Chapter 4

Tube With Apron Attached and Internal Pressure Remains Constant

4.1 Introduction
This chapter models the behavior of a tube with an apron attached, subjected to increasing exterior hydrostatic pressure on the apron side. The internal pressure within the tube remains constant during the analysis. The equations used to solve the model were derived by Dr. Plaut. Mathematica was utilized to solve for any unknowns and to compute the x and y coordinate values of the tube. Graphs from the resulting coordinate values were then generated in Excel.

4.2 Assumptions
Some basic assumptions were made in order to derive the equations that would model the behavior of a tube with an attached apron. The foundation was considered to be undeformable and the friction between it and the tube was neglected. The tube was assumed to be inextensible with no bending stiffness present. The weight of the tube material was neglected and the tube was assumed to be infinitely long. Any fluid within the tube was assumed to be incompressible. In order to achieve equilibrium, the exterior headwater was assumed to be above the point at which the apron begins to separate from the tube.

4.3 Derivation
Consider the diagram shown in Figure 4-1, a tube with an apron attached resting on a rigid foundation. The analysis is carried out in dimensionless terms. The horizontal position is x and the vertical position is y, with the origin placed at point A. The distance between A and G will be denoted b. The internal pressure head of the tube is represented by $h_{int}$ and the external hydrostatic pressure head by $h_{ext}$. The tension in the apron from point C to point E is $t_a$, and the tension in the portion of the tube from point A to point E is $t$. Tension in the tube from point E to point G will then be $t + t_a$. The vertical distance from the foundation to point E, denoted $y_E$, shall always be less than the external hydrostatic pressure $h_{ext}$. The basic derivation, equations, and variables from Chapter 3 apply in this chapter as well.
In section AE, equations 3.1-3, 3.8, and 3.10-12 apply.

Looking at the apron in section CE:

\[
\frac{dx_a}{ds_a} = \cos \theta_a, \quad \frac{dy_a}{ds_a} = \sin \theta_a, \quad \frac{d\theta_a}{ds_a} = \frac{h_{\text{ext}} - y_a}{t_a}, \quad (4.1, 4.2, 4.3)
\]

where \(x_a, y_a, s_a, \) and \(\theta_a\) are shown in Figure 4-2. At \(C, s_a = 0, x_a = 0, y_a = 0, \) and \(\theta_a = 0.\)

Letting

\[
u_a = \sqrt{h_{\text{ext}}^2 - 2t_a(1 - \cos \theta_a)}
\]

yields the following:

\[
y_a = h_{\text{ext}} - u_a, \quad x_a = t_a \int_0^{\theta_a} \frac{\cos \theta_a}{u_a} d\theta_a, \quad s_a = t_a \int_0^{\theta_a} \frac{1}{u_a} d\theta_a.
\]
In section EJ of the tube, the following can be derived:

\[
\frac{d\theta}{ds} = \frac{h_{\text{int}} - h_{\text{ext}}}{t + t_s},
\]

(4.6)

Let

\[
Q = \frac{h_{\text{int}} - h_{\text{ext}}}{t + t_s}.
\]

(4.7)

Then

\[
\theta = \theta_E + Q(s - s_t).
\]

(4.8)

The coordinate values can be derived as:

\[
x = x_E + \int_{s_t}^{s} \cos \theta \, ds, \quad y = y_E + \int_{s_t}^{s} \sin \theta \, ds.
\]

(4.9, 4.10)

Using equation 4.8 and letting

\[
P = \theta_E - Q s_t,
\]

(4.11)

where

\[
s_t = t \int_{0}^{\theta_E} \frac{1}{u} \, d\theta,
\]

(4.12)

produces the following equations:

\[
\int_{s_t}^{s} \sin \theta \, ds = \frac{1}{Q} \cos(P + Q s_t) - \frac{1}{Q} \cos(P + Q s) = y - y_E,
\]

(4.13)

\[
\int_{s_t}^{s} \cos \theta \, ds = \frac{1}{Q} \sin(P + Q s) - \frac{1}{Q} \sin(P + Q s_t) = x - x_E.
\]

(4.14)

Therefore,

\[
y_j = y_E + \frac{1}{Q} \cos(P + Q s_t) - \frac{1}{Q} \cos(P + Q s),
\]

(4.15)

\[
x_j = x_E + \frac{1}{Q} \sin(P + Q s_t) - \frac{1}{Q} \sin(P + Q s).
\]

(4.16)

In section JG of the tube, let
\[ v = \sqrt{(h_{\text{int}} - h_{\text{ext}})^2 - 2(t + t_a)(\cos \theta_{E} - \cos \theta)} \]  
\[ u_E = \sqrt{h_{\text{int}}^2 - 2t(1 - \cos \theta_E)} \] 
\[ u_{aE} = \sqrt{h_{\text{ext}}^2 - 2t_a(1 + \cos \theta_E)} \]  

With

\[ y_E = h_{\text{int}} - u_E \] 
\[ y_{aE} = y_E = h_{\text{ext}} - u_{aE} \]

one obtains the following:

\[ [(t + t_a) \cos \theta_E - (t - t_a)]^2 + 2t_a h_{\text{int}}^2 (1 + \cos \theta_E) + 2t h_{\text{ext}}^2 (1 - \cos \theta_E) + 2h_{\text{int}} h_{\text{ext}} [(t - t_a) \cos \theta_E \cdot (t + t_a)] = 0 \]  

At point G, using the conditions \( x = -b, s = 1 - b, \) and \( y = 0 \) gives:

\[-b = (t + t_a) \int_0^{\alpha} \cos \theta \frac{d\theta}{v} + t \int_0^{\theta_E} \cos \theta \frac{d\theta}{u} + \frac{1}{Q} \sin(P + Qs) - \frac{1}{Q} \sin(P - Qs) \]  

\[ 1 - b = (t + t_a) \int_0^{\alpha} \frac{d\theta}{v} + t \int_0^{\theta_E} \frac{d\theta}{u} + \frac{(\theta_J - \theta_E)}{Q} \]  

\[ t + t_a = \frac{h_{\text{ext}}(2h_{\text{int}} - h_{\text{ext}})}{2(1 - \cos \theta_J)} \]  

Using equilibrium and summing the forces in the horizontal direction yields:

\[ t_a = \frac{h_{\text{ext}}^2}{4} \]  

With \( y = h_{\text{ext}} \) at point J and combining equations 4.15 and 4.20, the following can be derived:

\[-h_{\text{ext}} + h_{\text{int}} - \sqrt{h_{\text{int}}^2 - 2t(1 - \cos \theta_E)} + \frac{1}{Q} \cos(P + Qs) - \frac{1}{Q} \cos(P - Qs) = 0 \]  

Subtracting equation 4.23 from 4.24 will eliminate the variable \( b \) to produce:
\[
\begin{align*}
\int_0^{\theta_j} \frac{(1 - \cos \theta)}{u} \, d\theta + (t + t_a) \int_{\theta_j}^{2\pi} \frac{(1 - \cos \theta)}{v} \, d\theta - \frac{1}{Q} \sin(P + Q si) + \frac{1}{Q} \sin(P + Q se) \\
+ \frac{(\theta_j - \theta_E)}{Q} \cdot 1 = 0
\end{align*}
\]  
(4.28)

Considering Figure 4-2, the variable \(x_{aE}\) will be defined as the horizontal distance from point C to point E and is calculated as

\[
x_{aE} = t_a \int_0^{\theta_E} \frac{\cos \theta}{\sqrt{h_{ext}^2 - 2t_a(1 - \cos \theta)}} \, d\theta.
\]  
(4.29)

### 4.4 Analysis

As in Chapter 3, Mathematica was utilized to solve the system of equations and compute data needed to plot the behavior of the tube subjected to hydrostatic pressure. Excel was used to produce visual representations of the tube’s behavior as it deforms and relationships between some of the different variables.

In order to compute data needed to generate graphs for the tube with an attached apron model, the “FindRoot” command in Mathematica was carried out. Placing equations 4.27 and 4.28 as the conditional equations and specifying an internal hydrostatic pressure \(h_{int}\) and an external hydrostatic pressure head \(h_{ext}\), the angles of tangents \(\theta_E\) and \(\theta_J\) were calculated. Once these angles were known, the remaining variables could then be computed.

There may be multiple solutions for any specified internal hydrostatic pressure \(h_{int}\), depending on the external hydrostatic pressure head \(h_{ext}\) indicated. For this model, a specific internal hydrostatic pressure to be observed is selected and set constant during the analysis. The external hydrostatic pressure head is then increased in intervals starting at the foundation until it equals the height of the tube, at which point it can no longer be increased.

Data are computed at every level of the rising pressure head and investigated. It is possible that the program will produce solutions for \(\theta_E\) and \(\theta_J\) that physically do not make sense. For this reason, it is important that the initial guesses used in the “FindRoot” command are reasonably
close to the correct values. All solutions were verified by checking equations 4.27 and 4.28. An example of one of the Mathematica programs can be found in Appendix B.

Once Mathematica solved for the unknowns and calculated the remaining variables, the data were then transferred to Excel, where plots could be generated in order to observe the behavior of the tube. Several different cases with an internal hydrostatic pressure ranging from 0.2 to 0.5 were studied. The behavior for each case was similar. Table 4-1 presents a list of the variables for the case \( h_{\text{int}} = 0.2 \). The variable \( x_{\text{CIE}} \) is the horizontal distance between points C and E. From the table, it is apparent that as \( h_{\text{ext}} \) increases, the height \( h \) of the tube decreases. Figure 4-3 is a visual representation of the tube’s behavior as it is subjected to a rising external pressure head. Figure 4-4 is a plot of the tension versus \( h_{\text{ext}} \). It can be seen that as the external hydrostatic pressure head \( h_{\text{ext}} \) increases, the tension \( t \) in the tube from point A to point E decreases and the tension \( t_a \) in the apron increases. The tension \( t + t_a \) in the tube from point E to point G remains nearly constant. In Figure 4-5, the vertical coordinate \( y_E \) of point E (point at which apron begins to separate from the tube) is plotted against the external hydrostatic pressure head \( h_{\text{ext}} \). The curve in the graph shows \( y_E \) increasing until the external hydrostatic pressure head reaches approximately 0.11, at which it then begins to decrease. Looking at \( h_{\text{ext}} = 0.11 \) in Table 4-1, \( \theta_E \) was found to be a little less than \( \pi/2 \). Therefore, \( y_E \) increases until the external hydrostatic pressure head reaches near the ninety-degree tangent angle of the tube, at which it then begins to decrease. Figures 4-6 through 4-8 are graphs showing the tube deformation for other internal hydrostatic pressure cases as the external hydrostatic pressure head increases.
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Table 4-1: Tube with Apron Attached; Constant $h_{int} = 0.2$

Figure 4-3: Tube with Apron Attached and $h_{int} = 0.2$ is Constant
Figure 4-4: Tension vs. $h_{\text{ext}}$ for a Tube with Apron Attached; $h_{\text{int}} = 0.2$ is Constant

Figure 4-5: $y_E$ vs. $h_{\text{ext}}$ for a Tube with Apron Attached; $h_{\text{int}} = 0.2$ is Constant
Figure 4-6: Tube with Apron Attached and $h_{int} = 0.3$ is Constant

Figure 4-7: Tube with Apron Attached and $h_{int} = 0.4$ is Constant
Figure 4-8: Tube with Apron Attached and $h_{\text{int}} = 0.5$ is Constant