$. Since $x'$ is not a solution, $V(x'') > V(x')$.

Since all the conditions of Theorem 1 are satisfied a potential game with a compact action space converges to the Nash equilibria of the game under a best response dynamic.

5.4.2 Better Response Convergence

Better response dynamic refers to an update procedure where players choose actions that increase their utilities as opposed to maximizing their utilities in the best response dynamic. Theorem 2 can again be used to investigate the convergence of a potential game with a compact action space under a better response dynamic.

In the context of this game, let $\Phi$ represent a map from the action space to the set of all better responses after one round-robin iteration of an ordinal or exact potential game. As before, since the action space is compact, condition 1 is satisfied. Let $S^n$ the set of all Nash equilibria. By the definition of the NE and the better response dynamic, the potential function satisfies the properties required of function $V(.)$ in conditions 2 and 3. However, a general better response can generate an infinite sequence of points, which are not a solution and which do not satisfy condition 4. For example, consider a better response scheme in which improvements occur in increasingly smaller step sizes. Then, given any convergent sequence $x^t \rightarrow x'$, $t \in T$, all subsequences $\{x^{t+q}\}_{t \in T}$ with $q$ a positive integer, also converge to $x'$. Hence there might exist no subsequence that converges to a point $x''$ such that $V(x'') > V(x')$. The solutions set of the scheme could thus be larger than the set of Nash equilibria, i.e., the scheme could converge to sub-optimal fixed points which are not NE. Hence additional properties such are required to establish the convergence of a better response scheme to fixed points that are also NE. For example, a better response procedure with a finite minimum step size can avoid convergence to the above-mentioned sub-optimal fixed points which are not NE. Alternatively, a random better response procedure can be used. In the next section (Section 5.5), it is shown that a potential game converges under a random better response to the NE of the game.

5.4.3 Noisy Best Response Convergence

Let $\Phi : A \rightarrow P(A)$ map an action profile $x \in A$ to the set of all possible best responses after one round robin iteration (Refer to Appendix E for a more detailed definition). Since, in general, there might be more than one such best response, $\Phi$ is a correspondence [61]. In a noisy best response iteration (NBRI), the best response of each player is perturbed by bounded noise, with bound $\delta > 0$. For each $\chi \in A$, where $A$ is the action space of the game, let $z(\chi)$ be a random vector with arbitrary joint probability density function $p_z(z; \chi, \delta)$. It is required that $p_z(z; \chi, \delta)$ be positive everywhere on $N_\delta(\chi)$ and zero outside $N_\delta(\chi)$. For instance, $p_z(z; \chi, \delta)$ could be a uniform distribution on $N_\delta(\chi)$. The $\delta$-NBRI is defined below:

Given a noise bound $\delta > 0$ and $x[0] \in A$, for each round robin iteration, $t$ (a positive integer),

1. Choose $\chi[t] \in \Phi(x[t])$
2. $x[t+1] = z(\chi[t])$

Here, $\chi[t]$ is the sequence of chosen best responses. The index $t \in \mathbb{N}_0$ indexes one round robin iteration.

Let $F_{\Phi, \Gamma}$ be the fixed points for a best response dynamic of $\Gamma$ or equivalently the Nash Equilibria of $\Gamma$. 

A continuous function \( V : A \to \mathbb{R} \), on a compact set \( A \) is called Nash Separable if:

1. there are no suboptimal local maxima on \( A \).
2. its maximum \( V_{\text{max}} \) is isolated from the image of other fixed points i.e., \( \exists \epsilon_m > 0 : N_{\epsilon_m} (V_{\text{max}}) \cap V (F_{\Phi, \Gamma}) = \emptyset \).
3. best response iterations are strictly improving in a neighborhood of the maximum i.e., \( \forall \epsilon > 0 \), with \( \epsilon < \epsilon_m \),
   \[ \Phi (V^{-1}([V_{\text{max}} - \epsilon, V_{\text{max}}])) \subset V^{-1}([V_{\text{max}} - \epsilon, V_{\text{max}}]). \]

A potential game is NS if its potential function is NS. The following theorem is a generalization of Theorem 8 in [62] for the \( \delta - \text{NBRI} \). The following theorem says that even if a NS game has suboptimal NE, arbitrarily small noise will asymptotically ensure the convergence of the game to a neighborhood of the global optima.

**Theorem 3** Consider a NS potential game with potential function \( V \). Then \( \forall \epsilon > 0, \exists \delta_0 > 0 \), such that \( \forall \delta \) with \( 0 < \delta < \delta_0 \) and \( \forall x \in A \), the \( \delta - \text{NBRI} \) with iterates \( \{x[t]\} \) obeys \( \lim_{t \to \infty} \inf_{\text{a.s.}} V(x[t]) \geq V_{\text{max}} - \epsilon \).

**Proof** Given in [63].

The theorem says that even if a NS game has suboptimal NE, arbitrarily small noise will asymptotically ensure the convergence of the game to a neighborhood of the global optima.

### 5.5 Random Better Response Convergence in Potential Games

A random better response dynamic refers to an update procedure, where a player select an action with uniform probability from the set of available actions that increase its utility function. In this section, it is established that, while following a random better response dynamic, an ordinal or exact potential game with a compact action space converges to the NE of the game.

In the context of the game, let \( \Phi \) represent a map from the action space to a random better response after one round robin iteration of an ordinal or exact potential game (a detailed definition of a random better response iteration is given in Appendix E.). Again, as before, conditions 1, 2 and 3 are satisfied. The following theorem illustrates condition 4.

**Theorem 4** Consider an ordinal or exact potential game with potential function \( V \) and a compact action space. Given a point \( x^0 \), let the random better response algorithm generate a sequence \( \{x^n\} \), where \( x^n \) is the output of the \( n \)th round-robin iteration. Then, if there exists a sequence \( \{x^n\}_{n \in \mathbb{N}} \) such that \( x^n \to x \), \( \text{Pr} \left[ x \notin S^* \right] = 0 \). Suppose that \( x \notin S^* \). Then there exists at least one player \( i \) with an action \( x'_i \) such that \( V(x'_i, x_{-i}) > V(x) \). Let \( \epsilon = V(x'_i, x_{-i}) - V(x) \). By theorem 4.19 in [64], a continuous function in a compact metric space is also uniformly continuous. Since the parameter space we consider is compact, the continuous function \( V(\cdot) \) is also a uniformly continuous function. Then, from the definition of a uniformly continuous function, there exists a \( \delta \) such that \( d(V(y), V(z)) < \epsilon \) for all \( y, z \) such that \( d(y, z) < \delta \). Here, \( d(\cdot) \) is a distance measure on a metric space, for example, the Euclidean distance.

\( x^n \) is the action-profile at the end of the \( n \)th round-robin iteration. The round-robin iteration is defined such that the the \( j \)th player takes the \( j \)th turn to update. Let \( \bar{x}^n \) be the action profile in the \( (n+1) \)th iteration.
after the \((i - 1)^{th}\) player has updated. Then \(\tilde{x}^n = [x_{i-1}^{n+1}, \ldots, x_{i-1}^{n+1}, x_i^n, x_{i+1}^n, \ldots, x_K^n]\). At the \(n^{th}\) iteration player \(i\) updates to an action \(\pi_i\) if \(V(\pi_i, \tilde{x}_{-i}^n) > V(x_i^n, \tilde{x}_{-i}^n)\).

Since \(x^n \to x\), there exists a \(N\) such that for all \(n > N\), \(d(x^n, x) < \frac{\delta}{2K}\). It follows that \(d(x^{n+1}, x) < \frac{\delta}{2K}\). Consequently, \(d(x^n_k, x_k) < \frac{\delta}{2K}; \forall k \in \{1, \ldots, K\}\) and \(d(x^{n+1}_k, x_k) < \frac{\delta}{2K}; \forall k \in \{1, \ldots, K\}\). Therefore, \(d(\tilde{x}_{-i}^n, x_{-i}) < \frac{\delta}{2}\).

Now, for all \(n > N\) and all \(\tilde{x}_i^n\) such that \(d(\tilde{x}_i^n, x_i^n) < \frac{\delta}{2}\), we have,

\[
d \left( \left( \tilde{x}_i^n, \tilde{x}_{-i}^n \right), (x_i^n, x_{-i}) \right) \leq d \left( \left( \tilde{x}_i^n, \tilde{x}_{-i}^n \right), (x_i^n, \tilde{x}_{-i}^n) \right) + d \left( \left( \tilde{x}_i^n, \tilde{x}_{-i}^n \right), (x_i^n, x_{-i}) \right) \]
\[
< \frac{\delta}{2} + \frac{\delta}{2} \]
\[
= \delta.
\]

Therefore, \(d \left( V(\tilde{x}_i^n, \tilde{x}_{-i}^n), V(x_i^n, x_{-i}) \right) < \epsilon\). Thus

\[
V(\tilde{x}_i^n, \tilde{x}_{-i}^n) > V(x) \quad (5.8)
\]

It follows that for all \(n > N\), there exists a neighborhood around \(x_i^n\) in which \(V(\tilde{x}_i^n, \tilde{x}_{-i}^n) > V(x)\).

Let \(\mu\) be the measure of the ball of radius \(\frac{\delta}{2}\) around \(x_i^n\) and let \(\mu_{tot}\) be the full measure of player \(i\)'s strategy space. Then for each \(n > N\) when \(i\) makes a choice, the player will choose to move to a point \(\tilde{x}\) with \(V(\tilde{x}) > V(x)\) with probability \(\frac{\mu}{\mu_{tot}} = p\). Given the present position, each of the choices is independent of the others (The system evolves as a Markov Chain.). Therefore as \(x^n \to x\),

\[
\sum_{n>N} \Pr(\left( x^n > V(x) \right)) > \sum_{n>N} p = \infty \quad (5.9)
\]

By using the second Borel-Cantelli lemma, \(\Pr(\left( x^n > V(x) \right)) = 1\). This contradicts the fact that \(x^n \to x\). Hence the probability that \(x^n \to x\) where \(x \notin S^*\) is zero. Therefore, if \(x^n \to x\), \(\Pr[ x \in S^* ] = 1\).

It is thus shown that a convergent sequence generated by a random better response dynamic almost surely converges to a NE \((S^*)\) for an exact or ordinal potential game with a compact action space.

### 5.6 Conclusions

In this chapter, we briefly introduced the potential game model and described its properties. We also discussed some convergence results for potential games which could be useful in evaluating the performance of waveform adaptation techniques, developed in later chapters.

The original contribution in this chapter is as follows:

- A new convergence result for potential games with respect to the random-better response dynamic was established. This result is found (in later chapters) to be useful in constructing reduced-feedback implementations for waveform adaptation algorithms.
Chapter 6

Game-theoretic framework for Waveform Adaptation in Centralized Networks

6.1 Introduction and Problem Statement

The investigation of desirable features for DSS in Chapters 3 and 4 show that spreading-based underlay techniques which also implement some form of interference avoidance are beneficial with respect to both the impact on the legacy system and the performance of a SS network. An example of such a technique is a spreading-sequence-based scheme that implements waveform (sequence) adaptation for IA.

In this chapter, we investigate a game-theoretic framework that could be used for the construction of waveform adaptation (WA) algorithms in networks with centralized or co-located receivers. Various algorithms for WA in centralized networks have been previously proposed in the literature. The game-theoretic framework accommodates most of this literature. In addition, the game-theoretic framework enables the formulation of some new results for WA. This study is used as the starting point for the development of WA algorithms for distributed networks in subsequent chapters.

6.2 Related Literature and Motivation

The channel characteristics of networks with co-located receivers are similar to multiple access channels which consider multiple access communication to a centralized receiver. Consequently, a majority of the capacity results on multiple access channels is directly relevant to networks with co-located receivers. The maximum sum capacity of multiple access channels was characterized in [65] and properties of optimal sequence multisets for synchronous CDMA systems were identified. It was shown that when the signals received from users have equal powers, sum capacity is achieved by real-valued signature sequences for users that satisfy the Welch Bound Equality (WBE). A characterization of maximum sum capacity and optimal sequences for asymmetric received power constraints was given in [20]. It was shown that when no user is oversized (a user is oversized if its received power constraint is large relative to the received power constraints of the other users), the optimal sequences are WBE sequences. However, if oversized users are present in the system, an optimum assignment of sequences is achieved by allocating orthogonal sequences to oversized users and WBE sequences to non-oversized users. A recursive algorithm for constructing optimal sequences was also outlined in the paper. It was shown in [66] that the sequences identified above are optimal even when user capacity (defined as the maximum number of users per unit processing gain such that the quality-of-service (QoS) requirement of each user is satisfied) is used as the performance metric.
In [21], an iterative algorithm that is amenable to a distributed implementation to generate the optimal signature sequences was proposed. In this algorithm, each user sequentially replaces its signature sequence with its normalized Minimum Mean Square Error (MMSE) filter. This algorithm (called the MMSE iteration) is shown to iteratively decrease the total squared correlation (TSC) of the sequence set and empirically converge to a set of optimum sequences when starting from random initial sequences. In [62], the fixed points of this algorithm were studied. It was shown that the sub-optimal fixed points of the MMSE iteration are not stable and hence the introduction of small perturbations, by the addition of noise at the end of each iteration, guaranteed almost sure convergence to the optimal sequence set. In [14], a general signal space formulation of the WA problem was described and an alternate distributed algorithm based on eigen-iterations is proposed. The fixed points of this algorithm were studied in [67] and a technique called class warfare was used to ensure convergence to the optimal set of sequences. However, unlike the approach in [62], this technique is not amenable to a distributed implementation.

The WA algorithms were extended to multi-carrier systems in [68] and to multi-cell systems in [28] and [69]. Other extensions of WA algorithms include adaptations in asynchronous CDMA systems [70] and multipath channels [71] [72]. Feedback is a significant issue in the implementation of distributed WA algorithms since the signature sequence can be calculated only at the receiver. Hence the real-valued sequence needs to be fed back to the user in each iteration. This could lead to a large increase in the network overhead. A reduced feedback mechanism was investigated in [73] where user’s sequence is restricted to a subspace of the sequence’s original signal space. However, this scheme might not lead to the optimal sequences. A different approach is to quantize the real-valued signal ([74] and [75]), thereby reducing the feedback information.

The WA problem was cast as a potential game (described in Chapter 5) in [27]. It was shown that the negated TSC is a potential function for a large class of WA algorithms that includes some new algorithms where users do not need to have identical utility functions. A new convergence property (described in Chapter 5) which establishes the convergence of the best response iterations of a class of potential games to the global solution when noise is added in a manner similar to [62] was derived. This result was used to show convergence of a subset of WA games (WA in levelable signal environment) to a neighborhood of the globally optimum solution.

In this chapter, we review the game-theoretic framework developed in [27]. In addition, we show a new result which can be used to establish the convergence of a larger subset of WA games (as compared to the subset of WA games considered in [27]) to a neighborhood of the globally optimum solution. We also use the new random-better response result derived in Chapter 5, to develop feedback schemes for WA that reduce the adaptation overhead in the network.

The rest of the chapter is organized as follows: The system model for the network under consideration is described in Section 6.3. In Section 6.4, the WA problem is cast as a potential game and the Nash equilibria of the game are identified. The best response and better response convergence of the WA game are investigated in Section 6.5. In Section 6.6, it is shown that noisy best response iterations lead to almost sure convergence to the global optimum for certain classes of WA algorithms. Reduced feedback mechanisms based on better response iterations are investigated in Section 6.7. Finally Section 6.8 summarizes the results and original contributions in this chapter.

### 6.3 System Model

We consider a network where multiple user-nodes communicate with receivers which are co-located. Interference caused to the transmission of a particular user-node due to other user transmissions is influenced by the correlation between the waveforms of users, transmit power levels and the channel characteristics.
We use a signal space characterization to represent the waveforms of nodes [14]. This signal space representation specifies the waveform of a node in orthogonal signal dimensions (for instance, time, frequency, or spreading code) and is referred to as the signature sequence of the node. Let $N$ be the number of signal dimensions available for transmission and $K$ be the number of transmitting nodes in the network. We denote the $N$-dimensional signature sequence associated with transmitting node $k$ by $s_k$. The signature sequences are real valued (as opposed to bi-polar sequences). Without loss of generality, the signature sequences are assumed to have unit norm and hence are constrained to the $N - 1$ dimensional sphere $S = \{ s_k \in \mathbb{R}^{N \times 1} : \|s_k\|^2 = 1 \}$. The received power level of the $k^{th}$ transmit node at its receiver is denoted by $p_k$. The received power level is a function of the transmit power level of the $k^{th}$ node and the fading and path loss of the channel from the $k^{th}$ transmit node to its receiver. The transmit power levels of users are assumed to be fixed by a process that is independent of the waveform adaptation process. The channel is assumed to be constant over all signal dimensions and also constant over the time required for the adaptation process. The received signal is assumed to incur no frequency selective multipath. It is also assumed that the signature sequences are synchronized at the receiver. However, the framework can be easily extended to include multipath channels and asynchronous systems (similar to the analysis in [71] and [70] respectively). The data symbol transmitted from the $k^{th}$ transmit node is denoted by $b_k$. It is assumed that the symbols sent by each transmitter are independent, have zero mean and unit variance. The signal at the receiver is given by

$$r = \sum_{k=1}^{K} \sqrt{P_k} s_k b_k + z. \quad (6.1)$$

Here, $r \in \mathbb{R}^{N \times 1}$ and the vector, $z \in \mathbb{R}^{N \times 1}$, models additive Gaussian noise with covariance matrix $R_{zz} = E[zz^T]$. If the noise process is white, $R_{zz}$ is a multiple of the identity matrix. The received vector can be written in matrix form as

$$r = s \sqrt{P} b + z. \quad (6.2)$$

Here, $s$ is a $N \times K$ matrix with the signature sequences of different radios as columns, $s = [s_1, \ldots, s_K]$ such that $s \in \times_{k \in K} S_k$. $p$ is a $K \times K$ diagonal matrix whose $k^{th}$ entry is the power level $p_k$ and the square root is over each individual entry. $b$ is a column vector with $b = [b_1, \ldots, b_K]^T$. The received cross-correlation matrix can now be written as

$$R_{rr} \Delta = E[rr^T] = sps^T + R_{zz}. \quad (6.3)$$

Let $s_{-k}$ denote the signature matrix $s$ without the $k^{th}$ column, i.e., $s_{-k} = [s_1, \ldots, s_k-1, s_{k+1}, \ldots, s_K]$. Let $p_{-k}$ denote the diagonal matrix $p$ without its $k^{th}$ row and column. Then the interference plus noise cross correlation matrix at the receiver for the $k^{th}$ user is given by

$$R_{ii,k} = s_{-k} p_{-k} s_{-k}^T + R_{zz} = R_{rr} - s_k p_k s_k^T. \quad (6.4)$$

The $k^{th}$ user’s receiver is assumed to be able to perfectly estimate $R_{ii,k}$ and $R_{rr}$. The receivers do not directly communicate with each other. However, they are assumed to be able to coordinate enough to allow their respective users to update their waveforms in a round-robin fashion.

### 6.4 Potential Game Formulation for WA

In this section, we interpret the WA problem as a game. The user-nodes in the network are the players of the game ($K = \{1, \ldots, K\}$). The transmit waveforms available to the user-nodes ($A_k = S$, $\forall k \in K$) are the action sets. An algorithm or game for WA can be formulated by allowing each user-node to iteratively adapt its waveform such that each adaptation improves (better response dynamic) or maximizes (best response dynamic) the utility function associated with the user-node.
As mentioned before, the optimal signature sequences/waveforms (i.e., sequences that achieve the maximum sum capacity) are shown to also maximize the negated generalized total squared correlation (NTSC) function defined as

\[ V(s) = -\|sp^T + R_{zz}\|_F^2. \]  

(6.5)

Here, \( \|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} \) is the Frobenius matrix norm. Distributed algorithms are proposed in [21] and [14], where users iteratively replace their sequence with their corresponding MMSE filter sequence and the inferior eigenvector of the corresponding interference cross correlation matrix, \( R_{ik} \), respectively, in a round robin fashion. The MMSE-iterations and the eigen-iterations are shown to increase the NTSC function at each update. This property suggests the existence of utility functions that allow a potential function, given by the NTSC function, for WA.

### 6.4.1 WA Algorithm as a Potential Game

Consider a network game (or an iterative algorithm) where the utility function for the \( k^{th} \) user is given by

\[ u_k(s_k, s_{-k}) = -2p_k s_k^T R_{ik} s_k. \]  

(6.6)

Note that the utility function is the negative of a weighted measure of the interference seen at the receiver corresponding to the \( k^{th} \) user. Expanding the NTSC function defined in (6.5), we have

\[ V(s) = -\|sp^T + R_{zz}\|_F^2 = -\|R_{ii,k} + p_k s_k s_k^T\|_F^2 \]

\[ = -\|R_{ii,k}\|_F^2 - p_k^2 - 2p_k s_k^T R_{ii,k} s_k. \]  

(6.7)

Here, the last step follows from the fact that \( s_k^T s_k = 1 \). The first two terms are independent of the choice of signature sequence by the \( k^{th} \) user and the last term is the same as the utility function given in (6.6). If user \( k \) replaces its sequence \( s_k \) by a new sequence \( s_k^* \), the change in potential function is given by

\[ V(s) - V(s_k, s_{-k}) = -\|R_{ii,k} + p_k s_k s_k^T\|_F^2 + \|R_{ii,k} + p_k s_k^* s_k^T\|_F^2 \]

\[ = -2p_k s_k^T R_{ii,k} s_k + 2p_k s_k^T R_{ii,k} s_k^* = u_k(s_k, s_{-k}) - u_k(s_k^*, s_{-k}). \]  

(6.8)

Therefore, the network game for WA, \( \Gamma_{WA} = \langle N, \{S\}, \{u_k(s)\} \rangle \), is an exact potential game with \( V(s) \) as its exact potential function. This shows that \( \Gamma_{WA} \) is also best and better-response-equivalent to the coordination game \( \Gamma_{WAc,co} = \langle N, \{S\}, \{V(s)\} \rangle \). Note that since \( R_{ii,k} \) is a symmetric positive semidefinite matrix, the utility function is a negative weighted Rayleigh quotient of \( R_{ii,k} \). This is maximized by the eigenvector corresponding to the minimum eigenvalue of \( R_{ii,k} \). Hence the best response for each user in the WA process is the inferior eigenvector of \( R_{ii,k} \).

Some other utility functions that allow a potential game formulation with \( V(s) \) as the potential function are described below. An algorithm for WA with the properties of a potential game can be constructed using any of these utility functions. The nature of these described utility functions depends upon two factors: receiver type and the user’s end performance metric. The following naming convention is used: in game A/B each player measures its signal quality with criterion A on the output of receiver B.

### SINR/Correlator Game

A natural choice for the utility function is the SINR at the output of the correlation receiver given by

\[ u_k(s) = SINR_k(s) = \frac{p_k}{s_k^T R_{ii,k} s_k}. \]  

(6.9)
An ordinal transformation of this utility function is given by
\[ \hat{u}_k(s) = \frac{-2p_k^2}{u_k(s)} = -2p_k s_k^T R_{ii,k} s_k. \] (6.10)

This is the same as the utility function for the exact potential game, \( \Gamma_{WA} \), given in (6.6). Hence an \( WA \) game with this utility function is an ordinal potential game with the NTSC function as the ordinal potential function. This game is thus also better-response-equivalent to the coordination game \( \Gamma_{WA,co} \).

**MSE/Correlator Game**

The mean squared error (MSE) at the output of a correlation receiver is given by
\[ MSE(s, \alpha) = 1 - 2\alpha \sqrt{p_k} + \alpha^2 s_k^T R_{rr} s_k, \] (6.11)

where \( \alpha \) is some scaling factor for the receiver. MSE is minimized when \( \alpha = \sqrt{p_k / s_k^T R_{rr} s_k} \). Hence a natural choice for a utility function is given by
\[ u_k(s) = -MMSE(s) = 1 - 2p_k s_k^T R_{rr} s_k + p_k s_k^T R_{rr} s_k - p_k s_k^T R_{ii,k} s_k. \] (6.12)
The last step follows from the fact that \( R_{rr} = R_{ii,k} + p_k s_k^T R_{rr} s_k \) and \( s_k^T s_k = 1 \). Consider the transformation of the utility function given by
\[ \hat{u}_k(s) = \frac{2p_k u_k(s)}{u_k(s) + 1} = -2p_k s_k^T R_{ii,k} s_k. \] (6.13)
This is an ordinal transformation since the function \( x/(x + 1) \) is monotonically increasing for \( x > -1 \) and \( -1 < u_k(s) < 0 \). It is thus seen that the utility function is an ordinal transformation of the utility function for the exact potential game \( \Gamma_{WA} \). Therefore, this game is also better-response-equivalent to the coordination game \( \Gamma_{WA,co} \).

**SINR/MSINR Game**

For the MSINR receiver, the linear filter, \( w_k \), for the \( k^{th} \) user is given by \( w_k = (R_{ii,k})^{-1} s_k \). The utility function, given by the SINR at the output of the receiver, is then
\[ u_k(s) = p_k s_k^T R_{ii,k}^{-1} [k] s_k. \] (6.14)
The \( k^{th} \) user’s best response is the normalized inferior eigenvector of \( R_{ii,k} \). Hence the WA game with this utility function is a best response potential game with the NTSC as the best response potential function. This game is thus best-response-equivalent to the coordination game \( \Gamma_{WA,co} \).

**MSE/MSINR Game**

The MSE at the output of the MSINR receiver is a natural choice of the utility function and is given by
\[ u_k(s) = -MSE(s) = \frac{-1}{p_k s_k^T R_{ii,k}^{-1} [k] s_k + 1}. \] (6.15)
This is again maximized by the inferior eigenvector of \( R_{ii,k} \). Hence the WA game with this utility function is a best response potential game with the NTSC as the best response potential function. This game is thus also best-response-equivalent to the coordination game \( \Gamma_{WA,co} \).
Hybrid Game

It is to be noted that the best response for a user is the inferior eigenvector of $R_{ii,k}$ for any of the utilities considered in the games discussed above. Hence any combination of the utility functions also results in a best response potential game. This says that if each user independently chooses its receiver type and performance criteria from the set of discussed utilities without regard to the choice of other users, the resulting game/algorithm is still a best response potential game and the game is best-response-equivalent to the coordination game $\Gamma_{WA,co}$.

6.4.2 Nash Equilibria of the WA Game

The Nash equilibria of a game are, by definition, the fixed points of the game under a best response dynamic. Let the fixed points of the WA game, $\Gamma_{WA} = \langle K, \{S\}, \{u_k(s)\}\rangle$, or equivalently the coordination game, $\Gamma_{WA,co} = \langle K, \{S\}, \{V(s)\}\rangle$, under a best response dynamic be given by $F_\Phi$. As mentioned before, the maximizers of the potential function are NE. Since the WA game has a potential function that is continuous, bounded and defined over a compact set, the potential function is guaranteed to have at least one maximum. Consequently, the game has at least one NE. The following lemma gives a characterization of the Nash equilibria of the game.

**Lemma 1:** Let $s \in \times_{k \in K} S_k$. If $s \in F_\Phi$, then for all $k \in \{1, \ldots, K\}$, $s_k$ is an eigenvector of $sp^T + R_{zz}(R_{rr})$.

**Proof:** It is shown in [14] that the fixed points of the distributed WA algorithms under eigen-iterations are given by sequences that satisfy Lemma 1. Since the WA game given here is best-response-equivalent to the distributed algorithm in [14], the lemma follows. □

An adaptation of theorem 2 in [62] can be used to show that the set of eigenvalue configurations for sequence multisets, $s$, such that $s \in F_\Phi$ is finite. It is also shown in [62], that all the local maxima of the NTSC function are also global maxima. Hence the fixed points of the game consist of globally optimal sequence configurations that lead to the global maximum value for the potential function (given by the NTSC function) and some sub-optimal points. The existence of such sub-optimal points with examples is shown in [67].

6.5 Convergence of WA Potential Games

This section investigates the convergence of the WA potential game to the NE, identified in the previous subsection, under best and better response dynamics.

6.5.1 Best Response Convergence

Note that the action-space for a user corresponds to the signature sequence space, $S$, which is compact. Since $S$ is a compact set, $S^K$ is also a compact set. The results in Section 5.4.1 shows that potential games with compact action spaces converge to the NE while following a best response dynamic. Hence, the WA potential game, $\Gamma_{WA,co}$, converges to the Nash equilibria of the game under a best response dynamic. This convergence property also holds for all WA games which are best-response-equivalent and better-response-equivalent to the WA potential game, $\Gamma_{WA,co}$ (Section 5.3). Therefore, an WA algorithm that can be constructed using any of the utility functions described in Section 6.4 converges to the NE for the network, when users iteratively update their signature sequences by a sequence that maximizes their respective utility functions (in other words, follow a best response dynamic).
6.5.2 Better Response Convergence

The result in Section 5.4.2 shows that potential games with compact action spaces converge while following a better response dynamic. As shown above, this result can be directly applied to show the convergence of the WA potential game, $\Gamma_{WA,co}$ under a better response dynamic. However, note that convergence to the NE is not guaranteed.

Section 5.5 shows that random better response iterations can be used to assure convergence to the NE. Again, this result can be directly applied to the WA potential game, $\Gamma_{WA,co}$. Note that the convergence property of the random better response dynamic holds for WA games which are better-response-equivalent to $\Gamma_{WA,co}$ (Section 5.3). Therefore, an WA algorithm that can be constructed using the utility functions described in Section 6.4 except the utility functions of games 3 and 4 converges to the NE for the network, when users iteratively update their signature sequences by their respective random better responses.

6.6 Noisy Best Response Convergence for WA

As seen in the previous section, under a best response dynamic, the WA potential game $\Gamma_{WA,co}$ (and therefore any of the WA algorithms that can be constructed using the utility functions in Section 6.4) converge to the NE of the game given by the set of eigenvectors of $R_r$. However, as mentioned before, all the NE are not globally optimal and the game might converge to sub-optimal NE. Empirical results suggest that when the initial sequence are chosen at random, the MMSE iterations (a better response adaptation procedure) and eigen iterations (a best response adaptation procedure) always converge to the optimal sequence configurations ([21] and [14] respectively). In [62], it is analytically shown that w.r.t. the MMSE iteration, the sub-optimal points are unstable and arbitrary small perturbations can lead the game to a neighborhood of the globally optimal configurations. A generalization of this result is developed in [27] and [63], where it is shown that for a special class of Nash Separable (NS) potential games, noisy best response iterations ($\delta$-NBRI) asymptotically converge to a neighborhood of the global optima (Section 5.4.3). In this section, this property is used to establish the convergence w.r.t. best-response iterations (including the eigen-iterations) of a large subset of WA games to a neighborhood of an optimal sequence set, without class warfare or any other form of coordination.

The following two theorems identify network scenarios in which the WA game, $\Gamma_{WA}$, is NS.

**Theorem 5** The WA game, $\Gamma_{WA}$, or equivalently the WA coordination game, $\Gamma_{WA,co}$, in a levelable signal environment is NS [27].

**Proof** A levelable signal environment is one in which the optimum sequences whiten the spectrum of $R_r$. This, for example, occurs in an WA system where $K = N$ and all signals are received with equal power in AWGN.

In [62], it is shown that $V(s)$ has no sub-optimal local maxima (i.e., all local maxima are also global maxima). Also, $F_{\Phi}$ is a finite set. Hence $V(F_{\Phi})$ is also a finite set. This fact can be used to choose a $0 < \epsilon < \min_{k \in K} p_k^2/2$ such that $s \in F_{\Phi}$ implies $V(s) = V_{\max}$ on $W_{\epsilon m} = V^{-1}(N_{\epsilon m}(V_{\max}))$. Hence conditions 1 and 2 for a NS game is satisfied.

Given, $0 < \epsilon < \epsilon_m$ and $W_{\epsilon} = V^{-1}(N_{\epsilon m}(V_{\max}))$, assume there is a non-improving sequence of best responses in a round-robin round. This occurs if $s \in W_{\epsilon}$ and $\inf V(\Phi(s)) = V(s)$. Since $s \in W_{\epsilon}$, $V(s) < V_{\max}$. Note that $s \notin F_{\Phi}$ by choice of $\epsilon_m$. Hence, $\exists j \in K$ such that $V(\Phi_j(s)) > V(s)$. Choose the smallest such $j$. This occurs when player $j$’s response is blocked by a player $k < j$ who changes its response. Consider the smallest such $k$. However, by choice of $j$, player $k$’s signature was already a best
response. Hence the two smallest eigenvalues of $R_{ii} [k]$ are identical and the inferior eigenspace of $R_{ii} [k]$ has a dimension of at least 2. Let $\lambda (R_{ii} [k]) = [\lambda_1, \ldots, \lambda_N]$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Also let $\lambda_m = \lambda_N = \lambda_{N-1}$. Let $s^* \in S^K$ be an optimal signature set. Since the WA game is levelable, $\exists \lambda > 0$ such that $R_{rr}^* = s^* p^* s^* T + R_{zz} = \lambda I$.

Thus,

$$V_{max} - V (s) = -\|R_{rr}^2 \|_F^2 + \|sps^T - s^* p^* s^* T + R_{rr}\|_F^2$$

$$= \|sps^T - s^* p^* s^* T\|_F^2 + 2\lambda Tr \{sps^T - s^* p^* s^* T\}$$

$$= \|R_{ii} [k] + p_k s_k s_k^T - \lambda I\|_F^2.\tag{6.16}$$

Using Fact 9.10.9 in [76],

$$V_{max} - V (s) \geq \sum_{n=1}^{n=N-2} (\lambda_n - \lambda)^2 + (\lambda_m + p_k - \lambda)^2 + (\lambda_m - \lambda)^2$$

$$\geq \frac{p_k^2}{2} > \epsilon.\tag{6.17}$$

This is a contradiction by initial choice of $\epsilon_m$. Hence there exists a neighborhood around the maximum of the potential function in which the best response iterations are strictly improving.

**Theorem 6** The WA game, $\Gamma_{WA}$, or equivalently the WA coordination game, $\Gamma_{WA,co}$ with less than (N-1) oversized users is NS.

**Proof** A user $i$ be oversized [20] if

$$p_i > \frac{\sum_{k \in \mathcal{K}; p_k < p_i} P_k}{N - \sum_{k \in \mathcal{K}; p_k \geq p_i} P_k} \tag{6.18}$$

Let $\mathcal{Q}$ be the set of oversized users in $\mathcal{K}$. It is shown in [20] that the optimal eigenvalues of $s^* p^* s^* T + R_{zz}$ are given by $[p_i; i \in \mathcal{Q}, \frac{\sum_{k \in \mathcal{Q}} P_k}{N - |\mathcal{Q}|}, \ldots, \frac{\sum_{k \in \mathcal{Q}} P_k}{N - |\mathcal{Q}|}]$. Let $[\lambda_1, \ldots, \lambda_N]$, be the set of eigenvalues of $s^* p^* s^* T + R_{zz}$ in decreasing order. Then if, $|\mathcal{Q}| < N - 1$, $\lambda_{N-1} = \lambda_N = \frac{\sum_{k \in \mathcal{Q}} P_k}{N - |\mathcal{Q}|}$. Let $y = \lambda_{N-1} - \lambda_N$. Following the steps of the proof for theorem 3, it can be shown that for an WA game, $V_{max} - V (S) \geq \frac{(p_k - y)^2}{2}$. When $|\mathcal{Q}| < N - 1$, $y = 0$ and a non-zero $\epsilon$ can be chosen.

It is thus shown in this section that for the WA potential game $\Gamma_{WA,co}$ (and therefore any of the WA algorithms that can be constructed using the utility functions in Section 6.4) in a levelable signal environment and or in a network with less than $N - 1$ oversized users, the $\delta$--NBRI almost surely converges to globally optimal configurations. Note that there can only be a maximum of $N - 1$ oversized users in any WA game. Hence, this result holds for a large percentage of possible WA games.

### 6.7 Reduced Feedback Schemes based on Better Response Convergence

Note that the implementation of the WA potential game $\Gamma_{WA,co}$ (or equivalently any of the WA algorithms that can be constructed using the utility functions in Section 6.4) under a best response dynamic requires knowledge of the covariance matrix, $R_{ii,k}$ at the adapting node (since the best response for a user is inferior eigenvector of $R_{ii,k}$). However, $R_{ii,k}$ is only available at the receiver. Hence, the signature sequence has to be computed at the receiver and then fed back to the transmitter. Since the signature sequence is real-valued, this feedback introduces a significant overhead in the network. In this section, we exploit the better response
convergence properties for exact and ordinal potential games, derived in Section 5.4.2 and Section 5.5, to design two reduced feedback schemes for WA. These schemes are applicable to WA games or algorithms that are better-response-equivalent to the coordination game $\Gamma_{\text{WA,co}}$. Note that we consider a synchronous network here. However, similar to the analysis in [70], the potential game formulation for WA and, consequently, the reduced feedback schemes can be easily adapted to more practical asynchronous networks albeit with more notational complexity.

### 6.7.1 Random Better Response

Consider the exact or ordinal potential WA games described in section 6.4. In a random better response update procedure, a user randomly chooses a signature sequence. The receiver indicates if the signature sequence increases the SINR of the user. The user reverts back to the old sequence if it does not. This process is repeated iteratively for each user in a round robin manner until convergence is achieved. This random better response procedure is assured to converge to a NE of the ordinal or exact potential game by the arguments described in Section 6.5.2. However, since the sequence space for the game is large, convergence could be very slow. This is illustrated in Figure 6.1 which shows the adaptation iterations in a network with 4 users sharing 3 signal dimensions. Convergence speed can be increased by having a directed better response scheme as described in the next subsection.

### 6.7.2 Gradient-based Better Response

In this update scheme, the receiver finds the $q$ ($q \in \{1, \ldots, N\}$) dimensions in which the gradient of the utility function has the largest magnitude. The variable $q$ can be used to control the amount of feedback from the receiver to the transmitter. The receiver then finds the step size $\lambda$ that maximizes the utility function in the direction specified by the $q$ chosen dimensions with the largest magnitude (referred to here as the ascent direction). For example, if the gradient of the utility function is a 4-dimensional vector given by $d =$

![Figure 6.1: Random better response update procedure. Network has 4 users sharing 3 signal dimensions. Large number of iterations required for convergence.](image_url)
[1 - 3 4 2]\text{T} and \(q = 2\), the receiver finds the optimum step size along the ascent direction \(d_q = [0 - 3 4 0]\text{T}\).

To illustrate the scheme, consider the WA algorithm with the utility function specified by (6.6). The utility function is modified below to incorporate the constraint that the signature sequences have unit norm.

\[
    u_k(s_k, s_{-k}) = -\frac{s^T_k R_{ii,k} s_k}{s_k^T s_k}. \tag{6.19}
\]

Note that this does not change the value of the utility function for a given unit norm signature sequence. The gradient of the utility function is given by

\[
    \frac{du_k(s_k, s_{-k})}{s_k} = -2(s^T_k R_{ii,k} s_k - 2 (s^T_k R_{ii,k} s_k) s_k (s_k^T s_k)^2). \tag{6.20}
\]

Let \(d_q\) be the ascent direction. The optimum step-size in this direction is given by the solution to the following optimization problem:

\[
    \max_{\lambda \neq 0} \frac{(s_k + \lambda d_q)^T R_{ii,k} (s_k + \lambda d_q)}{(s_k + \lambda d_q)^T (s_k + \lambda d_q)}. \tag{6.21}
\]

The optimal step-size can be computed by a simple line search procedure.

By the arguments in subsection 6.5.2, the gradient based better response algorithm converges. However, as mentioned before, the set of fixed points of the algorithm might be larger than the set of NE. The fixed points of the algorithm are characterized as follows: The utility function for each user is bounded. Hence, in each iteration, the gradient of the utility function, \(\frac{du_k(s_k, s_{-k})}{s_k}\), is zero in the ascent dimension. Since dimensions with the largest gradients are chosen for the ascent direction in each iteration, \(\frac{du_k(s_k, s_{-k})}{s_k}\) is zero at the convergence point. However, from (6.20), it can be observed that \(\frac{du_k(s_k, s_{-k})}{s_k}\) is zero for any eigenvector of \(R_{ii,k}\) and not just for the inferior eigenvector of \(R_{ii,k}\). Hence, the iterations could lead to fixed points where the signature sequence of some user is not the inferior eigenvector of its interference-plus-noise cross-correlation matrix, i.e., fixed points that are not a NE.

To steer the gradient-based iterations from these sub-optimal fixed points and to aid convergence to the Nash equilibria of the game, a random better response spacer step is added. If the receiver notices that the convergent sequence is not the minimum eigenvector of \(R_{ii}\), it instructs the transmitter to randomly choose a signature sequence that improves utility (a sequence that causes an increase in SINR at the receiver). NE are the only fixed points of a random better response. Hence the gradient based algorithm with the spacer steps also theoretically converges to the NE.

Figure 6.2 illustrates the convergence of the gradient-based better response scheme with \(q = 1\) using two example network scenarios. All the users are assumed to have an equal transmit power of 1 watt and to transmit in an AWGN channel with SNR of 10dB. Each iteration corresponds to a complete round-robin update by all the users in the system. It is seen that when 10 users share 6 signal dimensions, the best response takes 2 iterations to converge and the better response scheme takes 4 iterations to converge. Let \(r\) be the number of bits used to quantize a real value. Then the number of bits required to be fed back in the per user transmit-receive pair for the adaptations is \(2 \times 6r = 12r\) (feedback in the network is \(12r \times 10 = 120r\)) in the case of the best response scheme and \(4 \times r = 4r\) (feedback in the network is \(4r \times 10 = 40r\)) in the case of the better response scheme. Similarly, when 20 users share 12 signal dimensions, the number of bits to be fed back per user transmit-receive pair is \(2 \times 12r = 24r\) (feedback in the network is \(24r \times 20 = 480r\)) in the case of the best response adaptation scheme and \(7 \times r = 7r\) (feedback in the network is \(7r \times 20 = 140r\)) in the case of the better response adaptation scheme. Hence the better response scheme results in considerable reduction of the adaptation overhead in the network. It is to be noted that the speed of convergence of the better response scheme depends upon the number of available signal dimensions and the number of users.
Figure 6.2: WA with gradient-based better response. Channel is additive white Gaussian noise with SNR = 10dB.

Figure 6.3: WA with gradient-based better response. Network has 20 users sharing 12 signal dimensions. Channel is additive white Gaussian noise with SNR = 10dB. The users start from the same initial sequences.
Figure 6.4: WA with gradient-based better response and different values for $q$. Network has 20 users sharing 12 signal dimensions. Channel is additive white Gaussian noise with SNR = 10dB.

However, it has been observed that in general, the increase in the number of feedback bits due to the increase in the number of iterations required for convergence for the better response scheme is less than the increase in the number of feedback bits due to the increase in the number of signal dimensions for the best response scheme.

Figure 6.3 illustrates the convergence to a non-optimal solution when all users start adapting from the same initial sequence and when the random better response spacer step is not used. It has, however, been noticed empirically that when starting from random initial sequences, the users converge to the optimal NE even when the spacer step is not used.

Figure 6.4 shows the convergence of the better response scheme for different values of $q$. It is seen that increasing $q$ decreases the number of iterations required for convergence. However, increasing $q$ also increases the feedback in each iteration. Hence, the optimal $q$ for a given network size can be found by optimizing for the minimum amount of feedback in the network and the required convergence time for the network.

6.8 Conclusions and Contributions

In this chapter, a game-theoretic framework that could be used for the design of waveform adaptation algorithms in centralized networks was studied. The framework was originally proposed in [27]. In the framework, waveform adaptation is cast as a potential game. This formulation is shown to enable the design of new WA algorithms with non-identical utility functions for users in addition to accommodating previously proposed algorithms. Convergence properties for potential games were then used to show that a large subset of WA games, arbitrarily small noise assures the convergence of best response iterations WA algorithms, including the eigen iterations investigated in [14], to an arbitrarily small neighborhood of the globally optimal signature sequence set. In addition, it was shown that better response convergence properties of poten-
tial games could be leveraged to construct practical distributed implementations of WA that require limited feedback in the network and that converge to desirable network solutions. Two specific reduced feedback schemes that illustrate this were also constructed and described in the chapter.

Note that we considered a synchronous system in our analysis. However, note that this is done only for the sake of notational simplicity. Similar to the analysis in [70], the game theoretic framework for WA including the reduced feedback schemes can easily be adapted to asynchronous systems.

Until now the majority of the research on WA has focused on networks with a centralized receiver or co-located receivers. Direct extensions of WA techniques to networks with distributed receivers do not lead to convergent solutions due to the asymmetry of the mutual interference between users at different receivers. Game theory, in general, and potential games, in particular, could aid in the design of WA techniques for decentralized networks. The review of the game-theoretic framework for WA in this chapter thus provides the background and is used as a starting point for the development of WA techniques for interference avoidance in distributed networks, discussed in later chapters.

The original contributions in this chapter are as follows:

- It was shown that the WA game in a network with less than \( N - 1 \) over-sized users is Nash separable (NS). Note that the maximum possible number of over-sized users is \( N - 1 \). Hence a majority of WA games are NS. This result thus establishes the convergence of a majority of WA games to a neighborhood of the globally optimum solution (as compared to the small subset of levelable WA games in [27]).

- New algorithms for WA that reduce the feedback in the network were constructed by leveraging better response convergence properties of potential games.

- Convergence of the proposed reduced-feedback algorithms to desirable network states was established using the random-better-response result derived in Chapter 5.

The following publication resulted from this chapter:

Chapter 7

Waveform Adaptation in Distributed Networks

7.1 Introduction and Problem Statement

In this chapter, we develop a waveform adaptation framework, based on potential game theory, that could be used to construct convergent interference avoidance algorithms in networks with multiple distributed receivers (as in ad hoc networks). This is motivated by the fact that direct extensions of distributed greedy IA techniques for centralized networks to these de-centralized networks do not always lead to convergence. Some channel conditions that lead to non-convergence are also identified in the chapter. The proposed adaptation algorithms lead to desirable network solutions and are amenable to a distributed implementation that involves limited feedback in the network. Variations of IA algorithms, based on the framework, including IA with respect to legacy systems and IA with combined transmit-power and WA adaptations are also investigated.

7.2 Related Work and Motivation

Distributed WA techniques for interference avoidance (IA) have been extensively investigated for networks with a centralized receiver (or equivalently networks with co-located receivers) ([21], [14], [27] and references within, discussed in Chapter 6). Due to the inherent structure of these networks, the interference profiles for different users are symmetric. This property leads to the convergence of simple iterative WA schemes where each user greedily adapts to maximize or improve the signal to interference and noise ratio (SINR) at its receiver. In addition, this property ensures that each greedy adaptation decreases the total-sum-correlation (TSC) of the network. Minimization of TSC is equivalent to the maximization of sum-capacity. Hence greedy adaptations could lead to globally optimal waveforms in centralized networks. In [69] and [28], WA in multi-cell networks with multiple base-stations that collaborate with each other are investigated. There is considerable synergy between a centralized receiver and collaborative base-stations. Hence, as is also shown in the papers, WA algorithms for centralized systems can be extended to this scenario, if the user adaptations are based on the composite of received signals at all base-stations.

In networks where users communicate with non-colocated receivers (as in an ad hoc network), direct application of greedy WA algorithms might not lead to convergence. This is caused by the asymmetry of the mutual interference between users at different receivers, leading the users to adapt their sequences in conflicting ways and resulting in resource allocation cycles. Since, each adaptation, in general, requires considerable feedback from the receiver to the transmitter, these allocation cycles are expensive with respect
to the network overhead and are undesirable from a network performance perspective. Some distributed network scenarios in which such situations arise have been identified in [28]. We also identify other scenarios in this chapter.

Game theory is a branch of applied mathematics that models interactions between rational decision makers with formalized incentive (preference) structures and provides tools to predict and analyze the outcome of these interactions. Game theoretic models can hence be used for the design and analysis of distributed algorithms in which individual nodes adapt their actions and contend for a common resource. The optimality of these adaptations, convergence, steady-states and stability of these algorithms can then be investigated by using properties of game models. A survey on the use of game theory to analyze wireless ad-hoc networks is presented in [29].

In this chapter, we draw on concepts from game theory to develop iterative WA techniques for IA that converge to a desirable state in networks (including networks with non-colocated receivers). Specifically, we design an adaptation framework based on potential game theory. Potential games are chosen as these provide a framework where each user’s adaptation can be designed to iteratively increase a global network performance measure, leading to algorithms that are simple and easy to analyze and are useful from a network perspective. A similar game model has been used in [58] for design of dynamic frequency selection algorithms. However, the algorithms in this paper are based on greedy adaptations and only converge in networks with symmetric channels. In our proposed formulation on the other hand, as opposed to greedy adaptations, each user adapts to improve some measure of the influence caused by its actions on the other users in the network in addition to improving its own performance. This ensures the convergence of the adaptation techniques in networks with any structure. We then exploit the properties of potential games to construct distributed implementations of the proposed WA algorithm. We also present some variations of the IA algorithm based on the proposed framework. These include IA with respect to legacy systems with which the spectrum sharing systems might coexist. We also analyze a multi-parameter IA scheme where users adapt both their transmit power and their waveforms based on the proposed framework. Such combined power and WA adaptation algorithms have been previously analyzed only for centralized networks ([77], [78], [79] and references within).

The rest of the chapter is organized as follows: The system model for the network under consideration is described in Section 7.3. Section 7.4 describes some channel conditions under which greedy IA games do not converge. Section 7.5 develops a general framework based on potential games for adaptation of transmission parameters in a distributed network. In Section 7.6, the WA for IA problem is cast into this framework and the Nash equilibria of the game are identified. Based on the potential game formulation, a best response algorithm for WA is designed in 7.7. However, it is seen that the distributed implementation of the algorithm requires considerable feedback in the network. Properties of potential games are used to design distributed implementations of the algorithm that require limited feedback in the network in Section 7.8. Some variations of the IA algorithm including IA with respect to legacy systems and IA with combined transmit-power and WA adaptations are presented in Section 7.10. Finally, Section 7.11 concludes the paper.

### 7.3 System Model

We consider a distributed network, made up of a cluster of transmit and receive node-pairs. Figure 7.1 shows an example network with transmit-receive node-pairs indicated by arrows. This network model is a generalization of a network with co-located or centralized receivers and a network with multiple collaborative base-stations. Hence the results presented here are applicable to these network scenarios as well.

Let $N$ denote the number of transmission dimensions available to the network and $K$ denote the number of transmitting nodes in the network. As in Chapter 6, vector $s_k \in \mathbb{R}^{N \times 1}$ is used to denote the signature sequence or waveform associated with transmitting node $k$. The signature sequences are assumed to have
unit norm and hence constrained to the $N-1$ dimensional sphere $S = \{ s_k \in \mathbb{R}^{N \times 1} : \|s_k\|^2 = 1 \}$. The transmit power level of the $k^{th}$ node is denoted by $p_k$ and the fading coefficient of the channel between the $k^{th}$ transmit node and the $j^{th}$ receive node is denoted by $g_{kj}$. The channel is assumed to be constant over all signal dimensions and also constant over the time required for the adaptation process. The data symbol (assumed to be of zero-mean and unit-variance) transmitted from the $k^{th}$ transmit node is denoted by $b_k$. The received signal at the $j^{th}$ receive node is then given by

$$r_j = \sum_{k=1}^{K} \sqrt{p_k} g_{kj} s_k b_k + z,$$

where $r_j \in \mathbb{R}^N$, and the vector $z \in \mathbb{R}^N$ models zero mean additive Gaussian noise with variance $\sigma^2$. In the analysis in this chapter, the signature sequences from multiple users are assumed to be synchronized at the receivers. However, this is done only for the sake of notational simplicity. Similar to the analysis in [70] (which considers WA in centralized networks), the scheme can be directly and easily extended to an asynchronous system. Interference is caused at a receive node by transmissions from nodes different from the one associated with the particular receive node. Interference caused is influenced by the correlation between the waveforms of user nodes, transmit power levels and the channel characteristics.

### 7.4 Non-convergence of Greedy Best Response Games

An example scenario is used to show that greedy adaptation procedures do not always converge in decentralized networks (as is the case in ad hoc networks). Greedy adaptation refers to algorithms in which each user myopically tries to maximize its own Signal-to-Interference Ratio (SIR) or some other measure of link capacity in response to adaptations by other users in the network (e.g., the iterative water-filling algorithm).

Consider the cluster of three transmit-receive node-pairs shown in Figure 7.2. The transmit power of all user nodes is assumed to be equal. Also, the nodes are assumed to have only two signal dimensions available.
for transmission. Let the channel gains in the network be ordered as follows,

\[ g_{21} > g_{11} > g_{31}, \]

\[ g_{13} > g_{33} > g_{23}, \]

\[ g_{32} > g_{22} > g_{12}. \]  \hspace{1cm} (7.2)

Now consider an adaptation process in which each user tries to maximize its SINR by water-filling over the interference and noise it sees, in a round-robin fashion. Assume that at the start of the adaptation process, transmit node 2 uses dimension 1 and transmit node 3 uses dimension 2. Transmit node 1 chooses dimension 2 as the interference seen from node 3 is smaller than that from node 2 at receive node 1. However, at receive node 3, more interference is seen from transmit node 1 than transmit node 2 and hence transmit node 3 moves to dimension 1. Receive node 2 sees more interference from transmit node 3 than transmit node 1. Hence transmit node 2 shifts to dimension 2. However, as the interference seen from node 3 is smaller than that from node 2, transmit node 1 chooses dimension 1 now. This cycle thus continues with each node choosing the two dimensions alternately. This process is illustrated in Figure 7.3.

These results can be easily extended to over-loaded networks (networks with more users than available signal dimensions) with more than three users to show that allocation cycles or non-convergence of greedy resource allocations occur when the channel gains between multiple users are ordered cyclically, similar to the ordering for three users given in (7.2). Since, each adaptation, in general, requires considerable feedback from the receiver to the transmitter, these allocation cycles are expensive with respect to the network overhead and are undesirable from a network performance perspective.

### 7.5 Framework for Convergent Interference Avoidance Algorithms

A game-theoretic model for interference avoidance games in distributed networks based on potential game theory is suggested here. Potential games are chosen as these are easy to analyze and give a framework
where users maximize a global network function by only trying to maximize their own utilities, leading to simple game formulations.

Let the user node-pairs be the players of the game. The signature sequences (or waveforms) of users constitute the actions for the players in the game. Let the utility associated with a particular user be given by the following general expression:

\[
    u_k (s_k, s_{-k}) = f_1 (s_k, p_k, g_{kk}) - \sum_{j \neq k, j=1}^{K} f_2 (I (s_j, s_k), p_j, p_k, g_{jk}, g_{kk}) - \sum_{j \neq k, j=1}^{K} \gamma_{kj} f_3 (I (s_k, s_j), p_k, p_j, g_{kj}, g_{jj}).
\]

(7.3)

Here, function \( f_1 \) quantifies the benefit associated with a particular choice of signature sequence and power. Function \( f_2 \) is a measure of the interference due to the other users present in the system perceived at the receive node for user node \( k \). Function \( I \) is some function of two signature sequences \( s_k \) and \( s_j \) (e.g. the correlation between the sequences). Function \( f_3 \) is a measure of interference caused by a particular user at the receivers associated with other users in the network and coefficient \( \gamma_{kj} \) is a weighting factor.

A simple formulation of a candidate potential function is given by

\[
    V (s) = \sum_{k=1}^{K} \left( f_1 (s_k, p_k, g_{kk}) - \alpha \sum_{j \neq k, j=1}^{K} f_2 (I (s_j, s_k), p_j, p_k, g_{jk}, g_{kk}) - \beta \gamma_{kj} f_3 (I (s_k, s_j), p_k, p_j, g_{kj}, g_{jj}) \right).
\]

(7.4)

Coefficients \( \alpha \) and \( \beta \) are weighting factors and matrix \( s = [s_1, \ldots, s_K] \). The terms involving the \( k^{th} \) user are delineated below to derive conditions required for the formulation of exact and ordinal potential games.

\[
    V (s_k, s_{-k}) = f_1 (s_k, p_k, g_{kk}) - \sum_{j \neq k, j=1}^{K} f_2 (I (s_j, s_k), p_j, p_k, g_{jk}, g_{kk}) - \sum_{j \neq k, j=1}^{K} \gamma_{kj} f_3 (I (s_k, s_j), p_k, p_j, g_{kj}, g_{jj}).
\]

(7.5)

\[
    Non\text{-}contributing~Terms
\]

7.5.1 Exact Potential Game

The above game is an exact potential game if the following condition is satisfied:

\[
    u_k (s_k, s_{-k}) - u_k (\hat{s}_k, s_{-k}) = V (\hat{s}_k, s_{-k}) - V (s_k, s_{-k}).
\]

(7.6)

Here, \( \hat{s}_k \) is a signature sequence for user \( k \) different from \( s_k \). Examining (7.5), it is seen that condition (7.6) is satisfied and (7.4) forms an exact potential function under the two scenarios listed below.
Scenario-1:

\[
\begin{align*}
\phi_2 (I(s_j, s_k), p_j, p_k, g_{jk}, g_{kk}) &= \phi_2 (I(s_k, s_j), p_k, p_j, g_{kj}, g_{jj}) \\
\phi_3 (I(s_j, s_k), p_j, p_k, g_{jk}, g_{kk}) &= \phi_3 (I(s_k, s_j), p_k, p_j, g_{kj}, g_{jj})
\end{align*}
\]  
\[\alpha = \beta = \frac{1}{2}, \gamma_{kj} = \gamma_{jk} \forall k,j \]  
(7.7)

This scenario occurs when the interference caused by a transmit-node A, to a receive-node B, is the same as the interference caused by the transmit-node associated with receive-node B, to the receive-node associated with transmit-node A. An example is a network with co-located transmitters and co-located receivers. However, such symmetric links are not very likely in ad hoc networks.

Scenario-2:

\[
\phi_2 (\bullet) = \phi_3 (\bullet) 
\]  
(7.8)

This scenario requires that the interference caused at the receive-node of a particular transmit-node, be measured in the same manner as the interference caused by the transmit node at other receive-nodes. This scenario is thus more realistic and will be used for the game formulation in this paper. Let \( \tilde{\phi}_V (\bullet) = \phi_2 (\bullet) = \phi_3 (\bullet) \). Then, when Scenario-2 holds, the potential function reduces to the following,

\[
\begin{align*}
V (s) &= \sum_{k=1}^{K} \left( f_1 (s_k, p_k) - \sum_{j \neq k,j=1}^{K} f_V (I(s_j, s_k), p_k, p_j, g_{jk}, g_{kk}) \right),
\end{align*}
\]  
(7.9)

### 7.5.2 Ordinal Potential Game

Formulation of the game as an ordinal potential game requires the following condition to be satisfied:

\[
\begin{align*}
\phi_k (s_k, s_{-k}) &\geq \phi_k (s_k, s_{-k}) &\iff V (s_k, s_{-k}) &\geq V (s_k, s_{-k}).
\end{align*}
\]  
(7.10)

This is possible when, \( \tilde{f}_k (\bullet) = \tilde{f}_k (\bullet) = f_{uk} (\bullet) \), where \( f_{uk} (\bullet) \) is an ordinal transformation of \( f_V (\bullet) \) and the utility function of each user is given by

\[
\begin{align*}
\phi_k (s_k, s_{-k}) &= f_1 (s_k, p_k, g_{kk}) - \sum_{j \neq k,j=1}^{K} f_{2k} (I(s_j, s_k), p_k, p_j) - \sum_{j \neq k,j=1}^{K} f_{3k} (I(s_k, s_j), p_k, p_j).
\end{align*}
\]  
(7.11)

The ordinal potential function for the game is given by

\[
\begin{align*}
V (s) &= \sum_{k=1}^{K} f_1 (s_k, p_k) - \sum_{j \neq k,j=1}^{K} f_V (I(s_k, s_j), p_k, p_j).
\end{align*}
\]  
(7.12)

Under this game formulation, it is possible to construct convergent adaptation games with each user trying to maximize a different utility function as long as the functions are ordinal transformations of each other.

### 7.6 Potential Game Formulation for WA

In this section, we cast the WA problem in ad hoc networks as an exact potential game. The node-pairs in the network are the players of the game \( \mathcal{K} = \{1, \ldots, K\} \). The transmit waveforms available to the transmit nodes \( A_k = S, \forall k \in \mathcal{K} \) are the action sets. We now seek to design a utility function for the nodes that leads to a potential function which is also desirable from a network perspective.
7.6.1 WA as a Potential Game

The SINR at a receive-node is a good indicator of the throughput and performance of the particular user node-pair. Hence, the sum of Inverse SINRs (SISINR) of users (or in other words, weighted sum-interference-and-noise, wherein the interference at each user’s receive-node is divided by the power received from its transmitter) in a distributed network is a pertinent measure of network performance.

The interference and noise seen at the $k^{th}$ receiver in a distributed network with $K$ user node-pairs is given by

$$i_k = \sum_{j=1, j\neq k}^{K} \sqrt{p_j g_{jk}} s_j b_j + z.$$  \hfill (7.13)

The inverse SINR at the $k^{th}$ receiver, assuming a matched-filter, is given by

$$I_k (s_k, s_{-k}) = \frac{s_k^H \left( \sum_{j=1, j\neq k}^{K} s_j s_j^H p_j g_{jk}^2 + R_{zz} \right) s_k}{p_k g_{kk}^2} = \frac{s_k^H R_{ii,k} s_k}{p_k g_{kk}^2}. \hfill (7.14)$$

Here, $R_{ii,k}$ is the interference-plus-noise-crosscorrelation matrix given by $R_{ii,k} = E [i_k i_k^T]$ and $R_{zz} = E [zz^T]$ is the noise covariance matrix. If the noise process is white, $R_{zz}$ is a multiple of the identity matrix. The SISINR of the network is given by

$$I_{sum}(s) = \sum_{k=1}^{K} \frac{s_k^H R_{ii,k} s_k}{p_k g_{kk}^2}. \hfill (7.15)$$

To allow a WA update by each user in the network to reduce the above function (the weighted sum-interference-and-noise in the network), the negative of the SISINR function is taken to be the potential function of the game.

$$V(s) = -I_{sum}(s) \hfill (7.16)$$

From the framework (using the formulation in Subsection 7.5.1 - scenario-2), function $f_V (\bullet) (= f_2 (\bullet) = f_3 (\bullet))$ is given by

$$f_V (I (s_j, s_k), p_j, p_k, g_{jk}, g_{kj}) = -\frac{s_k^H s_j s_j^H s_k p_j g_{jk}^2}{p_k g_{kk}^2} \hfill (7.17)$$

and function $f_1 (\bullet)$ is given by

$$f_1 (s_k, p_k, g_{kk}) = \frac{s_k^H R_{zz} s_k}{p_k g_{kk}^2}. \hfill (7.18)$$

Function $I$ is thus the correlation between two sequences. The utility function of a user node-pair $k$, such that the negative of the SISINR function is an exact potential function of the game, is then given by

$$u_k (s_k, s_{-k}) = -\sum_{j\neq k, j=1}^{K} \frac{s_k^H \left( s_j s_j^H p_j g_{jk}^2 + R_{zz} \right) s_k}{p_k g_{kk}^2} - \sum_{j\neq k, j=1}^{K} \frac{s_j^H s_k s_k^H s_j p_k g_{kj}^2}{p_j g_{jj}^2}. \hfill (7.19)$$

Note that for a unilateral deviation by the $k^{th}$ user, from signature sequence $s_k$ to sequence $\hat{s}_k$, $u_k (s) - u_k (\hat{s}_k, s_{-k}) = V(s) - V(\hat{s}_k, s_{-k})$. The utility function of the $k^{th}$ user can be re-written as

$$u_k (s_k, s_{-k}) = -s_k^T X_k s_k, \hfill (7.20)$$

where

$$X_k = \frac{R_{ii,k}}{p_k g_{kk}^2} + \sum_{j\neq k, j=1}^{K} \frac{s_j s_j^T p_k g_{kj}^2}{p_j g_{jj}^2}. \hfill (7.21)$$
The utility for a user is thus seen to be made of two terms: the first term is the inverse-SINR of the user at its receive node and the second term is the interference caused by the user to all the other users in the network. Thus a user achieves benefit by reducing the interference caused to the other users in the network in addition to reducing the interference at its own receiver. Thus each user’s utility function incorporates a measure of the influence of its actions on the other users in the system as opposed to utility functions for users in greedy IA games.

7.6.2 Nash Equilibria of the Game

The utility function for the \( k \)th user (Equation (7.20)) can be re-written as follows since the sequences are normalized \( \left( s_k^T s_k = 1 \right) \):

\[
u_k(s_k, s_{-k}) = -s_k^T X_k s_k \frac{s_k^T s_k}{s_k^T s_k}.
\]

(7.22)

Matrix \( X_k \) can be observed to be a symmetric matrix since it consists of terms that are the weighted cross-correlations of the transmit sequences of users and which are hence symmetric. It is also positive definite since the diagonal terms are positive and also greater than zero due to the inclusion of the non-zero noise density terms. Hence the utility function can be identified to be a negative weighted Rayleigh quotient of \( X_k \). This is maximized by the eigenvector corresponding to the minimum eigenvalue of \( X_k \) [80]. The best response of the user to the current state of the network is, therefore, given by the inferior eigenvector of \( X_k \). At the NE, by definition, each user’s current action is equal to the best response of the user to its utility function (in other words the NE is equivalent to the fixed points of a best-response algorithm). The NE of the game can thus be characterized by the following expression:

\[
X_k s_k = a_{\text{min},k} s_k, \quad k \in \{1, 2, \ldots, K\}.
\]

(7.23)

Here, \( a_{\text{min},k} \) is the minimum eigenvalue of matrix \( X_k \).

As mentioned before, the NE of a potential game include the maximizers of the potential function. Since the potential function given by Equation (7.16) is continuous and bounded, the potential function is guaranteed to have at least one maximum (Weierstrass theorem [81]) and hence at least one NE. The following theorem characterizes the global maximizers of the potential function for a sub-set of possible network scenarios.

**Theorem 7** In an under-loaded \((K < N)\) and equally-loaded \((K = N)\) network scenario with a white noise process \(\text{\( R_{zz} = \sigma^2 I_{N \times N}\))\), the potential function \(V(s)\) (7.16) is maximized by a set of orthogonal sequences.

**Proof** Let \( s^O \) be an orthogonal sequence set with \( K \leq N \) sequences (i.e., \( s_i^T s_j = 0, \forall i, j \in \mathcal{K} \) and \( i \neq j \)). Then, the value of the potential function in a network with a white noise process is given by

\[
V(s^O) = -\sum_{k=1}^{K} \frac{\sigma^2}{p_k g_{kk}^2}.
\]

(7.24)

Now consider a sequence set \( s \) that is not orthogonal. There exists at least two sequences \( s \) and \( s_j \) in the sequence set such that \( s_i^T s_j = a \neq 0 \). Therefore the value of the potential function is given by

\[
V(s) \leq -\sum_{k=1, k \neq i, k \neq j}^{K} \frac{\sigma^2}{p_k g_{kk}^2} - \frac{\sigma^2 p_j g_{jj}^2 + \sigma^2}{p_i g_{ii}^2} - \frac{\sigma^2 p_i g_{jj}^2 + \sigma^2}{p_j g_{jj}^2} < -\sum_{k=1}^{K} \frac{\sigma^2}{p_k g_{kk}^2} = V(s^O).
\]

(7.25)
This shows that the set of orthogonal sequences maximize the potential function for the given network scenario. As an example if the signal dimensions denote different frequency bands, the best allocation when \( K < N \), is to choose orthogonal frequencies. Note that since the potential function is given by the negative of the weighted sum interference in the network, the global maximizer of the potential maximizer corresponds to a desirable solution for the network.

7.7 Algorithm with Complete Information

As mentioned before, an algorithm can be formulated for WA by using a decision timing rule to allow users to update their waveforms according to a decision making rule with respect to the utility function designed in the previous section. In this section, we design an algorithm where users update their transmit waveforms according to the best response dynamic using an asynchronous timing rule.

7.7.1 Algorithm Description

The best response of a user to the current network state with respect to the utility function (7.20) is the inferior eigenvector of \( X_k \). The WA algorithm for IA can therefore be formally written as follows:

**Best-response-based SISINR WA Algorithm**

1. Fix the transmit-power levels and initialize codeword \( s_k \) for each user.
2. Set \( k = 1 \).
3. while \( k \leq K \)
   
   (a) Let \( a_k \) be the inferior eigenvector of \( X_k \). If \( a_k \neq s_k \), replace \( s_k \) by \( a_k \).
   
   (b) \( k = k + 1 \)
4. Repeat step 2 until a fixed point or some termination criteria is reached

7.7.2 Convergence and Fixed Points

It can be seen from the analysis that each user update increases the value of the potential function and hence iteratively decreases the weighted sum interference in the network. As mentioned before, EPGs exhibit best response convergence to the NE of the game and the proposed EPG has at least one NE. Consequently, at least one fixed point exists for the proposed algorithm and the fixed points of the ISINR WA algorithm are characterized by Equation 7.23. Also, as mentioned before, the NE of a potential game include the maximizers of the potential function. Therefore, the proposed algorithm could lead to solutions that minimize the weighted interference in the network and hence are desirable from the network perspective.

The following theorem shows that for a subset of network scenarios, the proposed algorithm leads to the optimal network solution (or the global maximizer of the potential function). The performance of the algorithm in other network scenarios will be analyzed via simulations.

**Theorem 8** In an under-loaded \( (K < N) \) and equally-loaded \( (K = N) \) network scenario with a white noise process \( R_{zz} = \sigma^2 I_{N \times N} \), the fixed point of the Best-response-based SISINR WA algorithm corresponds to a set of orthogonal sequences for the users in the network, which is the global maximizer of the potential function and an optimal solution for the network.
Proof Let \( s^* \) be a fixed point of the algorithm (equivalent to the NE of the game under a best response as is the case here). Then by (7.23), the following equation holds:

\[
X_k s^*_k = a_{\min,k} s^*_k, \quad k \in \{1, \ldots, K\}
\]  

(7.26)

Here, \( a_{\min,k} \) is the minimum eigenvalue of matrix \( X_k \). Let \( \hat{X}_k = X_k - \frac{R_{kk}}{p_k g_{kk}} \). Then, when the noise is white, the fixed point can be characterized by the following Equation:

\[
\hat{X}_k s^*_k = \hat{a}_{\min,k} s^*_k, \quad k \in \{1, \ldots, K\}
\]  

(7.27)

Here, \( \hat{a}_{\min,k} \) is the minimum eigenvalue of matrix \( \hat{X}_k \). If \( K \leq N \) and if the sequence set is orthogonal, \( a_{\min,k} = 0, \forall k \) since the cross-correlation between any two sequences is zero. We shall prove by contradiction that if the sequence set is not orthogonal or in other words, if \( a_{\min,k} \neq 0, \forall k \), the sequence set is not a fixed point.

Since the solution sequence set is not orthogonal, there exists a user \( k \in \{1, \ldots, K\} \) for whom \( X_k s^*_k = as^*_k \), where \( a \) is a positive number that is not zero. Consider this user \( k \) and the matrix \( X = \hat{X}_k + s^*_k s^*_k T \). Since the \( K \) sequences are not orthogonal and \( K \leq N \), the matrix has less than \( N \) linearly independent rows or columns. Therefore, the matrix has at least one eigenvalue that is zero [80]. Also since the matrix is symmetric with positive diagonal values, all the eigenvalues of the matrix are non-negative. Therefore zero is the smallest eigenvalue of the matrix \( X \). It follows that zero is also the smallest eigenvalue of matrix \( \hat{X}_k \). This is due to the fact that \( \hat{X}_k \) is a symmetric matrix with positive diagonal elements and hence has non-negative eigenvalues. In addition, since \( \hat{X}_k = X_k - s^*_k s^*_k T \), the eigenvalues of \( \hat{X}_k \) are lesser than or equal to the eigenvalues of \( X_k \).

Since \( a > 0 \), user \( k \) can switch to the eigenvector of \( \hat{X}_k \) that corresponds to the eigenvalue of zero. However, \( s^* \) is then not a fixed point of the algorithm which contradicts our initial assumption.

Note that, in the proposed algorithm, we use a round-robin decision update scheme. However, the algorithm can converge with any asynchronous decision update rule. This allows easier implementations of the proposed algorithm in practical networks.

7.7.3 Simulation-based Performance Evaluation

A distributed network is simulated by placing \( K \) transmit and receive nodes uniformly in a circular region with radius \( R \) (\( R = 5m \) in the simulations). The power at a receive node from a transmit-node at a distance of \( rm \) from the transmit node is assumed to be given by \( \frac{P_w}{r^\alpha} \), where \( p_k \) is assumed to be the power received from a transmit node at a distance of \( 1m \) and \( \alpha \) is the path-loss exponent (\( \alpha = 3 \) in the simulations). All user-nodes are assumed to transmit at the same power-level of 100mW. The path loss at a distance of \( 1m \) is assumed to be \( 40dB \) (therefore \( p_k = -50dBw \)). The received signals are assumed to be corrupted by additive white Gaussian noise with power spectral density of \( -70dBw \). Note that one iteration of the algorithm in the simulation results corresponds to one waveform adaptation by a single user-node unless indicated otherwise.

Figure 7.4 illustrates the convergence of the algorithm in an example network scenario with \( K = 30 \) and \( N = 10 \). For an equally or under-loaded network scenario, simulation results corroborate Theorem 8 and the algorithm always converges to the orthogonal sequence configuration that results in the least interference in the network. An example with \( K < N \) is illustrated in Figure 7.5 which plots the convergence of the weighted sum-interference-plus-noise function from different random initial choice of signature sequences to the optimal solution for an equally-loaded network scenario. In an over-loaded scenario, on the other hand, multiple fixed points are seen to exist for the WA algorithm. This is illustrated in Figure 7.6. To evaluate the
Figure 7.4: Convergence of ISINR WA algorithm with 30 user node-pairs sharing 10 dimensions. Node-pairs are distributed in a circular region with radius 5m.

Figure 7.5: Weighted sum-interference-plus-noise function for a ISINR WA algorithm with 6 user node-pairs sharing 6 dimensions. Plots shows convergence to a global potential maximizer from different random initial choice of waveforms.
Figure 7.6: Weighted sum-interference-plus-noise function for a ISINR WA algorithm with 12 user node-pairs sharing 6 dimensions. Plots shows convergence from different random initial choice of waveforms to different fixed points.

quality of the fixed points, we compare them with solutions numerically obtained by a Lagrangian global search algorithm described below.

The optimization problem to find a sequence set that minimizes the sum interference function (7.15) with constraints on the power of the sequences can be written as follows:

\[
P1: \min_{\mathbf{s}} \sum_{k=1}^{K} \frac{s_k^H R_{ii,k} s_k}{p_k g_{kk}^2} \quad (7.28)
\]

subject to: \( s_k^T s_k = 1 \), \( \forall k \)

This can reformulated as the following Lagrangian function

\[
f_L = \sum_{k=1}^{K} \frac{s_k^H R_{ii,k} s_k}{p_k g_{kk}^2} + \sum_{k=1}^{K} \lambda_k (s_k^T s_k - 1)^2 . \quad (7.29)
\]

Here, \( \lambda_k, \ k \in \mathcal{K} \) are the Lagrangian multipliers. A gradient search algorithm similar to that in [82] can be used to find the stationary point of the Lagrangian function. The derivative of \( f_L \) with respect to \( \lambda_k \) and of \( f_L \) with respect to \( s_k \) for \( k \in \mathcal{K} \) are

\[
\frac{df_l}{d\lambda_k} = (s_k^T s_k - 1)^2 \quad \text{and} \quad \frac{df_L}{ds_k} = \frac{2R_{ii,k}s_k}{p_k g_{kk}^2} + \sum_{j=1,j \neq k}^{K} \frac{2g_{kj}^2 p_k s_j s_k^T s_k}{p_j g_{jj}^2} + 4\lambda_k (s_k^T s_k - 1) s_k \quad (7.30)
\]

respectively. At each iteration of the gradient algorithm, \( s_k \) and \( \lambda_k \) for \( k \in \mathcal{K} \) are updated according to the following relations:

\[
s_k \leftarrow s_k - \mu \frac{df_L}{ds_k} \quad \text{and} \quad \lambda_k \leftarrow \lambda_k + \mu \frac{df_L}{d\lambda_k} \quad (7.31)
\]

Here, \( \mu \) is a step size for the updates.
Figure 7.7: Plot of the values of the Weighted sum-interference-plus-noise function at the end of the proposed SISINR WA algorithm divided by the values of the function that result from the Lagrangian search algorithm over 100 realizations of an overloaded-network with 5 users sharing 4 signal dimensions. Also plotted are the normalized values of the function for the random sequences used to initialize the algorithms (normalized by dividing the values by the values that result at the end of the Lagrangian search algorithm).

Figure 7.7 plots the normalized value of the sum interference function (7.15) for independent instantiations of the network (Note that for the Lagrangian search algorithm only solutions that satisfy the constraints are retained.). It is seen that the proposed algorithm and the Lagrangian search algorithm significantly reduce the interference in the network. In addition, the performance of the proposed algorithm is very similar to that obtained by the Lagrangian algorithm (the solutions are equivalent in 95% of the runs) indicating that the performance of the proposed algorithm is near-optimal.

The waveform adaptation algorithm involves considerable network overhead. Hence, it is important to evaluate the gains provided by the algorithm over a simple random-access scheme which does not require any network overhead. This is done by setting up the following network simulation: We consider a network with $K$ users and $N$ dimensions. A user transmits in a slot with a probability of $\lambda$. A transmission by user is assumed to be successful if the SINR at its receiver is greater that 0dB. Retransmissions of unsuccessful packet transmissions are ignored. In the random-access-scheme (similar to a multi-band ALOHA scheme discussed in the literature [83]), a user that has a packet to transmit, randomly chooses a transmission dimensions from the $N$ available dimensions. Nodes that transmit on the same dimension interfere with each other. In the WA scheme, a user that has a packet to transmit, transmits using its current signature sequence. The signature sequences of all $K$ users are adapted continually for the first $K \times 10$ slots (i.e., the WA algorithm is run for 10 round-robin iterations). Throughput results are obtained over 1000 time-slots (Note that in the first $K \times 10$ slots of the 1000 slots, all users also adapt their waveforms in the WA scheme). Results are then averaged over multiple network instantiations. Figures 7.8 and 7.9 plot the results for an equally-loaded network (with $K = 5$ and $N = 5$) and an over-loaded network respectively. It is seen that for small-transmission probabilities (small value of $\lambda$), the performance of both schemes are similar (Note that the plots also include a reduced-feedback WA scheme. This scheme will be discussed in the next section.). This is due to the fact that for lower transmission probabilities, the probability of finding an unused dimension is larger while using the random-access scheme. However, when the transmission
Figure 7.8: Throughput in a network with $K = 5$ users and $N = 5$ dimensions. The plot is generated by averaging over multiple instantiations of a network.

Figure 7.9: Throughput in a network with $K = 10$ users and $N = 6$ dimensions. The plot is generated by averaging over multiple instantiations of a network.
Figure 7.10: Comparison of the SINRs of users for the ISINR WA algorithm and the greedy IA algorithm. The network has 6 users sharing 3 signal dimensions.

probability increases, the WA scheme, especially in the equally-loaded scenario, offers considerable gains over the random-access scheme. Hence WA scheme are useful in high-traffic networks.

Note that all users adapt in a round-robin fashion in the simulations. However, as mentioned before, since each user adaptation leads to an increase in the potential function, the algorithm can converge even with non-sequential updates.

7.7.4 Comparison with Greedy WA Game

In this sub-section, we compare the performance of the proposed ISINR WA algorithm with the performance of a greedy IA algorithm (similar to the algorithm suggested in [84]). In the greedy IA algorithm, the utility function of each user is given by the negative of the inverse SINR at its receiver, i.e.,

$$ u^g_k (s_k, s_{-k}) = -s_k^H R_{ii,k} s_k. $$

The user’s utility function increases when the interference at its receiver decreases. Note that the utility function does not incorporate the effect of the user’s action on the other users in the network. The best response of the $k^{th}$ user is given by the eigenvector corresponding to the minimum eigenvalue of the interference-plus-noise cross-correlation matrix $R_{ii,k}$ at the $k^{th}$ receive-node. As mentioned, the greedy algorithm does not converge for all network scenarios (Section 7.4).

Figure 7.11 shows the weighted sum-interference-plus-noise function for the proposed algorithm and the greedy algorithm over independent instantiations of the network. The simulation parameters are the same as used before. It is observed that, in general, the proposed ISINR WA algorithm results in lower interference in the network as compared to the greedy IA algorithm. It can also be seen that the greedy IA algorithm leads to a cyclic allocation of resources in some network scenarios. As noted before, non-convergence or allocation cycles result in a large network overhead.

As mentioned before, the utility function for a user in the proposed SISINR WA algorithm takes into account the effect of the particular user’s actions on the other users in the network. Hence, in general, the
The proposed algorithm is seen to lead to fairer allocation of resources than the greedy IA algorithm. This is illustrated in Figure 7.10 which plots the SINRs of the users in the network for the two adaptation algorithms. It is seen that the SINRs of the users are more closely distributed in the case of the proposed algorithm. In the case of the greedy IA algorithm, the SINRs of some users are very high while the SINRs of other users are very low (Note that a network scenario where the greedy algorithm converges is chosen for this particular simulated illustration example). Theil’s entropy measure, [85] an inequality index (defined below), is used to compare the fairness of the two algorithms.

**Theil’s entropy measure - Inequality Index:** This is a measure of inequality proposed by Theil that derives from the notion of entropy in information theory. The entropy measure, $T$, is given by:

$$
T = \sum_{i=1}^{L} q_i \left( \log q_i - \log \left( \frac{1}{L} \right) \right)
$$

(7.33)

where $q_i$ is the share of the $i^{th}$ group, and $L$ is the total number of groups. The index has a potential range from zero to infinity (for very large values of $L$), with higher values indicating more unequal distribution of resources.

For the computation of Theil’s entropy measure, let

$$
q_i = \frac{r_i}{\sum_{j=1}^{K} r_j}
$$

(7.34)

where, $r_i$ is defined as

$$
r_i \leq \log_2 \left( 1 + \frac{p_{ii}}{\sum_{j=1, j \neq i}^{K} s_j^H s_j p_{ji} + I} \right)
$$

(7.35)
When averaged over allocations in different arbitrary network scenarios, the proposed ISINR algorithm results in a Theil’s measure of 1.3998 while the greedy IA algorithm results in a measure of 2.9691. Hence it can be concluded that the proposed ISINR algorithm indeed results in a fairer allocation of resources than the greedy IA algorithm.

### 7.7.5 Performance in the Presence of Noise

In this subsection the effect of noise on the proposed algorithm is studied. It has been established that NE form the fixed points or steady states of the algorithm and the algorithm converges to these steady states under a best response dynamic. However the steady states are not necessarily stable. This is illustrated in Figure 7.12 which shows the adaptations of the algorithm in the presence of Gaussian noise with power spectral density of $-90\,\text{dB}$. It is seen that the interference in the network is substantially reduced. However, the algorithm continually adapts. Threshold improvement steps can be used to stabilize the algorithm and ensure convergence. This is done by modifying the decision rule for adaptations such that a user does not adapt to a different signature sequence if the its improvement in utility is less than some fixed threshold. The performance of the proposed algorithm with the fixed threshold improvement rule is illustrated in Figure 7.13. It is seen that the adaptations converge in this scenario.

### 7.7.6 Implementation of Algorithm

The proposed SISINR best-response algorithm can be implemented at a centralized controller for the network which has access to information about all the transmit-receive node-pairs in the network. The proposed algorithm is preferable compared to a global search algorithm since it is much simpler to implement (for example the Lagrangian-search algorithm discussed in the previous section requires many more iterations than the proposed algorithm and might not always converge to solutions that satisfy the constraints) and, as shown in the previous section, it leads to the optimal or near-optimal network solutions. Further, the pro-
posed iterative algorithm can easily incorporate variations in channel conditions or network characteristics. A global search will require the entire procedure to be repeated for each variation.

Alternatively, in the absence of a centralized controller (as is common in distributed ad hoc type networks), the algorithm can be implemented at each node-pair in the network. However, this would require any node making adaptation decisions to have access to the signature sequences of all the transmit-nodes in the network and the received power levels at all the receive-nodes from each transmit-node. This can be accomplished by requiring each transmit node to broadcast its sequence and transmit power level and each receive-node to broadcast the channel coefficients for all links at the beginning of the adaptation process and requiring the adapting node to broadcast its new signature sequence after each adaptation. However, this process considerably increase the overhead of the network especially since the signature sequences can have any real-value. This motivates the investigation of reduced feedback schemes for the SISINR game in the next section.

7.8 Algorithm with Incomplete Information

In this section, properties of potential games, especially the better response convergence properties, are exploited to design alternate implementations of the WA algorithm that result in reduced feedback in the network.

7.8.1 Random Better Response Scheme

In this scheme, a transmit-node adapts to a signature sequence chosen at random and sticks to the adaptation if it decreases the weighted sum-interference in the network. This algorithm is formally stated as follows:
Random Better Response SISINR Waveform Adaptation Algorithm

1. Fix the transmit-power levels and initialize codewords $s_k$ for each user.
2. Set $k = 1$
3. while $k \leq K$
   (a) Set count $i = 0$.
   (b) Choose a random sequence $\hat{s}_k \in S$. Adapt transmit-node $k$ to sequence $\hat{s}_k$.
   (c) All receivers in the network send change in ISINR due to sequence adaptation by transmit-node $k$.
   (d) If sum change in ISINR is positive, set $s_k = \hat{s}_k$ and go to step 3.e.
      Else set $i = i + 1$. If $i \leq T$ (some positive number), repeat steps 3.b to 3.d.
   (e) $k = k + 1$
4. Repeat step 2 and 3 until a fixed point is reached or some termination criteria is met.

Convergence and Fixed Points

Since NE are the only fixed points of a random better response 5, the random better response based SISINR WA algorithm theoretically converges to a NE of the game. However, since the signature sequences are real-valued, the action space for the random better response scheme is very large and convergence could be very slow. The slow convergence speed could make this scheme impractical though it involves reduced feedback in the network in each iteration. Convergence speed can be increased by having a directed better response scheme as described in the next subsection.

7.8.2 Gradient-based Better Response Scheme

In this scheme each transmit-node adapts according to the interference environment at its corresponding receiver and does not require information of the interference profiles at other receivers. The interference power at the $k^{th}$ receive-node is given by $I_k (s_k, s_{-k})$. The gradient of $I_k (s_k, s_{-k})$ with respect to sequence $s_k$ is given by

$$d_k (s_k, s_{-k}) = \frac{du_k (s_k, s_{-k})}{ds_k} = 2(s_k^T R_{ii,k} s_k) - 2(s_k^T R_{ii,k} s_k) s_k (s_k^T s_k)^2.$$

(7.36)

Note that the signature sequence $s_k$ is usually known at the receiver corresponding to user $k$. Also, the interference-plus-noise cross-correlation matrix, $R_{ii,k}$, can be expressed as $R_{ii,k} = E [r_k r_k^H] - p_k g_k s_k s_k^H$, assuming bit transmissions from multiple users are uncorrelated. Hence $R_{ii,k}$, and consequently, $d_k (s_k, s_{-k})$, can be estimated by computing and averaging the received correlation matrix with no additional overhead in the network.

In this reduced feedback scheme, the $k^{th}$ receive-node finds $q (q \in \{1, \ldots, N\})$ dimensions in which vector $d_k (s_k, s_{-k})$ has the largest magnitude. The variable $q$ can be used to control the amount of feedback from the receive-node to the transmit-node. The receive-node then finds the step size $\lambda$ that maximizes $I_k (s_k, s_{-k})$ in the direction specified by the $q$ chosen dimensions with the largest magnitude (referred to here as the ascent direction and denoted by $a_q$). For example, if the $d_k (s_k, s_{-k})$ is a 4-dimensional vector given by $d_k = [1 -3 4 2]^T$ and $q = 2$, the receive-node finds the optimum step size along the ascent
direction \( a_q = [0 - 3 4 0]^T \). The optimum step-size \( \epsilon_k \) along \( a_q \) is the solution to the following optimization problem and can be solved using a simple line search procedure:

\[
\max_{\lambda \neq 0} - \frac{(s_k - \lambda a_q)^T R_{ii,k}(s_k - \lambda a_q)}{(s_k - \lambda a_q)^T (s_k - \lambda a_q)}.
\] (7.37)

The transmit-sequence of the user is then adapted along this direction if interference in the network is decreased. This ensures that each adaptation reduces the potential function (Equation (7.16)). The algorithm can be formally stated as follows:

**Gradient-based Better Response SISINR Waveform Adaptation Algorithm**

1. Fix the transmit-power levels and initialize codeword \( s_k \) for each user. Also choose a value for variable \( q \).
2. Set \( k = 1 \)
3. while \( k \leq K \)
   (a) Set count \( i = 0 \).
   (b) Calculate gradient \( d_k (s_k, s_{-k}) \), corresponding ascent direction \( a_q \) and the optimum step-size \( \epsilon_k \) along \( a_q \) at receive-node \( k \).
   (c) Feedback \( a_q \) and \( \epsilon_k \) to the transmit-node \( k \).
   (d) Adapt transmit-node \( k \) to sequence \( \hat{s}_k = s_k + \epsilon_k a_q \sqrt{(s_k + \epsilon_k a_q)^T(s_k + \epsilon_k a_q)} \)
   (e) All receivers in the network that are in transmit-node-\( k \)'s transmission range (i.e., can hear transmit-node-k), send change in ISINR at the receive-node due to sequence adaptation by transmit-node \( k \).
   (f) If the sum change in ISINR is positive, set \( s_k = \hat{s}_k \) and go to step 3.g. Else set \( i = i + 1 \). If \( i \leq T \) (some positive number), set \( \epsilon_k = 0.5 \epsilon_k \) and repeat steps 3.b to 3.f.
   (g) \( k = k + 1 \)
4. Repeat step 2 and 3 until a fixed point is reached.

It is thus seen that for the implementation of this scheme, only feedback of a \( q + 1 \)-dimensioned \( (q < N) \) vector from the receive-node corresponding to an adapting transmit-node and negligible feedback from the other nodes in the network is required. This scheme thus substantially decreases the overhead in the network compared to the best-response iteration based ISINR WA algorithm.

**Convergence and Fixed Points**

As mentioned before, a potential game with a better response dynamic converges and hence the gradient-based better response SISINR WA algorithm also converges. However, the set of fixed points of the algorithm might be larger than the set of NE. This due to the fact that the gradient (7.36) is zero for all eigenvectors (not just for the inferior eigenvector) and the fact the gradient of only a part the utility function of a user is taken into consideration for the waveform adaptation process. Hence the adaptations can get stuck at sub-optimal points. Convergence to the NE can be forced by using a random better response spacer step (wherein each user randomly chooses a sequence that improves its utility function). In the under-loaded or equally-loaded network scenario with white-noise, the optimal configuration is to assign orthogonal signature sequences to each user. Hence, the random better response spacer step can be used
whenever a receive-node notices that after the convergence of the WA algorithm, the signature sequence of its corresponding transmit-node is not orthogonal to the interference sub-space at the receive-node. In the over-loaded scenario, the random-better response step can be added at regular pre-determined intervals. Since NE are the only fixed points of a random better response, the gradient-based algorithm with the spacer steps also theoretically converges to a NE.

7.8.3 Simulation-based Performance Analysis

Figure 7.14 shows the performance of both better-response schemes in an equally-loaded distributed network. Multiple runs of the better response schemes from different initial sequences are illustrated in the plot. In all different runs, the random-better response scheme is observed to have not yet converged. Nevertheless, both schemes are seen to substantially reduce the interference in the network. The gradient-based scheme (with the random better response spacer step) is seen to converge to the optimal orthogonal configuration of signature sequences for users. Figure 7.15 illustrates the convergence of the two better response schemes in an over-loaded network. It is seen that the random better response schemes perform slightly worse that the previous scenario. It is also seen that the gradient-based better response algorithm (without the random spacer step) converges to a sub-optimal fixed point. Figure 7.16 shows the convergence of the gradient-based algorithm for two different values of $q$ and from different initial sequences. It is seen that, in general, increasing $q$ increases the rate at which the interference in the network is reduced (number of iterations required for a specific decrease in the interference level). It is also seen that increasing $q$ decreases the amount of interference in the network after convergence. However increasing $q$ also increases the feedback in each iteration. Hence, the optimal $q$ for a given network size can be found based on the requirements of the amount of feedback that can be accommodated in the network, the convergence time for the network and the amount of interference that can be tolerated in the network.
Figure 7.15: Convergence comparison of the best-response based, the gradient-better-response-based and the random-better-response-based ISINR WA algorithms. The network has 10 users sharing 6 signal dimensions.

Figure 7.16: Convergence of the gradient-based better response ISINR WA algorithm for different values of $q$ and from different random initial sequences for the users. The network has 10 users sharing 6 signal dimensions. It is shown that increasing $q$ increases the rate at which the weighted sum-interference in the network is reduced.
Figure 7.17: The weighted sum-interference-plus-noise function for the ISINR WA algorithm with feedback from a limited number of nodes in the network. The network has 40 users, distributed in a circular region of radius 5m, sharing 10 signal dimensions.

Figures 7.8 and 7.9 compare the gradient-based algorithm (with $q = N - 2$) with the random-access scheme discussed before. The gradient-based scheme is run for 20 round-robin iterations. It is seen that the gradient-based scheme offers considerable gains, especially in the first scenario, over the random-access-scheme for large transmission probabilities. Also note that the gradient-based scheme involves much lesser feedback in the network that the best-response WA algorithm.

7.8.4 Feedback from a Limited Number of Nodes

An alternate way to reduce the feedback in the network is to reduce the number of user-nodes whose effect is included in the utility function of a user-node. This can be done by taking into consideration the fact that receive-nodes are not much affected by transmissions from far away transmit nodes (or transmissions from nodes whose channel coefficients are very small). The utility function of a user-node can thus be modified as follows:

$$
\hat{u}_k(s_k, s_{-k}) = -s_k^H R_{ii,k} s_k - s_k^H \left( \sum_{j \in J_k} s_j s_j^H \frac{p_k g_{kj}^2}{p_j g_{jj}^2} \right) s_k.
$$

(7.38)

Here, $R_{ii,k}$ is the interference-plus-noise cross-correlation term that can be measured at the $k^{th}$ user’s receive-node and $J_k$ is the set of receive-nodes which are close to the transmit-node corresponding to the $k^{th}$ user and hence are most affected by transmissions from the $k^{th}$ user. Hence, the utility function of the $k^{th}$ user can be calculated with the help of feedback from only the nodes in set $J_k$.

Figure 7.17 shows the convergence of the algorithm where users adapt by implementing a best response with respect to the modified utility function specified by Equation (7.38). In the simulations, 40 users sharing 10 signal dimensions are distributed in a circular region with radius 5m. The set $J_k$ for the $k^{th}$ user comprises of receive-nodes such that the distance between the receive node and the $k^{th}$ transmitter is less than the distance between the receive-node and its corresponding transmit-node. The path-loss exponent,
$\alpha = 3$. It is seen that this algorithm with limited feedback also substantially reduces the interference in the network though it does not perform as well as the best response algorithm with complete feedback from all affected nodes in the network. It should also be noted that the weighted sum-interference-plus-noise function does not monotonically decrease, since Equation (7.16) is no longer an exact potential function for the modified utility function in Equation (7.16). However, by judicious choice of the set $J_k$, the fluctuations in the interference function can be made negligible.

Note that this scheme can be used in conjunction with the gradient-based scheme, to further reduce the feedback in the network.

### 7.9 Comparison with Narrow-band Interference Avoidance

In this section, we compare, via simulations, the spreading-based waveform adaptation approach to a narrowband dynamic frequency allocation (DFS) approach where users in the network choose from the available frequency dimensions such that interference in the network is reduced. The objective of this comparison is to re-validate the results in the initial chapters of the dissertation, where it was shown that a spreading-based underlay approach which implements IA gives better network performance than a narrowband overlay approach that implements IA.

In an underloaded or equally-loaded network scenario, optimal allocation for both schemes consists of assigning orthogonal dimensions (orthogonal sequences in the case of WA and orthogonal frequency dimensions in the case of DFS). Hence we consider an over-loaded network scenario with $K = 20$ users sharing $N = 6$ dimensions. We use the same simulation parameters as before. For WA, we use the best-response-based SISINR WA algorithm (Section 7.7). For DFS, we use a modified version of the best-response-based SISINR WA algorithm, where in step 3.a, we choose a frequency that maximizes the utility
function $u_k$ (Equation (7.20)), instead of the minimum eigenvector of $X_k$. Note that the reduced feedback algorithm in Section 7.8 cannot be directly applied to DFS. Figure 7.18 illustrates the CDF of the value for the sum of inverse SINRs in the network (the negative of the potential function). Reducing the SISINR function reduces the interference in the network and hence improves the performance of the network. It is seen from the figure that WA leads to a smaller value of the SISINR and is thereby more preferable from a network perspective. This confirms results from earlier chapters, which show that a spreading-based underlay approach which implements IA is preferable to a NB overlay approach that implements IA.

### 7.10 Variations of Interference Avoidance Algorithms

#### 7.10.1 Variation of Weighted Interference Algorithm

The framework can be used to construct other WA algorithms for interference avoidance. For instance, a WA algorithm, which reduces the sum interference in the network weighted by the received power of individual users, defined below, can easily be constructed using an EPG framework.

$$
V(s) = -\sum_{k=1}^{K} s_k^T \left( \sum_{j=1,j\neq k}^{K} s_j s_j^H p_j g_{jk}^2 p_k g_{kk}^2 \right) s_k.
$$

(7.39)

Note that this function is different in spirit to the function given in Equation (7.16), since here, there is more incentive to provide lower interference to users with larger received powers while in the previous game, stronger users are assumed to be able to tolerate more interference. Therefore, this game might not lead to fair resource allocations but could result in larger sum capacities for the network.

A simple formulation of the utility function for user $k$, such that function (7.39) is an EPG is:

$$
u_k(s_k,s_{-k}) = -s_k^H \left( \sum_{j\neq k,j=1}^{K} s_j s_j^H p_j g_{jk}^2 p_k g_{kk}^2 \right) s_k - s_k^H \left( \sum_{j\neq k,j=1}^{K} s_j s_j^H p_k g_{kj}^2 p_j g_{jj}^2 \right) s_k.
$$

(7.40)

As in the ISINR game, the minimizers of the sum of weighted interferences (Equation (7.39)) form NE for the game.

In [28], a waveform adaptation game is described for a network where each transmit node has multiple collaborating receivers (multi-base networks). Each user iteratively finds sequences that minimizes the sum of interferences at all its receivers weighted by the received power of the user at these receivers. This game is similar to the weighted interference game. Hence, as also mentioned before, the proposed framework can be directly used to construct waveform adaptation games for multiple base networks as well.

#### 7.10.2 IA with respect to Legacy System

Consider a legacy static radio system with which the adaptive distributed network co-exists. Then the ISINR WA game can be directly extended to avoid the interference from or to avoid interfering with the legacy radio system. Let $R_{L,k}$ be the weighted covariance matrix of the sum of the signals received from the legacy transmitters at either the legacy receivers (if its possible to access this information) or the $k^{th}$ receive-node. The utility function of $k^{th}$ user can be modified as follows to incorporate the effect of the legacy transmission:

$$
u_k^L(s_k,s_{-k}) = -s_k^H R_{L,k} s_k - s_k^H \frac{R_{ki,k} s_k}{p_k g_{kk}^2} - s_k^H \left( \sum_{j\neq k,j=1}^{K} s_j s_j^H p_k g_{kj}^2 p_j g_{jj}^2 \right) s_k.
$$

(7.41)
The exact potential function corresponding to this modified utility function is given by

\[ V_L(s) = -\sum_{k=1}^{K} s_k^H \left( \frac{R_{i,k}^2 + R_{L,k}}{p_k g_{kk}^2} \right) s_k. \]  

(7.42)

Each user now adapts to a signature sequence that maximizes the utility function given by Equation (7.41). The existence of a potential function guarantees that each user adaptation monotonically increases the potential function and consequently decreases the interference in the network (which now constitutes the legacy system as well). Since the sum interference in a network is bounded from below, at least one fixed point (formed by a maximizer of the potential function which is also a NE) exists for the game and the algorithm is guaranteed to converge to a NE for the game.

Note that avoiding the interference to a legacy receiver requires information of the interference profile at the legacy receiver unless the channel between the legacy system and the adapting node-pair is symmetric. However, interference from a legacy system to an adaptive distributed network can easily be avoided by using measurements at the receive-nodes corresponding to the adapting transmit-nodes.

### 7.10.3 Algorithm for Combined Power and WA

By studying the utility function for the \( k^{th} \) user in Equation 7.20 and the negative of the potential function given by Equation 7.16, it can be observed that the ISINR game is an exact potential game with respect to the signature sequences \( s_k, k \in \{1, \ldots, K\} \) as well as the transmit power-levels \( p_k, k \in \{1, \ldots, K\} \) of the user node-pairs. In this section, motivated by the existence of an exact potential game, we investigate a joint power and WA algorithm for interference avoidance.

In this algorithm, each user adapts both its power and waveform to derive the best utility for the current state of the network. The game can be played as follows: Each user, in an asynchronous fashion, chooses a transmit waveform and power level that maximizes its utility for the current state of the network. The game can be played by assuming a finite number of transmit power levels for the users or by assuming the power level to range between a minimum (PL) and maximum (PU) power level. In the former case, a user finds a waveform corresponding to each of its power levels which maximizes its utility function for the current state of the network. The user then selects the waveform and power level that corresponds to the maximum utility. In the latter case, the user adapts to a power-level \( \hat{p}_k \) which is the power level that maximizes the utility of user-node \( k \) the current waveform and the current network state and which is defined as follows:

Let \( \bar{p} \) be given by

\[ \bar{p} = \sqrt{\frac{X_1}{X_2}} \]  

(7.43)

where,

\[ X_1 = s_i^H \left( \sum_{j \neq i, j=1}^{K} \frac{s_j s_j^H p_j g_{ji}^2}{g_{ii}^2} \right) s_i \] and \[ X_2 = s_i^H \left( \sum_{j \neq i, j=1}^{K} \frac{s_j s_j^H g_{ii}^2}{p_j g_{jj}^2} \right) s_i \]  

(7.44)

Then \( \hat{p}_k \) is given by

\[ \hat{p}_k = \begin{cases} P_L, & \text{if } \bar{p} \leq P_L \\ P_U, & \text{if } \bar{p} \geq P_U \\ \bar{p}, & \text{otherwise} \end{cases} \]  

(7.45)

The power and waveform adaptation algorithm can be formally stated as follows:

**Best-response-based ISINR Power and Waveform Adaptation Algorithm**

1. Initialize the transmit-power levels codeword \( s_k \) for each user.
Figure 7.19: Convergence of the ISINR power and WA algorithm in a network with 12 users sharing 6 signal dimensions from multiple random initial sequences. $P_U = 500\text{mW}$ and $P_L = 100\text{mW}$. Performance of the algorithm is also compared with that of a ISINR pure WA algorithm.

2. Set $k = 1$

3. while $k \leq K$

   (a) Replace $s_k$ by the eigenvector corresponding to the minimum eigenvalue of $X_k$
   (b) Replace $p_k$ by $\hat{p}_k$
   (c) Repeat steps 3.a and 3.b until a fixed point or a termination criterion is reached.
   (d) $k = k + 1$

4. Repeat step 2 and 3 until a fixed point is reached.

**Convergence**

This utility maximization process (steps 1-3) can be viewed as an extension of the standard "cyclic coordinate" optimization procedure [81] and iteratively each adaptation iteratively increases the potential function. It is hence assured to converge. However, the algorithm could converge to points which are not the NE.

Figure 7.19 illustrates the convergence of the joint power and waveform adaptation algorithm. The transmit power level of a user node is allowed to vary from 100mW to 500mW. Each iteration represents 10 waveform and power adaptations by a user (steps 3.1 and 3.2 is repeated 10 times). It is seen that the algorithm substantially decreases the interference in the network. The figure also compares the joint power and waveform adaptation algorithm with the pure WA algorithm (where the user’s transmit power levels were fixed at 100mW). Multiple runs illustrate the convergence of the algorithms from random initial choice of waveforms. It is observed that the combined waveform and power algorithm leads to better solutions (lesser interference in the network) than the pure WA algorithm. This has also been observed through simulations in different network scenarios and different fixed transmit power levels for the users in the pure
WA algorithm. However, it is to be noted, that the joint power and WA algorithm takes a larger number of iterations than the pure WA algorithm to converge.

7.11 Conclusions and Contributions

A waveform adaptation framework based on potential game theory to construct convergent interference avoidance algorithms in networks with multiple distributed receivers (as in ad hoc networks) was developed in this paper. This is motivated by the fact that direct extensions of myopic greedy IA algorithms do not lead to convergence in these networks. The proposed algorithms lead to solutions that are desirable from a network perspective. Reduced feedback implementations based on properties of potential games are then developed for the IA algorithms. Finally, several variations of IA algorithms based on the framework are also designed. These include IA with respect to legacy systems and joint power and waveform adaptation algorithms for IA. Note that the framework presented in the paper can be utilized to construct adaptations algorithms with other desirable network performance measures as well.

The performance of the waveform adaptation algorithm was also compared with a narrowband dynamic frequency allocation scheme that implements IA. It was shown that the waveform adaptation scheme results in larger performance gains for the network. This re-iterates our investigations in earlier chapters which showed that IA-based underlay techniques are preferable compared to IA-based overlay techniques with respect to the performance of a network that implements these techniques.

The original contributions in this chapter are as follows:

- A framework based on potential game theory that could be used to develop convergent waveform adaptation algorithms in distributed networks was developed.
- A waveform adaptation algorithm that reduces the sum-interference in the network was proposed.
- Reduced-feedback schemes were constructed for the waveform adaptation algorithm by leveraging better response convergence properties of potential games.
- A joint power and waveform adaptation algorithm that reduces the sum-interference in the network was proposed.

The publications that resulted from this chapter are as follows:


Chapter 8

Target-based Joint Power Control and Waveform Adaptation in Distributed Networks

8.1 Introduction and Problem Statement

In the previous chapter, we developed distributed algorithms, where users adapted their transmission parameters to maximize their utility functions. An alternate approach is to develop algorithms, where users try to achieve some target-performance at their receivers (as opposed to maximizing their performance). To this end, in this chapter, we design a joint power control and waveform adaptation algorithm that enables users to meet their target Signal to Interference plus Noise Ratio (SINR) requirements while reducing the transmit power-levels in the network. The algorithm is based on potential game theory and is amenable to a distributed implementation in networks with distributed receivers (as in an ad hoc network).

8.2 Related Work and Motivation

As discussed in previous chapters, distributed WA techniques, which involve the design of signature sequences or waveforms by users such that multi-user interference is reduced, have been extensively investigated for networks with a centralized receiver (or equivalently networks with co-located receivers). The symmetry of interference between users in these networks, lead to very simple greedy adaptation WA schemes. However, these same techniques lead to non-convergence in networks with non-colocated receivers since the mutual interference between users is asymmetric [28]. Hence convergent WA techniques in these networks require some feedback in the network and some possible schemes are proposed in Chapter 7.

WA reduces the interference in the network. However, to ensure that nodes in the network achieve a target-performance requirement, implementation of power control (PC) is also vital. One of the seminal investigations of the design of PC algorithms for achieving target-SINRs was presented by Yates in [86]. In this paper, a “standard power control” framework based on the “standard interference function update” is developed. A function that measures the interference in the network is called “standard” when it satisfies positivity, monotonicity, and scalability properties defined in [86]. Under the framework, users independently adapt their transmit power according to the standard interference function. An example user-update is as follows: User-k updates its power level at stage $n+1$ by evaluating the following update function

$$p_k (n + 1) = p_k (n) I (p) \tag{8.1}$$
Here, $I(p)$ is the standard interference function formed by the ratio of the target SINR and the effective SINR. It is shown that standard power control iterations converge to a unique solution. Also a variety of standard interference functions are identified in the paper. Extensions of the above framework with power limitations and a pricing constraint on the battery life or the power available to the device were developed in [87]. Potential game theory can also be used to develop PC algorithms. In [58], a set of possible utility functions for users that could lead to potential games were identified. By using the properties of potential games, algorithms designed using the identified utility functions are guaranteed to converge to steady states for the network.

Since both transmit power-levels and transmit waveforms influence the interference in the network, it is believed that joint PC and WA can lead to more efficient network solutions. Some papers from the existing literature that address the joint problem are discussed below:

In [20], the optimal sequence and power allocation for a given target SINR requirement in the uplink of a centralized spreading-based system was identified. In [77], a combined PC and sequence adaptation algorithm was presented where users attempt to achieve a target SINR in the uplink of a CDMA network. In the algorithm each user in the network replaces its signature sequence by the minimum eigenvector of the interference plus noise cross correlation matrix. If the achieved SINR is less than the target SINR the user adapts its power to meet the target SINR. Else it leaves it power level unchanged. This procedure is iteratively repeated for all the users in the system. This sequence and power adaptation procedure was shown to iteratively decrease a normalized total weighted squared correlation (NTWSC) function and hence converge. The paper mentioned that the algorithm empirically converged to points where the target SINR constraints are met. However, it is to be noted that the fixed points of the iterative adaptation scheme depend upon the sequence and power vectors with which the system was initialized and hence the algorithm might not converge to the optimal configurations identified in [20].

A different version of a joint algorithm was presented in [78]. This algorithm was formulated on the basis of a Lagrangian-based optimization framework where each user tries to maximize its spectral efficiency (given by the log of 1 plus the SINR of the user at its receiver) subject to the conditions that the target SINR is met and the signature sequence has unit norm. In the algorithm each user in the network replaces its signature sequence by the minimum eigenvector of the interference plus noise cross correlation matrix. It then updates its transmit power by a small increment using a gradient-based approach (unlike the approach in [77], where the powers are adjusted to meet the target SINR). This adaptation is iteratively repeated by all the users in the network until convergence. The convergence of the proposed algorithm to the optimal solution set was not explicitly investigated though it was mentioned that convergence can be established. The paper also stated that when starting from random sequences, the algorithm empirically converged to the optimum configuration (identified in [20]) within some tolerance limits.

In [82], power and sequence control for the downlink of a CDMA system was analyzed. A power and sequence allocation scheme that minimizes the extended total squared correlation (E-TSC) was identified. In this scheme, over-faded users are allocated orthogonal sequences while under-faded users are allocated WBE sequences. The optimality of this scheme, with respect to minimizing the total power consumption, was investigated by comparing it with sequence and power allocations produced by a Lagrangian search method. It was shown that for $K > N$, the allocations are not optimal. However, they are very close to those produced by the Lagrangian method. An iterative algorithm for combined power and waveform adaptations was also presented. However, the convergence of the algorithm was not established.

In [79], the stability of joint power and sequence adaptation algorithms was investigated using game-theoretic techniques. Consider the joint control game. Let the PC sub-game, $G_p(s)$ be the game obtained by fixing the sequences to a fixed set of sequences denoted by $s$. Let the sequence control sub-game, $G_s(p)$, be the game obtained by fixing the power levels for all the users. An important theorem is formulated in the paper which shows that a point $(p, s)$ in the parameter space for a separable game is a Nash equilibria (NE) if and only if $p$ is a NE of sub-game $G_p(s)$ and $s$ is a NE of sub-game $G_s(p)$. Hence a joint game converges
(has a stable point) if and only if the sub-games also converge. The paper also identified an iterative joint PC and sequence adaptation game that converged to the optimal allocations in centralized networks.

It is seen from the above discussion that joint algorithms have been previously analyzed only for networks with centralized or colocated receivers. However, as is the case with pure WA, direct extensions of these algorithms to networks with distributed receivers do not lead to convergence (also indicated in [79] by the fact that if the WA sub-game does not converge, the joint game also does not converge).

In this chapter, we propose a joint algorithm that converges in networks with distributed receivers and which allows users to achieve target SINRs while reducing the sum of transmit power-levels in the network. The algorithm utilizes a potential game framework and the convergent WA schemes for distributed networks, developed in Chapter 7. Note that this is the second joint PC and WA that we propose. An alternate joint algorithm, that minimized the sum-interference in the network (as opposed to achieving target SINRs) and that followed directly from the WA framework, was formulated in the previous chapter (Chapter 7).

The rest of the chapter is organized as follows: The system model is described in Section 8.3. Section 8.4 provides the background and motivation for the joint algorithm. The proposed algorithm is discussed in Section 8.5. An extension of the proposed algorithm with two stages that leads to lower transmit power-levels in the network is described in Section 8.6. Finally, Section 8.7 summarizes the chapter.

### 8.3 System Model

The system model we consider is the same as in the previous chapter (Chapter 7), albeit with slight additions in notation. We re-state the same here for convenience.

We consider a distributed network, made up of a cluster of transmit and receive node-pairs (as in an ad hoc network). This network model is a generalization of a network with co-located or centralized receivers and hence the results presented here are applicable to the centralized network scenario as well.

A signal-space characterization is used to represent the transmit waveform of a node in orthogonal signal dimensions and is referred to as the signature sequence of the node. Let $N$ denote the number of transmission dimensions available to the network and $K$ denote the number of transmitting nodes in the network. Vector $s_k \in \mathbb{R}^{N \times 1}$ is used to denote the signature sequence associated with transmitting node $k$. The signature sequences are allowed to have real values and are assumed to have unit norm. They are hence constrained to the $N - 1$ dimensional sphere $S = \{ s_k \in \mathbb{R}^{N \times 1} : \|s_k\|^2 = 1 \}$. The set of allowable transmit power-levels are denoted by $P$ and the transmit power level of the $l^{th}$ node is denoted by $p_l \in P$. $P$ is assumed to be a compact set with $p_{\text{max}} = \max\{P\}$ and $p_{\text{min}} = \min\{P\}$. The fading coefficient of the channel between the $k^{th}$ transmit node and the $j^{th}$ receive node is denoted by $g_{kj}$. The channel is assumed to be constant over all signal dimensions and also constant over the time required for the adaptation process. The data symbol (assumed to be of zero-mean and unit-variance) transmitted from the $k^{th}$ transmit node is denoted by $b_k$. The received signal at the $j^{th}$ receive node is then given by

$$ r_j = \sum_{k \in \mathcal{K}} \sqrt{p_k} g_{kj} s_k b_k + z, \quad (8.2) $$

where $\mathcal{K} = \{1, \ldots, K\}$, $r_j \in \mathbb{R}^{N \times 1}$, and the vector $z \in \mathbb{R}^{N \times 1}$ models zero mean additive Gaussian noise with variance $\sigma^2$. Again, for notational simplicity, signature sequences from multiple users are assumed to be synchronized at the receivers. Interference is caused at a receive node by transmissions from nodes different from the one associated with the particular receive node. Interference caused is influenced by the correlation between the waveforms of user nodes, transmit power levels and the channel characteristics. Note that we implicitly assume that soft decisions are made which implies that correlations between waveforms (as opposed to Hamming distance) are important.
8.4 Formulation of Target-SINR Problem

Our goal in this paper is to design an adaptation framework where users in the network are able to achieve a feasible target SINR while at the same time minimizing the required transmit power levels in the network. The SINR of the $k^{th}$ user in the network is given by

$$\gamma_k = \frac{p_k g_k^2}{I_k(p,s)}.$$  \hspace{1cm} (8.3)

Here, $I_k(p,s)$ is the interference-plus-noise at the $k^{th}$ user’s receive-node and is given by $I_k(p,s) = s_k^T R_k s_k + \sigma^2$, where $R_k = \sum_{j \in \mathcal{K}, j \neq k} s_j^T s_j s_k H_j g_{jk}^2$. A target SINR vector $\bar{\gamma} = [\bar{\gamma}_1, \ldots, \bar{\gamma}_K]^T$ is feasible if there exists a vector $p = [p_1, \ldots, p_K]^T \in P^K$ and a sequence set $s = [s_1, \ldots, s_K]^T \in S^K$ such that $\gamma_k \geq \bar{\gamma}_k$, \forall $k \in \mathcal{K}$. A approach to identify feasible SINRs for a network can be found in [88].

The adaptation objective can be formulated as the following optimization problem:

$$P1: \begin{cases} \min_{p,s} \sum_{k \in \mathcal{K}} p_k \\
\text{subject to:} \quad \frac{p_k g_k^2}{I_k(p,s)} \geq \bar{\gamma}_k \quad \forall k \\
\quad s_k^T s_k = 1 \quad \forall k \end{cases}$$  \hspace{1cm} (8.4)

Now, consider the following optimization problem:

$$P2: \begin{cases} \min_{p,s} \sum_{k \in \mathcal{K}} I_k(p,s) \frac{\gamma_k}{g_k} \\
\text{subject to:} \quad p_k \geq \gamma_k \frac{I_k(p,s)}{g_k} \quad \forall k \\
\quad s_k^T s_k = 1 \quad \forall k \end{cases}$$  \hspace{1cm} (8.5)

It can be seen that problems $P1$ and $P2$ are closely related and are equivalent if the condition $p_k = \gamma_k \frac{I_k(p,s)}{g_k} \quad \forall k$ can be satisfied for the target power-levels and signature sequences at the optimal point. The combined power and WA framework we develop is a sub-optimal iterative approach to solving problem $P2$ and hence problem $P1$. The framework consists of a WA component that iteratively reduces the objective function of $P2$ and a PC component that ensures that the transmit power levels are feasible (i.e., the power constraints, $p_k \geq \gamma_k \frac{I_k(p,s)}{g_k} \quad \forall k$, for $P1$ and $P2$ are satisfied).

The Waveform Adaptation Component

Let us first consider the design of the WA component. The objective function of $P2$ can be written as follows:

$$I_{sum}(s) = \sum_{k \in \mathcal{K}} \frac{\gamma_k s_k^H R_k s_k}{g_k^2}.$$  \hspace{1cm} (8.6)

The terms involving the $k^{th}$ user can be separated to yield

$$I_{sum}(s) = \frac{\gamma_k s_k^H \left( \sum_{j \in \mathcal{K}, j \neq k} s_j s_j^H p_j g_{jk}^2 + \sigma^2 I \right) s_k}{g_k^2} + \sum_{j \in \mathcal{K}, j \neq k} \frac{\gamma_j s_j^H s_k s_k^H s_j p_k g_{jk}^2}{g_{jj}} + \sum_{j \in \mathcal{K}, j \neq k} \frac{\gamma_j s_j^H \left( \sum_{l \in \mathcal{K}, l \neq k,j} s_l s_l^H p_l g_{lj}^2 + \sigma^2 I \right) s_j}{g_{jj}}.$$  \hspace{1cm} (8.7)

Non-contributing terms
The effects of the actions of the \( k^{th} \) user are only perceived in the first two terms: the first term is the inverse-SINR of the user at its receive node and the second term is the interference caused by the user to all other users in the network. Therefore, the user can iteratively decrease \( I_{sum}( s ) \) if it adapts its waveform to decrease function \( u_k( s ) = s_k^T X_k s_k \), where

\[
X_k = \frac{\gamma_k R_k}{g_{kk}} + \sum_{j \in K \neq k} \gamma_j s_j s_j^T p_k g_{kj}^2 g_{jj}.
\]  

(8.8)

A user's adaptation thus incorporates the effect of the user's waveform on other users in the network. It can be observed that matrix \( X_k \) is a symmetric matrix since it consists of terms that are weighted cross-correlations of the transmit sequences of users and which are hence symmetric. It is also positive definite since the diagonal terms are positive and greater than zero due to the inclusion of the non-zero noise density terms. Hence the utility function can be identified to be a negative weighted Rayleigh quotient of \( X_k \) [80]. Therefore a possible way for a user-\( k \) to adapt its waveform such that it reduces \( u_k( s ) \) is to replace \( s_k \) by the minimum eigenvector of \( X_k \) (denoted by \( s_k, EVI \)). This waveform iteration is referred to in this chapter as the eigenvalue-iteration (EVI). Note that the waveform adaptation discussed in Section 7.7 in Chapter 7, is very similar to this iteration. The only difference being, the \( X_k \) term in the current scenario incorporates the target SINR-level \( \bar{\gamma}_k \) instead of the power-level \( p_k \).

As mentioned in Subsection 7.7.6, to implement the EVI, each adapting node requires access to the signature sequences of all transmit-nodes in the network and the received power levels at all receive-nodes from each transmit-node. This can be accomplished by requiring each transmit node to broadcast its sequence and transmit power level and each receive-node to broadcast the channel coefficients at the beginning of the adaptation process and requiring the adapting node to broadcast its new signature sequence after each adaptation. However, this process could considerably increase network overhead. This motivates the gradient-based-iteration (GI), an extension of the gradient-based better response scheme proposed in Section 7.8.2, which involves limited feedback in the network. In this approach, a user adapts to a waveform that improves its SINR (using a gradient-scheme which only utilizes local information available at the receiver) and sticks to the adaptation if the sum-change in weighted inverse SINR at all the other receivers (a measure of interference in the network) is negative. Hence feedback in the network comprises only of receivers that can hear the adapting user sending their change in weighted inverse SINR due to the adaptation. This also deals with the security problem of broadcasting sequences. This scheme is described in detail as follows:

The gradient of \( I_k( s ) \) with respect to sequence \( s_k \) is

\[
d_k( s ) = \frac{d u_k( s )}{d s_k} = \frac{2 ( s_k^T R_k s_k ) R_k s_k - 2 ( s_k^T R_k s_k ) s_k}{( s_k^T R_k s_k )^2}.
\]

(8.9)

Note that the signature sequence \( s_k \) and \( R_k \) are usually known at the receiver corresponding to user \( k \) and consequently, \( d_k( s_k, s_{-k} ) \), can be estimated no additional overhead in the network. The \( k^{th} \) receive-node then finds \( q ( q \in \{1, \ldots, N \} ) \) dimensions in which vector \( d_k( s_k, s_{-k} ) \) has the largest magnitude (referred to here as the ascent direction and denoted by \( a_q \)). The variable \( q \) is used to control the amount of feedback in the network. The transmit-sequence of the user is then adapted along this direction if interference in the network is decreased. The GI algorithm component for user-\( k \) can be written as follows:

1. Calculate \( d_k( s_k, s_{-k} ) \) and \( a_q \) at receive-node \( k \).
2. Feedback \( a_q \) to the transmit-node \( k \).
3. Adapt transmit-node \( k \) to sequence \( \tilde{s}_k = \frac{s_k + \epsilon_k a_q}{\sqrt{(s_k + \epsilon_k a_q)^H (s_k + \epsilon_k a_q)}} \), where \( \epsilon_k \) is the adaptation constant.
4. All receivers in the network send change in weighted inverse SINR \( \frac{\Delta I}{g_{jj}} \).
5. If the sum change in weighted inverse SINR is negative (interference in the network is reduced), set $s_{k,GI} = \tilde{s}_k$. Else set $s_{k,GI} = s_k$.

It can easily be shown that the above adaptation reduces the value of function $u_k$. More details can be found in Section 7.8.2.

8.5 Algorithm for Joint Power Control and WA

8.5.1 Algorithm Description

As mentioned earlier, the iterative algorithm (stated below) has two components: the WA component which reduces the correlation between user transmissions and the PC component which ensures that the target SINRs of users are met.

**Target-based Power and Waveform Adaptation Algorithm**

1. Initialize transmit power of all users to $p_k = p_{ini,k} = \frac{\bar{\gamma}_k \sigma^2}{g_{kk}}$.
2. Assign random transmit sequences to all users.
3. For $k \in K$
   
   (a) Replace $s_k$ with $\hat{s}_k$.
   
   (b) If $p_k < \frac{\bar{\gamma}_k s_k^T R_k s_k}{g_{kk}}$, replace $p_k$ by $\hat{p}_k = \min \left\{ \frac{\bar{\gamma}_k s_k^T R_k s_k}{g_{kk}}, p_{max} \right\}$.
4. Repeat step 3 until a termination criterion is reached.

Here, $\hat{s}_k = s_k, EVI$ if EVI is used and $\hat{s}_k = s_k, GI$ if GI is used (both are defined in the previous Section 8.4). Note that the PC component of the algorithm (Step 3.b) is based only on SINR measurements at a user’s receiver. Hence most of the network overhead required to implement the algorithm is due to the WA component (Step 3.a). This overhead can be substantially reduced by using GI instead of EVI.

8.5.2 Convergence and Fixed Points

Consider the following function:

$$V(p, s) = \frac{I_{sum}(p, s) + C}{\sum_{k \in K} p_k}. \quad (8.10)$$

where $C$ is a constant with $C > K_{p_{max}} \left( \sum_{i \in K} \sum_{j \in K} g_{ij}^2 \bar{\gamma}_j \right)^2$. The following two theorems show that at each iteration, the waveform and power adaptation by a user monotonically decreases the above function. Note that the function is similar to the NTWSC function discussed for the centralized scenario in [77] and hence can also be considered to be a measure of network performance.

**Theorem 9** Each WA, where user-$k$ ($k \in K$) replaces sequence $s_k$ by $\hat{s}_k$ decreases function $V(p, s)$. 

Theorem 10 Each power adaptation, where user-\(k\) \((k \in K)\) replaces \(s_k\) by \(\hat{s}_k\) is given by

\[
V (p, s) - V (p, (\hat{s}_k, s_{-k})) = \frac{s_k^t X_k s_k - s_k^t X_k \hat{s}_k}{\sum_{k \in K} p_k}.
\]  

(8.11)

Since EVI or GI is used, \(s_k^t X_k \hat{s}_k < s_k^t X_k s_k\). Therefore, \(V (p, s) - V (p, (\hat{s}_k, s_{-k}))) > 0\).

Proof The change in function \(V (p, s)\) when user-\(k\) \((k \in K)\) replaces \(s_k\) by \(\hat{s}_k\) is given by

\[
V (p, s) - V (p, (\hat{s}_k, s_{-k})) = \frac{s_k^t X_k s_k - s_k^t X_k \hat{s}_k}{\sum_{k \in K} p_k}.
\]  

(8.11)

Proof The change in the value of function \(V (\cdot)\) due to the power adaptation by user \(k\) is given by

\[
V (p, s) - V ((\hat{p}_k, p_{-k}), s) = \frac{1}{\hat{p}_k + D} \left( \hat{p}_k + p_k B + \hat{C} \right) - \frac{1}{\hat{p}_k + D} \left( \hat{p}_k + \hat{p}_k B + \hat{C} \right)
\]  

(8.12)

Here, \(\hat{C} = C + \sum_{j \neq k} \gamma_j \sum_{s \in \mathbb{S}_j} s_j s_j^t s_j g_{jj}^2, D = \sum_{j \neq k} p_j\) and \(B = \sum_{j \neq k} s_j s_j^t s_j g_{jj}^2 \gamma_j\). Let \(\bar{p}_k = \frac{s_k^t R_k s_k \gamma_k}{g_{kk}^2}\). Substituting,

\[
V (p, s) - V ((\hat{p}_k, p_{-k}), s) = \frac{\hat{p}_k - \hat{p}_k}{\hat{p}_k + D} + \frac{\hat{p}_k B + \hat{C}}{\hat{p}_k + D} - \frac{\hat{p}_k B + \hat{C}}{\hat{p}_k + D} - \frac{\hat{p}_k B + \hat{C}}{\hat{p}_k + D} \frac{p_k B}{(p_k + D) (\hat{p}_k + D)}
\]  

(8.13)

If \(C\) is chosen such that \(C > K_{p_{max}} \left( \sum_{k \in K} \sum_{j \in K} g_{jj}^2 \right) \) (Note that such a choice is possible since the all values are bounded and can easily be estimated for a given system),

\[
V (p, s) - V ((\hat{p}_k, p_{-k}), s) > 0.
\]  

(8.14)

Hence power adaptation by a user decreases the value of function \(V\).

The target-based power and WA algorithm thus iteratively decreases the value of function \(V (p, s)\), where \(s \in S^K\) and \(p \in P^K\). Therefore, \(-V\) is a potential function for the algorithm and since each iteration of the algorithm monotonically increases the potential function, the algorithm follows a better-response dynamic. It is shown in Chapter 5, that potential games with compact action spaces converge while following the better response dynamic. Since, \(S^K\) and \(P^K\) are both compact sets, the proposed target-based power and WA algorithm converges.

8.5.3 Simulation Analysis

A distributed network is simulated by placing \(K\) transmit and receive nodes uniformly in a circular region with radius \(R\) \((R = 5m\) in the simulations). The power at a receive node from a transmit-node at a distance
Figure 8.1: Convergence of user-SINRs in a network with 7 users sharing 5 signal dimensions and target SINRs, $\bar{\gamma} = [5111111]$ dB. Single-stage Algorithm is employed. The number of bits feedback for GI is $1 (q = 1)$.

of $r$ from the transmit node is assumed to be given by $\frac{p_r}{p}$, where $p_r$ is assumed to be the power received from a transmit node at a distance of 1m and $\alpha$ is the path-loss exponent ($\alpha = 3$ in the simulations). The path loss at a distance of 1m is assumed to be 40dB and $p_{\text{max}} = 500$ mwatts. Noise power is assumed to be $-90$dBw per signal dimension.

Figure 8.1 illustrates the SINRs of the users that over multiple iterations of the joint algorithm. The simulated network has 7 users ($K = 7$) sharing 5 signal dimensions ($N = 5$) and the target SINRs for the users are given by $\bar{\gamma} = [5111111]$ dB. It is seen that the proposed algorithm (with both EVI and GI) converges and users are able to achieve SINRs that are greater than the required target SINRs. Note that users over-achieve the required SINRs since the algorithm does not allow transmit power-levels to be reduced. It can also be seen that using GI as opposed to EVI increases the number of iterations required for convergence. This is due to the fact that EVI corresponds to the optimal WA at each iteration. Figure 8.2 shows the value of function $V(p, s)$ for multiple iterations of the algorithm. It can be observed, as expected, that the algorithm iteratively decreases this function.

The performance of the proposed joint algorithm is now compared with a pure WA algorithm (which is essentially the proposed algorithm without the power update step 3.b and an extension of the pure WA algorithm with in Section 7.7, Chapter 7) and the pure PC algorithm proposed in [86]. Two cases of the pure PC algorithm are considered: in the first case, users transmit using random sequences and in the second case, users choose a signal-dimension randomly from the $N$ available signal dimensions. Figure 8.3 plots histograms of the number of users that achieve target-SINRs resulting from different algorithms. In the simulations, to ensure a fair comparison, the transmit power-level for all users in the pure WA algorithm, is fixed to be the average transmit-power level resulting from the joint algorithm. It is found that the proposed joint algorithm leads to a larger number of users achieving their target SINRs than both the pure PC and the pure WA algorithm. In addition, as can be observed from Figure 8.4, which plots the cumulative distribution of the transmit power-levels resulting from different algorithms, the proposed algorithm leads to lower transmit
Figure 8.2: Plot to illustrate that function $V(s, p)$ monotonically decreases. Network has 7 users sharing 5 signal dimensions and target SINRs, $\bar{\gamma} = [5111111]$dB. Single-stage Algorithm is employed. The number of bits feedback for GI is $1(q = 1)$.

Figure 8.3: Normalized histogram of the number of users that achieve their Target-SINRs that result from different algorithms, over 500 realizations of a random network and with random initial signature sequences. Networks have 7 users sharing 5 signal dimensions. Target SINRs, $\bar{\gamma} = [5111111]$dB.
power-levels than the pure PC algorithms. (Note that we should look at 1-stage for the joint algorithms)

It can also be observed from Figure 8.3 that in algorithms which incorporate WA, users are able to achieve their target SINRs in a larger percentage of network scenarios, as compared to in pure PC algorithms. This is due to the fact that WA reduces the correlation between sequences and hence the effective interference at receivers. Therefore the RHS of the constraint \( p_k \geq \gamma_k \frac{f_k(p,s)}{g_{kk}} \) in problem \( P2 \) is reduced, leading to a larger set of feasible power-vectors. Also, note that a majority of the overhead in the network for the joint algorithm results from the WA component. The PC component for a user is based only SINR measurement at the user’s receiver. Hence the joint algorithm further (and substantially) improves the network solution that results from a pure WA algorithm with minimal additional feedback.

8.6 Two-stage Algorithm for Joint Power Control and WA

Note that, in the proposed algorithm, users might over-achieve their target-SINRs. This can be remedied by including a second stage, where when starting from feasible target SINRs, user power-levels are decreased such that the achieved SINRs are equivalent or nearer to the target requirements.

8.6.1 Algorithm Description

Two-stage Target-based Power and Waveform Adaptation Algorithm

1. Initialize transmit power of all users to \( p_k = p_{ini,k} \).
2. Assign random transmit sequences to all users
3. For \( k \in \mathcal{K} \)
   
   (a) Replace \( s_k \) with \( \widehat{s}_k \).
   
   (b) If \( p_k < \frac{\gamma_k s_k^H R_k s_k}{g_{kk}^2} \), replace \( p_k \) by \( \widehat{p}_k \).

4. Repeat step 3 until a termination criterion is reached.

Stage-2

5. For \( k \in \mathcal{K} \)
   
   (a) If \( p_k > \frac{\gamma_k s_k^H R_k s_k}{g_{kk}^2} \), let \( \widehat{p}_k = \max \left\{ \frac{\gamma_k s_k^H R_k s_k}{g_{kk}^2}, p_{\min} \right\} \).

6. Repeat step 5 until a termination criterion is reached.

8.6.2 Convergence and Fixed Points

The first phase of the algorithm is the same as that proposed in the previous section. Hence the first phase converges by the same arguments. The convergence of the second-stage of the algorithm can be established using the following theorem.

**Theorem 11** Each power adaptation in the second stage of the algorithm, where user- \( k \) \( (k \in \mathcal{K}) \) replaces transmit power level \( p_k \) by \( \widehat{p}_k \in P \) does not increase function \( I_{sum} \) (in other words the objective function of \( P_2 \)).

**Proof** The change in function \( I_{sum} \) when the \( k^{th} \) user adapts from power-level \( p_k \) to power-level \( \widehat{p}_k \) is given by

\[
I_{sum}(s, p) - I_{sum}(s, (p_{-k}, \widehat{p}_k)) = \frac{\gamma_k s_k^H R_k s_k}{g_{kk}^2} - \frac{\gamma_k s_k^H R_k s_k}{g_{kk}^2} + \sum_{j \in \mathcal{K}, j \neq k} \frac{p_k g_{kj}^2 s_j^H s_k s_k^H s_j}{p_j g_{jj}^2} - \sum_{j \in \mathcal{K}, j \neq k} \frac{\widehat{p}_k g_{kj}^2 s_j^H s_k s_k^H s_j}{p_j g_{jj}^2} \tag{8.15}
\]

The first two terms are independent of the transmit power-level of the \( k^{th} \) user. Since \( \widehat{p}_k \leq p_k \), the sum of the last two terms is positive and hence the power-adaptations in the second-stage do not increase \( I_{sum} \). This is also intuitive since a reduction in the transmit power-level of a user decreases the interference it causes to all the other users in the network.

From the above theorem it can be seen that the second stage of the joint algorithm monotonically decreases function \( I_{sum} \) which is bounded from below. Therefore, \( -I_{sum} \) is a potential function for the second stage of the game. Again, since the action set \( P^\mathcal{K} \) is compact, the 2-stage also converges. The corollary below shows that when starting from a feasible solution, the second-stage of the algorithm maintains the feasibility constraint for all users.

**Corollary 1** When starting from feasible power-levels, each power adaptation in the second stage of the algorithm maintains the feasibility condition of the power levels of all users i.e., \( p_k \geq \frac{\gamma_k I_k(p,s)}{g_{kk}^2} \), \( \forall k \).

**Proof** It follows from the proof of Theorem 11, that a power-adaptation does not decrease function \( \frac{\gamma_k I_k(p,s)}{g_{kk}^2} \) \( \forall k \). Thus when starting from a feasible solution, the condition \( p_k \geq \frac{\gamma_k I_k(p,s)}{g_{kk}^2} \) is maintained \( \forall k \).
The following theorem characterizes the solution of the two-stage algorithm for a specific network scenario.

**Theorem 12** In an under-loaded ($K < N$) or equally-loaded ($K = N$) network scenario with a white noise process, the joint two-stage algorithm with EVI as the WA component converges to the optimal solution.

**Proof** The optimal solution in an under-loaded or equally-loaded network corresponds to an allocation of orthogonal sequences to users and transmit power-levels given by $p_k = p_{\text{ini},k}$, $\forall k$ (provided these power-levels are within the transmission constraints.).

Let $(s^*, p^*)$ be the fixed point of the joint power and WA algorithm. Let $A_w(p)$ be the WA component of the algorithm when the power-levels are fixed at $p$ and let $A_p(s)$ be the PC component when the waveforms are fixed at $s$. Then it can easily be shown that $s^*$ is a fixed point of $A_w(p)$ and $p^*$ is a fixed point of $A_p(s)$. It is shown in Theorem 8 that orthogonal sequences are the fixed points of EVI. Hence $s^*$ corresponds to a set of orthogonal sequences. Since orthogonal sequences are allocated to different users, the interference between users is zero and the first-stage of the joint algorithm results in transmit-power levels that satisfy $p_k \geq p_{\text{ini},k}$, $\forall k$ (provided these power-levels are within the transmission constraints.). The second stage of the algorithm then reduces the transmit power-levels of algorithms that over-achieve the required target SINRs resulting in $p_k = p_{\text{ini},k}$, $\forall k$.

### 8.6.3 Simulation Analysis

The SINRs of users over multiple iterations of the joint two-stage algorithm is illustrated in Figure 8.5. The simulated network has the same parameters as that in the previous section. The two stages of the algorithm can be clearly identified in the figure. In the second stage of the algorithm, power-levels of users (with feasible transmit power-levels) are reduced such that target-SINRs are just met. It can also be observed from Figure 8.4 that, as expected, the two-stage algorithm results in lower transmit power-levels than the
Single-stage algorithm. However, the two-stage algorithm requires some additional iterations as compared to the single-stage algorithm.

### 8.7 Conclusions and Contributions

Single-stage and two-stage joint power-control and waveform adaptation algorithms that converge in networks with non-colocated receivers and which are amenable to a distributed implementation, were developed in this paper. Previously, convergent joint power and WA algorithms have been developed only for networks with centralized or co-located receivers. The proposed algorithms allow users to achieve their target-SINRs while reducing the transmit power-levels in the network. The convergence and performance of both algorithms were investigated via theoretical-analysis and simulations. It was shown that the joint algorithms result in better solutions than a pure PC or a pure WA algorithm. In addition, in under-loaded and equally-loaded scenarios, the two-stage algorithm with eigen-value iterations converges to the optimum allocation. Note that the joint algorithm requires additional network overhead as compared to a pure PC algorithm. However, as was shown in the paper, the use of gradient iterations can considerably reduce this overhead.

The original contributions in this chapter are as follows:

1. Single-stage and two-stage joint power-control and waveform adaptation algorithms were proposed that converge in networks with distributed receivers and allow users to achieve their target-SINRs while reducing the transmit power-levels in the network.

2. The proposed algorithms are shown to result in better network solutions that a pure power control or a pure waveform adaptation algorithm.

The following publication resulted from this chapter

Chapter 9

Waveform Adaptation in Ultra Wideband Systems

9.1 Introduction and Problem Statement

The investigations in Chapters 3 and 4 show that spreading-based underlay techniques which also implement some form of interference avoidance are desirable from the perspective of both, reducing interference to a legacy system as well as improving the performance of a network that implements the scheme. It is also shown that the performance benefit offered by the technique is accentuated by an increase in spreading-factor or transmission bandwidth available to the system. This motivates the use of UWB-based underlay schemes that implement interference avoidance.

To this end, in this chapter, we propose the use of a novel sequence-optimization-based UWB scheme which can be used to maximize energy capture from the channel as well as mitigate interference. The proposed scheme also allows a reduction in receiver complexity. We then investigate transmission as well as adaptation strategies for these schemes in multi-user networks. We also also discuss how the scheme could be used to reject or avoid narrowband legacy interferers.

9.2 Related Work and Motivation

The large operation bandwidth of UWB systems [9] leads to a received signal very rich in resolvable multipath components. This renders a simple transmit pulse-shape correlator highly inefficient in these systems. Receiver structures, such as template-based receivers [89] and Rake receivers [90] [91], can improve the energy capture and have been widely researched for UWB. However, it is found that to get adequate performance, these schemes require substantial complexity at the receiver [92]. Receiver complexity can be reduced by using a transmit precoding technique called time-reversal (TR), where the transmit signal consists of the time-reversed channel impulse response (CIR) convolved with the transmit pulse-shape ([93], [94] and reference within). This technique leads to a focusing of the energy at the receiver and results in a significant energy capture with very few correlation taps at the receiver. Note that the TR technique with a single tap at the receiver can perform like the perfect Rake or the perfect template-based receiver and is equivalent to pre-Rake diversity combining [95].

An alternate strategy in multipath channels, as suggested in [96], is to precode the transmit sequence using a spreading sequence. The detection at the receiver is based on correlation with a fixed spreading sequence. The transmitted spreading sequence is optimized in order to maximize the energy output after
correlation. A multipath beamforming effect is achieved, where the energy from several multipath components is coherently added at the receiver. However, the energy capture in this method is limited by the use of a fixed sequence at the receiver. In [97] and [98], the receiver and transmitter sequences were jointly optimized to maximize the output Signal-to-Noise-Ratio (SNR) in a CDMA-based spread spectrum system. The joint sequence optimization (SO) scheme was extended to a UWB system in [99]. It was shown that the scheme exploits the fine time resolution of the UWB pulse (that results in a large number of resolvable multipath components) and achieves better performance than all the three UWB schemes discussed in the previous paragraph. However, the number of correlation taps (or length of correlation sequence) at the receiver required for optimal energy capture is dependent upon the length of the CIR and, in general, a larger number of correlation taps are required than the TR scheme. In our work (Section 9.4.1), we propose a new generalized SO scheme which exploits the benefits of SO and at the same time allows simpler receivers.

In the presence of multiuser interference, the transmit and receive sequences for a user could be optimized to improve the energy capture from the channel as well as to reject interference from the other users in the system. Design of such transmit and receive sequences which minimize the Mean Square Error (MSE) at a user’s receiver in a CDMA system was presented in [71], [72] and references within. However, note that a change in the transmit sequence for a user itself changes the interference environment for the other users in the system, prompting the other users in the system to also adapt their transmit sequences. Iterative sequence adaptations for such a scenario have also been investigated in the literature. [100] develops an iterative transmit and receive sequence adaptation scheme which minimizes the sum-MSE in the network. In [72], sequence adaptation schemes which greedily minimize the MSE for individual users and schemes which minimize the sum-MSE in the network were investigated. However, neither paper establishes the convergence of the adaptation procedures. This issue was addressed in [71], which develops and establishes the convergence of three different sequence adaptation schemes for three different criteria: minimization of the sum-MSE in the network, minimization of a measure of total-sum-correlation (TSE) in the network and maximization of the capacity of the network. All the above mentioned papers consider CDMA networks with co-located or centralized receivers. Sequence adaptation schemes for distributed multiuser networks with multipath channels have not been investigated.

In this chapter, we develop a generalized sequence-optimization scheme that maximizes the energy capture from the rich UWB multipath channel and at the same time allows simpler multiuser receivers. We then extend this scheme to incorporate the multiuser rejection techniques discussed above, which were previously proposed for centralized CDMA networks with multipath channels. Results in Chapter 7 can be used to show that the adaptations for centralized networks, proposed in the papers discussed above, do not lead to convergence in distributed networks. Hence, alternate algorithms for sequence adaptation in UWB networks with distributed receivers, based on the potential game formulation in Chapter 7, are developed. An alternate strategy, which does not require users in the network to iteratively adapt their transmit sequences, yet still has the potential to generate performance gains in distributed networks, is also investigated.

The rest of the chapter is organized as follows: The UWB system model is described in Section 9.3. The generalized sequence-optimization scheme which maximizes energy capture from the multipath channel is described in Section 9.4. Sections 9.5 and 9.6, investigate sequence adaptations for centralized and distributed networks respectively. Sequence adaptation with respect to narrowband legacy interference is discussed in Section 9.7. Finally, Section 9.8 concludes the chapter and delineates original contributions.
### 9.3 System Model

We consider a sequence-based impulse-radio UWB transmission scheme with bi-phase modulation. The transmit signal consists of a sequence convolved with the transmit pulse-shape and can be written as

$$s(t) = \sqrt{p} \sum_{i=-\infty}^{\infty} b_i \sum_{n=0}^{N_x-1} x_n w (t - nT_s - iT_f), \quad (9.1)$$

where $b_i = \pm 1$ are the data bits, $X = [x_0 \ x_1 \ldots \ x_{N_x-1}]^T$ is the real-valued transmit signal sequence, $T_s$ is the spacing between consecutive chips of the sequence and $w(t)$ is the unit-energy transmit pulse-shape of width $T_w$ with autocorrelation function

$$R(\tau) = \int_{0}^{T_w} w(t)w(t-\tau)dt. \quad (9.2)$$

It is assumed that the effect of the transmit and receive antennas are incorporated into $w(t)$, $p$ is the average transmit power of the user and $T_f$ is the symbol length (sometimes referred to as the frame duration [9]). Let $T_c$ be the maximum excess delay of the channel. The effects of inter-symbol-interference (ISI) are ignored in the analysis by assuming that $T_f > (N_x - 2)T_s + T_c + T_w$ (length of a received symbol). However, note that the proposed technique can be easily adapted to mitigate ISI in a manner similar to [93] which adapts the TR scheme for ISI. We assume the following $L$-path channel tap-delay model:

$$h(t) = \sum_{l=1}^{L} \alpha_l \delta (t - \tau_l), \quad (9.3)$$

where $\alpha_l$ and $\tau_l$ are the amplitude (including polarity) and delay of the $l^{th}$ multipath component, respectively. The received signal in the time interval $[0, T_f]$ can then be written as:

$$r(t) = \sqrt{p}b_0 \sum_{l=1}^{L} \alpha_l \sum_{n=0}^{N_x-1} x_n w (t - nT_s - \tau_l) + n(t), \quad (9.4)$$

where $n(t)$ is an zero-mean additive white Gaussian noise (AWGN) process with power spectral density (PSD) $\frac{N_0}{2}$. The received signal is first convolved with the time-reversed pulse-shape (pulse-match-filtered) and then sampled with a period of $T_w$. The samples are then correlated with a unit energy sequence $Y = [y_1, \ldots, y_{N_y-1}]^T$. Note that to capture maximum energy from the channel, $N_y = \left\lceil \frac{(N_x - 2)T_s + T_c + T_w}{T_w} \right\rceil$. The two operations can be represented by a correlation with the following unit-energy template:

$$s_0(t) = \sum_{m=0}^{N_y-1} y_m w (t - mT_w). \quad (9.5)$$

The decision statistic can be written as follows:

$$z = \sqrt{p}b_0 \sum_{l=1}^{L} \alpha_l \sum_{n=0}^{N_x-1} x_n \int_{0}^{T_f} w (t - nT_s - \tau_l) s_0(t)dt + \int_{0}^{T_f} n(t)s_0(t)dt \quad (9.6)$$

where $d$ is the desired signal term and $n$ is the noise term. Since the correlation template is normalized to be of unit energy, it can easily be shown that $n$ is a zero mean Gaussian random variable with variance $\frac{N_0}{2}$. The signal-to-noise-ratio (SNR) of the received signal is therefore given by

$$SNR = \frac{2pd^2}{N_0} \quad (9.7)$$
Now, the expression for \( d \) can be rearranged as follows:

\[
d = \sqrt{pb_0} \sum_{m=0}^{N_y-1} y_m \sum_{n=0}^{N_x-1} x_n \sum_{l=1}^{L} \alpha_l R(nT_s + \tau_l - mT_w).
\] (9.8)

For mathematical simplicity, assume that \( T_w = N_w T_s \), where \( N_w \) is a positive integer and let

\[
h_k = \sum_{l=1}^{L} \alpha_l \tau_l - (k - N_w)T_s.
\] (9.9)

Then \( d \) can be re-written as

\[
d = \sqrt{pb_0} \sum_{m=0}^{N_y-1} y_m \sum_{n=0}^{N_x-1} x_n R((m+1)N_w - n). \] (9.10)

Note that \( \{h_k\} \) are the equivalent channel coefficients. These can be constructed from the original channel coefficients \( \{\alpha_l\} \) by using Equation (9.9) or alternatively, can be directly obtained by transmitting pulse-shape \( w(t) \) and sampling at the receiver every \( T_s \) seconds after convolution with the time-reversed pulse-shape \( (\text{pulse-match-filtering}) \).

To allow the transmit sequence to exploit the time resolution of the channel, \( T_s \) can be chosen to be the channel sampling period (or, in other words, the sampling rate of the channel estimates). Without loss of generality, it can be assumed that the delay spread \( T_c = N_c T_s \), where \( N_c \) is a positive integer. In this scenario, the length of \( \{h_k\} \) is \( N_h = N_c + 2N_w - 2 \). The expression for \( d \) can now be written in matrix form as

\[
d = Y^T H X, \] (9.11)

where \( H \) is an \( N_y \times N_x \) matrix whose \( m \)th row \( H[p] \) is given by \( H[p] = [h_{(m+1)N_w-1}, \ldots, h_{(m+1)N_w-N_x}] \).

Note again that \( h_k = 0 \) when \( k < 0 \) and when \( k > N_h - 1 \). Also note that maximum energy can be captured by allowing \( N_x = N_c \) and, as mentioned earlier, by allowing \( N_y = \left\lceil \frac{(N_x - 2) + N_c + N_w}{N_w} \right\rceil \).

### 9.4 Sequence Optimization for Multipath Channels

In this section, we jointly design the transmit sequence \( X \) and the received sequence \( Y \) such that the energy capture from the channel is maximized.

#### 9.4.1 Solution with Optimal Correlator

We desire to find \( Y \) and \( X \) that maximize the SNR at the detector. It is well known that the matched filter achieves maximum SNR. Thus, the optimal sequence used at the receiver is matched to the transmit signal convolved with the channel. Then, \( Y = HX \), and \( d = X^T H^T H X \). Now, we wish to find the sequence \( X \) which maximizes SNR. We also require the transmitted signal (the transmit sequence convolved with the pulse-shape) to be normalized to one. Let \( w = [w_0, \ldots, w_{N_w-1}]^T \) represent the pulse-shape sampled with a period equal to that of the channel sampling period (=\( T_s \) in the present scenario). Then the transmitted signal can be represented by \( WX \), where \( W \) is a \( (N_W + N_x - 1) \times N_x \) matrix whose \( p \)th column is given by \([0_{1 \times p-1} \ w^T \ 0_{1 \times N_w-1-p+1-N_w}]^T \). Note that if the spacing between consecutive chips of the transmit sequence is \( T_w \) instead of the channel sampling period, \( X = WX \).

The sequence optimization process can now be formulated as the following optimization problem,

\[
P1 : \max_{X \neq 0} d(X) = \max_{X} \frac{X^T H^T H X}{X^T W^T W X}. \] (9.12)
Since $W^TW$ is a symmetric matrix, it can be diagonalized as $W^TW = U^T\Lambda U$, where $U$ is a unitary matrix and $\Lambda$ is a diagonal matrix. Let $C = \Lambda^{\frac{1}{2}}UX$. Then

$$d(X) = \frac{X^TH^THX}{X^TW^WX} = \frac{C^T\Lambda^{-\frac{1}{2}}UH^THU^T\Lambda^{-\frac{1}{2}}C}{C^TC} = \frac{C^TR_{hh}C}{C^TC}. \quad (9.13)$$

The matrix $R_{hh}$ is symmetric and positive semi-definite. Hence, $d(X)$ is the Rayleigh quotient of $R_{hh}$ and is maximized when $C$ corresponds to the eigenvector corresponding to the maximum eigenvalue of $R_{hh}$ [80], which is denoted by $C_{opt}$. The optimal sequence $X_{opt}$ that solves $P_1$ is therefore given by $U^T\lambda^{-\frac{1}{2}}C_{opt}$. Note that in time-varying channels, the channel coefficients ($\{\alpha_l\}$) or alternatively the equivalent channel coefficients ($\{\alpha_l\}$), the transmit sequence $X$ and the correlation sequence $Y$ at the receiver need to be estimated periodically.

### 9.4.2 Solution for Sub-optimal Correlator with Reduced Receiver Complexity

Receiver complexity can be reduced by using a shorter correlation sequence at the receiver of length $N \leq N_y$. From the structure of matrix $H$ and Equation (9.11), it can be seen that maximum energy can be captured if the correlation taps are towards the center of a received symbol period where the rows of the $H$ matrix have the largest number of non-zero terms. Let the shortened receiver correlation sequence be given by $Y_s = [y_0, \ldots, y_{N_S-1}]^T$. The decision portion of the received signal can then be re-written as

$$d = Y^TH(s)X. \quad (9.14)$$

Here, $H(s) = [H[N - \lfloor \frac{N_y}{2} \rfloor]^T, \ldots, H[N]^T, \ldots, H[N + \lfloor \frac{N_y}{2} \rfloor]^T]^T$ and $N = \lfloor \frac{N_h}{N_w} \rfloor$. The optimal receiver is the matched filter and the optimal transmit sequence can be computed in a manner similar to that in the previous section to be $X_s = U^T\Lambda^{-\frac{1}{2}}C_{opt,s}$, where $C_{opt,s}$ is the maximum eigenvector of matrix $\Lambda^{-\frac{1}{2}}UH(s)^TH(s)U^T\Lambda^{-\frac{1}{2}}$.

### 9.4.3 Relationship with Existing UWB Receiver Techniques

In the SO scheme proposed in [99], the chips of the transmit sequence are separated by a pulse width and hence the transmit sequence length is given by $N_{x, Ib} = \lceil \frac{T_w}{T_s} \rceil$. The proposed technique, on the other hand, allows the chips to be separated by the channel sampling period. The length of the transmit sequence is therefore given by $N_x = \lceil \frac{T_w}{T_s} \rceil > N_{x, Ib} = \lceil \frac{T_w}{T_f} \rceil$. Also, the time resolution of the transmit sequence in the proposed scheme is equal to the time resolution of the channel estimates. These factors result in a potential for larger energy capture for the proposed scheme than the scheme in [99]. Also, the proposed technique optimizes the transmit waveform for a “desired” length of the received correlation sequence (length can be less than $N_y$) as opposed to a correlation sequence of fixed length ($N_y, Ib = 2N_{x, Ib} - 1 \approx N_y$) that has to be used in [99]. Hence the proposed technique could be used to construct receivers with lesser complexity than the SO scheme in [99].

In the TR scheme [93], the time-reversed CIR is used as the transmit sequence. Let the CIR with channel sampling period of $T_s$ be represented by $[\check{\alpha}_0, \ldots, \check{\alpha}_{N_c-1}]$. Then the transmit sequence for TR is given by $X_{TR} = [\check{\alpha}_{N_c-1}, \ldots, \check{\alpha}_0]^T$. The coefficients of the received correlator $Y_{TR}$ are again chosen such that they are matched to the received signal after correlation with the time-reversed pulse-shape. Hence the decision statistic for TR is given by $d = \frac{X_{TR}^TH^THX_{TR}}{X_{TR}^TW^WX_{TR}}$. Note that the proposed SO scheme uses a sequence $X$ that maximizes the decision statistic $d$ instead of using $X_{TR}$. The proposed scheme can therefore be expected to perform as well as TR or to outperform TR.
When only one correlation tap is used at the receiver, it can easily be shown that the optimal transmit sequence corresponds to the time-reversed CIR. In this scenario, the performance of the optimal scheme is therefore equivalent to that of TR which is in turn similar to the performance of a perfect Rake receiver and a perfect template-based receiver. Note that all schemes require channel estimation. However, all precoding schemes could perform channel estimation at the transmitter (assuming TDD operation) while the Rake and template-based schemes need to estimate the channel at the receiver.

### 9.4.4 Performance Analysis

Simulations were run to examine the performance of the scheme and were based on indoor non-line-of-sight CIRs recorded from measurements taken at MPRG [101]. A Gaussian monocycle of duration $T_w = 3$ nanoseconds is used. The sampling period $T_s = 150$ picoseconds. Results are averaged over multiple channel measurements. Therefore, the number, amplitudes and delays of the multipath components as well as the sequence length $N_x$ change over each measurement. The average transmit sequence length is around $N_x = 450$.

The performance of the proposed SO scheme is plotted in Figure 9.1. It is seen that the proposed scheme with the optimal correlator ($N_y \approx 50$) outperforms the perfect-Rake-receiver by about 3 dB. This is due to the fact that the optimized transmit and receive sequences lead to a boost in the received power level due to coherent multipath combining in the channel. This effect is similar to the gains seen when using transmit beamforming. Also note that the inter-chip separation in the scheme proposed in this paper is equivalent to the sampling period, whereas the inter-chip separation in the SO scheme proposed in [99] (which outperforms the perfect Rake by 2dB) is equivalent to the pulse-duration. Therefore, the proposed scheme has a longer transmit sequence and is able to exploit a finer time-resolution of the multipath components as compared to the latter scheme, resulting in a larger energy capture. In addition, it is seen from the figure that the performance of the proposed scheme with 11 receiver taps ($N_y = 11$) is similar to the performance of the SO scheme proposed in [99] (with $N_y,ib \approx N_y \approx 50$) though the receiver complexity of the latter scheme is greater than the former scheme. The use of additional taps at the transmitter thus reduces the number of correlation taps required at the receiver. The plot also illustrates that the use of a sub-optimal correlator with only half the number of required taps ($N_y = 21$) results in near-optimal energy capture. In fact, performance similar to the perfect-Rake can be achieved by the use of just a single-tap correlator at the receiver.

The performance of the proposed scheme is compared with the TR scheme in Figure 9.2. In the simulations, the TR scheme with the suboptimal correlator chooses $N_y$ taps with the largest amplitude from the available $N_y$ taps. As expected, the performance of the TR scheme and the SO scheme when only a single tap is used at the receiver are equivalent and similar to the performance of the Rake receiver or the performance in AWGN (Note that the performance is not equivalent to that of AWGN since only the actual channel coefficients $\{\alpha_l\}$ were normalized to one; the energy of the equivalent channel coefficients $\{h_k\}$, obtained from expression (9.9), that incorporates the effect of pulse-shape correlations need not be one.). However, when the number of taps is greater than one, SO outperforms a TR scheme with the same receiver complexity. In addition, it can be seen that the SO scheme with just 11 taps outperforms the TR scheme which uses the optimum ($N_y$) number of taps. Hence the proposed technique can be used to construct receivers with less complexity than the TR scheme.

### 9.5 Sequence Optimization for a Centralized Multiuser Networks

In this section, we jointly design transmit and receive sequences, which are optimized to improve the energy capture from the channel as well reject multiuser interference, in a centralized UWB multiuser system. Note that transmit sequence adaptation by a user also effects the interference in the system and consequently
Figure 9.1: Performance of the proposed SO scheme. Performance is averaged over multiple NLOS measured channel profiles.

Figure 9.2: Performance comparison of the proposed SO and the TR scheme where the transmitted sequence is the inverse CIR. Performance is averaged over multiple NLOS measured channel profiles.
the choice of transmit and receive sequences for the other users in the systems. Hence a potential game
framework to establish the convergence of the sequence adaptation in the network is also investigated.

9.5.1 System Model

We consider a multiple access channel where \( K \) users communicate with a centralized receiver. The transmit
waveform of each user is assumed to be sequence-based. For simplicity of notation, the users’ received
signals are assumed to be synchronized. However, note that the scheme can be directly and easily be
extended to an asynchronous system. Following the notation in the previous section, let \( H_k \) be the channel
matrix from the \( k^{th} \) user to the receiver, \( p_k \) be the signal power of the \( k^{th} \) user, \( X_k \) be the transmit sequence
vector of the \( k^{th} \) user and \( b_k \) be the data bit transmitted by the \( k^{th} \) user. Let \( \overline{X}_k = \frac{X_k}{\sqrt{(X_k^T W^T W X_k)}} \) be the
normalized transmit sequence. The normalization is done to ensure that the signal transmitted by a user has
unit energy. As before, the effects of ISI are ignored by assuming the symbol length to be larger than the
length of a received symbol. The signal vector at the receiver in a single symbol interval after pulse-matched
filtering and sampling can be represented as

\[
R = \sum_{k=1}^{K} \sqrt{p_k} H_k \overline{X}_k b_k + N. \tag{9.15}
\]

Here, \( R \) is a column vector of length \( N_y \) and \( N \) is also a column vector of length \( N_y \) of zero-mean Gaussian
random variables. The decision statistic for \( k^{th} \) user is constructed by correlating the received sample vector
\( R \) with a unit-energy correlation template \( Y_k \) and is given by

\[
Y_k^T R = Y_k^T \sqrt{p_k} H_k \overline{X}_k b_k + Y_k^T \sum_{j=1, j \neq k}^{K} \sqrt{p_j} H_j \overline{X}_j b_j + Y_k^T N. \tag{9.16}
\]

Here, \( d_k \) is the desired signal term, \( i_k \) is the interference term and \( n_k \) is the noise term. Since the correlation
template is normalized to be of unit energy, \( n_k \) is a zero mean Gaussian random variable with variance
\( \sigma^2 = \frac{N_0}{2} \). The SINR for the \( k^{th} \) user is given by

\[
\text{SINR}_k = \frac{p_k d_k^2}{\mathbb{E}\{i_k^2\} + \frac{N_0}{2}}. \tag{9.17}
\]

In the above expression, the expectation term is over multiple bit transmissions. Assuming binary transmis-
sions with \( b \in \{1, -1\} \), \( \mathbb{E}\{b_k^2\} = 1 \) and assuming that bit transmissions from multiple users are uncorrelated,
it can be shown that \( \mathbb{E}\{i_k^2\} = Y_k^T \left( \sum_{j \neq k} p_j H_j \overline{X}_j \overline{X}_j^T H_j^T \right) Y_k \). Let \( Z_k = \sum_{j \neq k} p_j H_j \overline{X}_j \overline{X}_j^T H_j^T + \sigma^2 I \). Then
the SINR for the \( k^{th} \) user can be re-written as

\[
\text{SINR}_k = \frac{p_k (Y_k^T H_k \overline{X}_k)^2}{Y_k^T Z_k Y_k}. \tag{9.18}
\]

9.5.2 Solution for Optimal Sequences

In the analysis in this section, the optimal receiver correlation sequence \( Y_k \) with respect to maximizing the
SINR of the \( k^{th} \) user is first designed as a function of the transmit sequence \( \overline{X}_k \). The optimal transmit
sequence \( X_k \) is then designed.
The optimal sequence $Y_k$ (that corresponds to the optimal linear receiver), which maximizes $SINR_k$, is the solution to the following maximization problem.

$$\max_{Y_k \neq 0} SINR_k = \max_{Y_k \neq 0} \frac{Y_k^T H_k X_k X_k^T H_k^T Y_k}{Y_k^T Z_k Y_k}. \quad (9.19)$$

$Z_k$ is a Hermitian matrix. Hence, it can be diagonalized as $Z_k = V_k \Lambda_k V_k^T$, where $V_k$ is a unitary matrix and $\Lambda_k$ is a diagonal matrix. Let $A_k = \Lambda_k^{1/2} V_k^T Y_k$. Then,

$$SINR_k = \frac{p_k A_k^T \Lambda_k^{-1/2} V_k^T H_k X_k X_k^T H_k^T V_k \Lambda_k^{-1/2} A_k}{A_k^T A_k}. \quad (9.20)$$

The sequence $A_k$ that maximizes $SINR_k$ is given by,

$$A_k = \Lambda_k^{-1/2} V_k^T H_k X_k. \quad (9.21)$$

The optimal $Y_k$ is therefore given by,

$$Y_k = V_k \Lambda_k^{-1} V_k^T H_k X_k = Z_k^{-1} H_k X_k. \quad (9.22)$$

The optimum receive-vector $Y_k$ is henceforth referred to as the Maximum SINR (MSINR) filter for the multi-user scenario and is denoted by $Y_k^{MSINR}$. Note that the normalization of the receiver correlation sequence $Y_k$ (such that it is of unit-norm) does not affect the SINR of a user (since both the numerator and the denominator contains the term $Y_k$ raised to the same exponent). Hence the optimization is independent of the normalization at the receiver.

The SINR of the $k^{th}$ user when the optimal sequence is employed at the receiver is given by

$$SINR_k = \frac{p_k (Y_k, MSINR) H_k X_k}{Y_k, MSINR Z_k Y_k, MSINR} \left( X_k^T H_k^T Z_k^{-1} H_k X_k \right)^2 = \frac{p_k X_k^T H_k^T Z_k^{-1} H_k X_k}{X_k^T W W^T X_k} \quad (9.23)$$

Note that since $Z_k$ is a symmetric matrix, $\left(Z_k^{-1}\right)^T = Z_k^{-1}$. Also $Z_k Z_k^{-1} = I$. The SINR is hence given by

$$SINR_k = \frac{p_k X_k^T H_k^T Z_k^{-1} H_k X_k}{X_k^T W W^T X_k}. \quad (9.24)$$

The solution for the optimal transmit sequence with respect to maximizing SINR can therefore be framed as the following optimization problem:

$$\max_{X_k \neq 0} SINR_k = \max_{X_k \neq 0} \frac{X_k^T H_k^T Z_k^{-1} H_k X_k}{X_k^T W W^T X_k}. \quad (9.25)$$

Matrix $H_k^T Z_k^{-1} H_k$ is symmetric and positive definite. The optimal transmit sequence can be calculated in a manner similar to Subsection 9.4.1 and is given by $X_k^{MSINR} = U^T \Lambda^{-0.5} C_{opt, MSINR}$, where $C_{opt, MSINR}$ is the maximum eigenvector of matrix $\Lambda^{-0.5} U H_k^T Z_k^{-1} H_k U^T \Lambda^{-0.5}$ and $U^T \Lambda U = W W^T$.

Note that $Z_k = E \left[ RR^T \right] - p_k H_k X_k X_k^T H_k^T$, assuming bit transmissions from multiple users are uncorrelated. Hence, in the absence of transmissions from the $k^{th}$ user, $Z_k$ can be estimated at the receiver by computing and averaging the received correlation matrices with no additional load on the system. If the interference seen at the transmitter and the receiver is the same, the optimum transmit sequence can be calculated at the transmitter. Else, it is calculated at the receiver. The optimum transmit sequence is then fed back to the transmit-node.
9.5.3 Solution for Sub-optimal Receiver with Reduced Receiver Complexity

Receiver complexity can be reduced by using a shorter receive sequence of length $N_y \leq N_y$. As before, the receiver taps could be positioned towards the center of a received symbol. However, if some information about the interference is available at the receiver, the receiver taps could also be positioned at a location in the received symbol period, where least interference is observed. For example, if the channel from an interfering user is short as compared to the channel from the desired transmitter and signals from both arrive at the same time, the correlation taps could be positioned towards the end of the received symbol. Let $Y_{(s),k} = [y_0, \ldots, y_{N_y-1}]^T$ be shortened receiver sequence for the $k^{th}$ user. Then the SINR for the $k^{th}$ user is given by

$$\text{SINR}_k = \frac{p_k \left( Y_{(s),k}^T H_{(s),k} \overline{X}_k \right)^2}{Y_{(s),k}^T Z_{(s),k} Y_{(s),k}}. \quad (9.26)$$

Here, $H_{(s),k} = \left[ H_k \left[ N - \left[ \frac{N}{2} \right] \right]^T, \ldots, H_k \left[ N \right]^T, \ldots, H_k \left[ N + \left[ \frac{N}{2} \right] \right]^T \right]^T$, $Z_{(s),k} = \left[ Z_k \left[ N - \left[ \frac{N}{2} \right] \right]^T, \ldots, Z_k \left[ N \right]^T, \ldots, Z_k \left[ N + \left[ \frac{N}{2} \right] \right]^T \right]^T$, and $N$ is the midpoint of the desired location for the correlation taps at the receiver. The optimum transmit and receive sequences can now be calculated in a manner similar to that in the previous subsection (Subsection 9.5.2).

Note that this technique could be also be used to further reduce multi-user interference in a centralized network. The transmit sequences for users could be designed such that the energy from different users are focused in different regions of the received symbol period.

9.5.4 Solution for Minimum Mean Square Error

In this section, the optimal transmit sequence and receive filter sequence with respect to minimizing the Mean Square Error (MSE) for a user transmission at the receiver in a multipath channel is investigated.

The optimization problem for the choice of receive filter and transmit sequence that minimizes the MSE of the $k^{th}$ user at the receiver in a multipath channel can be written as

$$\max_{Y_k \neq 0} MSE_k = \max_{Y_k \neq 0} E \left\{ \left( Y_k^T R - \sqrt{\lambda_k} b_k \right)^2 \right\}. \quad (9.27)$$

Here, $\lambda_k$ is the maximum eigenvalue of $\Lambda^{0.5} U H_k^T Z_k^{-1} H_k U^T \Lambda^{0.5}$ and is an upper bound on the maximum energy that can be collected from the multipath channel. The expression for the MSE of the $k^{th}$ user can be expanded to give

$$MSE_k = E \left\{ Y_k^T R R^T Y_k + \lambda_k b_k^2 - 2 \sqrt{\lambda_k} b_k Y_k^T R_k \right\}. \quad (9.28)$$

The expectation of the received cross-correlation matrix is given by, $Z = \sum_{j=1}^K p_j H_j^T X_j X_j^T H_j + \sigma^2 I$. Then, $MSE_k$ can be re-written as

$$MSE_k = Y_k^T Z Y_k + \lambda_k - 2 \sqrt{\lambda_k p_k} Y_k^T H_k \overline{X}_k. \quad (9.29)$$

The first derivative of $MSE_k$ is

$$\frac{dMSE_k}{dY_k} = 2ZY_k - 2\sqrt{\lambda_k p_k} H_k \overline{X}_k. \quad (9.30)$$

The second derivative of $MSE_k$ is $Z$ which is positive definite. Hence $MSE_k$ is a convex function of $Y_k$ and its global minimum is given by

$$Y_k,_{MMSE} = \sqrt{\lambda_k p_k} Z^{-1} H_k \overline{X}_k. \quad (9.31)$$
Note that the MMSE filter $Y_{k,MMSE} = \sqrt{\lambda_k p_k} Z^{-1} H_k \overline{X}_k$ is just a scaled version of the MSINR filter $Y_{k,MSINR} = Z^{-1} H_k \overline{X}_k$. This is shown as follows: $Y_{k,MMSE}$ can be written as

$$Y_{k,MMSE} = \sqrt{\lambda_k p_k} Z^{-1} H_k \overline{X}_k = \sqrt{\lambda_k p_k} \left( Z_k + p_k H_k \overline{X}_k \overline{X}_k^T H_k^T \right)^{-1} H_k \overline{X}_k. \quad (9.32)$$

Using the matrix inversion lemma [80] which states that

$$(A + xy^T)^{-1} = A^{-1} - A^{-1}xy^T A^{-1}, \quad (9.33)$$

it can be shown that

$$Y_{k,MMSE} = \sqrt{\lambda_k p_k} \left( Z_k + p_k H_k \overline{X}_k \overline{X}_k^T H_k^T \right)^{-1} H_k \overline{X}_k$$

$$= \left( Z_k^{-1} - \frac{p_k Z_k^{-1} H_k \overline{X}_k \overline{X}_k^T H_k^T Z_k^{-1} H_k \overline{X}_k}{1 + p_k \overline{X}_k^T H_k^T Z_k^{-1} H_k \overline{X}_k} \right) H_k \overline{X}_k$$

$$= \sqrt{\lambda_k p_k} \left( Z_k^{-1} H_k \overline{X}_k - \frac{p_k Z_k^{-1} H_k \overline{X}_k \overline{X}_k^T H_k^T Z_k^{-1} H_k \overline{X}_k}{1 + \overline{X}_k^T H_k^T Z_k^{-1} H_k \overline{X}_k} \right)$$

$$= \sqrt{\lambda_k p_k} \left( \frac{\sqrt{\lambda_k p_k} Z_k^{-1} H_k \overline{X}_k}{1 + \overline{X}_k^T H_k^T Z_k^{-1} H_k \overline{X}_k} \right) = \alpha Y_{k,MSINR}. \quad (9.34)$$

The denominator is constant for a given received cross-correlation matrix. Hence $Y_{k,MSINR}$ is just a scaled version of $Y_{k,MSINR}$.

Substituting the optimal MMSE receiver in the expression for $MSE_k$, we get

$$MSE_k = \lambda_k - \lambda_k p_k \overline{X}_k H_k^T Z^{-1} H_k \overline{X}_k. \quad (9.35)$$

Since the first term is a constant for a given channel, $MSE_k$ is minimized when the second term is maximized. The second term is maximized when the transmit sequence is the eigenvector corresponding to the maximum eigenvalue of $H_k^T Z^{-1} H_k$. Again, by using the matrix inversion lemma, $MSE_k$ can be expressed in terms of $Z_k$ as follows:

$$MSE_k = \lambda_k - \lambda_k \frac{p_k \overline{X}_k H_k^T Z_k^{-1} H_k \overline{X}_k}{1 + p_k \overline{X}_k H_k^T Z_k^{-1} H_k \overline{X}_k}. \quad (9.36)$$

The function $\frac{x}{1+x}$ is a monotonically increasing function of $x$ for $x > 0$, as is the case here. Hence $MSE_k$ is minimized by the maximum value possible value of $\overline{X}_k H_k^T Z_k^{-1} H_k \overline{X}_k$. This is the same condition derived for maximizing user-$k$’s SINR (Equation (9.25)). The optimal transmit sequence is therefore given by $X_{k,MMSE} = U^T \Lambda^{-0.5} C_{opt,MMSE}$, where $C_{opt,MMSE}$ is the maximum eigenvector of matrix $\Lambda^{-0.5} H_k^T Z_k^{-1} H_k U^T \Lambda^{-0.5}$ and $U^T \Lambda U = W^T W$.

It is thus established that the optimal transmit and receive sequences with respect to maximizing the MSINR for a user are also optimal with respect to minimizing the MSE of the user.

### 9.5.5 Sequence Adaptation Algorithm for the Network

In the previous subsection, we derived the optimum transmit sequence $X_k$ and the optimum MSINR receive filter sequence $Y_{k,MSINR}$ such that the SINR at the receiver for the transmission from user $k$ is maximized. However, note that the interference to a particular user’s transmission and consequently its SINR at the receiver is influenced by the transmit sequences employed by other users in the network. Hence, when multiple users in the network adapt their transmit sequence based on the multipath channel to the receiver and the interference perceived at the receiver, it becomes important to investigate the overall convergence and steady state characteristics of the adaptations in the network. A potential game model is used towards this end in this subsection.
Potential Game Formulation

The sum capacity of a multiple access multipath channel is given by [71]

\[ C_{\text{sum}} = \frac{1}{2} \log \det \left( 1 + \sigma^2 (Z - \sigma^2 I) \right) \]  

(9.37)

Here, \( Z \) is the received cross-correlation matrix defined as

\[ Z = \sum_{j=1}^{K} p_j H_j \bar{X}_j \bar{X}_j^T H_j^T + \sigma^2. \]

This can be simplified as follows.

\[ C_{\text{sum}} = \frac{1}{2} \log \det \left( 1 + \sigma^2 \left( Z_k - \sigma^2 I \right) \left( I + p_k (\sigma^2 I + (Z_k - \sigma^2 I))^{-1} H_k \bar{X}_k \bar{X}_k^T H_k^T \right) \right) \]

(9.38)

Using the fact that \( \det(AB) = \det(BA) \) and letting \( A = Z_k^{-1} H_k \bar{X}_k \) and \( B = \bar{X}_k^T H_k^T \), the above expression can be re-written as,

\[ C_{\text{sum}} = \frac{1}{2} \log \det \left( 1 + \sigma^2 \left( Z_k - \sigma^2 I \right) \right) + \frac{1}{2} \log \det \left( I + p_k Z_k^{-1} H_k \bar{X}_k \bar{X}_k^T H_k^T \right) \]

(9.39)

It can be seen by comparing the above expression and Equation (9.24) that the sum-capacity of the network is improved if user-\( k, \forall k \), chooses a sequence that improves its SINR (since \( \text{SINR}_k = p_k H_k \bar{X}_k \bar{X}_k^T H_k^T \)). Hence a game with utility function \( \text{SINR}_k \) for user-\( k, \forall k \) is an ordinal potential game with \( C_{\text{sum}} \) as the potential function. The existence of this function with the above property was indicated in [71] from the perspective of CDMA sequences in multipath channels. Also note that the utility function for a user-\( k \) is maximized if it adapts to the maximum SINR sequence \( \bar{X}_k,_{\text{MSINR}} \). Hence the adaptation where a user replaces its current sequence by \( \bar{X}_k,_{\text{MSINR}} \) is the best response dynamic for the game.

Adaptation Algorithm

The WA algorithm that can be constructed using the potential game framework described in the previous subsection is formally stated here:

**Distributed Multiuser Multipath Waveform Adaptation Algorithm**

1. Fix the transmit-power levels and initialize codeword \( \bar{X}_k \) for each user.
2. Set \( k = 1 \)
3. while \( k \leq K \)
   (a) Calculate \( \bar{X}_k,_{\text{MSINR}} \) based on the channel and interference cross-correlation matrices (Section 9.5.2).
   (b) Set \( \bar{X}_k = \bar{X}_k,_{\text{MSINR}} \) (at transmit-node)
   (c) At the receiver, for \( j \in \{1, \ldots, K\} \), replace receive filter \( Y_j \) by \( Y_j,_{\text{MSINR}} \).
   (d) \( k = k + 1 \)
4. Repeat step 2 and 3 until a fixed point is reached.

Note that if the interference seen at the transmitter and the receiver is the same, \( \bar{X}_k,_{\text{MSINR}} \) for user-\( k \) can be calculated at the transmitter. Else, it is calculated at the receiver. The sequence can then be fed back to the transmit-node.
Convergence of Algorithm and Fixed Points

The NE of the game are given by transmit sequences that satisfy the following expression:

\[ X_k = U^T \Lambda^{-0.5} C_k, \text{ where} \]
\[ \Lambda^{-0.5} U H_k^T Z_k^{-1} H_k U^T \Lambda^{-0.5} C_k = a_{\text{max},k} C_k, \quad k \in \{1, 2, \ldots, K\} \]  

(9.40)

Here, \( a_{\text{max},k} \) is the maximum eigenvalue of matrix \( \Lambda^{-0.5} U H_k^T Z_k^{-1} H_k U^T \Lambda^{-0.5} \) and \( U^T \Lambda U = W^T W \). As established before (Chapter 5), a potential game following a best-response dynamic converges to the NE of the game. Hence, the fixed points of the multiuser multipath WA algorithm are also given by sequence multisets satisfying Equation (9.40). Also, the maximizers of the potential function constitute a subset of the NE for the game. Since the potential function for the multiuser multipath WA algorithm, given by the sum-capacity function, is continuous and bounded (the capacity of a multiuser multipath channel cannot increase to infinity), the potential function has at least one maximizer. Hence the game has at least one NE and consequently the algorithm has at least one fixed point and is guaranteed to converge. Note that the fixed point of the algorithm could be a maximizer of the sum-capacity in the network. In any case, the algorithm iteratively improves the sum-capacity of the network.

Note that each iteration of the algorithm only requires the user to know the interference profile at its receiver. Hence the implementation of the algorithm involves minimal network overhead.

9.5.6 Performance Analysis

The performance of a single user that employs the optimum transmit-sequence and receive sequence derived in Section 9.5.2 is illustrated in Figure 9.3. As before, the simulations are performed with indoor NLOS CIRs recorded from measurements. A Gaussian monocycle of duration \( T_w = 500 \) picoseconds and a channel sampling period of 12.5 picoseconds is used. For ease of implementation, the inter-chip separation is assumed to be equal to the width of the pulse-shape \( T_s = T_w \) (this reduces the length of the transmit sequence and hence the processing complexity). A load of 160 equal-power users using 160-length transmit sequences (which are optimized for the multipath channel as in Section 9.4.1) is assumed. The max SNR transmit sequence (again from Section 9.4.1) with a MSINR receive filter fails, because performance is dominated by the high cross-correlation between the channels of the users and the MSINR receiver is unable to reject all the multiuser interference. A CDMA-like scheme where each user is assigned a length-160 random binary spreading sequence is also simulated. Despreading is followed by a perfect Rake receiver with perfect channel knowledge. This system also fails, because the multipath structure combined with the high multi access interference overcomes the spreading gain. The designed optimal transmit and receive sequence for multiple users (Subsection 9.5.2) effectively mitigates interference. Performance is close to that of a single-user system with a perfect Rake receiver.

Figure 9.4 illustrates the convergence the distributed WA algorithm described in Section 9.5.5 where all users adapt to their respective optimal transmit and receive sequences. At the beginning of the adaptation process, the transmit sequence of each user is initialized with the sequence optimized for the multipath channel (from Section 9.4.1). The receive-sequence of each user is the corresponding MSINR filter sequence. Each iteration in the plot represents an adaptation by a single user. The number of users in the network is assumed to be 10. Noise variance is chosen such that \( \frac{2E_s}{N_0} = 10\text{dB} \). It is seen that each iteration increases the value of the sum-capacity of the network, illustrating the existence of a potential function. Also, the algorithm converges in around 10 round-robin iterations. However, note that the convergence time can grow with the number of adapting users in the system. This is illustrated in Figure 9.5 which shows the convergence of the WA algorithm in a network with 30 users. The system has not yet converged after 20 round-robin iterations. Nevertheless, it is seen that the SINRs of the users have been substantially improved.
Figure 9.3: Performance of sequence adaptation optimized for multiple-users. 160 equal-power users. Performance is averaged over multiple NLOS measured channel profiles.

Figure 9.4: Convergence of the distributed multiuser multipath WA algorithm. The number of users in the network is 10.
Figure 9.5: Convergence of the distributed multiuser multipath WA algorithm. The number of users in the network is 30.

Figure 9.6: Comparison of inverse-CDF of SINR of users with different transmission schemes in a centralized network with 10 users. Performance is averaged over multiple instantiations of the network which use NLOS channels generated using a modified Saleh-Valenzuela channel model for UWB. The length of the CIR is fixed at 30. For the WA algorithm, SINRs after 10 round-robin iterations are used.
Figure 9.7: Comparison of inverse-CDF of SINR of users with different transmission schemes in a centralized network with 30 users. Performance is averaged over multiple instantiations of the network which use NLOS channels generated using a modified Saleh-Valenzuela channel model for UWB. The length of the CIR is fixed at 30. For the WA algorithm, SINRs after 10 round-robin iterations are used.

Figures 9.6 and 9.7 illustrate the inverse-cumulative distribution of the SINRs of users for different transmission schemes in a network with 10 users (network-1) and 30 users (network-2) respectively. The distribution is generated by using multiple instantiations of the network with NLOS channels generated using a modified Saleh-Valenzuela channel model for UWB. For ease of implementation, the channel sampling period is chosen such that the length of each CIR is around 30. Also, the chip interval is chosen to be equal to the pulse duration which is assumed to be smaller than the channel sampling-period. Noise variance is chosen such that $\frac{2E_p}{N_0} = 10\text{dB}$. Note that in network-2, the number of users in the network is equal to the channel length and hence, comparable to the transmit-sequence length (a measure of the number of orthogonal transmission dimensions available in the network).

The performance of a user in the network with no multiple access interference (network with single user), using the transmit and receive sequences to maximize energy capture, forms an upper-bound on performance. It is seen that the WA algorithm, after just 10 iterations is able to achieve performance nearly identical to that of the single-user performance in network-1 and close to that of the single-user performance in network-2. In both scenarios, the resulting SINRs for users are greater than the performance of a perfect Rake. Using the SNR sequence with the MMSE receiver also offers some gains when the number of users is less as compared to the channel length (around 60% of the users achieve SINRs greater than the SINR in AWGN and almost all users have SINRs greater than 0dB.). This is due to the fact that energy capture from the channel (while using SNR sequences) compensates for the multi-user interference that is not rejected by the MMSE receiver. On the other hand, when the number of users is comparable to the channel length (network-2), the scheme fails (around 30% of the users achieve SINRs less than 0dB). In this scenario, since the number of users is comparable to the channel (and hence the length of the transmit and receive-filter sequences), the percentage of multi-user interference that can be rejected decreases. The gains from energy capture from the channel are not enough to offset the loss in performance due to unrejected interference. However, note that in general the length of the transmit sequence (which is dependent upon the length of
the CIR) is much greater than 30 (which is only used here for computational simplicity). The length of the sequence is around 160 when the inter-chip duration is assumed to equal to the pulse-width and the parameters in [99] are used. It is around around 400 when the inter-chip duration is assumed to equal to the channel-sampling period and the parameters from Section 9.4.4 are used. Hence the number of users that can be accomodated in the network, before the scheme fails is quite large. However, the WA algorithm offers substantial improvement over just using the SNR sequence with an MMSE filter, with very little additional overhead in the network. Note that the performance with just a matched-filter is poor in both network scenarios since it is overwhelmed by the multi-user interference.

9.5.7 Other Possible Potential Game Formulations

In this section, two alternate potential games for WA adaptations in multiuser multipath channels are investigated. The first game considers the sum MSE in the network as a potential function while the second game considers a measure of the total sum correlation in the network as the potential function.

Sum-MSE-based Potential Function

The sum-MSE (SMSE) of the centralized network can be expressed as

\[ \text{SMSE} = \sum_{k=1}^{K} \text{MSE}_{k} = \sum_{k=1}^{K} \lambda_{k} - \lambda_{k} p_{k} \bar{X}_{k}^{T} H_{k} H_{k}^{T} H_{k} \bar{X}_{k}. \]  

(9.41)

The matrix inversion lemma can be used to expand this in terms of the transmit sequence of the \( k \)th user as

\[ \text{SMSE} = \sum_{i=1}^{K} \lambda_{i} - \sum_{i=1, i \neq k}^{K} \left( \lambda_{i} p_{i} \bar{X}_{i}^{T} H_{i}^{T} Z_{k}^{-1} H_{i} \bar{X}_{i} - \lambda_{i} p_{k} p_{i} \frac{\bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{i} \bar{X}_{i} \bar{X}_{k}^{T} H_{i}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}}{1 + p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}} \right) \]

\[- \lambda_{k} \frac{p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}}{1 + p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}}. \]

(9.42)

Let \( \lambda_{i} = \lambda_{m} = \max \lambda_{i}, \forall i. \) Then,

\[ \text{SMSE} = K \lambda_{m} - \sum_{i=1, i \neq k}^{K} \left( \lambda_{m} p_{i} \bar{X}_{i}^{T} H_{i}^{T} Z_{k}^{-1} H_{i} \bar{X}_{i} \right) + \lambda_{m} \frac{p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} - \sigma_{k}^{2} Z_{k}^{-2} H_{k} \bar{X}_{k}}{1 + p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}} \]

\[- \lambda_{m} \frac{p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}}{1 + p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}} \]

\[ = K \lambda_{m} - \sum_{i=1, i \neq k}^{K} \left( \lambda_{m} p_{i} \bar{X}_{i}^{T} H_{i}^{T} Z_{k}^{-1} H_{i} \bar{X}_{i} \right) - \lambda_{m} \sigma_{k}^{2} \frac{p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-2} H_{k} \bar{X}_{k}}{1 + p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}}. \]

(9.43)

In the above Equation (9.43), it can be noted that only the last term depends upon the adaptations of the \( k \)th user. Hence, SMSE can be written as

\[ \text{SMSE} = A_{k} \left( \bar{X}_{-k} \right) - \lambda_{m} p_{k} \sigma_{k}^{2} \frac{\bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-2} H_{k} \bar{X}_{k}}{1 + p_{k} \bar{X}_{k}^{T} H_{k}^{T} Z_{k}^{-1} H_{k} \bar{X}_{k}}. \]

(9.44)
Here, \( A_k (\overline{X}_k) = K \lambda_m - \sum_{i=1,i \neq k}^K \left( \lambda_m p_i X_i^H Z_k^{-1} H_i X_i \right) \). Let \( S_k = D^\frac{1}{2} V^T \overline{X}_k \), where, \( V D V^T = 1 + H_k^T Z_k^{-1} H_k \) (\( V \) is a unitary matrix and \( D \) is a diagonal matrix). Then, (9.43) can be re-written as,

\[
SMSE = A_k - \lambda_m p_k \frac{S_k^T D^{-\frac{1}{2}} V^T H_k^1 Z_k^{-2} h_k V D^\frac{1}{2} S_k}{S_k^T S_k} \quad (9.45)
\]

This expression is minimized when \( S_k \) is the maximum eigenvector of \( D^{-\frac{1}{2}} V^T H_k^1 Z_k^{-2} h_k V D^\frac{1}{2} \). Hence the optimal sequence, \( \overline{X}_k \), for user-\( k \) that minimizes SMSE is given by

\[
\overline{X}_{k,MSMSE} = \frac{VD^{-\frac{1}{2}} S_k}{S_k^T D^{-1} S_k} \quad (9.46)
\]

The transmit sequence \( X_{k,MSMSE} \) for user-\( k \) that minimizes SMSE can be found using the relation \( \overline{X}_k = \sqrt{X_k^T W^T W X_k} \). Therefore, in each iteration when the \( k^{th} \) user updates its transmit sequence with \( X_{k,MSMSE} \), the sum MSE in the network is reduced. Consequently, a WA algorithm in which each user iteratively replaces its transmit sequence with \( X_{k,MSMSE} \) is an exact potential game with the SMSE function as the exact potential function. This formulation (though not mentioned as a potential game) is derived in [71]. Note that, as in the MSINR game, the transmit sequence adaptation by a particular user does not require knowledge of the transmit sequences and channel characteristics of other users.

**TSC-based Potential Function**

Now consider a potential function given by

\[
V (S) = \sum_{k=1}^K \sum_{j=1,j \neq k}^K \left( X_k^T H_k^1 H_j X_j \sqrt{p_k \sqrt{p_j}} \right)^2 - 2 \sum_{k=1}^K (1 - \sigma^2 p_k) X_k^T H_k^1 H_k X_k. \quad (9.47)
\]

The first term is the total sum cross-correlation (TSC) of the received sequences. Decreasing the TSC is known to decrease the interference at a centralized receiver. The second term corresponds to the sum of the maximum possible energy capture by user transmissions. Hence decreasing the interference in the network and increasing the energy capture by users, decreases the function \( V (S) \). Function \( V (S) \) can thus be used a useful measure of network performance.

Separating out the terms involving the \( k^{th} \) user, we get,

\[
V (S) = 2 \sum_{j=1,j \neq k}^K \left( X_k^T H_k^1 H_j X_j \sqrt{p_k \sqrt{p_j}} \right)^2 - 2 (1 - \sigma^2) X_k^T H_k^1 H_k X_k p_k + A_k. \quad (9.48)
\]

Now choose the utility function for the \( k^{th} \) user to be

\[
u_k (S) = 2X_k^T H_k^1 Z_k H_k \overline{X}_k p_k - 2X_k^T H_k^1 H_k \overline{X}_k p_k. \quad (9.49)\]

Then the transmit sequence that maximizes the utility function is the minimum eigenvector of

\[
\Lambda^{-\frac{1}{2}} U H_k^1 (Z_k - I) H_k U^T \Lambda^{-\frac{1}{2}}, \text{ where } U^T \Lambda U = W^T W. \]

This sequence can be shown to also minimize function \( V (S) \) for any particular choice of transmit sequences for other users in the network. Consequently a WA algorithm in which each user iteratively replaces its transmit sequence with a sequence that minimizes the above utility function (Equation 9.49) is an exact potential game with function \( V (S) \) as the exact potential function. Again note that, as in the MSINR game, the transmit sequence adaptation by a particular user does not require knowledge of the transmit sequences and channel characteristics of other users.
As illustrated before (Chapter 5), an exact potential game following either a best or better response
dynamic converges. The formulation of the exact potential game (as opposed to an ordinal potential game in
Section 9.5.5) in the above two algorithm thus guarantees convergence via a better response dynamic. This
facilitates the design of reduced feedback schemes that exploit the better response convergence property (as
has been done in the Chapter 6). It is also found that the performance of the above two algorithm is similar
to the algorithm in Section 9.5.5. Hence these algorithm can be used instead of the algorithm in Section
9.5.5 when the network overhead is to be reduced.

9.6 Sequence Adaptation for Distributed Multiuser Networks

In this section, we discuss extensions of the sequence adaptation scheme for UWB networks with dis-
tributed receivers. Recall from Chapter 7 that direct extensions of greedy sequence adaptation algorithms,
where each user adapts it sequence only on the basis of the interference seen at its receiver, could lead to
non-convergence in networks with distributed receivers. The potential-game-based algorithm framework
discussed in Chapter 7, Section 7.7, that uses information from other receivers in the network, could be used
to design convergent WA algorithms for distributed UWB networks. Such a development is presented in
Subsection 9.6.2. However, it is found that adaptation schemes in multipath channels could result in con-
siderable feedback in the network. Hence a reduced feedback adaptation approach and an alternate strategy,
which does not require users in the network to iteratively adapt their transmit sequences are presented in
Subsection 9.6.4 and Subsection 9.6.3 respectively.

9.6.1 System Model

We consider a network with \( K \) user-receive node-pairs. The channel matrix from the \( k^{th} \) transmit-node
to the \( j^{th} \) receive-node is assumed to be \( H_{kj} \). The other notations are the same as used in the previous
sections. Again, for mathematical simplicity, we assume the signals from different users are synchronized
at a receiver. The results can easily be extended to an asynchronous system. The signal vector at the \( j^{th} \)
receive-node after pulse-matched filtering and sampling can be represented by

\[
R_j = \sum_{k=1}^{K} \sqrt{p_k} H_{kj} X_k b_k + N. \quad (9.50)
\]

Here, \( R \) is a column vector of length \( N_y \) and \( N \) is also a column vector of length \( N_y \) of zero-mean Gaussian
random variables. The interference-plus-noise crosscorrelation matrix at the \( j^{th} \) receive-node is given by

\[
Z_j = \sum_{k=1, k \neq j}^{K} H_{kj} X_k X_k^T H_{kj}^T + \sigma^2 I. \quad (9.51)
\]

9.6.2 Potential Function-based Adaptation framework

The sum of the SINRs (SINRs) of the users in the network is used as the potential function (The WA game
in Section 7.7 uses the sum of ISINRS. However, the WA game in this section uses the sum of SINRs since
this is found to yield a relatively simpler formulation for the multipath scenario considered here.). Note that
while using an MMSE receiver, the SINR of the \( k^{th} \) user is \( p_k X_k^T H_{kk}^T Z_k^{-1} H_{kk} X_k \) (from Equation (9.24)).
The sum of SINRs (SSINR) is then given by

\[
V(X) = \sum_{k=1}^{K} p_k X_k^T H_{kk}^T Z_k^{-1} H_{kk} X_k. \quad (9.52)
\]
The terms involving the $k^{th}$ user can be separated as follows:

$$
V(X) = p_kX_k^TH_k^TZ_k^\Lambda^{-1}H_kX_k + \sum_{j \neq k} p_jX_j^TH_j^TZ_j^\Lambda^{-1}H_jX_j
$$

$$
= p_kX_k^TH_k^TZ_k^\Lambda^{-1}H_kX_k + \sum_{j \neq k} p_jX_j^TH_j^T\left(Z_{j,-k} + p_kX_kH_k^TH_k^TX_k^T\right)^{-1}H_jX_j
$$

$$
= p_kX_k^TH_k^TZ_k^\Lambda^{-1}H_jX_k + \sum_{j \neq k} p_jX_j^TH_j^T\left(Z_{j,-k}^{-1} - \frac{p_kZ_{j,-k}^{-1}X_kH_k^TH_k^TX_k^TZ_{j,-k}^{-1}}{1 + p_kX_kH_k^TH_k^TX_k^T}\right)H_jX_j
$$

(Using the Matrix inversion lemma)

$$
= p_kX_k^TH_k^TZ_k^\Lambda^{-1}H_jX_k - p_k\sum_{j \neq k} X_j^TH_k^T\left(p_jZ_{j,-k}^{-1}X_jH_j^TH_j^TX_j^T\right)H_jX_k
$$

$$
+ \sum_{j \neq k} p_jX_j^TH_j^T\left(Z_{j,-k}^{-1} - \frac{p_kZ_{j,-k}^{-1}X_kH_k^TH_k^TX_k^T}{1 + p_kX_kH_k^TH_k^TX_k^T}\right)H_jX_j
$$

Non-contributing terms

Here, $Z_{j,-k} = Z_j - p_jX_kH_k^TH_k^TX_k^T$. It is seen that only the first two terms of the above expression (denoted by $u(X_k, X_{-k})$) are a function of user-k’s transmit sequence. Hence, an exact potential game with $SSINR$ as the potential function could be formulated by using $u(X_k, X_{-k})$ as the utility function for the $k^{th}$ user. An algorithm could be formed by allowing each user in the network to sequentially adapt its waveform to a waveform that maximizes (forming a best-response dynamic) or reduces (forming a better-response dynamic) its utility function. The existence of an exact potential function would guarantee the convergence of such an algorithm. Also, when the best-response dynamic is used, the algorithm converges to the NE of the game. However, the evaluation of the above utility function at a user-node would require considerable feedback in the network, since the node is required to know the channel matrices and the interference profiles at all the other receiver-nodes and the transmit-sequences of all the other user-nodes in the network. Hence in the next section, we discuss a gradient-based reduced feedback implementation of this algorithm.

### 9.6.3 Reduced Feedback Sequence Adaptation Algorithm

In this scheme, each user forms an adaptation sequence without knowledge of the interference profiles at the other users in the system. From Equation (9.24), the SINR of the $k^{th}$ user is given by

$$
\text{SINR}_k = p_kX_k^TH_k^TZ_k^\Lambda^{-1}H_kX_k = \frac{p_kX_k^TH_k^TZ_k^\Lambda^{-1}H_kX_k}{X_k^TW^TWX_k}.
$$

The above expression can be rewritten as

$$
\text{SINR}_k = p_kC_k^T\Lambda^{-\frac{1}{2}}UH_k^TZ_k^\Lambda^{-1}H_kU^T\Lambda^{-\frac{1}{2}}C_k = p_kC_k^TR_{hz}C_k
$$

Here, $U^T\Lambda = W^TW$ (as in Section 9.4.1) and $C_k = \Lambda^{\frac{1}{2}}U X_k$. The gradient of SINR$_k$ with respect to sequence $X_k$ is then given by

$$
d_k(C_k) = \frac{\text{SINR}_k}{dC_k} = 2\left(C_k^T C_k\right)R_{hz}C_k - 2\left(C_k^T R_{hz}C_k\right)C_k
$$

(9.56)
Note that $R_{hz}$ can be estimated at the receiver corresponding to user-k (the $k^{th}$ receive-node) and hence $d_k(C_k)$ can be calculated at the $k^{th}$ receive-node.

In the reduced feedback scheme, the $k^{th}$ user-node finds $q$ ($q \in \{1, \ldots, N\}$) dimensions in which the gradient of the utility function has the largest magnitude. The variable $q$ can be used to control the amount of feedback from the receiver to the transmitter. The node then finds the step size $\lambda$ that maximizes the utility function in the direction specified by the $q$ chosen dimensions with the largest magnitude (referred to here as the ascent direction and denoted by $a_q$). The optimum step-size $\epsilon_k$ in this direction is given by the solution to the following optimization problem:

$$
\max_{\lambda \neq 0} -\left(\frac{(C_k + \lambda a_q)^T R_{hz}[k] (C_k + \lambda a_q)}{(C_k + \lambda a_q)^T (C_k + \lambda a_q)}\right)
$$

(9.57)

The optimal step-size can be computed by a simple line search procedure. The sequence $C_k$ and hence the transmit sequence of the user is adapted in this direction if the interference in the network is reduced. The algorithm can be formally stated as follows:

**Gradient-based Better Response SSINR Waveform Adaptation Algorithm**

1. Fix the transmit-power levels and initialize codeword $X_k$ for each user. Also choose a value for variable $q$.
2. Set $k = 1$
3. while $k \leq K$
   a. Set count $i = 0$.
   b. Calculate gradient $d_k(C_k)$, corresponding ascent direction $a_q$ and the optimum step-size $\epsilon_k$ along $a_q$ at receive-node $k$.
   c. Feedback $a_q$ and $\epsilon_k$ to the transmit-node $k$ if calculation is done at the receiver.
   d. Adapt transmit-node $k$ to sequence $\hat{X}_k = \frac{U^T \Lambda^{-\frac{1}{2}} \hat{C}_k}{\sqrt{C_k^T \Lambda^{-\frac{1}{2}} U^T \Lambda^{-\frac{1}{2}} \hat{C}_k}}$, where $\hat{C}_k = C_k + \epsilon_k a_q$.
   e. Set $j = 1$
   f. while $j \leq K$
      i. Replace $Y_j = Z_j^{-1} H_{jj} X_j$.
      ii. Calculate change in SINR and feedback to transmit-node $k$.
      iii. $j = j + 1$.
   g. If the sum change in SINR in the network is positive, set $X_k = \hat{X}_k$ and go to step 3.g.
      Else set $i = i + 1$. If $i \leq T$ (some positive number), set $\epsilon_k = 0.5 \epsilon_k$ and repeat steps 3.b to 3.f.
   h. $k = k + 1$
4. Repeat step 2 and 3 until a fixed point is reached.

Note that, as before, if the interference seen at the transmitter and the receiver is the same, $q_k$ and $\epsilon_k$ for user-k can be calculated at the transmitter. Else, it is calculated at the receiver and then fed back to the transmit-node.
Convergence of the algorithm

Each iteration of the above algorithm improves the Sum-SINR in the network (SSINR function $V(X)$ (9.52)). Hence SSINR is the potential function for the game and the algorithm follows a better response dynamic. Since the potential function is bounded, the sequence space is compact, the algorithm converges. However the set of fixed points for the algorithm might be larger than the set of NE (as is the case for a better response dynamic). Nevertheless, the algorithm iteratively improves network performance by iteratively increasing the sum of SINRs in the network. Note that the algorithm is a sub-optimal implementation of the best-response iterative algorithm discussed in the previous subsection since each user adapts to a sequence that improves (and not maximizes as in the best-response dynamic) its utility function.

9.6.4 SNR Sequence with MMSE Receiver

It is seen that convergent sequence adaptation for distributed networks especially in multipath channels required considerable feedback in the network. Hence an alternate (though suboptimal) strategy is to use the transmit sequence that maximizes the energy capture from the channel (referred to as the max-SNR sequence) with a MMSE filter (sequence $Y_{MMSE,k}$) at the receiver. The MMSE filter offers significant protection against multiuser interference. In addition, the improved energy capture that can be achieved by using a max-SNR sequence can offset the loss in performance due to multiuser interference that is not rejected by the MMSE filter. Further, the max-SNR sequence is independent of the transmit sequences of other users in the system and hence a sequence adaptation procedure is not required for the network.

9.6.5 Performance Analysis

The simulations in this section use NLOS UWB channels generated using a modified Saleh-Valenzuela channel model for UWB. For ease of implementation, the channel sampling period is chosen such that the length of each CIR is around 30. Also, the chip interval is chosen to be equal to the pulse duration which is assumed to be smaller than the channel sampling-period. Noise variance is chosen such that $2E_{n_0} = 10$ dB.

Figure 9.8 illustrates the WA algorithm proposed in Section 9.6.3 over 30 round-robin iterations in a distributed network with 10 users. The transmit sequences at the beginning of the algorithm are initialized with random sequences. It is seen that the algorithm has not yet converged. Nevertheless, the algorithm iteratively increases the sum of SINRs in the network (as designed).

Figures 9.6 and 9.7 illustrate the inverse-cumulative distribution of the SINRs of users for different transmission schemes (the sequences considered for the distributed algorithm are those generated after 10 round-robin iterations) in a distributed network with 10 users (network-1) and 30 users (network-2) respectively. Statistics are generated over multiple instantiations of the network. It is seen that the use of a SNR sequence with an MMSE receiver offers substantial gains in the network. The gains are decreased when the number of users is comparable to the channel length (Figure 9.7). However, as pointed out in the previous Section (Subsection 9.5.6), since the channel lengths for UWB systems are quite large, a large number of users can be accommodated in the network (and guaranteed good performance while using the SNR sequence with the MMSE receiver. Note that the performance offered by the distributed scheme proposed in Section 9.6.3 is quite poor. This is due to adaptations for the proposed scheme being constrained by the fact that the adaptations have to improve the sum of SINRs of users in the network. However, it is seen that the distributed scheme, in general, improves the minimum SINRs for users in the network. For example, in network-2, almost all users have SINRs greater than 2 dB and no user has SINR less than 0 dB. On the other hand, when using the SNR sequence with the MMSE receiver, only 80% of the users have SINRs greater than 2 dB and some users have SINRs less than 0 dB. Hence the distributed algorithm can be used in situations where a minimum performance is to be guaranteed to all users in the network. Note that
Figure 9.8: Iterations of a WA algorithm for distributed multipath networks. The number of users in the networks is 10.

Figure 9.9: Comparison of inverse-CDF of SINR of users with different transmission schemes in a distributed network with 10 users. Performance is averaged over multiple instantiations of the network which use NLOS channels generated using a modified Saleh-Valenzula channel model for UWB. The length of the CIR is fixed at 30. For the WA algorithm, SINRs after 10 round-robin iterations are used.
the performance of the distributed algorithm can be improved by allowing more iterations of the algorithm or allowing the algorithm to run until convergence.

### 9.7 Interference Avoidance with respect to Legacy System

The discussed sequence optimization schemes for UWB systems can be directly extended to avoid the interference from or to avoid interfering with a static legacy radio system. Assume that the legacy signal’s period is an integer multiple $B$ of the symbol duration of the UWB user ($B > 1$ is a valid assumption taking into consideration the bandwidth of a UWB system. If the legacy system is also an UWB system, $N = 1$). $B$ is assumed to be known at the receiver. Note that the interference in these $B$ symbol intervals can be viewed as transmissions from $B$ separate interferers, where only one interferer transmits in each interval. Let $I_j, \ j \in \{1,...,B\}$ be the interference vector in the $j^{th}$ symbol interval. The interference-plus-noise correlation matrix for the $k^{th}$ user can then be written as

$$Z_k^L = \sum_{j \neq k} p_j H_j X_j X_j^T H_j^T + \sum_{j=1}^{B} I_j I_j^T + \sigma^2.$$  \hfill (9.58)

The analysis in Subsection 9.5.2 can be directly used to show that the optimal transmit sequence is $U^T \Lambda^{-\frac{1}{2}} C_{k,\text{opt}}$, where $C_{k,\text{opt}}$ is the maximum eigenvector of $H_k^T (Z_k^L)^{-1} H_k$ and the optimal receiver correlation template $(Z_k^L)^{-1} H_k X_k$. Note that the legacy system does not usually adapt to changes in the interference environment. Hence, the potential game framework and the sequence adaptation algorithms in Subsections 9.5.5 and 9.6.3, can also be directly extended to the current scenario by substituting $Z_k^L$ instead of $Z_k$ for the $k^{th}$-user’s adaptation.
The performance of a single user in the presence of NBI with data modulation, $B = 5$, and whose received power is 100 dB above that of the UWB user, is shown in Figure 9.11. It is seen that NBI is effectively avoided, and the performance of the system is similar to that of the single user system with no interference in Figure 9.1.

### 9.8 Conclusions and Contributions

In this chapter, sequence-based transmission schemes were investigated for UWB systems. We first developed a general sequence-optimization technique for UWB systems that maximizes energy capture from the rich UWB multipath channel and which can be used with simple receiver structures. It was shown that, for the same receiver complexity, the proposed transmission scheme significantly out-performs time-reversal techniques that have been extensively investigated for UWB systems.

The sequence-optimization scheme was then extended to incorporate interference rejection w.r.t. other users in the network and legacy systems. The convergence of these sequence adaptations, when multiple users adapt their transmit sequences according to the interference perceived at their receivers (referred to as greedy adaptations), was established for networks with centralized receivers. However, results in Chapter 7 can be used to show that these greedy adaptations do not converge in networks with distributed receivers. Hence a different sequence adaptation algorithm, based on the reduced feedback potential game formulation in Chapter 7, was proposed. An alternate strategy, where users use transmit sequences only optimized to maximize the energy capture from the channel together with an MMSE receiver, was also investigated for distributed networks. It was shown that the latter scheme, which does not require iterative adaptations in the network, offers considerable performance gains in UWB channels.

The original contributions in this chapter are as follows:
• A generalized sequence optimization scheme for UWB systems, which maximizes energy capture from the channel and at the same time allows simpler receivers, was developed.

• Sequence adaptation algorithms for multiuser interference rejection, previously proposed for centralized CDMA networks with multipath channels, were adapted and investigated for centralized UWB networks.

• Reduced complexity receivers were investigated for the above mentioned algorithms. These reduced complexity receivers could be used to further alleviate multiuser interference.

• A new sequence adaptation algorithm for multiuser interference rejection that converges in distributed networks was designed. This algorithm increases the minimum SINRs of users in the network.

• A transmission strategy, based on the use of transmit sequences that maximize energy capture from the channel with MMSE receivers, was proposed and investigated for distributed networks. This scheme is shown to result in considerable performance gains in UWB channels, while not requiring any adaptation overhead in the network.

• A sequence-based scheme for UWB systems to mitigate narrowband interference was proposed.

The publications that resulted from this chapter are as follows:


Chapter 10

Conclusions and Future Work

10.1 Research Summary

Dynamic spectrum sharing (DSS) is a new paradigm for spectrum allocation that is expected to lead to more efficient spectrum usage and alleviate the spectrum-scarcity that has been perceived in recent years. The two main aspects of the design of a DSS system are: coexistence with legacy or incumbent devices and the multiple access control (MAC) design for distributed DSS systems. Our work has addressed some of the issues involved.

Specifically, the first part of our work involved the identification of DSS techniques that are desirable with respect to the impact on legacy radio systems as well as the performance of a network that implements the DSS technique. It was found that that interference avoidance (IA) based underlay techniques, which implement some form of interference avoidance to reject the strongest interferers while spreading to average over the remaining interferers, are beneficial with respect to both objectives. It was also found that in such techniques, spreading alleviates the burden on the IA scheme to some extent, making the technique more robust to imperfections in the system knowledge that is required for implementing IA. This simplifies the design of the MAC layer, reduces the accuracy required of spectrum-sensing mechanisms, decreases the required overhead and coordination in the network and hence, in general, increases the practicality of the deployment of DSS. A specific example of an IA-based underlay technique is a spreading-sequence-based scheme that implements sequence (also referred to as waveform) adaptation for IA. Another example is a fast frequency hopping scheme that avoids certain frequency bands.

The latter part of our work dealt with the design of specific DSS techniques that exhibit the desired features. We developed spreading-sequence-based waveform adaptation (WA) algorithms that are amenable to a distributed implementation in networks with non-colocated or distributed receivers. Two different objectives were considered for the adaptations: reduction of sum-interference in the network and target-performance realization. Game theory, which is particularly suited for analysis of algorithms where individual decision nodes (users) contend for the use of a common resource (spectrum in our case), was used for the design of the algorithms and also for establishing convergence results.

Our initial investigations also showed that an increase in the bandwidth available to the DSS system accentuates the performance benefit provided by the IA-based underlay technique. Hence, WA schemes were also investigated for UWB systems. The large available bandwidth and the large number of resolvable multipath-components dramatically increase the degrees of freedom available for WA in these systems, making these systems exceptionally suitable for accommodating multiple users. UWB systems thus show great promise with regard to DSS deployment.
10.2 Contributions

The main contributions from this thesis are listed below. The first three contributions are with respect to identifying features for DSS techniques. The remaining contributions are with respect to designing specific schemes that exhibit these desired features and which are amenable to a distributed implementation.

- Developed a framework to investigate the impact of DSS radio systems on legacy static radio systems with which they must coexist. The framework incorporates practical deployment issues such as, the hidden-node problem, imperfect sensing and out-of-band interference, that result in imperfect static system knowledge.

- Developed a framework to investigate the relative benefits of different DSS schemes with respect to the performance of a distributed network that implements the schemes. The framework incorporates an exclusion-region-based MAC, which is suggested to be the optimal MAC in distributed networks.

- Identified desirable features for DSS on the basis of the above investigations. It was found that an IA-based underlay scheme that rejects the strongest interferers in the system and averages the remaining interference is desirable with respect to the impact on legacy systems as well as the performance of DSS networks. The benefit of the scheme was shown to be accentuated by an increase in transmission bandwidth. The scheme is also more robust to inaccuracies in the system knowledge required for IA.

- Established a new convergence result regarding the random better response dynamic for potential games. This result is useful in constructing reduced-feedback implementations for algorithms based on potential games.

- Proposed a new algorithm that reduces the feedback in the network for spreading-sequence-based WA in centralized networks. Note that the spreading-sequence-based WA scheme is a specific example of an IA-based underlay technique.

- Developed a WA framework based on potential game theory that could be used to construct convergent spreading-sequence-based WA algorithms for IA in networks with distributed networks.

- Proposed a WA algorithm, including a reduced-feedback implementation, which converges in networks with distributed receivers and which reduces the sum-interference in the network. Variations of the algorithm, including a joint power and waveform adaptation algorithm and IA with respect to legacy devices, were also investigated.

- Proposed a joint power and waveform adaptation algorithm that converges in networks with distributed receivers and allows users to achieve their target-SINRs while reducing the transmit power levels in the network.

- Proposed a novel sequence-optimization-based WA scheme for UWB systems, which maximizes energy capture from the channel and at the same time allows low complexity receivers. Note that UWB systems are investigated since it was shown that an increase in transmission bandwidth improves the gains provided by the IA-based underlay technique.

- Adapted WA algorithms for multiuser interference rejection, previously proposed for centralized CDMA networks with multipath channels, to centralized UWB networks. These algorithms are shown to provide substantial gains in UWB systems due to the large number of resolvable multipath components.

- Investigated WA and alternate sequence-based transmission strategies for UWB multiuser networks with distributed receivers.

- Proposed a novel WA algorithm for UWB systems to reject narrowband interference.
10.3 Future Work

Our work showed that interference avoidance based underlay techniques are preferable with respect to the outage-capacity-based performance of distributed DSS networks. However, the identification of the optimum allocation of spectrum resources, with respect to maximizing the actual capacity of distributed networks remains an open problem. Closely tied to this problem is the characterization of the maximum capacity achievable in distributed networks, for which until now, only order bounds have been established.

DSS allows multiple uncoordinated systems to contend for spectrum resources. Concepts from Nash bargaining theory [102], cooperative game theory and reward-and-punishment-based game-theoretic models, directly lend themselves for the analysis and design of spectrum access strategies in such scenarios and could be an interesting direction for future research.
Appendix A

Poisson Distribution as the Limit of a Binomial Distribution

A Poisson distribution is the limit of a binomial distribution. Consider the probability of obtaining $n$ successes in $N$ trials given by,

$$P\{n|N\} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

(A.1)

The mean of the distribution is given by,

$$\lambda = Np$$

(A.2)

Re-writing the probability distribution in terms of $\lambda$, we get,

$$P_\lambda\{n|N\} = \frac{N!}{n!(N-n)!} \frac{\lambda^n}{N^n} (1 - \frac{\lambda}{N})^{N-n}$$

(A.3)

When the sample size, $N$, becomes large, the distribution can be evaluated as follows.

$$P_\lambda(n) = \lim_{N \to \infty} P_\lambda\{n|N\} = \lim_{N \to \infty} \frac{N(N-1) \ldots (N-n+1) \lambda^n}{n!} \left(1 - \frac{\lambda}{N}\right)^N \left(1 - \frac{\lambda}{N}\right)^{-n}$$

$$= \lim_{N \to \infty} \frac{N(N-1) \ldots (N-n+1) \lambda^n}{n!} \left(1 - \frac{\lambda}{N}\right)^N \left(1 - \frac{\lambda}{N}\right)^{-n}$$

(A.4)

$$= \lambda^n \frac{e^{-\lambda}}{n!}$$

The simplification in the last step uses the binomial expansion for $(1 - \frac{\lambda}{N})^N$ and the series expansion for $e^{-\lambda}$.

In the context of the distribution of SS radios with uniform density $n$ in a circular region with radius $r$ extending to infinity, the total number of radios is given by $K = n\pi r^2$. The probability of finding a radio in an area of measure $A$ is given by $p = \frac{A}{\pi r^2}$. As $r \to \infty$, $K \to \infty$. Hence the number of radios in an area of measure $A$ follows a Poisson distribution with parameter $\lambda = Kp = NA$. 

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Appendix B

Moments and Cumulants

Let \( \Phi(s) = \mathbb{E}\{e^{sx}\} \) be the moment generating function of a random variable \( x \). Then the \( k^{th} \) moment of the random variable \( x \) is given by

\[
\mu_k = \left[ \frac{d^k \Phi(s)}{ds^k} \right]_{s=0}.
\]  

(B.1)

On the other hand, the \( k^{th} \) cumulant of the random variable \( x \) is given by

\[
m_k = \left[ \frac{d^k (\ln \Phi(s))}{ds^k} \right]_{s=0}.
\]  

(B.2)

The first few moments and cumulants are therefore related by the following equalities:

\[
\begin{align*}
m_1 &= \mu_1 \\
m_2 &= \mu_2 - \mu_1^2 \\
m_3 &= 2\mu_1^3 - 3\mu_1\mu_2 + \mu_3 \\
m_4 &= -6\mu_1^4 + 12\mu_1^2\mu_2 - 3\mu_2^2 - 4\mu_1\mu_3 + \mu_4
\end{align*}
\]  

(B.3)

Note that the first two cumulants of random variable \( x \) correspond to the mean and variance of \( x \).

The text in this section is based on [46].
Appendix C

Gaussian Approximation for SINR at the Output of the MMSE Receiver

In [56], the random fluctuation of $\beta^{NB}$ around the limit $\beta^*$ is tackled by decomposing the expression into three terms

$$\beta^{NB} - \frac{P_1}{NB} \text{tr}(\Lambda + \sigma^2 I)^{-1}$$

and

$$\frac{P_1}{NB} \text{tr}(\Lambda + \sigma^2 I)^{-1} - \mathbb{E}[\beta^{NB}]$$

and

$$\mathbb{E}[\beta^{NB}] - \beta^*$$

Here, $\Lambda$ is the diagonal matrix of eigenvalues of $S_{-1}T_{-1}S_{-1}^T$. It is shown that the first term converges in the limit to a Gaussian distribution $\mathcal{N}(0, b_{NB})$ (fluctuates by order of $1/\sqrt{NB}$), where

$$b = 2 \int \left[ \frac{P_1}{\lambda_e + \sigma^2} \right]^2 dG^* (\lambda_e) + \left( \mathbb{E}(v_{11}) - 3 \right) \left[ \int \frac{P_1}{\lambda_e + \sigma^2} dG^* (\lambda_e) \right]^2.$$  \hspace{1cm} (C.4)

In other words,

$$\sqrt{NB} \left( \beta^{NB} - \frac{P_1}{NB} \text{tr}(\Lambda + \sigma^2 I)^{-1} \right) \xrightarrow{D} \mathcal{N}(0, b)$$  \hspace{1cm} (C.5)

The variance of the second term is decays by a factor of $1/N_B$ (the term fluctuates by order of $1/N_B$). In other words,

$$\lim_{N_B \to \infty} \sup \text{var} (\Lambda + \sigma^2 I)^{-1} < \infty.$$  \hspace{1cm} (C.6)

Also the deviation of the mean SINR from the limit fluctuates on the order of at most $1/N_B$. From the above discussion, since the fluctuations of the second and third terms decay at the rate of $1/N_B$, they are negligible compared to the fluctuations of the first term. The distribution of the first term thus dominates the fluctuation of $\beta^{NB}$ around $\beta^*$ or in other terms the distribution of $\beta^*$. We thus have that

$$\sqrt{NB} \left( \beta^{NB} - \beta^* \right) \xrightarrow{D} \mathcal{N}(0, b).$$  \hspace{1cm} (C.7)

In the case of unequal received powers at the receive-node, Equation (C.5) and the bound on the third term still hold. However, the derivation of the bound on the fluctuations of the second term is not straightforward. In the analysis here, we assume that these terms die out and that the distribution of SINR even in the unequal power case is given by Equation (C.7).
Appendix D

Proof of the Special Case of Modified Zangwill’s Theorem

If the algorithm stops, then by condition 2.(b), it terminates in a solution. Hence assume that an infinite sequence is generated. By condition 1, a convergent subsequence exists. Let this convergent subsequence be given by

\[ x^t \to x', \quad t \in \mathcal{T}. \]  

(D.1)

Since \( V \) is a continuous function,

\[ \lim_{t \in \mathcal{T}} V(x^t) = V(x'). \]  

(D.2)

Using condition 2, we see that \( V(x^{t+1}) \geq V(x^t) \). Now using the fact that if a sequence is monotonically non-decreasing and a subsequence converges to some limit point, then the entire sequence converges to that limit ([60] Lemma 4.1),

\[ \lim_{t \to \infty} V(x^t) = \lim_{t \in \mathcal{T}} V(x^t) = V(x'). \]  

(D.3)

Assume that \( x' \) is not a solution. Then using condition 3, there is a \( \mathcal{T}' \) such that (5.6) holds. Using compactness, we can find \( \mathcal{T}'' \subset \mathcal{T}' \) such that

\[ x^t \to x'', \quad t \in \mathcal{T}''. \]  

(D.4)

Again, by using continuity and the argument that if a sequence is monotonically non-decreasing and a subsequence converges to some limit point, then the entire sequence converges to that limit ([60] Lemma 4.1),

\[ \lim_{t \to \infty} V(x^t) = \lim_{t \in \mathcal{T}'} V(x^t) = \lim_{t \in \mathcal{T}''} V(x^t) = V(x''). \]  

(D.5)

From (D.3) and (D.5), it follows that

\[ V(x'') = V(x'). \]  

(D.6)

However, by condition 3,

\[ V(x'') > V(x'). \]  

(D.7)

Equation (D.7) contradicts (D.6). Hence \( x' \) is a solution. \( \square \)
Appendix E

A Discussion of Set-valued Functions and Some Definitions

Since the waveform adaptation algorithms discussed in out work could result in multiple best responses, all possible choices must be considered while establishing convergence results independent of the choices. To aid this approach, a discussion on set-valued functions or correspondences [57] is presented here.

A correspondence from metric space $A$ to metric set $S$ is a mapping, $\Phi: A \rightarrow P(S)$, where $P(S)$ denotes the power set of $S$. A correspondence is compact valued if the set, $\Phi(x)$, is compact for every $x \in A$. A correspondence is upper-semi-continuous (u.s.c) if for a given point $x \in A$, for every open neighborhood of $\Phi(x)$, $\Theta$, there is an open neighborhood of $x$, $U$, such that $\Phi(U) \subseteq \Theta$. The finite sum and composition of compact-valued u.s.c correspondences is compact-valued u.s.c. Also, the image of a compact set under a compact-valued u.s.c. correspondence is compact, and the pre-image of open sets are open.

The set of best responses for player, $k \in K$, is the correspondence $D^*_k : A_{-k} \rightarrow P(A_k)$.

$$D^*_k (x_{-k}) = \operatorname{arg} \max_{x_k \in A_k} V(x_k, x_{-k}).$$ (E.1)

The best response iteration for player $k$ is the correspondence $\Phi_k : A \rightarrow P(A)$,

$$\Phi_k (x) = (D^*_k (x_{-k}), x_{-k}).$$ (E.2)

The composite best response iteration is the composition of all players’ best responses after one round-robin iteration, $\Phi : A \rightarrow P(A)$,

$$\Phi (x) = \Phi_{|K|} (\ldots \Phi_2 (\Phi_1 (x)) \ldots).$$ (E.3)

For every $k \in K$, $V$ is a continuous function on the compact set $A_k$, $D^*_k (x_{-k}) \neq \emptyset \forall k \in K, x_{-k} \in A_{-k}$. The maximum theorem [61] states that for every $k \in K$, $D^*_k$ is a compact-valued u.s.c. correspondence. Hence, so are $\{\Phi_k\}$ and $\Phi$.

Consider an ordinal or exact potential game with potential function $V$. The set of better responses for player $k, k \in K$, is the correspondence $C^*_k : A \rightarrow P(A_k)$,

$$C^*_k (x) = \{x_k^* : x_k^* \in A_k \& V(x_k^*, x_{-k}) > V(x)\}$$ (E.4)

The random better response iteration for player $k$ is the mapping $\Psi_k : X \rightarrow X$,

$$\Psi_k (x) = (x_k^*, x_{-k}),$$ (E.5)
where $x^*_k$ is chosen with uniform probability from the set $C^*_k(x)$. The composite better response iteration is the composition of all player’s better response iterations after one round-robin iteration, $\Psi : X \rightarrow X$,

$$\Psi(x) = \Psi_{|K|} \left( \cdots \Psi_2 (\Psi_1 (x)) \cdots \right).$$

(E.6)
Bibliography


**Vita**

Rekha Menon was born in Avadi, a small town close to Chennai in India. She received her Bachelors degree in Electronics and Communication Engineering from Regional Engineering College, Trichy, India in 2000. She received her Masters degree in Electrical Engineering specializing in wireless communications from Virginia Tech in 2003 and started her Ph.D work in 2004. While at Virginia Tech, she was part of the Center for Wireless Telecommunication as well as part of the Mobile Portable and Radio Research Group (MPRG). She worked on various projects including the development of OSSIE (Open-Source SCA Implementation::Embedded), a Software Communication Architecture (SCA) tool, an ONR project for the game theoretic analysis of radio resource management for ad-hoc networks and a project sponsored by ETRI for the development of a cognitive engine. She also interned with the RF and Network Operations Division of Cingular Wireless in Hanover, MD. She is expected to receive her Ph.D. degree in May 2007 and will be thereafter joining M/A-Com in Lynchburg. Her research interests include cognitive radios systems and wireless physical-layer system design.