THE EFFECT OF WORK OF ADHESION ON CONTACT OF A PRESSURIZED BLISTER WITH A FLAT SURFACE

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Thesis submitted to the faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

Master of Science in Civil Engineering

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(ABSTRACT)

The ability to accurately measure surface and interfacial energies affects our understanding of friction, wear, bonding and adhesion. Although there are accurate ways to measure the surface energies of liquids, the surface energies of solids have been harder to characterize. In order to broaden the knowledge of adhesion of solids, a modification to the constrained blister test is proposed. Most of the previous work on constrained blisters has examined the debonding of the blister from the surface underneath as pressure is applied from below. In this thesis, the contact of the constrained blister with the flat surface above it is considered. In addition, the blister is given specified boundary conditions at its outer radius, which has a fixed value.

Three models of the blister behavior are considered: linear plate, nonlinear plate, and membrane. The contact of the blister with the substrate above it is modeled with no adhesion, the JKR-type of adhesion, and the DMT-type of adhesion. Several substrate heights are considered, along with several values for the work of adhesion in the JKR analysis, and several combinations of force magnitude and gap size in the DMT analysis. The effect of adhesion on the contact radius is investigated. Sometimes the contact radius changes discontinuously as the pressure is increased or decreased. Results from the three models of blister behavior and the different models of adhesion are compared.
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# Table of Contents

List of Figures vi

List of Tables x

Chapter 1 – Introduction and Literature Review ............................ 1
  A. Introduction ............................................. 1
  B. Literature Review ........................................ 2

Chapter 2 – Linear Plate, No Adhesion ................................. 7
  A. Formulation ............................................. 7
  B. Results .................................................. 10

Chapter 3 – Linear Plate, JKR .......................................... 13
  A. Formulation ............................................. 13
  B. Results .................................................. 15

Chapter 4 – Linear Plate, DMT ......................................... 20
  A. Formulation ............................................. 20
  B. Results .................................................. 22

Chapter 5 – Membrane, No Adhesion ..................................... 27
  A. Formulation ............................................. 27
  B. Results .................................................. 30

Chapter 6 – Membrane, JKR ........................................... 32
  A. Formulation ............................................. 32
  B. Results .................................................. 33

Chapter 7 – Membrane, DMT .......................................... 37
  A. Formulation ............................................. 37
  B. Results .................................................. 40

Chapter 8 – Nonlinear Plate, No Adhesion .............................. 42
  A. Formulation ............................................. 42
  B. Results .................................................. 47

Chapter 9 – Nonlinear Plate, JKR ..................................... 50
  A. Formulation ............................................. 50
  B. Results .................................................. 51

Chapter 10 – Nonlinear Plate, DMT .................................... 56
  A. Formulation .............................................. 56
  B. Results .................................................. 60
# Table of Contents, Continued

Chapter 11 – Conclusions and Suggestions for Future Research .......................... 63  
   A. Conclusions .................................................................................. 63  
   B. Suggestions for Future Research ................................................. 70  

References .............................................................................................. 71  

Appendix A – Linear Plate, No Adhesion ................................................. 75  
   A.1: Solution Procedure for the Governing Equation When $p > 64h$ ...... 75  
   A.2: Mathematica Program for Linear Plate with No Adhesion ............ 76  

Appendix B – Linear Plate, JKR .............................................................. 77  
   B.1: Mathematica Program for Linear Plate with JKR Analysis ............ 77  

Appendix C – Linear Plate, DMT ............................................................. 78  
   C.1: Solution of the System Equations for the Linear Plate with No  
       Adhesion ...................................................................................... 78  
   C.2: Mathematica Program for Linear Plate with DMT ......................... 80  

Appendix D – Membrane, No Adhesion ..................................................... 81  
   D.1: Derivation of the Governing Equations ......................................... 81  
   D.2: Description of the Shooting Method ............................................. 83  
   D.3: Mathematica Program for Membrane with JKR Analysis .............. 84  
       D.3.1: Finding the Height at a Given Contact Radius ....................... 84  
       D.3.2: Finding the Contact Radius at a Given Value of $\xi h$ ............. 85  

Appendix E – Membrane, JKR ................................................................. 86  
   E.1: Mathematica Program for Membrane with JKR Analysis ............... 86  

Appendix F – Membrane, DMT ............................................................... 88  
   F.1: Mathematica Program for Membrane with DMT ............................ 88  

Appendix G – Nonlinear Plate, No Adhesion ........................................... 90  
   G.1: Mathematica Program for Nonlinear Plate with No Adhesion ....... 90  

Appendix H – Nonlinear Plate, JKR ....................................................... 92  
   H.1: Mathematica Program for Nonlinear Plate with JKR .................... 92  

Appendix I – Nonlinear Plate, DMT ....................................................... 94  
   I.1: Mathematica Program for Nonlinear Plate with DMT Analysis ........ 94  

Vita ............................................................................................................ 96
**List of Figures**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Blister (cross-section)</td>
<td>4</td>
</tr>
<tr>
<td>1.2</td>
<td>Constrained Blister (cross-section)</td>
<td>5</td>
</tr>
<tr>
<td>1.3</td>
<td>Blister Studied in the Present Research (cross-section)</td>
<td>6</td>
</tr>
<tr>
<td>2.1</td>
<td>Perspective of the Problem</td>
<td>7</td>
</tr>
<tr>
<td>2.2</td>
<td>Geometry for the Linear Plate</td>
<td>8</td>
</tr>
<tr>
<td>2.3</td>
<td>Moment and Shear on the Blister</td>
<td>8</td>
</tr>
<tr>
<td>2.4</td>
<td>Nondimensionalized Geometry</td>
<td>8</td>
</tr>
<tr>
<td>2.5</td>
<td>Blister Shape – No Adhesion, h = 0.01</td>
<td>11</td>
</tr>
<tr>
<td>2.6</td>
<td>Contact Radius Versus Height at Constant Pressures – No Adhesion</td>
<td>12</td>
</tr>
<tr>
<td>2.7</td>
<td>Contact Radius Versus Pressure at Constant Heights – No Adhesion</td>
<td>12</td>
</tr>
<tr>
<td>3.1</td>
<td>Total Energy $U_T$ versus Contact Radius b at $h = 0.01$, $p = 2.0$, $\Delta \gamma = 0.01$ - Special Case Where $U_T$ Has Both a Minimum and a Maximum</td>
<td>14</td>
</tr>
<tr>
<td>3.2</td>
<td>Contact Radius versus Height at Pressure $p = 0.5$</td>
<td>16</td>
</tr>
<tr>
<td>3.3</td>
<td>Contact Radius versus Height at Pressure $p = 2$</td>
<td>16</td>
</tr>
<tr>
<td>3.4</td>
<td>Contact Radius versus Pressure at Height $h = 0.01$</td>
<td>17</td>
</tr>
<tr>
<td>3.5</td>
<td>Contact Radius versus Pressure at Height $h = 0.02$</td>
<td>17</td>
</tr>
<tr>
<td>3.6</td>
<td>Blister Shapes at Stable and Unstable Configurations</td>
<td>18</td>
</tr>
<tr>
<td>3.7</td>
<td>Path Followed in the JKR Case When Height is Sufficiently Small and $\Delta \gamma$ is Sufficiently Large</td>
<td>19</td>
</tr>
<tr>
<td>4.1</td>
<td>Geometry and Location of the DMT Force</td>
<td>20</td>
</tr>
<tr>
<td>4.2</td>
<td>Nondimensional Geometry and Location of the DMT Force</td>
<td>20</td>
</tr>
<tr>
<td>4.3</td>
<td>Dugdale Pressure</td>
<td>21</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4.4</td>
<td>Nondimensional Dugdale Pressure</td>
<td>21</td>
</tr>
<tr>
<td>4.5</td>
<td>Contact Radius Versus Height for $p = 0.64$ and $\alpha = 10^{-6}$</td>
<td>22</td>
</tr>
<tr>
<td>4.6</td>
<td>Contact Radius Versus Height for $p = 1.00$ and $\alpha = 10^{-6}$</td>
<td>23</td>
</tr>
<tr>
<td>4.7</td>
<td>Contact Radius Versus Height for $p = 2.00$ and $\alpha = 10^{-6}$</td>
<td>23</td>
</tr>
<tr>
<td>4.8</td>
<td>Contact Radius Versus Pressure for $h = 0.01$ and $\alpha = 10^{-6}$</td>
<td>24</td>
</tr>
<tr>
<td>4.9</td>
<td>Contact Radius Versus Pressure for $h = 0.01$ and $\alpha = 10^{-5}$</td>
<td>25</td>
</tr>
<tr>
<td>4.10</td>
<td>Contact Radius Versus Pressure for $h = 0.01$ and $\alpha = 10^{-4}$</td>
<td>25</td>
</tr>
<tr>
<td>4.11</td>
<td>Path Followed in the DMT Case When $\alpha$ is Sufficiently Small and $f_0$ is Sufficiently Large</td>
<td>26</td>
</tr>
<tr>
<td>5.1</td>
<td>Geometry of the Membrane</td>
<td>27</td>
</tr>
<tr>
<td>5.2</td>
<td>Nondimensional Geometry of the Membrane</td>
<td>27</td>
</tr>
<tr>
<td>5.3</td>
<td>Blister Shape - Normalized Blister Height versus Radius</td>
<td>30</td>
</tr>
<tr>
<td>5.4</td>
<td>Contact Radius versus Pressure</td>
<td>31</td>
</tr>
<tr>
<td>5.5</td>
<td>Contact Radius versus $\xi h$</td>
<td>31</td>
</tr>
<tr>
<td>6.1</td>
<td>Membrane Geometry for JKR Analysis</td>
<td>32</td>
</tr>
<tr>
<td>6.2</td>
<td>Nondimensionalized Membrane Geometry for JKR Analysis</td>
<td>32</td>
</tr>
<tr>
<td>6.3</td>
<td>Contact Radius versus Pressure with Relatively Strong Adhesion When $\xi h = 0.3$</td>
<td>34</td>
</tr>
<tr>
<td>6.4</td>
<td>Contact Radius versus Pressure with Relatively Weak Adhesion When $\xi h = 0.3$</td>
<td>35</td>
</tr>
<tr>
<td>6.5</td>
<td>Contact Radius versus Substrate Height with Relatively Strong and Weak Adhesion When $\xi = 30$ and $p = 2$</td>
<td>36</td>
</tr>
<tr>
<td>7.1</td>
<td>Membrane Geometry for DMT Analysis</td>
<td>37</td>
</tr>
<tr>
<td>7.2</td>
<td>Nondimensionalized Membrane Geometry for DMT Analysis</td>
<td>37</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>-------</td>
</tr>
<tr>
<td>7.3</td>
<td>Dimensions for the Membrane DMT Analysis</td>
<td>38</td>
</tr>
<tr>
<td>7.4</td>
<td>Contact Radius versus Pressure for Relatively Small Values of $f_0$ when $\alpha = 10^{-5}$, $\xi = 30$, and $h = 0.03$</td>
<td>40</td>
</tr>
<tr>
<td>7.5</td>
<td>Contact Radius versus Pressure for Relatively Large Values of $f_0$ when $\alpha = 10^{-5}$, $\xi = 30$, and $h = 0.03$</td>
<td>41</td>
</tr>
<tr>
<td>7.6</td>
<td>Contact Radius versus Pressure when $\alpha = 10^{-5}$, $\xi = 30$, and $h = 0.05$</td>
<td>41</td>
</tr>
<tr>
<td>8.1</td>
<td>Geometry for the Nonlinear Plate with No Adhesion</td>
<td>42</td>
</tr>
<tr>
<td>8.2</td>
<td>Nondimensionalized Geometry for the Nonlinear Plate with No Adhesion</td>
<td>45</td>
</tr>
<tr>
<td>8.3</td>
<td>Blister Shapes for Nonlinear Plate with No Adhesion for $h = 0.03$ and $\xi = 30$</td>
<td>48</td>
</tr>
<tr>
<td>8.4</td>
<td>Contact Radius versus Pressure for $h = 0.03$</td>
<td>48</td>
</tr>
<tr>
<td>8.5</td>
<td>Contact Radius versus Pressure for $h = 0.05$</td>
<td>49</td>
</tr>
<tr>
<td>9.1</td>
<td>Nondimensional Geometry for the Nonlinear Plate for JKR Analysis</td>
<td>50</td>
</tr>
<tr>
<td>9.2</td>
<td>Contact Radius versus Pressure When $h = 0.03$, $\xi = 20$</td>
<td>51</td>
</tr>
<tr>
<td>9.3</td>
<td>Contact Radius versus Pressure When $h = 0.03$, $\xi = 30$</td>
<td>52</td>
</tr>
<tr>
<td>9.4</td>
<td>Contact Radius versus Pressure When $h = 0.05$, $\xi = 20$</td>
<td>52</td>
</tr>
<tr>
<td>9.5</td>
<td>Contact Radius versus Pressure When $h = 0.05$, $\xi = 30$</td>
<td>53</td>
</tr>
<tr>
<td>9.6</td>
<td>Contact Radius versus $\Delta \gamma$ When $h = 0.03$, $\xi = 20$</td>
<td>54</td>
</tr>
<tr>
<td>9.7</td>
<td>Contact Radius versus $\Delta \gamma$ When $h = 0.03$, $\xi = 30$</td>
<td>54</td>
</tr>
<tr>
<td>9.8</td>
<td>Contact Radius versus $\Delta \gamma$ When $h = 0.05$, $\xi = 20$</td>
<td>55</td>
</tr>
<tr>
<td>9.9</td>
<td>Contact Radius versus $\Delta \gamma$ When $h = 0.05$, $\xi = 30$</td>
<td>55</td>
</tr>
<tr>
<td>10.1</td>
<td>Nondimensional Geometry and Location of the DMT Force</td>
<td>56</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>-------</td>
<td>-----------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>10.2</td>
<td>Dimensions for the Membrane DMT Analysis</td>
<td>56</td>
</tr>
<tr>
<td>10.3</td>
<td>Contact Radius Versus Pressure for $h = 0.03$, $\alpha = 10^{-6}$, and $\xi = 20$</td>
<td>61</td>
</tr>
<tr>
<td>10.2</td>
<td>Contact Radius Versus Pressure for $h = 0.03$, $\alpha = 10^{-6}$, and $\xi = 30$</td>
<td>61</td>
</tr>
<tr>
<td>10.3</td>
<td>Contact Radius Versus Pressure for $h = 0.05$, $\alpha = 10^{-6}$, and $\xi = 20$</td>
<td>62</td>
</tr>
<tr>
<td>10.4</td>
<td>Contact Radius Versus Pressure for $h = 0.05$, $\alpha = 10^{-6}$, and $\xi = 30$</td>
<td>62</td>
</tr>
<tr>
<td>11.1</td>
<td>Comparison of Results for the Linear Plate for $h = 0.03$</td>
<td>63</td>
</tr>
<tr>
<td>11.2</td>
<td>Comparison of JKR and DMT for the Linear Plate for $h = 0.01$</td>
<td>64</td>
</tr>
<tr>
<td>11.3</td>
<td>Comparison of the Membrane Models for $h = 0.03$ and $\xi = 30$</td>
<td>65</td>
</tr>
<tr>
<td>11.4</td>
<td>Comparison of the Nonlinear Plate Models for $h = 0.03$ and $\xi = 30$</td>
<td>66</td>
</tr>
<tr>
<td>11.5</td>
<td>Blister Shape Comparison when $h = 0.03$, $b = 0.1$, and $\xi = 10$</td>
<td>66</td>
</tr>
<tr>
<td>11.6</td>
<td>Blister Shape Comparison when $h = 0.03$, $p = 3$, and $\xi = 10$</td>
<td>67</td>
</tr>
<tr>
<td>11.7</td>
<td>Comparison of the Models with No Adhesion for $h = 0.03$</td>
<td>67</td>
</tr>
<tr>
<td>11.8</td>
<td>Comparison of the Models with No Adhesion for $h = 0.05$</td>
<td>68</td>
</tr>
<tr>
<td>11.9</td>
<td>Comparison of the Models with no adhesion for $h = 0.03$, $\xi = 30$, and $\Delta \gamma = 0.01$</td>
<td>69</td>
</tr>
<tr>
<td>11.10</td>
<td>Comparison of DMT when $h = 0.03$, $\xi = 30$, $f_0 = 1000$, and $\alpha = 10^{-5}$</td>
<td>69</td>
</tr>
</tbody>
</table>
## List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Nondimensional Quantities for the Linear Plate with No Adhesion</td>
<td>9</td>
</tr>
<tr>
<td>2.2</td>
<td>Boundary Conditions for the Linear Plate with No Adhesion</td>
<td>10</td>
</tr>
<tr>
<td>2.3</td>
<td>Contact Radius at Various Pressures</td>
<td>11</td>
</tr>
<tr>
<td>4.1</td>
<td>Nondimensional Quantities for the Plate with DMT Analysis</td>
<td>21</td>
</tr>
<tr>
<td>4.2</td>
<td>Boundary and Transition Conditions</td>
<td>22</td>
</tr>
<tr>
<td>5.1</td>
<td>Nondimensional Quantities for the Membrane with No Adhesion</td>
<td>28</td>
</tr>
<tr>
<td>5.2</td>
<td>Boundary Conditions for the Membrane with No Adhesion</td>
<td>29</td>
</tr>
<tr>
<td>7.1</td>
<td>Definitions for the Dimensions of the Membrane DMT Analysis</td>
<td>38</td>
</tr>
<tr>
<td>7.2</td>
<td>Nondimensional Quantities for the Membrane DMT Analysis</td>
<td>38</td>
</tr>
<tr>
<td>7.3</td>
<td>Boundary Conditions for the Membrane DMT Analysis</td>
<td>39</td>
</tr>
<tr>
<td>7.4</td>
<td>Boundary Conditions for the Membrane DMT Analysis</td>
<td>39</td>
</tr>
<tr>
<td>8.1</td>
<td>Nondimensional Variables for the Nonlinear Plate with No Adhesion</td>
<td>45</td>
</tr>
<tr>
<td>8.2</td>
<td>Definition of Variables for the Nonlinear Plate with No Adhesion</td>
<td>46</td>
</tr>
<tr>
<td>8.3</td>
<td>Boundary Conditions for the Nonlinear Plate with No Adhesion</td>
<td>47</td>
</tr>
<tr>
<td>10.1</td>
<td>Definitions of Variables in Region 2-3</td>
<td>57</td>
</tr>
<tr>
<td>10.2</td>
<td>Definitions of Variables in Region 1-2</td>
<td>58</td>
</tr>
<tr>
<td>10.3</td>
<td>Boundary Conditions for the Nonlinear Plate with DMT Adhesion</td>
<td>60</td>
</tr>
</tbody>
</table>
Chapter 1 – Introduction and Literature Review

A. Introduction

The characterization of surface and interfacial energies is of interest to both the commercial and scientific communities. Such knowledge can lead to a better understanding and advanced technology in fields such as friction, wear, lubrication, mechanics, adhesive bonding, coating, and cleaning of materials. Many of these industries would also benefit from an increased understanding of adhesion with regard to surface and interfacial energies. While there is accurate information available for measuring the surface energy of liquids, knowledge of these energies for solid materials continues to challenge researchers.

Adhesion is defined as the molecular attraction exerted between the surfaces of bodies in contact. However, adhesive forces exist in materials that are not in contact with another body in the form of surface energies. Surface energies of liquids can be reliably measured using several techniques. Capillary rise, the maximum bubble pressure method, and the droplet or shape technique can all be used to determine the surface energies of liquids. However, the surface energies of solids are much harder to measure because their large stiffnesses can often prevent the material from measurably changing shape when adhesive forces are present.

Several techniques have been developed which are intended to measure the surface energies of solids while averting the limitations posed by the material stiffness. Initially, measurements of the surface energy were obtained by performing tests on the liquid form of the material, assuming that the material properties would remain the same in both the liquid and solid states. This assumption is no longer accepted as true for most materials. Additionally, this measurement can only be performed on materials that can be handled in the liquid form, eliminating many materials. Another method of measuring surface energies involves the cleavage of brittle materials. This technique has the limitation that the material must be brittle. Since many of the materials for which scientists and industries desire knowledge of adhesive properties are not brittle, the method has limited acceptability for measuring the surface energies of solids. These and
other methods that have limited applicability to the desired materials are reviewed by Adamson (1967).

More recently, new methods have been devised to determine the surface energies of solids. These include the JKR technique and the DMT theory. These methods will be discussed in detail in the following literature review.

Research has been conducted to study the energy of adhesion in the classical blister test and the constrained blister test using the JKR and DMT techniques. Both of these tests have attempted to measure the energy of adhesion between the blister and the material underneath it as the pressure is applied from below and the blister debonds at its edge. However, the present research, proposed by D. A. Dillard, addresses the contact between the blister and a flat substrate above it.

The blister behavior is modeled with no adhesion, the JKR-type of adhesion, and the DMT-type of adhesion. Several substrate heights are considered, along with several values for the work of adhesion in the JKR analysis, and several combinations of force magnitude and gap size in the DMT analysis.

B. Literature Review

A modification to the constrained blister test is proposed in order to study the surface and interfacial energies between solids. Previous research has included studies of the energy of adhesion between the blister and the material underneath it in the blister test and the constrained blister test. However, research has not been conducted to study the energy of adhesion for the contact of the blister with the rigid substrate above it. Following is a review of the studies leading to the present research.

The first modern attempt to measure the energy of adhesion between solids was proposed by Hertz in 1881. Hertz proposed a solution for the energy between two elastic spheres brought into contact with one another, but ignored friction and adhesion in his solution. Bradley (1932) also studied the force required to separate two spheres, but did not include interfacial interactions in his solution.

A more recent solution was developed by Johnson, Kendall, and Roberts (1971) and has now become known as the JKR theory. Using two spheres with frictionless
surfaces, Johnson et al. (1971) showed that the contact region between the two spheres would be larger than Hertz had predicted. When the spheres are brought into contact via a normal force, a flat circular region of contact develops as the applied force increases. The forces of adhesion that occur outside of the contact region are ignored, and at the outer edge of the contact zone, an infinite tensile stress and sharp discontinuity are assumed to cause a larger contact area than Hertz had originally predicted (Horn et al., 1987). In order to find the contact area using the JKR theory, the total energy of the system, consisting of the mechanical, surface, and elastic energies, is minimized. A discussion of the history of the JKR theory is found in Kendall (2001).

A modification to the JKR technique used on spheres was studied by Shanahan (1997). In Shanahan’s approach, a spherical membrane under pressure is brought into contact with a flat surface. This balloon test allowed Shanahan to measure the contact area more accurately because the energy of adhesion is proportional to the square of the contact area. For the sphere, the energy is proportional to the cube of the contact area, so errors are more easily propagated in the calculation of the energy. The separation energies observed for the balloon test are about 33% higher than those predicted using the JKR model. The advantage of using the balloon test is that the pressure can be varied; however, the results are only accurate for small pressures.

A second method for predicting the energy of adhesion of solids was developed by Derjaguin, Muller, and Toporov (1975). They derived a solution for a sphere contacting a rigid surface. When contact occurs, an attractive force is assumed to act in the annular region just outside the contact zone, whereas the JKR adhesion occurs only within the contact area. The contact areas are larger than the Hertzian contact areas, which is attributed to the introduction of van der Waal’s forces. However, the Hertzian model was still assumed to apply. Muller et al. (1983) and Pashley (1984) developed additions and corrections to Derjaguin et al. (1971) and the analysis became known as the DMT theory.

Since different models were used to develop the JKR and DMT theories, they were originally believed to be conflicting. However, according to Tabor (1977), the two theories apply to different materials. The JKR technique should be used for soft solids with large radii of curvature and high energy of adhesion. The DMT theory applies when
the material is a hard solid with a small radius and low energy. This theory was presented by Muller et al. (1980) using a Lennard-Jones potential to consistently combine the theories and abandon the Hertzian theory (Maugis, 2000). Following Muller’s prescription, the JKR and DMT techniques can be used as limiting cases for measuring the work of adhesion. Maugis (1992), Maugis and Gauthier (1994), and Baney and Hui (1997) developed this compatibility theory.

Maugis (1992) used the Dugdale model to develop a general theory. Forces in the Dugdale model are assumed to be uniform and to act in an annular region outside the contact area, known as the cohesive zone. In this thesis, after Chapter 1, “DMT” refers to the Dugdale model.

Several experimental tests have attempted to verify the predicted JKR and DMT theories including alterations in the geometry, surface roughness, material ductility, and oxidized surfaces.

The classic blister test was originally proposed by Dannenberg (1961) to study the debonding (or peeling) from the rigid surface underneath the blister. This setup is shown in Figure 1.1, with dots representing locations of debonding.

![Figure 1.1: Blister (cross-section)](image)

The constrained blister test is another method used to measure the work of adhesion between solids. A review of this test with analytical and experimental results is found in Napolitano et al. (1988). The difference between the blister test and the constrained blister test is that a flat surface is placed above the blister and, under pressure, the blister is allowed to contact the surface. Figure 1.2 shows the constrained blister setup, and, again, the dots represent locations of debonding. The same debonding
is studied as in the blister test: the underside of the blister peels away from the bottom surface as the pressure is increased. The strain energy release rate for the constrained blister test was predicted by Lai and Dillard (1990a) and Chang et al. (1989), and a numerical analysis confirmed the theory (Lai and Dillard, 1990b).

![Figure 1.2: Constrained Blister (cross-section)](image)

More recently, energy release rates were studied by Williams (1997) for both of these blister tests. Although many studies have contributed knowledge to the field of adhesion, there is much more information and understanding to be gained from further analysis and experiments.

C. Contact of Blister with Substrate

Unlike the previously described blister and constrained blister tests that predict the blister’s behavior with respect to the lower surface, the present research investigates the behavior of the constrained blister with respect to the upper surface, as shown in Figure 1.3. The outer edge of the circular blister is constrained to have no deflection in all of the models, and no slope in the plate models.
Three models are considered, with three types of adhesion applied to each. The linear plate is described first, followed by the membrane, and finally, the linear plate. The results are presented for each of these models under no adhesion, the JKR-type of adhesion, and the DMT-type of adhesion. Finally, conclusions are drawn and future research is suggested.
Chapter 2 – Plate with No Adhesion

A. Formulation

The blister test is used in order to study the effects of surface energy on the adhesion of solids. In this thesis, the test consists of a thin, flexible, circular membrane or plate pressurized from below and coming into contact with a rigid substrate (Figure 2.1). The blister is clamped at its outer edge. In this chapter, the blister is considered to be a plate with fixed edge. The problem is axisymmetric. The displacements are assumed to be small in this chapter, and a linear analysis is carried out. The plate is linearly elastic, homogeneous, isotropic, and has constant thickness.

Figure 2.1: Perspective of the Problem

The radius of the blister is $R_0$, the height of the substrate is $H$, and the thickness of the blister is $T$. The blister is pressurized from below with a pressure $P$, which causes circular contact of radius $B$ with the substrate. The radius at any point along the deflection curve is $R$, and the shape of the blister is defined by the function $W(R)$. The blister is clamped at $R=R_0$, such that the vertical deflection and the slope at $R_0$ are zero. Young’s modulus is E and Poisson’s ratio is $\nu$. The flexural rigidity of the blister, $D$, is defined as $ET^3/[12(1-\nu^2)]$ (Szilard, 1974). The cross section along a diameter is shown in Figure 2.2.
As a simplification during the solution, \( Y(R) \) is defined as \( H - W(R) \). The moment and shear per unit length are \( M_r \) and \( Q_r \), respectively, as shown in Figure 2.3.

In order to simplify the results for a variety of specimen sizes and material properties, the dimensions and forces are nondimensionalized. Figure 2.4 shows the nondimensional geometry and pressure acting in this blister test. Table 2.1 gives the nondimensional conversion. Since \( r \) is equal to \( R / R_0 \), when \( R \) equals \( R_0 \), \( r \) is equal to one.
Table 2.1: Nondimensional Quantities for the Linear Plate with No Adhesion

<table>
<thead>
<tr>
<th>Nondimensional Quantity</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>W / R₀</td>
</tr>
<tr>
<td>p</td>
<td>PR₀² / D</td>
</tr>
<tr>
<td>r</td>
<td>R / R₀</td>
</tr>
<tr>
<td>b</td>
<td>B / R₀</td>
</tr>
<tr>
<td>h</td>
<td>H / R₀</td>
</tr>
<tr>
<td>y</td>
<td>h – w(r)</td>
</tr>
<tr>
<td>mᵣ</td>
<td>Mᵣ R₀ / D</td>
</tr>
<tr>
<td>σᵣ</td>
<td>σᵣ R₀² / D</td>
</tr>
</tbody>
</table>

The blister is analyzed using linear plate bending theory in this chapter. Neglecting membrane forces, the nondimensionalized governing differential equation is (Szilard, 1974)

\[ w'''' + \left(\frac{2}{r}\right) w''' - \left(\frac{1}{r^2}\right) w'' + \left(\frac{1}{r^3}\right) w' = p \]  
(2-1)

where \( b < r < 1 \). The function \( y(r) \) is defined as

\[ y(r) = h - w(r) \]  
(2-2)

as seen in Figure 2.4. The general solution of the governing equation for \( y(r) \) is

\[ y(r) = A₁ + A₂ \ln r + A₃ r² + A₄ r² \ln r - \left(\frac{pr^4}{64}\right) \]  
(2-3)

The boundary conditions (Lai and Dillard, 1990) are listed in Table 2.2. The last boundary condition is due to \( m_r = 0 \) at \( r = b \), since \( m_r = 0 \) for \( 0 \leq r < b \) and \( m_r \) is continuous at \( r = b \).
Table 2.2: Boundary Conditions for the Linear Plate with No Adhesion

<table>
<thead>
<tr>
<th>a</th>
<th>y = h</th>
<th>r = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>y' = 0</td>
<td>r = 1</td>
</tr>
<tr>
<td>c</td>
<td>y  = 0</td>
<td>r = b</td>
</tr>
<tr>
<td>d</td>
<td>y' = 0</td>
<td>r = b</td>
</tr>
<tr>
<td>e</td>
<td>y'' = 0</td>
<td>r = b</td>
</tr>
</tbody>
</table>

For the case where p is not large enough to establish contact between the blister and the substrate, the deflection is (Szilard, 1974)

\[ y(r) = h - \frac{p (1 - r^2)^2}{64} \]  

(2-4)

and at \( r = 0 \), the deflection becomes

\[ y(0) = h - \frac{p}{64} \]  

(2-5)

Therefore, the blister just touches the substrate when \( p \) reaches 64h.

The procedure to solve the governing equation when \( p > 64h \) was derived by Professor Plaut and is described in Appendix A. In order to solve the nonlinear equation for the contact radius \( b \), the FindRoot program in Mathematica is used. Values are chosen for the height \( h \) and pressure \( p \), and the contact radius \( b \) is found.

B. Results

The shape of the blister is plotted for \( h = 0.01 \) and for several values of \( p \). Some examples of the blister shape are shown in Figure 2.5, and the contact radius at different pressures is shown in Table 2.3.
Figure 2.5: Blister Shape – No Adhesion, $h = 0.01$

<table>
<thead>
<tr>
<th>$p$ (pressure)</th>
<th>$b$ (contact radius)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.65</td>
<td>$3.57 \times 10^{-29}$</td>
</tr>
<tr>
<td>1.20</td>
<td>0.079349</td>
</tr>
<tr>
<td>2.00</td>
<td>0.193171</td>
</tr>
<tr>
<td>5.00</td>
<td>0.365418</td>
</tr>
</tbody>
</table>

Then the contact radius versus the height is plotted for constant pressures, as well as the contact radius versus the pressure for constant heights. These plots are shown in Figures 2.6 and 2.7, respectively. For a given pressure $p$, the contact radius $b$ decreases as the substrate height increases. Also, the contact radius $b$ increases as the pressure $p$ increases. For a given pressure $p$, the contact radius $b$ hits zero when the substrate height reaches $p/64$, and the curve is quite flat at very low contact radii.
Figure 2.6: Contact Radius Versus Height at Constant Pressures – No Adhesion

Figure 2.7: Contact Radius Versus Pressure at Constant Heights – No Adhesion
Chapter 3 – Plate with JKR Analysis

A. Formulation

The JKR analysis measures the work done by adhesion within the contact area. In this analysis, the work of adhesion (surface energy per unit area) is denoted \((\Delta \gamma)_d\), which is then multiplied by the contact area to get the total surface energy \(U_T\). The total energy has a local minimum at stable equilibrium states.

For the JKR analysis on the linear plate, the strain energy due to bending, in nondimensional terms, is (Szilard, 1974, p. 218)

\[
U_E = \pi \int_b^1 \left[ r \left( w'' + \frac{w'}{r} \right)^2 - 2(1-\nu)w'w'' \right] dr \tag{3-1}
\]

but the last term is zero if \(w' = 0\) at \(r = b\) and \(r = 1\), so

\[
U_E = \pi \int_b^1 \left[ r \left( w'' + \frac{w'}{r} \right)^2 \right] dr \tag{3-2}
\]

The potential of the pressure is the negative of the work done, so that

\[
U_P = -\pi b^2 h p - 2\pi p \int_b^1 r w dr \tag{3-3}
\]

The surface energy of adhesion for the JKR-type analysis is

\[
U_A = -\pi b^2 \Delta \gamma \tag{3-4}
\]

where the nondimensional work of adhesion is

\[
\Delta \gamma = (\Delta \gamma)_d \frac{R_0^2}{D} \tag{3-5}
\]
The total energy is

$$U_T = U_E + U_P + U_A.$$  \hspace{1cm} (3-6)

A moment $m_b$ is allowed at the edge of the contact radius to represent the effect of adhesion. The magnitude of this moment is such that the total energy $U_T$ is minimized. The boundary condition $y''(b) = 0$ for the case of no adhesion is replaced by

$$y''(b) = m_b$$ \hspace{1cm} (3-7)

The location of a relative minimum or maximum value of the total energy is found by choosing a starting contact radius slightly smaller than the expected contact radius at the maximum or minimum, and an ending value slightly larger than the expected value. A sample plot is shown in Figure 3.1, and the Mathematica program used for this case is found in Appendix B. The value of $b$ corresponding to a maximum or minimum is found numerically to an accuracy of three significant figures.

![Figure 3.1: Total Energy $U_T$ versus Contact Radius $b$ at $h = 0.01$, $p = 2.0$, $\Delta \gamma = 0.01$ - Special Case Where $U_T$ Has Both a Minimum and a Maximum](image-url)
B. Results

The contact radius $b$ is plotted versus the substrate height $h$ at constant pressures $p$, and versus the pressure at constant heights. The plots are generated for $\Delta \gamma$ values of 0.0000, 0.0025, 0.0050, 0.0075, and 0.0100. Generally, at constant pressure, as the height increases, the contact radius decreases, and at constant substrate height, as the pressure increases, the contact radius also increases, as seen in Figures 3.2 through 3.5. In some cases, there is more than one mathematical solution, but only one of the solutions is physically possible. The solutions that are unstable or are mathematically, but not physically, possible are shown as dashed lines. The equilibrium of the blister is stable when the energy has a minimum, and unstable otherwise (e.g., when the energy has a maximum). In Figure 3.3, the solid lines depict the portion of the curve for which the total energy has a minimum, and the dashed lines that continue from the vertical tangent down to $b = 0$ are the unstable equilibrium solutions whose total energy is at a maximum. The whole curve for $\Delta \gamma = 0$ is comprised of stable equilibrium states. The dashed lines extending from the point where $b = 0$ and $h = 0.0078$ back to $h = 0$ are solutions for which the total energy has a minimum, but the shape of the curve is not possible. For these physically impossible solutions, the height of the blister near the contact radius is greater than the height of the substrate. A sample of this behavior is shown in Figure 3.6, which shows the stable, unstable, and physically impossible blister shapes at $p = 2$, $\Delta \gamma = 0.01$, and $h = 0.01$.

In Figure 3.3, the substrate height at which the contact radius is zero is 0.03125. In Figures 3.4 and 3.5, the substrate heights at which the contact radius is zero are 0.64 and 1.28, respectively.
Figure 3.2: Contact Radius versus Height at Pressure $p = 0.5$

Figure 3.3: Contact Radius versus Height at Pressure $p = 2$
Figure 3.4: Contact Radius versus Pressure at Height $h = 0.01$

Figure 3.5: Contact Radius versus Pressure at Height $h = 0.02$
Figure 3.6: Blister Shapes at Stable and Physically Impossible Configurations

For the cases in which there is more than one physically possible solution, the contact radius will follow the path shown in Figure 3.7 as the pressure is increased and decreased. As the pressure increases, the contact radius starts at 0 and reaches point A. Then, the blister “jumps” to point B and continues out to point C. As the pressure decreases, the contact radius will start at point C and go all the way down to point D, passing point B. Then it will “jump” back down to point E and continue back to 0. For these cases, the blister “jumps” onto or off of the substrate as the pressure is varied, instead of having the contact radius increase continuously from zero or decrease continuously to zero. A similar process will happen as the substrate height is changed for a case with more than one physically possible solution. The dashed line in the figure is still the solution where the total energy has a local maximum.
Figure 3.7: Path Followed in the JKR Case When Height is Sufficiently Small and $\Delta \gamma$ is Sufficiently Large
Chapter 4 – Plate with DMT Analysis

A. Formulation

Now the linear plate is analyzed with a DMT adhesion model. In this analysis, a Dugdale attractive pressure, which is a constant pressure, acts from the contact radius $B$ for a distance $D$ beyond the contact radius as seen in Figure 4.1. The nondimensional geometry is shown in Figure 4.2. The upward force on the plate ends when the gap between the substrate and the plate reaches $Y = \alpha^0$. The adhesion pressure $F(Y)$ is shown in Figure 4.3a, and nondimensionally in Figure 4.3b. The contact radius $B$ and the outer radius $C$ of the annular region in which the Dugdale force acts are determined from the solution of the equilibrium equation.

![Figure 4.1: Geometry and Location of the DMT Force](image1)

![Figure 4.2: Nondimensional Geometry and Location of the DMT Force](image2)
The system nondimensionalization is shown in Table 4.1.

Table 4.1: Nondimensional Quantities for the Plate with DMT Analysis

<table>
<thead>
<tr>
<th>Nondimensional Quantity</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>W / R₀</td>
</tr>
<tr>
<td>p</td>
<td>PR₀^2 / D</td>
</tr>
<tr>
<td>r</td>
<td>R / R₀</td>
</tr>
<tr>
<td>b</td>
<td>B / R₀</td>
</tr>
<tr>
<td>h</td>
<td>H / R₀</td>
</tr>
<tr>
<td>y</td>
<td>h – w(r)</td>
</tr>
<tr>
<td>f</td>
<td>FR₀ / D</td>
</tr>
<tr>
<td>c</td>
<td>C / R₀</td>
</tr>
<tr>
<td>α</td>
<td>α₀/R₀</td>
</tr>
</tbody>
</table>

The shape of the blister y(r) in the DMT analysis is separated into two parts. When the radius r is between b and c, the blister shape is y₁(r), and when it is between c and 1, the shape is y₂(r). The equations for the shape are

\[
y_1(r) = B_1 + B_2 \ln r + B_3 r^2 + B_4 r^2 \ln r - (p + f_o) r^4/64 \tag{4-1}
\]

and

\[
y_2(r) = E_1 + E_2 \ln r + E_3 r^2 + E_4 r^2 \ln r - pr^4/64. \tag{4-2}
\]

The solutions of these equations were derived by Dr. Plaut and are found in Appendix C.

The boundary and transition conditions are shown in Table 4.2.
Table 4.2: Boundary and Transition Conditions

<table>
<thead>
<tr>
<th></th>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$y_2 = h$</td>
<td>$r = 1$</td>
</tr>
<tr>
<td>b</td>
<td>$y_2' = 0$</td>
<td>$r = 1$</td>
</tr>
<tr>
<td>c</td>
<td>$y_1 = 0$</td>
<td>$r = b$</td>
</tr>
<tr>
<td>d</td>
<td>$y_1' = 0$</td>
<td>$r = b$</td>
</tr>
<tr>
<td>e</td>
<td>$y_1'' = 0$</td>
<td>$r = b$</td>
</tr>
<tr>
<td>f</td>
<td>$y_2 = \alpha$</td>
<td>$r = c$</td>
</tr>
<tr>
<td>g</td>
<td>$y_1' = y_2'$</td>
<td>$r = c$</td>
</tr>
<tr>
<td>h</td>
<td>$y_1'' = y_2''$</td>
<td>$r = c$</td>
</tr>
<tr>
<td>i</td>
<td>$y_1''' = y_2'''$</td>
<td>$r = c$</td>
</tr>
<tr>
<td>j</td>
<td>$y_1 = \alpha$</td>
<td>$r = c$</td>
</tr>
</tbody>
</table>

B. Results

In Figures 4.4, 4.5, and 4.6, respectively, the contact radius $b$ is plotted versus the substrate height $h$ for pressures $p$ of 0.64, 1.00, and 2.00, at $\alpha = 10^{-6}$. The $f_0$ values are 250, 500, 750, and 1000. The case of no adhesion is also plotted. Figures 4.4 through 4.6 show that as the substrate height increases, the contact radius decreases. As $f_0$ increases, the contact radius at a given pressure also increases.

![Figure 4.4: Contact Radius Versus Height for $p = 0.64$ and $\alpha = 10^{-6}$](image_url)
Figure 4.5: Contact Radius Versus Height for $p = 1.00$ and $\alpha = 10^{-6}$

Figure 4.6: Contact Radius Versus Height for $p = 2.00$ and $\alpha = 10^{-6}$
Next the contact radius $b$ is plotted versus the pressure $p$ for substrate height $h = 0.01$ and $\alpha$ values of $10^{-6}$, $10^{-5}$, and $10^{-4}$, respectively, in Figures 4.7, 4.8, and 4.9. The values of $f_0$ are shown on each plot, and are based on the “area” of $f_0\alpha$ equal to 0.0025, 0.005, 0.0075, and 0.01. Figures 4.7 through 4.9 show that as the pressure increases, the contact radius also increases.

**Figure 4.7: Contact Radius Versus Pressure for $h = 0.01$ and $\alpha = 10^{-6}$**
Figure 4.8: Contact Radius Versus Pressure for $h = 0.01$ and $\alpha = 10^{-5}$

Figure 4.9: Contact Radius Versus Pressure for $h = 0.01$ and $\alpha = 10^{-4}$
In Figure 4.9, when $f_0 = 10$, the contact radius is greater than zero when the pressure is zero. This is because the adhesion force is so large that it causes contact to occur between the blister and the substrate without any pressure.

As in the JKR results, some cases have more than one solution at a given pressure. For such cases, the contact radius will change as pressure is increased or decreased, as shown in Figure 4.10. As the pressure increases from zero, contact first occurs at point A, and the contact radius follows the solid line to B, and then “jumps” to C and continues to D. As the pressure is decreased, the contact radius will begin at D and move to E, and then “jump” down to F and follow back to 0. The dashed line in this case is the portion of the solution path that corresponds to unstable equilibrium states.

**Figure 4.10: Path Followed in the DMT Case When $\alpha$ is Sufficiently Small and $f_0$ is Sufficiently Large**
Chapter 5 – Membrane with No Adhesion

A. Formulation

The second type of blister studied is the membrane, in which bending resistance is neglected and only stretching occurs. Linearly elastic behavior is considered. As shown in Figures 5.1 and 5.2, the membrane shape is different from the plate shape discussed in Chapters 2 through 4 because although the membrane is immovable at the outer radius, it does not have a slope of zero there. The membrane geometry is shown both dimensionally and nondimensionally. Friction between the membrane and the substrate is neglected, as well as the energy dissipation due to plasticity and viscoelasticity. The membrane is homogeneous, isotropic, and has constant thickness. The mathematical model for the membrane allows additional change of strain in the contact region as pressure is applied.

Figure 5.1: Geometry of the Membrane

Figure 5.2: Nondimensional Geometry of the Membrane
The substrate height is \( H \), the membrane thickness is \( T \), and the pressure is \( P \). The radius of the blister is \( R_0 \). The modulus of elasticity is \( E \) and Poisson’s ratio is \( \nu \). The displacement components along radial and transverse directions are \( U(R) \) and \( W(R) \), respectively, and the strains in polar coordinates are \( \varepsilon_r \) and \( \varepsilon_\theta \) (Kao and Perrone, 1971; Lai and Dillard, 1994, 1996). There is a constant prestress \( \sigma_0 \) and the displacements and strains due to the pressure occur after the prestress. The numerical examples will not include prestress. The radial and circumferential stresses are \( \sigma_r \) and \( \sigma_\theta \) respectively. The strains due to the pressure are assumed to be small. The nondimensionalization of the geometry is shown in Table 5.1.

### Table 5.1: Nondimensional Quantities for the Membrane with No Adhesion

<table>
<thead>
<tr>
<th>Nondimensional Quantity</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_m )</td>
<td>( \left( \frac{W}{R_0} \right) \left( \frac{ET}{PR_0} \right)^{1/3} )</td>
</tr>
<tr>
<td>( p )</td>
<td>( \frac{PR_0^3}{D} )</td>
</tr>
<tr>
<td>( r )</td>
<td>( \frac{R}{R_0} )</td>
</tr>
<tr>
<td>( b )</td>
<td>( \frac{B}{R_0} )</td>
</tr>
<tr>
<td>( h_m )</td>
<td>( \left( \frac{H}{R_0} \right) \left( \frac{ET}{PR_0} \right)^{1/3} )</td>
</tr>
<tr>
<td>( y_m )</td>
<td>( h - w_m(r) )</td>
</tr>
<tr>
<td>( u_m )</td>
<td>( \left( \frac{U}{R_0} \right) \left( \frac{ET}{PR_0} \right)^{2/3} )</td>
</tr>
<tr>
<td>( \sigma_{0m} )</td>
<td>( \left( \frac{\sigma_0}{E} \right) \left( \frac{ET}{PR_0} \right)^{2/3} )</td>
</tr>
</tbody>
</table>

The governing equations for the system are
\[ r^2 u_m'' + r^2 w_m' w_m'' = u_m - ru_m' - \left(1 - v\right) r \left(w_m'ight)^2 \]  
\[ \text{(5-1)} \]

and
\[ 2 r w_m' u_m'' + \left[ 3 r \left(w_m'ight)^2 + 2ru_m' + 2v u_m + 2 \left(1 - v^2\right) \cdot r \sigma_0 \right] w_m'' = -\left(w_m'ight)^3 - 2 \left(1 + v\right) \cdot u_m' w_m' - 2 \left(1 - v^2\right) \cdot \sigma_0 w_m' - 2 \left(1 - v^2\right) \cdot r \]
\[ \text{(5-2)} \]

The pressure is not a parameter in these nondimensionalized equations. It only appears if the results are put in terms of the dimensional variables, or in terms of the variables used for the plate model. Equation (5-1) is valid when \(0 < r < b\), where \(w_m = h_m\), and the solution is

\[ u_m = kr \]
\[ \text{(5-3)} \]

where \(k\) is a constant. The derivation of the governing equations is shown in Appendix D.

Since the substrate cannot have an upward concentrated force, the slope at \(r = b\) is zero. Based on Equation (5-3) and continuity, \(u_m = kb\) and \(u_m' = k\) at \(r = b\). The boundary conditions for the case with no adhesion and \(b < r < 1\) are shown in Table 5.2 in terms of \(u_m\) and \(y_m\).

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>(y_m = 0)</td>
<td>(r = b)</td>
</tr>
<tr>
<td>b</td>
<td>(y_m' = 0)</td>
<td>(r = b)</td>
</tr>
<tr>
<td>c</td>
<td>(u_m = u_m' b)</td>
<td>(r = b)</td>
</tr>
<tr>
<td>d</td>
<td>(y_m = h_m)</td>
<td>(r = 1)</td>
</tr>
<tr>
<td>e</td>
<td>(u_m = 0)</td>
<td>(r = 1)</td>
</tr>
</tbody>
</table>

Using the shooting method in Mathematica described in Appendix D, the system is analyzed. Poisson’s ratio is assumed to be 0.3 and \(\sigma_0 = 0\). Some examples of the blister shape are shown in Figure 5.3. In order to normalize the curves, the blister deflection \(w_m\) is divided by the substrate height \(h_m\).
Figure 5.3: Blister Shape - Normalized Blister Height versus Radius

Based on Table 5.1, the equation for the pressure $p$ is

$$p = \left( \frac{\xi h}{h_m} \right)^3$$  \hspace{1cm} (5-4)

where

$$h = \frac{H}{R_0}$$  \hspace{1cm} (5-5)

and

$$\xi = \left[ \frac{12(1-v^2)\left(\frac{R_0}{T}\right)^2}{3} \right]^{1/3}$$  \hspace{1cm} (5-6)

The parameter $\xi$ increases if the blister’s radius increases or the thickness decreases.

B. Results

The contact radius $b$ is plotted versus the pressure $p$ in Figure 5.4 for three different values of $\xi h$. For example, if $\xi = 30$, then the curves correspond to substrate heights $h = 0.01$, 0.02, and 0.03. As the pressure increases, the contact radius increases for all cases.

Next, in Figure 5.5, the contact radius $b$ is plotted versus $\xi h$ for pressures $p$ of 0.5, 1.0, and 2.0. In each of these cases, as $\xi h$ increases, the contact radius decreases.
Figure 5.4: Contact Radius versus Pressure

Figure 5.5: Contact Radius versus $\xi_h$
Chapter 6 – Membrane with JKR Analysis

A. Formulation

The geometry and the nondimensionalized geometry for the membrane with the JKR type of adhesion are shown in Figures 6.1 and 6.2, respectively. It is assumed for this model that adhesion causes a concentrated upward line force $Q$ (nondimensionally, $q$) per unit length around the edge of the contact area at $r = b$. This allows the slope to be discontinuous at $r = b$, so that $w_m'(b) < 0$.

The nondimensional work of adhesion $\Delta \gamma_m$ for the membrane is defined as

$$\Delta \gamma_m = (\Delta \gamma_d) \left( \frac{ET}{PR_0} \right)^{\frac{1}{4}} \left( R \right)^{\frac{3}{2}}$$

(6-1)
The total energy of the system for a particular contact radius consists of the strain energy $U_{Em}$ for $b < r < 1$, potential energy $U_{Pm}$ of the pressure, adhesion energy $U_{Am}$, and stretching energy $U_{Sm}$ for $0 < r < b$. These energies are

$$U_{Em} = \int_{r}^{1} \left[ \frac{1}{1 - \nu^2} \left( \frac{y_4}{2} \right) + \frac{\nu}{2} \right] dy_3 dr$$

$$+ \int_{0}^{r} \left[ \frac{1}{1 - \nu^2} \left( \frac{y_3}{r} + \nu y_4 + \nu \frac{y_2}{2} \right) + \sigma_0 \right] dy_3 dr$$

(6-2)

$$U_{Pm} = -h_m + 2 \int_{0}^{r} ry_3 dr$$

(6-3)

$$U_{Am} = -\Delta \gamma_m b^2$$

(6-4)

and

$$U_{Sm} = \frac{k^2 b^2}{1 - \nu}$$

(6-5)

where

$$y_1 = y_m, \ y_2 = y_m', \ y_3 = u_m, \ y_4 = u_m'$$

(6-6)

and $k$ is defined in Equation (5-3). Adding equations 6-2 through 6-5, the total energy is then

$$U_{Tm} = U_{Em} + U_{Pm} + U_{Am} + U_{Sm}$$

(6-7)

The procedure is similar to that used for the linear plate. Using Mathematica, the total energy is minimized for a given $\Delta \gamma_m$ and $h_m$.

B. Results

The contact radius is plotted versus the pressure for both relatively large and relatively small values for the work of adhesion. Figures 6.3 and 6.4 include plots for $\Delta \gamma$ values of 0, 0.00001, 0.00005, 0.0001, 0.0025, 0.005, and 0.01. The $\xi$ value is 30 and the
substrate height $h$ is 0.01, resulting in a $\xi h$ value of 0.3 for all of the plots shown. In general, as the pressure $p$ increases, the contact radius $b$ also increases. However, a behavior similar to that displayed in the linear plate with the JKR type of adhesion occurs. For smaller pressures, there is sometimes more than one solution, where the total energy has a local maximum and a local minimum at the same pressure. The solid portion of the curves is the solution where the total energy is at a local minimum, and the dashed portion of the curves is the portion where the total energy is at a local maximum. These latter solutions are not physically stable when the pressure is increased and decreased, and the behavior of the membrane blister is the same in this case as in the linear plate shown in Figure 3.7. For the case where $\Delta \gamma = 0.01$, the JKR forces cause the blister to have an initial contact radius of 0.256 when the pressure is zero.

![Figure 6.3: Contact Radius versus Pressure with Relatively Strong Adhesion When $\xi h = 0.3$](image)

Figure 6.3: Contact Radius versus Pressure with Relatively Strong Adhesion When $\xi h = 0.3$
Next, the contact radius $b$ is plotted versus the substrate height for $\Delta \gamma = 0.01$, 0.005, 0.00001, and 0 for $\xi = 30$ and $p=2$. Figure 6.5 shows that, generally, as the substrate height increases, the contact radius decreases. Again, if $\Delta \gamma$ is sufficiently large, there is more than one solution for contact radius $b$, for some of the substrate heights. These are the cases where the total energy has both a local maximum and a local minimum. Again, the lower value of $b$ is associated with an unstable equilibrium shape.
Figure 6.5: Contact Radius versus Substrate Height with Relatively Strong and Weak Adhesion When $\xi = 30$ and $p = 2$
Chapter 7 – Membrane with DMT Analysis

A. Formulation

The DMT analysis for the membrane involves allowing an upward pressure $F(Y)$ just as in the linear plate with DMT type adhesion. This pressure occurs in the annular region when $B < R < C$, or $b < r < c$ nondimensionally. At $r = c$, the distance from the substrate to the blister is $Y = \alpha^0$, or $y_m(r) = \alpha_m$ nondimensionally. The blister geometry for the nonlinear membrane with the DMT type of adhesion is shown in Figure 7.1, and nondimensionally in Figure 7.2.

![Figure 7.1: Membrane Geometry for DMT Analysis](image1)

![Figure 7.2: Nondimensionalized Membrane Geometry for DMT Analysis](image2)

For the membrane with the DMT analysis, the shooting method is used in both directions from the outer circle of the region where the pressure $f(y)$ is acting, which
corresponds to the radius $c$. The directions and their notations are shown in Figure 7.3, and their definitions are given in Table 7.1. The nondimensional quantities for this model are shown in Table 7.2.

![Figure 7.3: Dimensions for the Membrane DMT Analysis](image)

**Table 7.1: Definitions for the Dimensions of the Membrane DMT Analysis**

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>$c - r$</td>
</tr>
<tr>
<td>$x$</td>
<td>$r - c$</td>
</tr>
<tr>
<td>$y_0$</td>
<td>$y_m - \alpha_m$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$\alpha_m - y_m$</td>
</tr>
</tbody>
</table>

**Table 7.2: Nondimensional Quantities for the Membrane DMT Analysis**

<table>
<thead>
<tr>
<th>Nondimensional Quantity</th>
<th>Definition</th>
<th>Relation to Plate Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{0m}$</td>
<td>$\xi^2 F_0/P$</td>
<td>$f_0/p$</td>
</tr>
<tr>
<td>$\alpha_m$</td>
<td>$\alpha^0 (ET/P)^{1/3} / R_0^{4/3}$</td>
<td>$\alpha \xi^2 / p^{1/3}$</td>
</tr>
<tr>
<td>$h_m$</td>
<td>$H (ET/P)^{1/4} / R_0^{4/3}$</td>
<td>$h \xi / p^{1/3}$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>$[12(1-v^2)(R_0/T)^2]^{1/3}$</td>
<td></td>
</tr>
</tbody>
</table>

In region 2-3, the governing equations are equations (5-1) and (5-2), where $y_m = y_0 + \alpha$, $r = x + c$, and $u_m$ is denoted $v_m$. The equations are the same in region 1-2 with $y_m = \alpha_m - \eta$, $r = c - z$, and the last term in (5-2) is multiplied by $(p+f_0)/p$. Using the geometric relationships and the boundary and transition conditions at points 1, 2, and 3 on the blister, the system boundary conditions are found. These conditions are listed in Table 7.3.
Table 7.3: Boundary Conditions for the Membrane DMT Analysis

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$r = b$</td>
<td>$\eta = \alpha_m$</td>
</tr>
<tr>
<td>b</td>
<td>$r = b$</td>
<td>$b\frac{du_m}{dz} = -u_m$</td>
</tr>
<tr>
<td>c</td>
<td>$r = b$</td>
<td>$\frac{d\eta}{dz} = 0$</td>
</tr>
<tr>
<td>d</td>
<td>$r = c$</td>
<td>$y_0 = 0$</td>
</tr>
<tr>
<td>e</td>
<td>$r = c$</td>
<td>$\eta = 0$</td>
</tr>
<tr>
<td>f</td>
<td>$r = c$</td>
<td>$\frac{dy_0}{dx} = \frac{d\eta}{dz}$</td>
</tr>
<tr>
<td>g</td>
<td>$r = c$</td>
<td>$v_m = u_m$</td>
</tr>
<tr>
<td>h</td>
<td>$r = c$</td>
<td>$\frac{dv_m}{dz} = -\frac{du_m}{dx}$</td>
</tr>
<tr>
<td>i</td>
<td>$r = 1$</td>
<td>$y_0 = h_m - \alpha_m$</td>
</tr>
<tr>
<td>j</td>
<td>$r = 1$</td>
<td>$u_m = 0$</td>
</tr>
</tbody>
</table>

In order to solve the governing equations, the shooting method is used. The quantities $f_0$, $h$, $\alpha$, and $\xi$ are chosen. For the numerical solution, the lengths are scaled in both directions from $c$ so that the two total lengths are unity. They are both denoted $t$, so $t = z/(c - b)$ in 1-2 and $t = x/(1 - c)$ in 2-3. The “initial conditions” at $t = 0$ are listed in Table 7.4, where $y_{02}$, $y_{03}$, and $y_{04}$ are unknown values, and $y_1 = y_0$, $y_2 = dy_0/dr$, $y_3 = u_m$, $y_4 = du_m/dr$, $y_5 = \eta$, $y_6 = d\eta/dr$, $y_7 = v_m$, and $y_8 = dv_m/dr$. Therefore, $y_1$ to $y_4$ are defined in region 2-3 and $y_5$ to $y_8$ in region 1-2.

Table 7.4: Boundary Conditions for the Membrane DMT Analysis

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$y_1 = 0$</td>
<td>$t = 0$</td>
</tr>
<tr>
<td>b</td>
<td>$y_2 = y_{02}$</td>
<td>$t = 0$</td>
</tr>
<tr>
<td>c</td>
<td>$y_3 = y_{03}$</td>
<td>$t = 0$</td>
</tr>
<tr>
<td>d</td>
<td>$y_4 = y_{04}$</td>
<td>$t = 0$</td>
</tr>
<tr>
<td>e</td>
<td>$y_5 = 0$</td>
<td>$t = 0$</td>
</tr>
<tr>
<td>f</td>
<td>$y_6 = -y_{02}$</td>
<td>$t = 0$</td>
</tr>
<tr>
<td>g</td>
<td>$y_7 = y_{03}$</td>
<td>$t = 0$</td>
</tr>
<tr>
<td>h</td>
<td>$y_8 = y_{04}$</td>
<td>$t = 0$</td>
</tr>
</tbody>
</table>

The lengths $b$ and $c$ and parameters $y_{02}$, $y_{03}$, and $y_{04}$ are varied until $y_1 = h_m - \alpha_m$, $y_3 = 0$, $y_5 = \alpha_m$, $y_6 = 0$, and $by_8 - y_7 = 0$ at $t = 1$. The Mathematica program for this solution using the shooting method is shown in Appendix F.1.

B. Results
The contact radius is plotted versus the pressure for $\alpha = 10^{-5}$ and $\xi = 30$. Figures 7.4 and 7.5 contain solutions for several values of $f_0$ when $h = 0.03$. Figure 7.6 also contains solutions for several values of $f_0$, but when $h = 0.05$. As the pressure increases on all of these curves, the contact radius also increases, except for the portion where the contact radius is relatively small. As seen in previous cases, the dashed portion of the curves corresponds to solutions that are mathematically possible but physically unstable. As the pressure increases from zero, the blister will not have a contact radius until it reaches the pressure where $b = 0$ on the curves, and (if the slope of the curve is negative there) will then “jump” vertically to the pressure on the solid portion of the curve. This is illustrated in Figure 3.7 for the linear plate with the JKR type of adhesion.

![Figure 7.4: Contact Radius versus Pressure for Relatively Small Values of $f_0$ when $\alpha = 10^{-5}$, $\xi = 30$, and $h = 0.03$](image-url)
Figure 7.5: Contact Radius versus Pressure for Relatively Large Values of $f_0$
when $\alpha = 10^{-5}$, $\xi = 30$, and $h = 0.03$

Figure 7.6: Contact Radius versus Pressure when $\alpha = 10^{-5}$, $\xi = 30$, and $h = 0.05$
Chapter 8 – Nonlinear Plate with No Adhesion

A. Formulation

The last type of blister studied is the nonlinear plate, which is a combination of the plate model and the stretching from the membrane forces. The von Karman equations are used (Szilard, 1974; Ugural, 1999; Chia, 1980; Timoshenko and Woinowsky-Krieger, 1959). The geometry of the nonlinear plate with no adhesion is shown in Figure 8.1.

A stress function $F$ is defined such that the radial and circumferential stress resultants are given by

$$ N_r = \frac{T}{R} \frac{dF}{dR} \quad (8-1) $$

and

$$ N_\theta = T \frac{d^2 F}{dR^2} \quad (8-2) $$

where $T$ is the thickness of the plate. The radial displacement can be written in terms of $F$ as

$$ U = \frac{1}{E} \left[ R \frac{d^2 F}{dR^2} - \nu \frac{dF}{dR} \right] \quad (8-3) $$

It is assumed that $dU/dR$ and $d^2W/dR^2$ can be neglected in comparison with 1.
The von Karman equations in $W$ and $F$ are

$$D\left[ \frac{d^4W}{dR^4} + \frac{2}{R} \frac{d^3W}{dR^3} - \frac{1}{R^2} \frac{d^2W}{dR^2} + \frac{1}{R^3} \frac{dW}{dR} \right] - P - \frac{T}{R} \frac{d}{dR} \left( \frac{dW}{dR} \frac{dF}{dR} \right) = 0 \quad (8-4)$$

and

$$\frac{d^4F}{dR^4} + \frac{2}{R} \frac{d^3F}{dR^3} - \frac{1}{R^2} \frac{d^2F}{dR^2} + \frac{1}{R^3} \frac{dF}{dR} + \frac{E}{R} \frac{dW}{dR} \frac{d^2W}{dR^2} = 0 \quad (8-5)$$

The derivative of the stress function $F$ with respect to the radius $R$ is

$$\frac{dF}{dR} = \frac{R}{T} N_r = \frac{RE}{(1-\nu^2)} \left[ \frac{dU}{dR} + \frac{1}{2} \left( \frac{dW}{dR} \right)^2 + \frac{U}{R} \right] \quad (8-6)$$

and from equation (8-2)

$$\frac{d^2F}{dR^2} = \frac{N_\theta}{T} = \frac{E}{(1-\nu^2)} \left[ \frac{U}{R} + \nu \left( \frac{dW}{dR} \right)^2 + \nu \frac{dU}{dR} \right] \quad (8-7)$$

Using equation (8-6) in equation (8-4) gives

$$D\left[ \frac{d^4W}{dR^4} + \frac{2}{R} \frac{d^3W}{dR^3} - \frac{1}{R^2} \frac{d^2W}{dR^2} + \frac{1}{R^3} \frac{dW}{dR} \right] - P$$

$$- \frac{ET}{(1-\nu^2)} \frac{d}{dR} \left[ \frac{dW}{dR} \frac{dU}{dR} + \left( \frac{dW}{dR} \right)^3 \frac{R}{2} + \nu U \frac{dW}{dR} \frac{d^2W}{dR^2} \right] = 0 \quad (8-8)$$

where the first term in brackets is $\nabla^4 W$.

Differentiating equation (8-6) and comparing the result to equation (8-7) gives

$$R \frac{d^2U}{dR^2} + \frac{dU}{dR} - \frac{U}{R} + \frac{(1-\nu)}{2} \left( \frac{dW}{dR} \right)^2 + R \frac{dW}{dR} \frac{d^2W}{dR^2} = 0 \quad (8-9)$$
which is the same as the nondimensional equation (5-1) for the membrane with no
adhesion.

Differentiating in equation (8-8) leads to

\[
-\frac{RT^2}{12} \nabla^4 W + \frac{PR(1-v^2)}{ET} + \frac{dU}{dR} \frac{dW}{dR} + R \frac{d^2U}{dR^2} \frac{dW}{dR} + R \frac{dU}{dR} \frac{d^2W}{dR^2} + \frac{3}{2} R \left( \frac{dW}{dR} \right)^2 \frac{d^2W}{dR^2} + \frac{1}{2} \left( \frac{dW}{dR} \right)^3 + \nu \frac{dU}{dR} \frac{dW}{dR} + \nu U \frac{d^2W}{dR^2} = 0
\]

(8-10)

which agrees with equation (5.2) if there were no \( \nabla^4 W \) term. Rearranging equations
(8-9) and 8-10) gives

\[
\frac{T^2}{12} \frac{d^4W}{dR^4} = \frac{T^2}{12} \left[ \frac{-2}{R} \frac{d^3W}{dR^3} + \frac{1}{R^2} \frac{d^2W}{dR^2} - \frac{1}{R^3} \frac{dW}{dR} \right] + \frac{P(1-v^2)}{ET}
\]

\[
+ \frac{1}{R} \frac{dU}{dR} \frac{dW}{dR} + \frac{d^2U}{dR^2} \frac{dW}{dR} + \frac{dU}{dR} \frac{d^2W}{dR^2} + \frac{3}{2} \left( \frac{dW}{dR} \right)^2 \frac{d^2W}{dR^2} + \frac{1}{2R} \left( \frac{dW}{dR} \right)^3
\]

(8-11)

\[
+ \frac{\nu}{R} \left( \frac{dU}{dR} \frac{dW}{dR} + U \frac{d^2W}{dR^2} \right)
\]

and

\[
\frac{d^3U}{dR^3} = -\frac{1}{R} \frac{dU}{dR} + \frac{U}{R^2} \left( \frac{1-v}{2R} \left( \frac{dW}{dR} \right)^2 - \frac{dW}{dR} \frac{d^2W}{dR^2} \right)
\]

(8-12)

The nondimensional geometry for the nonlinear plate with no adhesion is shown
in Figure 8.2, and the variables are shown in Table 8.1.
Table 8.1: Nondimensional Variables for the Nonlinear Plate with No Adhesion

<table>
<thead>
<tr>
<th>Nondimensional Quantity</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>W/R₀</td>
</tr>
<tr>
<td>u</td>
<td>U/R₀</td>
</tr>
<tr>
<td>p</td>
<td>PR₀³/D</td>
</tr>
<tr>
<td>r</td>
<td>R/R₀</td>
</tr>
<tr>
<td>b</td>
<td>B/R₀</td>
</tr>
<tr>
<td>h</td>
<td>H/R₀</td>
</tr>
<tr>
<td>y</td>
<td>h - w</td>
</tr>
<tr>
<td>ξ³</td>
<td>ETR₀³/D</td>
</tr>
</tbody>
</table>

where $D = \frac{ET³}{12(1-v²)}$.

In terms of $u$ and $w$, equations (8-11) and (8-12) become

\[
\begin{align*}
\frac{d}{dr} \left( \frac{dw}{dr} \right) & = -2 w'' - \frac{1}{r} w' - \frac{1}{r^2} w + p \\
& + \frac{ξ³}{(1-v²)} \left[ \frac{3}{2} w'^{2} w'' + \frac{1}{2r} w'^{3} + \frac{(1+v)u'w'}{r} + \frac{vw''}{r} + u''w' + uu'' \right] \\
\end{align*}
\]

and

\[
\begin{align*}
u'' & = -\frac{1}{r} u' + \frac{u}{r^2} + \frac{(v-1)}{2r} w'^{2} - w'w'' \\
\end{align*}
\]

where $' = \frac{d}{dr}$. Using $u''$ from (8.14) in (8.13) and combining terms gives
\[ w''' = p - \frac{2}{r} w'' + \frac{1}{r^2} w' - \frac{1}{r^3} w' \]
\[ + \left[ u w' + \frac{\nu u w'}{r} + \frac{u w'}{r^2} + \frac{\nu w'}{2r} w'^3 + \frac{1}{2} w'^2 w'' \right] \left[ \frac{\xi^3}{(1 - \nu^2)} \right] \]  

(8-15)

Letting \( w = h - y \) gives

\[ y''' = -p - \frac{2}{r} y'' + \frac{1}{r^2} y'' - \frac{1}{r^3} y' \]
\[ + \left[ u y' + \frac{\nu u y'}{r} + \frac{uy'}{r^2} + \frac{\nu y'}{2r} y'^3 + \frac{1}{2} y'^2 y'' \right] \left[ \frac{\xi^3}{(1 - \nu^2)} \right] \]

(8-16)

and

\[ u'' = -\frac{1}{r} u' + \frac{u'}{r^2} + \frac{(\nu - 1)}{2r} y'^2 - y'y'' \]

(8-17)

The governing equations for the system are equations (8-14) and (8-16). The Mathematica shooting method is used to solve the system. This program is given in Appendix G.1. The variables for the Mathematica program are defined in Table 8.2.

### Table 8.2: Definition of Variables for the Nonlinear Plate with No Adhesion

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>( y )</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>( y' )</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>( y'' )</td>
</tr>
<tr>
<td>( y_4 )</td>
<td>( y''' )</td>
</tr>
<tr>
<td>( y_5 )</td>
<td>( u )</td>
</tr>
<tr>
<td>( y_6 )</td>
<td>( u' )</td>
</tr>
<tr>
<td>( r )</td>
<td>( t )</td>
</tr>
</tbody>
</table>

The first-order equations are

\[ y_1' = y_2 \]  
(8-18)

\[ y_2' = y_3 \]  
(8-19)

\[ y_3' = y_4 \]  
(8-20)
\[
y_4' = -p - \frac{2}{t} y_4 + \frac{1}{t^2} y_3 - \frac{1}{t^3} y_2 \\
+ \left[ y_6 y_3 + \frac{\nu}{t} \left( y_6 y_2 + y_5 y_3 \right) + \frac{1}{t^2} y_5 y_2 + \frac{\nu}{2t} y_2^3 + \frac{1}{2} y_2^2 y_3 \right] \frac{\xi^3}{(1-\nu^2)}
\] (8-21)

\[
y_5' = y_6
\] (8-22)

and

\[
y_6' = -\frac{1}{t} y_6 + \frac{1}{t^2} y_5 + \frac{(\nu-1)}{2t} y_2^2 - y_2 y_3
\] (8-23)

For \(0 < r < b\), \(y = 0\), \(u = kr\), and \(u' = k\). The boundary conditions are shown in Table 8.3.

**Table 8.3: Boundary Conditions for the Nonlinear Plate with No Adhesion**

<table>
<thead>
<tr>
<th></th>
<th>y_1 = 0</th>
<th>t = b</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>y_2 = 0</td>
<td>t = b</td>
</tr>
<tr>
<td>c</td>
<td>y_3 = 0</td>
<td>t = b</td>
</tr>
<tr>
<td>d</td>
<td>y_4 = y_{04}</td>
<td>t = b</td>
</tr>
<tr>
<td>e</td>
<td>y_5 = kb</td>
<td>t = b</td>
</tr>
<tr>
<td>f</td>
<td>y_6 = k</td>
<td>t = b</td>
</tr>
<tr>
<td>g</td>
<td>y_1 = h</td>
<td>t = 1</td>
</tr>
<tr>
<td>h</td>
<td>y_2 = 0</td>
<td>t = 1</td>
</tr>
<tr>
<td>i</td>
<td>y_5 = 0</td>
<td>t = 1</td>
</tr>
</tbody>
</table>

Instead of specifying the pressure \(p\) and solving for the contact radius \(b\) (and other unknowns), it is easier in the numerical procedure to fix \(b\) and find the corresponding pressure \(p\). The values of \(\nu\), \(h\), \(b\), and \(\xi\) are chosen. The unknown values \(k\), \(y_{04}\), and \(p\) are varied until the conditions at \(t = 1\) (conditions g through i, Table 8.3) are met.

**B. Results**

Samples of the blister shape for the nonlinear plate with no adhesion are shown in Figure 8.3 for \(h = 0.03\) and \(\xi = 30\). As the pressure increases, the contact radius also increases.
Figure 8.3: Blister Shapes for Nonlinear Plate with No Adhesion for $h = 0.03$ and $\xi = 30$

Next the contact radius $b$ is plotted versus the pressure $p$ for $\xi$ values of 0, 20, 25, and 30 with $h = 0.03$ in Figure 8.4 and $h = 0.05$ in Figure 8.5. The case $\xi = 0$ corresponds to the linear plate. As expected, the contact radius increases as the pressure increases.
Figure 8.5: Contact Radius versus Pressure for $h = 0.05$
Chapter 9 – Nonlinear Plate with JKR Analysis

A. Formulation

The forces due to the JKR type of adhesion are added to the formulation for the nonlinear plate with no adhesion. The geometry remains the same as for no adhesion, except that the JKR adhesion forces are assumed to cause a moment that acts at the contact radius \( b \) (see Figure 9.1).

![Figure 9.1: Nondimensional Geometry for the Nonlinear Plate for JKR Analysis](image)

In nondimensional terms, the total energy \( U_T \) of the system consists of the energy due to bending \( V_B \), stretching \( V_S \), pressure \( V_P \), and adhesion \( V_A \). The equations for these energies are

\[
V_B = \int_0^1 \left[ t y_3^2 + \frac{1}{t} y_2^2 \right] dt \quad (9.1)
\]

\[
V_S = \frac{\varepsilon^3 k^2 b^2}{1 - \nu} + \frac{\varepsilon^3}{(1 - \nu^2)} \int_0^1 \left\{ \left( y_6 + \frac{1}{2} y_2^2 \right)^2 + \frac{y_5^2}{t^2} + 2 \nu \left( y_6 + \frac{1}{2} y_2^2 \right) \frac{y_5}{t} \right\} dt \quad (9.2)
\]

\[
V_P = -p h + 2 p \int_0^1 y_1 dt \quad (9.3)
\]

and

\[
V_A = -b^2 \Delta \gamma \quad (9.4)
\]

where the variables have the same definitions as for the nonlinear plate with no adhesion. The total energy is
The program for the nonlinear plate with the JKR type of adhesion is given in Appendix H.

B. Results

The contact radius $b$ is plotted versus the pressure $p$ in Figures 9.2 through 9.5 for substrate heights $h$ of 0.03 and 0.05, $\xi$ values of 20 and 30, respectively, and various values of the work of adhesion $\Delta \gamma$, including the case of no adhesion. In general, as the pressure increases, the contact radius also increases. As seen in the figures, some cases exist where there are two solutions for the contact radius at relatively small pressures, similar to those occurring in the linear plate and the nonlinear membrane with the JKR type of adhesive forces. When $\xi = 30$, larger pressures are required to achieve the same contact radius as when $\xi = 20$. Also, when the substrate height is $h = 0.05$, a higher pressure is required to achieve the same contact radius than when $h = 0.03$.

$$U_T = V_k + V_s + V_p + V_A$$

(9.5)

Figure 9.2: Contact Radius versus Pressure When $h = 0.03, \xi = 20$
Figure 9.3: Contact Radius versus Pressure When $h = 0.03$, $\xi = 30$

Figure 9.4: Contact Radius versus Pressure When $h = 0.05$, $\xi = 20$
The next four figures (Figures 9.6 through 9.9) present plots of the contact radius $b$ versus $\Delta \gamma$, for the same combinations of substrate height and $\xi$, and at various pressures appropriate for each combination. When the pressure is the same, a larger contact radius occurs for the smaller value of $\xi = 20$. A larger contact radius also results for the smaller substrate height of $h = 0.03$. Naturally the contact radius increases as the work of adhesion increases.
Figure 9.6: Contact Radius versus $\Delta \gamma$ When $h = 0.03$, $\xi = 20$

Figure 9.7: Contact Radius versus $\Delta \gamma$ When $h = 0.03$, $\xi = 30$
Figure 9.8: Contact Radius versus $\Delta \gamma$ When $h = 0.05$, $\xi = 20$

Figure 9.9: Contact Radius versus $\Delta \gamma$ When $h = 0.05$, $\xi = 30$
Chapter 10 – Nonlinear Plate with DMT Analysis

A. Formulation

The DMT solution for the nonlinear plate is similar to the membrane DMT solution. The nondimensional geometry, along with the DMT force for this case, is shown in Figure 10.1.

![Figure 10.1: Nondimensional Geometry and Location of the DMT Force](image)

The dimensions and coordinates are similar to those in the membrane DMT solution and are shown with the nonlinear plate geometry in Figure 10.2.

![Figure 10.2: Dimensions for the Membrane DMT Analysis](image)
For the DMT solution, \( p \) becomes \( p + f_0 \) for \( b < r < c \), and the origin moves to point 2. The two regions of interest are region 1-2 and region 2-3. The lengths of these two regions, \( c-b \) and \( 1-c \), are scaled to be 1, so that both can be represented by \( t \), where \( 0 < t < 1 \).

From the case with no adhesion, equations (8.13) and (8.14), in terms of \( w \) and \( u \), and equations (8.16) and (8.17), in terms of \( y \) and \( u \), are applicable.

The nondimensional quantities used in region 2-3 are defined in Table 10.1.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( \frac{x}{1-c} )</td>
</tr>
<tr>
<td>( r )</td>
<td>( c + t - ct )</td>
</tr>
<tr>
<td>( y_0 )</td>
<td>( y - \alpha )</td>
</tr>
<tr>
<td>( y_1 )</td>
<td>( y_0 )</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>( \frac{dy_0}{dr} )</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>( \frac{d^2 y_0}{dr^2} )</td>
</tr>
<tr>
<td>( y_4 )</td>
<td>( \frac{d^3 y_0}{dr^3} )</td>
</tr>
<tr>
<td>( y_5 )</td>
<td>( u )</td>
</tr>
<tr>
<td>( y_6 )</td>
<td>( \frac{du}{dr} )</td>
</tr>
</tbody>
</table>

In region 2-3, the first-order equations are

\[
\begin{align*}
y_1' &= (1-c)y_2 \\
y_2' &= (1-c)y_3 \\
y_3' &= (1-c)y_4
\end{align*}
\]
\[
y_4' = (1 - c) \left\{ \frac{-2}{t} y_4 + \frac{1}{t^2} y_3 - \frac{1}{t} y_2 - p \\
+ \left[ y_3 y_6 + \frac{V}{t} (y_2 y_6 + y_3 y_3) \right] \frac{\xi^3}{(1 - \nu^2)} \right\}
\]

(10-4)

\[
y_5' = (1 - c) y_6
\]

(10-5)

and

\[
y_6' = (1 - c) \left\{ \frac{-1}{t^2} y_6 + \frac{1}{t^2} y_5 + \frac{(\nu - 1)}{2t} y_2^2 - y_2 y_3 \right\}
\]

(10-6)

where \( ' = \frac{d}{dt} \). In region 1-2, the quantities used are given in Table 10.2.

**Table 10.2: Definitions of Variables in Region 1-2**

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>( \frac{z}{c - b} )</td>
</tr>
<tr>
<td>z</td>
<td>( c - r )</td>
</tr>
<tr>
<td>r</td>
<td>( c + bt - ct )</td>
</tr>
<tr>
<td>( \eta )</td>
<td>( \alpha - y )</td>
</tr>
<tr>
<td>y_7</td>
<td>( \eta )</td>
</tr>
<tr>
<td>y_8</td>
<td>( \frac{d\eta}{dr} )</td>
</tr>
<tr>
<td>y_9</td>
<td>( \frac{d^2\eta}{dr^2} )</td>
</tr>
<tr>
<td>y_10</td>
<td>( \frac{d^3\eta}{dr^3} )</td>
</tr>
<tr>
<td>y_11</td>
<td>u</td>
</tr>
<tr>
<td>y_12</td>
<td>( \frac{du}{dr} )</td>
</tr>
</tbody>
</table>
The first-order equations in region 1-2 are

\[ y_7' = (b - c)y_8 \tag{10-7} \]
\[ y_8' = (b - c)y_9 \tag{10-8} \]
\[ y_9' = (b - c)y_{10} \tag{10-9} \]

and

\[ y_{10}' = (b - c)\frac{d^4y}{dr^4} \tag{10-10} \]

In equations (8-16) and (8-17), \( p \) becomes \( p + f_0 \), \( r \) becomes \( c + bt - ct \), \( y' = -y_8 \), \( y'' = -y_9 \), \( y''' = -y_{10} \), \( u = y_{11} \), and \( u' = y_{12} \), where \( ' = d/dr \). Therefore, with \( ' = d/dt \),

\[
y_{10}' = (c-b) \left\{ -p-f_0 + \frac{2y_{10}}{(c+bt-ct)} - \frac{y_9}{(c+bt-ct)^2} + \frac{y_8}{(c+bt-ct)^3} \right\}
\[ - \frac{(c-b)\xi^3}{(1-\nu^2)} \left\{ \frac{v(y_8y_{12} + y_{11}y_9) + 0.5vy_8^3}{(c+bt-ct)} + \frac{y_8y_{11}}{(c+bt-ct)^2} + \frac{1}{2}y_8^2y_9 + y_8y_{12} \right\} \tag{10-11} \]
\[ y_{11}' = (b-c)y_{10} \tag{10-12} \]
\[ y_{12}' = (b-c) \left\{ - \frac{y_{12}}{(c+bt-ct)} + \frac{y_{11}}{(c+bt-ct)^2} + \frac{(v-1)y_8^2}{2(c+bt-ct)} - y_8y_9 \right\} \tag{10-13} \]

At 2, the functions \( y \), \( y' \), \( y'' \), \( y''' \), \( u \), and \( u' \) are continuous, where \( ' = d/dr \). The boundary conditions are shown in Table 10.5.
Table 10.5: Boundary Conditions for the Nonlinear Plate with DMT Adhesion

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$y_1 = 0$</td>
<td>$t = 0$ (r = c)</td>
</tr>
<tr>
<td>b</td>
<td>$y_2 = -y_8$</td>
<td>$t = 0$ (r = c)</td>
</tr>
<tr>
<td>c</td>
<td>$y_3 = -y_9$</td>
<td>$t = 0$ (r = c)</td>
</tr>
<tr>
<td>d</td>
<td>$y_4 = -y_{10}$</td>
<td>$t = 0$ (r = c)</td>
</tr>
<tr>
<td>e</td>
<td>$y_5 = y_{11}$</td>
<td>$t = 0$ (r = c)</td>
</tr>
<tr>
<td>f</td>
<td>$y_6 = y_{12}$</td>
<td>$t = 0$ (r = c)</td>
</tr>
<tr>
<td>g</td>
<td>$y_7 = 0$</td>
<td>$t = 0$ (r = c)</td>
</tr>
<tr>
<td>h</td>
<td>$y_1 = h - \alpha$</td>
<td>$t = 1$ (r = 1)</td>
</tr>
<tr>
<td>i</td>
<td>$y_2 = 0$</td>
<td>$t = 1$ (r = 1)</td>
</tr>
<tr>
<td>j</td>
<td>$y_5 = 0$</td>
<td>$t = 1$ (r = 1)</td>
</tr>
<tr>
<td>k</td>
<td>$y_7 = \alpha$</td>
<td>$t = 1$ (r = b)</td>
</tr>
<tr>
<td>l</td>
<td>$y_8 = 0$</td>
<td>$t = 1$ (r = b)</td>
</tr>
<tr>
<td>m</td>
<td>$y_9 = 0$</td>
<td>$t = 1$ (r = b)</td>
</tr>
<tr>
<td>n</td>
<td>$y_{11} = b y_{12}$</td>
<td>$t = 1$ (r = b)</td>
</tr>
</tbody>
</table>

The parameters $\nu$, $\xi$, $h$, $p$, $f_0$, and $\alpha$ are set. Using the Mathematica shooting method, b and c, along with the values of $y_2$ through $y_6$ at $t = 0$, are found by shooting for the boundary conditions at points 2 and 4. This program is given in Appendix I.

B. Results

The contact radius is plotted versus the pressure in Figures 10.3 through 10.6 for $\alpha = 10^{-5}$, several values of $f_0$, $\xi = 20$ and 30, and $h = 0.03$ and 0.05. The case $f_0 = 0$ corresponds to no adhesion. As the pressure increases in all the cases, the contact radius also increases. In addition, as $f_0$ increases, the contact radius increases as expected. For the nonlinear plate with no adhesion, no cases are plotted where more than one solution exists for the contact radius at a given pressure, as found in the linear plate and the membrane solutions.
Figure 10.3: Contact Radius Versus Pressure for $h = 0.03$, $\alpha = 10^{-5}$, and $\xi = 20$

Figure 10.4: Contact Radius Versus Pressure for $h = 0.03$, $\alpha = 10^{-5}$, and $\xi = 30$
Figure 10.5: Contact Radius Versus Pressure for $h = 0.05$, $\alpha = 10^{-5}$, and $\xi = 20$

Figure 10.6: Contact Radius Versus Pressure for $h = 0.05$, $\alpha = 10^{-5}$, and $\xi = 30$
Chapter 11 – Conclusions and Recommendations

A. Conclusions

This chapter contains a comparison of the results for the various models studied. The JKR and DMT results can be compared through the work of adhesion. The JKR work of adhesion $\Delta \gamma$ is equivalent to the DMT “area” found by multiplying $f_0$ and $\alpha$ (Barthel, 1998; Greenwood and Johnson, 1998). For example, the JKR case where $\Delta \gamma = 0.01$ corresponds to the DMT cases where the product of $f_0$ and $\alpha$ is 0.01. The linear plate results are shown in Figure 11.1 for $h = 0.03$. The contact radii for the JKR and DMT models are very close to each other for most of the range shown.

![Figure 11.1: Comparison of Results for the Linear Plate for h = 0.03](image)

Next, the JKR and DMT results are compared in Figure 11.2 for $h = 0.01$, showing several combinations of $f_0$ and $\alpha$ for the DMT analysis, whose product is equal to the $\Delta \gamma$ for the JKR curve. The cases shown are $f_0 = 10$ and $\alpha = 10^{-4}$, $f_0 = 100$ and $\alpha = 10^{-5}$, and $f_0 = 1000$ and $\alpha = 10^{-6}$, the last of which most closely follows the JKR curve for
$\Delta \gamma = 0.01$ in Figure 4.11. This might be expected, i.e., that the correlation between the JKR and DMT types of analysis would tend to improve as the gap in which the DMT forces are active becomes smaller.

![Comparison of JKR and DMT for the Linear Plate for $h = 0.01$](image)

**Figure 11.2: Comparison of JKR and DMT for the Linear Plate for $h = 0.01$**

The membrane results are shown next in Figure 11.3 for $h = 0.03$. The JKR and DMT models result in larger contact radii for the same pressure than the no-adhesion model, but they are not as close together as they are for the linear plate. The DMT model results in a significantly higher contact radius than the JKR model.
Next, in Figure 11.4, a comparison of the models for the nonlinear plate is shown for $h = 0.03$. For the nonlinear plate, the DMT solution does not contain any unstable solutions; therefore, the shape is not the same as in the previous comparisons. The DMT and the JKR solutions are higher than the no-adhesion solution, and are relatively close together in the stable region of the JKR curve.
Figure 11.4: Comparison of the Nonlinear Plate Models for $h = 0.03$ and $\xi = 30$

The shapes of the three blister models are shown for the same contact radius in Figure 11.5 and for the same pressure in Figure 11.6. The nonlinear plate requires the highest pressure to achieve the same contact radius, followed by the linear plate and then the membrane. The shapes of the linear and nonlinear plate models are almost identical at the same pressure for the conditions shown in Figure 11.6, while the membrane has a much larger contact radius.

Figure 11.5: Blister Shape Comparison when $h = 0.03$, $b = 0.1$, and $\xi = 10$
Figure 11.6: Blister Shape Comparison when $h = 0.03$, $p = 3$, and $\xi = 10$

Figures 11.7 and 11.8 show comparisons of all three models with no adhesion for $h = 0.03$ and $h = 0.05$, respectively. In both of these plots, the following definitions are used for each curve: L - linear plate, M1 - membrane with $\xi = 20$, M2 - membrane with $\xi = 25$, M3 - membrane with $\xi = 30$, NP1 - nonlinear plate with $\xi = 20$, NP2 - nonlinear plate with $\xi = 25$, and NP3 - nonlinear plate with $\xi = 30$. The linear plate solution corresponds to the nonlinear plate solution with $\xi = 0$.

Figure 11.7: Comparison of the Models with No Adhesion for $h = 0.03$
In general for $h = 0.03$, the membrane curves show a higher contact radius at the same pressure than the linear and nonlinear plates. This is expected because the boundary conditions for the membrane involve an immovable outer edge where the slope is not zero, while the linear plate and the nonlinear plate have a “fixed” edge. At a higher substrate height of $h = 0.05$, this behavior lessens because the end conditions become less important at larger substrate heights.

The next plot is the JKR comparison for $h = 0.03$ and $\xi = 30$. Figure 11.9 presents a comparison of the three models when $\Delta \gamma = 0.01$. Figure 11.10 contains a comparison of the three models for the DMT type of adhesion when $h = 0.03$, $\xi = 30$, $f_0 = 1000$, and $\alpha = 10^{-5}$.

For each of the three models, no-adhesion, JKR, and DMT, the nonlinear plate solution results in the lowest contact radius at a given pressure for the same substrate height and other conditions that are chosen ($\Delta \gamma$, $f_0$, $\alpha$, and $\xi$). At relatively large pressures, the linear plate results in the next highest contact radius, with the membrane...
resulting in the largest contact radius. This is expected intuitively because the membrane has a “pinned” boundary condition while the linear and nonlinear plates are “fixed.”

**Figure 11.9:** Comparison of the Models with JKR adhesion for $h = 0.03$, $\xi = 30$, and $\Delta \gamma = 0.01$

**Figure 11.10:** Comparison of DMT when $h = 0.03$, $\xi = 30$, $f_0 = 1000$, and $\alpha = 10^{-5}$
An interesting behavior occurs in certain cases where the blister seems to “jump” onto the rigid substrate. As seen in Figure 3.7, the jumping is expected to occur because the adhesive forces act as a magnet drawing the blister up to the substrate when the adhesive forces can no longer be resisted by the blister. In these cases, the blister will start from no contact and “jump” onto the substrate as the pressure is increased, or “jump” off the substrate as the pressure is decreased. However, results are also shown where the contact radius begins to increase from zero as the pressure is increased, and then “jumps” to a new contact radius after already having contact. As the pressure is decreased in this case, the contact radius “jumps” down to a smaller contact radius and then continues to zero, as seen in Figure 4.10. This behavior may be a new discovery in adhesion research.

B. Recommendations for Future Research

Although this research has increased our knowledge about the measurement of the energy of adhesion, opportunities for future research on this type of blister test are abundant. First, experimental studies are needed to compare to the mathematical results. This would provide a better understanding and an indication of the accuracy of the results.

Other boundary conditions can be studied, such as the simply-supported plate. The boundary conditions for the simply-supported plate are like those for the membrane, and will result in a larger contact radius than for the plate with a fixed edge.

In addition, the DMT solution with a linearly-varying force can be added to this research (Dalrymple, 1999). The linearly-varying force would be at its maximum at the edge of the contact region, $r = b$, and decrease linearly to zero at the edge of the cohesive zone, $r = c$.

Finally, a broader range of substrate heights, pressures, and values of $\Delta \gamma$, $f_0$, $\alpha$, and $\xi$ can be studied. This might provide a better understanding of the comparison between the results of the three models.
References


References, Continued


References, Continued


Appendix A: Linear Plate with No Adhesion

A.1: Solution Procedure for the Governing Equation When \( p > 64h \)

Derivatives of the solution (2.3) for \( y(r) \):

\[
y'(r) = \left(\frac{1}{r}\right) A_2 + 2r A_3 + (2 \ln r + r) A_4 - \left(\frac{pr^3}{16}\right)
\]

\[
y''(r) = \left(-\frac{1}{r^2}\right) A_2 + 2 A_3 + (2 \ln r + 3) A_4 - \left(\frac{3pr^2}{16}\right)
\]

\[
y'''(r) = \left(\frac{2}{r^3}\right) A_2 + \left(\frac{2}{r}\right) A_4 - \left(\frac{3pr}{8}\right)
\]

Using the boundary conditions listed in Table 2.2, the following equations are obtained:

\[
A_1 + A_3 = h + \frac{p}{64}
\]

\[
A_2 + 2A_3 + A_4 = \frac{p}{16}
\]

\[
A_1 + A_2 \ln b + A_3 b^2 + A_4 b^2 \ln b = pb^4/64
\]

\[
A_2(1/b) + 2b A_3 + (1 + 2 \ln b) b A_4 = \frac{pb^3}{16}
\]

\[
(-1/b^2)A_2 + 2A_3 + (3 + 2 \ln b) A_4 = 3pb^2/16
\]

Solving equation (A.4) for \( A_1 \) in terms of \( A_3 \), equations (A.5) through (A.7) can be rewritten as:

\[
A_2 + 2A_3 + A_4 = \frac{p}{16}
\]

\[
A_2 \ln b + A_3(b^2-1) + A_4b^2 \ln b = (b^4 - 1)(p/64) - h
\]

\[
A_2/b + 2b A_3 + (1 + 2 \ln b) b A_4 = \frac{pb^3}{16}
\]

Equations (A.9) through (A.11) are solved for \( A_2 \), \( A_3 \), and \( A_4 \), in terms of the contact radius, pressure, and height, and \( A_1 \) is obtained from equation (A.4). Setting values for
height and pressure, equation (A.8) can then be written as a function of $b$ equal to zero,
$g(b) = 0$. The FindRoot program in Mathematica is then used to find the root of $g(b) = 0$
in the range $0 < b < 1$.

A.2: Mathematica Program for Linear Plate with No Adhesion

```mathematica
p = 5
h = 0.02
c11 = 1
c12 = 2
c13 = 1
c21 = Log[b]
c22 = (b^2) - 1
c23 = (b^2) Log[b]
c31 = 1/b
c32 = 2/b
c33 = b (1 + 2 Log[b])
d1 = p/16
d2 = (p/64) ((b^4) - 1) - h
d3 = p (b^3) / 16
\Delta 1 = c11 c22 c33 + c12 c23 c31 + c13 c32 c21 - c13 c31 c22 - c12 c21 c33 - c23 c32 c11
\Delta 2 = d1 c22 c33 + d2 c32 c13 + d3 c12 c23 - d3 c22 c13 - d2 c12 c33 - d1 c23 c32
\Delta 3 = d2 c11 c33 + d1 c23 c31 + d3 c21 c13 - d2 c13 c31 - d1 c21 c33 - d3 c23 c11
\Delta 4 = d3 c11 c22 + d2 c12 c31 + d1 c21 c32 - d2 c21 c31 - d1 c22 c31 - d2 c32 c11 - d3 c12 c21
A2 = \Delta 2 / \Delta 1
A3 = \Delta 3 / \Delta 1
A4 = \Delta 4 / \Delta 1
A1 = h + (p/64) - A3
g = -A2 / (b^2) + 2 A3 + (3 + 2 Log[b]) A4 - 3 p (b^2) / 16
FindRoot[g == 0, {b, 0.01}]
b = 0.23916
y = A1 + A2 Log[x] + A3 (x^2) + A4 (x^2) Log[x] - p (x^4) / 64
```
Appendix B: Linear Plate with JKR

B.1: Mathematica Program for Linear Plate with JKR

This program calculates \( b \) for different values of \( \Delta \gamma \) at a given pressure and height using the energy minimization of the JKR technique.

\[
\begin{align*}
    h &= 0.01; \ p = 2 \\
    c_{11} &= 1; \ c_{12} = 2; \ c_{13} = 1; \ c_{21} = \log[b]; \ c_{22} = (b^2) - 1; \ c_{23} = (b^2) \log[b] \\
    c_{31} &= 1/b; \ c_{32} = 2b; \ c_{33} = b(1 + 2 \log[b]) \\
    d_1 &= p/16; \ d_2 = (p/64)(b^4) - 1 - h; \ d_3 = p(b^3)/16 \\
    \Delta_1 &= c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{32}c_{21} - c_{13}c_{31}c_{22} - c_{12}c_{21}c_{33} - c_{23}c_{32}c_{11} \\
    \Delta_2 &= d_1c_{22}c_{33} + d_2c_{32}c_{13} + d_3c_{12}c_{23} - d_3c_{22}c_{13} - d_2c_{12}c_{33} - d_1c_{23}c_{32} \\
    \Delta_3 &= d_2c_{11}c_{33} + d_1c_{23}c_{31} + d_3c_{21}c_{13} - d_2c_{13}c_{31} - d_1c_{21}c_{33} - d_3c_{23}c_{11} \\
    \Delta_4 &= d_3c_{11}c_{22} + d_2c_{12}c_{31} + d_1c_{21}c_{32} - d_1c_{22}c_{31} - d_2c_{32}c_{11} - d_3c_{12}c_{21} \\
    A_2 &= \Delta_2 / \Delta_1; \ A_3 = \Delta_3 / \Delta_1; \ A_4 = \Delta_4 / \Delta_1; \ A_1 = h + (p/64) - A_3 \\
    g &= -A_2/(b^2) + 2A_3 + (3 + 2 \log[b]) A_4 - 3p(b^2)/16 \\
    y &= A_1 + A_2 \log[x] + A_3 x^2 + A_4 (x^2) \log[x] - p(x^4)/64 \\
    w &= h - y \\
    S &= \text{D}[\text{D}[w, x], x] + (\text{D}[w, x])/x \\
    \text{Simplify}[%] \\
    UE &= \pi \text{Integrate}[x S^2, \{x, 0, 1\}] \\
    \text{Factor}[%] \\
    UP &= -\pi (b^2) hp - 2 \pi p \text{Integrate}[x w, \{x, 0, 1\}] \\
    \text{Factor}[%] \\
    UA &= -\Delta \gamma \pi b^2 \\
    UT &= UE + UP + UA \\
    \text{Do}[\text{Print}[-UT, \{b, 0.1, 0, 0.5\}], \{\Delta \gamma, 0.0025, 0.01, 0.0025\}]
\end{align*}
\]
Appendix C: Linear Plate with DMT

C.1: Solution of the System Equations for the Linear Plate with No Adhesion

The shape of the blister in the DMT analysis, \( y \), is separated into two parts. When the radius is between \( b \) and \( c \), the blister shape is \( y_1 \), and when it is between \( c \) and 1, the shape is \( y_2 \). The equations for the shape are

\[
y_1(r) = B_1 + B_2 + B_3 r^2 + B_4 r^2 \ln r - (p + f_0) r^4 / 64 \tag{C-1}
\]

and

\[
y_2(r) = E_1 + E_2 \ln r + E_3 r^2 + E_4 r^2 \ln r - pr^4 / 64 \tag{C-2}
\]

From the boundary and transition conditions in Table 4.2, the ten equations are

\[
E_1 + E_3 = h + p / 64 \tag{C-3}
\]

\[
E_2 + 2E_3 + E_4 = p / 16 \tag{C-4}
\]

\[
B_1 + B_2 \ln b + B_3 b^2 + B_4 b^2 \ln b = (p + f_0) b^4 / 64 \tag{C-5}
\]

\[
B_2 / b + 2bB_3 + (1 + 2 \ln b)bB_4 = (p + f_0)b^3 / 16 \tag{C-6}
\]

\[
-B_2 / b^2 + 2B_3 + (3 + 2 \ln b)B_4 = 3(p + f_0)b^2 / 16 \tag{C-7}
\]

\[
E_1 + E_2 \ln c + E_3 c^2 + E_4 c^2 \ln c = \alpha + pc^4 / 64 \tag{C-8}
\]

\[
B_2 / c + 2cB_3 + (1 + 2 \ln c)cB_4 - E_2 / c - 2E_3 - (1 + 2 \ln c)cE_4 = f_0c^3 / 16 \tag{C-9}
\]

\[
-B_2 / c^2 + 2B_3 + (3 + 2 \ln c)B_4 + E_3/c^2 - 2E_4 - (3 + 2 \ln c)E_4 - 3f_0c^2 / 16 = 0 \tag{C-10}
\]

\[
2B_2/c^3 + 2B_4/c - 2E_2/c^3 - 2E_4/c - 3f_0c/8 = 0 \tag{C-11}
\]

and

\[
B_1 + B_2 \ln c + B_3 c^2 + B_4 c^2 \ln c = \alpha + (p + f_0)c^4 / 64 \tag{C-12}
\]

Letting

\[
h_1 = h + p / 64 \tag{C-13}
\]

and

\[
h_2 = (p + f_0)b^4 / 64 \tag{C-14}
\]
the first three equations can be simplified to

\[
E_1 = h_1 - E_3 \\
E_2 = p/16 - 2E_3 - E_4
\]
and

\[
B_1 = h_2 - B_2 \ln b - B_3 b^2 - B_4 b^2 \ln b
\]

Substituting equation (C-17) into equations (C-6), (C-7), and (C-8), the solution for \( B_2 \), \( B_3 \), and \( B_4 \) is

\[
B_2 = D_2/D_1 \\
B_3 = D_3/D_1 \\
B_4 = D_4/D_1
\]

where \( D_1, D_2, D_3, \) and \( D_4 \) are defined in Appendix A and

\[
c_{11} = 1/b \\
c_{12} = 2b \\
c_{13} = (1 + 2 \ln b)b \\
c_{21} = -1/b^2 \\
c_{22} = 2 \\
c_{23} = 3 + 2 \ln b \\
c_{31} = \ln c - \ln b \\
c_{32} = c^2 - b^2 \\
c_{33} = c^2 \ln c - b^2 \ln b \\
d_1 = (p + f_0)b^2/16 \\
d_2 = 3(p + f_0)b^2/16 \\
d_3 = \alpha - h_2 + (p + f_0)c^4/64
\]

Substituting equations (C-15) and (C-16) into equation (C-8), and letting
\[ h_3 = \alpha + p c^4/64 - p (\ln c)/16 - h_1 \]  \hspace{1cm} (C-33)

\[ h_4 = c^2 - 1 - 2 \ln c \]  \hspace{1cm} (C-34)

\[ h_5 = (c^2 - 1) \ln c \]  \hspace{1cm} (C-35)

\[ h_6 = h_3/h_5 \]  \hspace{1cm} (C-36)

\[ h_7 = h_4/h_5, \]  \hspace{1cm} (C-37)

\[ h_8 = 2 c h_7 \ln c - 2c - h_7/c + 2/c + ch_7 \]  \hspace{1cm} (C-38)

and

\[ h_9 = f_0 c^3/16 - B_2/c - 2cB_3 + (1 + 2 \ln c)(h_6 - B_4)c + p/16c - h_6/c \]  \hspace{1cm} (C-39)

the solution for \( E_1, E_2, E_3, \) and \( E_4 \) is

\[ E_3 = h_9/h_8 \]  \hspace{1cm} (C-40)

\[ E_1 = h_1 - E_3 \]  \hspace{1cm} (C-41)

\[ E_2 = E_3 (h_7 - 2) + p/16 - h_6 \]  \hspace{1cm} (C-42)

and

\[ E_4 = h_6 - h_7 E_3 \]  \hspace{1cm} (C-43)

Substituting the values for \( B_1, B_2, B_3, B_4, E_1, E_2, E_3, \) and \( E_4 \) into equations (C-10) and (C-11), and specifying particular values for \( p, h, f_0, \) and \( \alpha, \) results in two equations with two unknowns, \( b \) and \( c. \) The equations are solved using the FindRoot program of Mathematica. A sample of this program follows.
C.2: Mathematica Program for Linear Plate with DMT

This program is used to find the contact radius b and the outer radius c of the linear plate when DMT forces are included.

\[ p = 5; \ h = 0.01; \ \alpha = 0.001; \ fo = 10 \]
\[ c_{11} = 1/ b; \ c_{12} = 2/ b; \ c_{13} = (1 + 2 \log(b)); \ c_{21} = -1/ (b^2); \ c_{22} = 2 \]
\[ c_{23} = 3 + 2 \log(b); \ c_{31} = \log(c) - \log(b); \ c_{32} = (c^2) - (b^2); \]
\[ c_{33} = (c^2) \log(c) - (b^2) \log(b) \]
\[ h_1 = h + (p/64); \ h_2 = (p + fo) (b^4)/64 \]
\[ d_1 = (p + fo) (b^3)/16; \ d_2 = 3 (p + fo) (b^2)/16; \ d_3 = \alpha - h_2 + ((p + fo) (c^4)/64) \]
\[ \Delta_1 = c_{11} c_{22} c_{33} + c_{12} c_{23} c_{31} + c_{13} c_{21} c_{32} - c_{13} c_{31} c_{22} - c_{12} c_{21} c_{33} - c_{23} c_{32} c_{11} \]
\[ \Delta_2 = d_1 c_{22} c_{33} + d_2 c_{23} c_{31} + d_3 c_{21} c_{32} - d_3 c_{22} c_{31} - d_2 c_{12} c_{33} - d_1 c_{23} c_{32} \]
\[ \Delta_3 = d_2 c_{11} c_{33} + d_1 c_{23} c_{31} + d_3 c_{12} c_{31} - d_2 c_{13} c_{31} - d_1 c_{21} c_{33} - d_3 c_{23} c_{11} \]
\[ \Delta_4 = d_3 c_{11} c_{22} + d_2 c_{12} c_{31} + d_1 c_{21} c_{32} - d_1 c_{22} c_{31} - d_2 c_{23} c_{11} - d_3 c_{12} c_{21} \]
\[ B_2 = \Delta_2 / \Delta_1; \ B_3 = \Delta_3 / \Delta_1; \ B_4 = \Delta_4 / \Delta_1 \]
\[ B_1 = h_2 - B_2 \log(b) - B_3 (b^2) - B_4 (b^2) \log(b) \]
\[ h_3 = \alpha + (p (c^4)/64) - (p (\log(c))/16) - h_1 \]
\[ h_4 = (c^2) - 1 - 2 \log(c); \ h_5 = ((c^2) - 1) \log(c); \ h_6 = h_3 / h_5; \ h_7 = h_4 / h_5 \]
\[ h_8 = (2 c h_7 \log(c)) - (2 c) - (h_7 / c) + (2 / c) + ch_7 \]
\[ h_9 = (fo (c^3) / 16) - (B_2 / c) - (2 c B_3) + ((1 + 2 \log(c)) (h_6 - B_4)) + (p / (16 c)) - (h_6 / c) \]
\[ E_3 = h_9 / h_8; \ E_1 = h_1 - E_3; \ E_2 = E_3 (h_7 - 2) + (p / 16) - h_6; \ E_4 = h_6 - h_7 E_3 \]
\[ x = (-B_2 / (c^2)) + 2 B_3 + ((3 + 2 \log(c)) B_4) + (E_2 / (c^3)) - 2 E_3 - ((3 + 2 \log(c)) E_4) - (3 fo (c^2) / 16) \]
\[ z = (2 B_2 / (c^2)) + (2 B_4 / c) - (2 E_2 / (c^3)) - (2 E_4 / c) - (3 fo c / 8) \]
\[ \text{FindRoot} \{x == 0, \ z == 0, \ \{b, \{0.1, \ 0.2\}\}, \ \{c, \ \{0.3, \ 0.4\}\}, \ \text{MaxIterations} \to 500\} \]
Appendix D: Nonlinear Membrane with No Adhesion

D.1: Derivation of the Governing Equations

This derivation follows the work of Kao and Perrone (1971) and Lai and Dillard (1994, 1996). The radial and circumferential strains are

\[ \varepsilon_r = \frac{dU}{dR} + \frac{1}{2}\left( \frac{dW}{dR} \right)^2 \]  

(D-1)

and

\[ \varepsilon_\theta = \frac{U}{R} \]  

(D-2)

where terms higher than second order in \( \frac{dU}{dR} \) and \( \frac{dW}{dR} \) are neglected in equation (D-1). Hooke’s laws for elastic behavior say that the radial and circumferential stresses are

\[ \sigma_r = \frac{E}{(1-v^2)}\left[ \varepsilon_r + v\varepsilon_\theta \right] + \sigma_0 \]  

(D-3)

and

\[ \sigma_\theta = \frac{E}{(1-v^2)}\left[ \varepsilon_\theta + v\varepsilon_r \right] + \sigma_0 \]  

(D-4)

Using equations (D-1) and (D-2) in equations (D-3) and (D-4) yields

\[ \sigma_r = \frac{E}{(1-v^2)} \left[ \frac{dU}{dR} + \frac{1}{2}\left( \frac{dW}{dR} \right)^2 + \nu \frac{U}{R} \right] + \sigma_0 \]  

(D-5)

and
\[ \sigma_\theta = \frac{E}{(1-\nu^2)} \left[ \frac{U}{R} + \nu \frac{dU}{dR} + \frac{\nu}{2} \left( \frac{dW}{dR} \right)^2 \right] + \sigma_0 \]  \tag{D-6}

Using equilibrium in the radial direction gives

\[ \frac{d}{dR} (R \sigma_\theta) = \sigma_\theta \quad \tag{D-7} \]

and in the vertical direction gives

\[ \frac{d}{dR} \left( TR \sigma_\theta \frac{dW}{dR} \right) + PR = 0 \quad \tag{D-8} \]

Using equations (D-5) and (D-6) in equation (D-7) gives

\[ R^2 U'' + RU' - U + R^2 W''' + \frac{(1-\nu)}{2} R(W')^2 = 0 \quad \tag{D-9} \]

and using equations (D-5) and (D-6) in equation (D-8) gives

\[ W'' \left\{ 2\sigma_0 \frac{R(1-\nu^2)}{E} + 2RU' + 3R(W')^2 + 2\nu U \right\} + 2RW'' + (W')^3 \]
\[ + 2(1+\nu)U'W' + 2 \frac{(1-\nu^2)}{E} \sigma_0 W' + \frac{2PR}{TE} (1-\nu^2) = 0 \quad \tag{D-10} \]

The governing system equations [Equations (5-1) and (5-2)] are found by substituting the nondimensionalized variables defined in Table 5.1 into equations (D-9) and (D-10).
D.2: Description of the Shooting Method

Mathematica can solve a system of differential equations using the shooting method. The user inputs the differential equations and the boundary conditions, as well as all known constants and a guess for the variables. Mathematica then runs a continuous loop until the specified accuracy is obtained. This method is used exclusively to solve the membrane and the nonlinear plate problems for this thesis.

From the boundary conditions listed in Table 5.2, \( u_m' \) is unknown. However, at \( r = 1 \), \( u_m \) should be zero. Using the shooting method, a guess for \( u_m' \) is input and varied until \( u_m = 0 \) at \( r = 1 \). In sections D.3.1 and D.3.2, “x” is \( u_m' \) and “gx” is the initial guess for x in the programs. In addition, \( t \) is \( r \), \( y_1 \) is \( y_m \), \( y_2 \) is \( y_m' \), \( y_3 \) is \( u_m \), and \( y_4 \) is \( u_m' \). The programs output the substrate height at a particular given contact radius (D.3.1) or the contact radius at a given value of \( \xi h \) (D.3.2). These values are used to plot Figures 5.3 through 5.5.
D.3.1: Finding the Height at a Given Contact Radius

\[ b = 0.05; \quad gx = 0.25; \quad v = 0.3 \]
\[
\text{de}[y_2, \ y_3, \ y_4] :=
\]
\[
(y_1'[t] = y_2[t],
\]
\[
y_2'[t] =
\]
\[
((y_3[t] - t y_4[t] - 0.5 (1 - v) t y_2[t]^2) (2 t y_2[t]) -
\]
\[
(-y_2[t]^3 - 2 (1 + v) y_2[t] y_4[t] + 2 (1 - v^2) t^2 t^2) /
\]
\[
(2 t^3 y_2[t]^2 - t^2 (3 t y_2[t]^2 + 2 t y_4[t] + 2 v y_3[t]))), \]
\[
y_3'[t] = y_4[t],
\]
\[
y_4'[t] =
\]
\[
(t^2 y_2[t] (-y_2[t]^3 - 2 (1 + v) y_2[t] y_4[t] + 2 (1 - v^2) t) -
\]
\[
(y_3[t] - t y_4[t] - 0.5 (1 - v) t y_2[t]^2) (3 t y_2[t]^2 + 2 t y_4[t] + 2 v y_3[t])) /
\]
\[
(2 t^3 y_2[t]^2 - t^2 (3 t y_2[t]^2 + 2 t y_4[t] + 2 v y_3[t])))
\]
\[
\text{leftBC}[x_] := \{y_1[b] = 0, y_2[b] = 0, y_3[b] = x b, y_4[b] = x\}
\]
\[
\text{soln} := \text{NDSolve}[\text{Flatten}[\text{Append}[\text{de}[y_2, y_3, y_4, \text{leftBC}[x]]], \{y_1, y_2, y_3, y_4\},
\]
\[
\{t, b, 1\}, \text{MaxSteps} \to 10000] \}/. t \to 1;
\]
\[
\text{endpt}[x_] :=
\]
\[
\{y_1[t], y_2[t], y_3[t], y_4[t]\} /. \text{First}[\text{NDSolve}[\text{Flatten}[\text{Append}[\text{de}[y_2, y_3, y_4, \text{leftBC}[x]]], \{y_1, y_2, y_3, y_4\},
\]
\[
\{t, b, 1\}, \text{MaxSteps} \to 10000] \}/. t \to 1;
\]
\[
\text{endpt}[gx]
\]
\[
x = .
\]
\[
\text{rts} := \text{FindRoot}[\{\text{endpt}[x][3] = 0\}, \{x, \{gx, 0.99 gx\}\}, \text{AccuracyGoal} \to 10,
\]
\[
\text{MaxIterations} \to 200]
\]
\[
\text{rts}
\]
\[
\text{endpt}[x/. \text{rts}]
\]
\[
x = x/. \text{rts}
\]
\[
\text{Print}[
\]
\[
\text{First}[\text{soln}]
\]
\[
\{y_1[t], y_2[t], y_3[t], y_4[t]\} = \{y_1[t], y_2[t], y_3[t], y_4[t]\} /. \text{First}[\text{soln}]
\]
\[
h := \text{Part}[\text{endpt}[x/. \text{rts}], 1]
\]
\[
w[t] := h - y_1[t]
\]
\[
\text{shapes} = \text{TableForm}[\text{Table}[[a, S[a]], \{a, 0.7, 0.9, 0.1\}]]
\]
\[
\text{numbers} =
\]
\[
\text{TableForm}[[b, N[\text{Part}[\text{endpt}[x/. \text{rts}], 1]], 8], \text{Part}[\text{endpt}[x/. \text{rts}], 2],
\]
\[
\text{Part}[\text{endpt}[x/. \text{rts}], 4], x/. \text{soln} /. \text{rts}, \{b, 0.05, 0.7, 0.05\}],
\]
\[
\text{TableHeadings} \to \{\text{None}, \{"b", "y_1(l) = h", "y_2(l)", "y_4(l)"}, "x"\}]
\]
\[
\text{N}[h, 10]
\]
\[
\text{InputForm}[h]
\]
\[
\text{ParametricPlot}[\text{Evaluate}[[t, w[t]]]/. \text{soln} /. \text{rts}, \{t, b, 1\}, \text{PlotRange} \to \text{All},
\]
\[
\text{AspectRatio} \to \text{Automatic}. \text{PlotPoints} \to 3000]
D.3.2: Finding the Contact Radius at a Given Value of $\xi h$

\[
b = 0.2; \quad gx = 0.25; \quad \nu = 0.3
\]

\[
de[y2_, y3_, y4_] :=
\{y1'[t] = y2[t],
\quad y2'[t] =
\quad ((y3[t] - t y4[t] - 0.5 (1 - \nu) t y2[t]^2) (2 t y2[t]) -
\quad (-y2[t]^3 - 2 (1 + \nu) y2[t] y4[t] + 2 (1 - \nu^2) t t^2) /
\quad (2 t^2 y2[t]^2 - t^2 (3 t y2[t]^2 + 2 t y4[t] + 2 \nu y3[t])))\},
\quad y3'[t] = y4[t],
\quad y4'[t] =
\quad (t^2 y2[t] (-y2[t]^3 - 2 (1 + \nu) y2[t] y4[t] + 2 (1 - \nu^2) t) -
\quad (y3[t] - t y4[t] - 0.5 (1 - \nu) t y2[t]^2) (3 t y2[t]^2 + 2 t y4[t] + 2 \nu y3[t]))) /
\quad (2 t^3 y2[t]^2 - t^2 (3 t y2[t]^2 + 2 t y4[t] + 2 \nu y3[t])))
\]

leftBC[x_] := \{y1[b] = 0, y2[b] = 0, y3[b] = x b, y4[b] = x\}

\[soln := \text{NDsolve[Flatten[Append[de[y2, y3, y4], leftBC[x]]]}\,\cdot\,\text{t} \to 1;\]

\[
\text{endenpt[x]} :=
\quad \text{First[NDsolve[Flatten[Append[de[y2, y3, y4], leftBC[x]]]}\,\cdot\,\text{t} \to 1;\]
\]

\[
x = .
\]

\[rts := \text{FindRoot[endenpt[x][3] = 0, \{x, gx, 0.99 gx\}, AccuracyGoal -> 10,}
\quad \text{MaxIterations \to 2000}\];
\]

\[\text{endenpt[x] /. rts}\]

\[x = x /. \text{rts}\]

\[\text{Print[x /. soln /. rts]}\]

\[
\{yy1[t_], yy2[t_], yy3[t_], yy4[t_] =
\quad \text{First[soln]}\}
\]

\[h := \text{Part[endenpt[x] /. rts, 1]}\]

\[w[t] := h - yy1[t]\]

\[S[a_] = h - yy1[a]\]

\[\text{shapes} = \text{TableForm[Table[\{a, S[a]\}, \{a, 0.7, 0.9, 0.1\}]}\]

\[\text{Clear[UX]}\]

\[\text{U[x_] := \text{NIntegrate[yy1[t]], \{t, b, 1\}}\]

\[\text{Print[U[x]}]\]

\[\text{numbers} =
\quad \text{TableForm[}
\quad \quad \text{Table[\{N[Part[endenpt[x] /. rts, 1], 8], x /. soln /. rts, b,}
\quad \quad \quad (0.3/(N[Part[endenpt[x] /. rts, 1], 8])^3), \{b, 0.31, 0.32, 0.001}\},]
\quad \quad \text{TableHeadings \to \{None, \"y1(1) = h\", \"x\", \"b\", \"[0.3/(y1(1))^3]\}\]]\]

\[\text{ParametricPlot[Evaluate[\{t, w[t] \/. soln /\/. rts\}, \{t, b, 1\},}
\]

85
Appendix E: Nonlinear Membrane with JKR

E.1: Mathematica Program for Membrane with JKR

\[
\begin{align*}
\text{b} &= 0.005; \quad \text{p} = 2; \quad \Delta Y = 0.005; \quad h = 0.0279; \quad \xi = 30; \quad \nu = 0.3 \\
\text{ga} &= 0.5; \quad \text{gc} = 0.28 \\
\text{h}^0 &= \xi h (p^{-(1/3)}); \quad \Delta y^0 = \Delta Y (\xi^*(1/3) (p^{-(4/3)}) \\
\text{de}[y_2, y_3, y_4] &:= \\
\quad (y_3[t] - t y_4[t] - 0.5 (1 - \nu) \text{t} y_2[t])^2 (2 t y_2[t]) - \\
\quad (-y_2[t]^3 - 2 (1 + \nu) y_2[t] y_4[t] + 2 (1 - \nu^2) t) t^2) / \\
\quad (2 t^3 y_2[t]^2 - t^2 (3 t y_2[t] - 2 + 2 t y_4[t] + 2 \nu y_3[t])), \\
\quad y_3'[t] = y_4'[t], \\
\quad y_4'[t] = \\
\quad (t^2 y_2[t] (-y_2[t]^3 - 2 (1 + \nu) y_2[t] y_4[t] + 2 (1 - \nu^2) t) - \\
\quad (y_3[t] - t y_4[t] - 0.5 (1 - \nu) t y_2[t])^2 (3 t y_2[t] + 2 t y_4[t] + 2 \nu y_3[t])) / \\
\quad (2 t^3 y_2[t]^2 - t^2 (3 t y_2[t] + 2 t y_4[t] + 2 \nu y_3[t])) \\
\text{leftBC}[a_, c_] &:= (y_1[b] = a, y_2[b] = b, y_3[b] = bc, y_4[b] = c) \\
\text{soln} &:= \text{NDSolve}[\text{Flatten}[\text{Append}[\text{de}[y_2, y_3, y_4], \text{leftBC}[a, c]]], \\
\quad (y_1, y_2, y_3, y_4), (t, b, 1), \text{MaxSteps} \to 2200] \\
\text{endpt}[a_, c_, \_] &:= \\
\quad (y_1[t], y_2[t], y_3[t], y_4[t]) / . \\
\quad \text{Last}[\text{NDSolve}[\text{Flatten}[\text{Append}[\text{de}[y_2, y_3, y_4], \text{leftBC}[a, c]]], \\
\quad (y_1, y_2, y_3, y_4), (t, b, 1), \text{MaxSteps} \to 2200]] / . t \to 1; \\
\text{endpt}[\text{ga}, \text{gc}] \\
\text{Clear}[a, c] \\
\text{rts} &:= \text{FindRoot}[\{\text{endpt}[a, c][1] = h^0, \text{endpt}[a, c][3] = 0, \{a, \{\text{ga}, 0.99 \text{ga}\}\}, \\
\quad \{c, \{\text{gc}, 0.99 \text{gc}\}\}, 10, \text{MaxIterations} \to 200] \\
\text{endpt}[a_/., \text{rts}, c_/., \text{rts}] \\
a = a_/., \text{rts} \\
c = c_/., \text{rts} \\
\text{Clear}[\text{yy}] \\
\text{yy}[t_, \_] &:= (y_1[t], y_2[t], y_3[t], y_4[t]) / . \text{First}[\text{soln}] \\
\text{UE1}[a_, c_, \_] &:= \\
\quad \text{NIntegrate}\{ \\
\quad \text{Evaluate}[((1/ (1 - \nu^2)) (\text{yy}[t][[4]] + ((\text{yy}[t][[2]])^2 / 2 + \nu \text{yy}[t][[3]] / t)) \\
\quad (\text{yy}[t][[4]] + (\nu \text{yy}[t][[2]]^2 / 2) t), (t, b, 1)] \\
\}
\]
Clear[UE2]

UE2[a_, c_] :=
NIntegrate[
Evaluate[
(1/ (1 - (y)^2)) ((yy[t][[3]]/ t) + y yy[t][[4]] + y ((yy[t][[2]])^2 / 2))
], {t, b, 1}]

Clear[UE]

UE[a_, c_] := UE1[a, c] + UE2[a, c]

Clear[UP]

UP[a_, c_] := -h^0 + 2 NIntegrate[ Evaluate[ t yy[t][[1]] ], {t, b, 1}]

Clear[UA]

UA := -Δγ^0 (b^2)

Clear[UB]

UB[c_] := ((c^2) (b^2)) / (1 - y)

Clear[UT]

UT[a_, c_] := UE[a, c] + UP[a, c] + UA + UB[c]
Appendix F: Nonlinear Membrane with DMT

F.1: Mathematica Program

\begin{align*}
\nu &= 0.3; f = 10; \alpha = 0.001; h = 0.01; \xi = 30; p = 3; f^0 = \frac{f}{p}; \alpha^0 = \alpha \xi p^{-3}; \\
h^0 &= \xi h p^{-3} \\
gb &= 0.8; gc = 0.85; gy02 = 1; gy03 = 0.1; gy04 = -0.35 \\
de[y2_, y3_, y4_, y6_, y7_, y8_, b_, c_] := \\
\{y1'[t] = (1 - c) y2[t], \\
y2'[t] &= \frac{(1 - c) (c + t - ct)^2 (1 - \nu^2) - (c + t - ct) \nu y2[t] \left(y4[t] + \frac{y2[t]_2^2}{2}\right) - y2[t] y3[t]}{\left((c + t - ct)^2 \left(y4[t] + \frac{y2[t]_2^2}{2}\right) + (c + t - ct) \nu y3[t]\right)}, \\
y3'[t] &= (1 - c) y4[t], \\
y4'[t] &= (1 - c) \left[-\frac{y3[t]}{(c + t - ct)^2} - \frac{y4[t]}{(c + t - ct)} - \frac{(1 - \nu) y2[t]_2^2}{2 (c + t - ct)} - \\
y2[t] \left((c + t - ct)^2 \left(y4[t] + \frac{y2[t]_2^2}{2}\right) + (c + t - ct) \nu y3[t]\right)\right] / \left((c + t - ct)^2 \left(y4[t] + \frac{y2[t]_2^2}{2}\right) + (c + t - ct) \nu y3[t]\right), \\
y5'[t] &= (b - c) y6[t], \\
y6'[t] &= -\left((b - c) \left((c + bt - ct)^2 (1 - \nu^2) (1 + f^0) + (c + bt - ct) \nu y6[t] \left(y8[t] + \frac{y6[t]_2^2}{2}\right) + \\
y7'[t] &= (b - c) y8[t], \\
y8'[t] &= (b - c) \left[-\frac{y7[t]}{(c + bt - ct)^2} - \frac{y8[t]}{(c + bt - ct)} - \frac{(1 - \nu) y6[t]_2^2}{2 (c + bt - ct)} + \\
y6[t] \left((c + bt - ct)^2 \left(y8[t] + \frac{y6[t]_2^2}{2}\right) + (c + bt - ct) \nu y7[t]\right)\right] / \left((c + bt - ct)^2 \left(y8[t] + \frac{y6[t]_2^2}{2}\right) + (c + bt - ct) \nu y7[t]\right), \\
leftBC[b_, c_, y02_, y03_, y04_] := \\
\{y1[0] == 0, y2[0] = y02, y3[0] = y03, y4[0] = y04, y5[0] = 0, y6[0] = -y02, \\
y7[0] = y03, y8[0] == y04\}
\end{align*}

NDSolve[
Flatten[Append[de[y2, y3, y4, y6, y7, y8, b, c], leftBC[b, c, y02, y03, y04]]],
\{y1, y2, y3, y4, y5, y6, y7, y8\}, \{t, 0, 1\}, MaxSteps -> 2200]
endpt[b, c, y02, y03, y04] :=
{y[1][t], y[2][t], y[3][t], y[4][t], y[5][t], y[6][t], y[7][t], y[8][t]}/.

Last[
NDSolve[Flatten[Append[de[y2, y3, y4, y6, y7, y8, b, c],
leftBC[b, c, y02, y03, y04]]],
{y[1][t], y[2][t], y[3][t], y[4][t], y[5][t], y[6][t], y[7][t], y[8][t]}, {t, 0, 1},
MaxSteps → 2200] /._ → 1;
endpt[gb, gc, gy02, gy03, gy04]
Clear[b, c, y02, y03, y04]

rts := FindRoot[eq1, eq2, eq3, eq4][1] == h0 - α0,
endpt[b, c, y02, y03, y04] [3] == 0, endpt[b, c, y02, y03, y04] [5] == α0,
endpt[b, c, y02, y03, y04] [6] == 0,
endpt[b, c, y02, y03, y04] [7] == b endpt[b, c, y02, y03, y04] [8]],
{b, {gb, 0.99 gb}}, {c, {gc, 0.99 gc}}, {y02, {gy02, 0.99 gy02}},
{y03, {gy03, 0.99 gy03}}, {y04, {gy04, 0.99 gy04}}, AccuracyGoal → 7,
MaxIterations → 200]
endpt[b /. rts, c /. rts, y02 /. rts, y03 /. rts, y04 /. rts]
b = b /. rts
c = c /. rts
y02 = y02 /. rts
y03 = y03 /. rts
y04 = y04 /. rts
Clear[yy]

yy[t_] := {y[1][t], y[2][t], y[3][t], y[4][t], y[5][t], y[6][t], y[7][t], y[8][t]}/.First[soln]
Appendix G: Nonlinear Plate with No Adhesion

G.1: Mathematica Program for Membrane with No Adhesion

\[
\begin{align*}
\nu &= 0.3; \ h = 0.05; \ \xi = 20; \ b = 0.00151 \\
gk &= 0.00168; \ gp = 7.25; \ gy04 = 44 \\
de[y2\_ , y3\_ , y4\_ , y5\_ , y6\_ , p\_] := \\
\{y1'[t] &= y2[t], \ y2'[t] = y3[t], \ y3'[t] = y4[t], \\
y4'[t] &= - \frac{2}{t} y4[t] + \frac{1}{t^2} y2[t] - p + \\
(y3[t] y6[t] + \frac{y}{t} y2[t] y6[t] + \frac{y}{t} y5[t] y3[t] + \frac{1}{t^2} y2[t] y5[t] + \\
y^3 \frac{y}{t} (y2[t])^3 + \frac{1}{2} (y2[t])^2 y3[t] \xi^3 \ (1 - y^2), \\
y5'[t] &= y6[t], \\
y6'[t] &= - \frac{1}{t} y6[t] + \frac{1}{t^2} y5[t] + \frac{(y-1)}{2t} (y2[t])^2 - y2[t] y3[t] \\
leftBC[k\_ , p\_ , y04\_] := \\
\{y1[b] &= 0, \ y2[b] = 0, \ y3[b] = 0, \ y4[b] = y04, \ y5[b] = k b, \\
y6[b] &= k \\
soln := \\
Flatten[Append[de[y2, y3, y4, y5, y6, p], leftBC[k, p, y04]]], \\
{y1, y2, y3, y4, y5, y6}, \{t, b, 1\}, \text{MaxSteps} \rightarrow 2200] \\
endpt[k\_ , p\_ , y04\_] := \\
\{y1[t], y2[t], y3[t], y4[t], y5[t], y6[t] \}/. \\
\text{Last}[
\text{NDSolve}[Flatten[Append[de[y2, y3, y4, y5, y6, p], \\
leftBC[k, p, y04]]], \\
\{y1[t], y2[t], y3[t], y4[t], y5[t], y6[t]\}, \{t, b, 1\}, \\
\text{MaxSteps} \rightarrow 2200]]/. t \rightarrow 1; \\
endpt[gk, gp, gy04] \\
Clear[k, p, y04] \\
\text{rts} := \\
\text{FindRoot}\{\text{endpt}[k, p, y04][1] = h, \ \text{endpt}[k, p, y04][2] = 0, \\
\text{endpt}[k, p, y04][5] = 0, \ \{k, \{gk, 0.99 gk\}\}, \\
\{p, \{gp, 0.99 gp\}\}, \{y04, \{gy04, 0.99 gy04\}\}, \text{AccuracyGoal} \rightarrow 10, \text{MaxIterations} \rightarrow 200\] \\
endpt[k/\ . \ rts, p/. rts, y04/. rts] \\
k = k/\ . \ rts \\
p = p/\ . \ rts \\
y04 = y04/\ . \ rts \\
\text{Print}[40 - 10 p]
\end{align*}
\]
Clear[yy]
{yy1[t_] , yy2[t_] , yy3[t_] , yy4[t_] , yy5[t_] , yy6[t_]} = 
{y1[t] , y2[t] , y3[t] , y4[t] , y5[t] , y6[t]} /. First[soln]
w[t] := h - yy1[t]
S[a_] = h - yy1[a]
shapes = TableForm[Table[{a, S[a]}, {a, 0.7, 0.9, 0.1}]]
Clear[UX]
U[x_] := NIntegrate[yy1[t], {t, b, 1}]
Print[U[x]]
Appendix H: Nonlinear Plate with JKR Analysis

H.1: Mathematica Program for Nonlinear Linear Plate with JKR Analysis

\[ \nu = 0.3; \quad h = 0.05; \quad b = 0.005; \quad p = 16; \quad \Delta Y = 0.0001 \]

\[ g_k = 0.0015; \quad g_{y03} = 0.2; \quad g_{y04} = 0.1 \]

```mathematica
ν = 0.3; h = 0.05; ξ = 30; b = 0.005; p = 16; Δγ = 0.0001

gk = 0.0015; gy03 = 0.2; gy04 = 0.1

deq[y_2_, y_3_, y_4_, y_5_, y_6_] :=

\[ \{y_1'[t] = y_2[t], \quad y_2'[t] = y_3[t], \quad y_3'[t] = y_4[t], \]
\[ y_4'[t] = \frac{2}{t} y_4[t] + \frac{1}{t^2} y_3[t] - 1 y_2[t] - p + \]
\[ 1 y_5[t] y_3[t] + \frac{1}{t} y_2[t] y_5[t] + \frac{1}{t^2} (y_2[t])^3 + 1 \]
\[ \frac{1}{2} (y_2[t])^2 y_3[t] \]
\[ (1 - ν) , \quad y_5'[t] = y_6[t], \]
\[ y_6'[t] = -1 y_6[t] + \frac{1}{t} y_5[t] + \frac{1}{2t} (y_2[t])^2 - y_2[t] y_3[t] \]

leftBC[k_, y_03_, y_04_] :=

\{y_1[b] = 0, \quad y_2[b] = 0, \quad y_3[b] = y_03, \quad y_4[b] = y_04, \]
\[ y_5[b] = k b, \quad y_6[b] = k \]

soln :=

NDSolve[
Append[deq[y_2, y_3, y_4, y_5, y_6],
leftBC[k, y_03, y_04]],
\{y_1, y_2, y_3, y_4, y_5, y_6\},
\{t, b, 1\}, MaxSteps \rightarrow 2200]

endpt[k_, y_03_, y_04_] :=

\{y_1[t], y_2[t], y_3[t], y_4[t], y_5[t], y_6[t]\} /.

Last[
NDSolve[
Append[deq[y_2, y_3, y_4, y_5, y_6],
leftBC[k, y_03, y_04]],
\{y_1[t], y_2[t], y_3[t], y_4[t], y_5[t], y_6[t]\},
\{t, b, 1\}, MaxSteps \rightarrow 1000]] /. t \rightarrow 1;

endpt[gk, gy03, gy04]

Clear[k, y_03, y_04]

rts :=

FindRoot[\{endpt[k, y_03, y_04][1] == h,
endpt[k, y_03, y_04][2] == 0, \quad endpt[k, y_03, y_04][5] == 0, \}
(k, \{gk, 0.99 gk\}), \{y_03, \{gy03, 0.99 gy03\}],
\{y_04, \{gy04, 0.99 gy04\}]}, AccuracyGoal \rightarrow 10,
MaxIterations \rightarrow 200]

endpt[k/. rts, y_03/. rts, y_04/. rts]

k = k/. rts

y_03 = y_03/. rts

```
Clear[yy]

(yy1[t_], yy2[t_], yy3[t_], yy4[t_], yy5[t_], yy6[t_]) = 
(y1[t], y2[t], y3[t], y4[t], y5[t], y6[t]) /. First[soln]
Clear[UT]

UT[k_, y03_, y04_] :=

NIntegrate[Evaluate[t (yy3[t])^2 + 1/t (yy2[t])^2], {t, b, 1}] +

NIntegrate[

Evaluate[

1

1 - ν^2

ξ^3 t \left( (yy6[t] + 0.5 (yy2[t])^2 + \frac{(yy5[t])^2}{t^2} + \right.

2 \nu (yy6[t] + 0.5 (yy2[t])^2) \left( \frac{yy5[t]}{t} \right) \right) \right], {t, b, 1}] -

p h + 2 p NIntegrate[Evaluate[t yy1[t]], {t, b, 1}] +

\frac{ξ^3 k^2 b^2}{(1 - ν)} - b^2 \Delta y

Print[{-100000 UT[k, y03, y04] - 4970}]

93
Appendix I: Nonlinear Linear Plate with DMT Adhesion

I.1: Mathematica Program for the Nonlinear Plate with DMT adhesion

\[ v = 0.3; \quad h = 0.05; \quad \xi = 29.5; \quad p = 17.2; \quad \alpha = 0.00001; \quad f = 500; \quad b = 0.054; \]
\[ c = 0.073 \]
\[ gb = 0.054; \quad gc = 0.073; \quad gy02 = 0.00143; \quad gy03 = 0.116; \quad gy04 = 0.556; \quad gy05 = 0.000125; \]
\[ gy06 = 0.00172 \]
\[ \text{de[b_, c_, y2_, y3_, y4_, y5_, y6_, y8_, y9_, y10_, y11_, y12_] :=} \]
\[ \{y1'[t] = (1-c) y2[t], \quad y2'[t] = (1-c) y3[t], \quad y3'[t] = (1-c) y4[t], \]
\[ y4'[t] = \]
\[ \left(-\frac{2}{(c+h-ct)} y4[t] + \frac{1}{(c+h-ct)^2} y3[t] - \frac{1}{(c+h-ct)^3} y2[t] - p^+ \right) \]
\[ y3'[t] y6[t] + \frac{1}{(c+h-ct)} y2[t] y6[t] + \frac{1}{(c+h-ct)} y5[t] y3[t] + \]
\[ \left(\frac{1}{(c+h-ct)^2} y2[t] y5[t] + \frac{1}{2(c+h-ct)} (y2[t])^3 + \frac{1}{2} (y2[t])^2 y3[t] \right) \]
\[ \left(\frac{c^3}{1-v^2}\right) (1-c), \quad y5'[t] = (1-c) y6[t], \]
\[ y6'[t] = \]
\[ (1-c) \left(-\frac{1}{(c+h-ct)} y6[t] + \frac{1}{(c+h-ct)^2} y5[t] + \frac{(v-1)}{2(c+h-ct)} (y2[t])^2 - \right) \]
\[ y2[t] y3[t] \right), \quad y7'[t] = (b-c) y8[t], \quad y8'[t] = (b-c) y9[t], \]
\[ y9'[t] = (b-c) y10[t], \]
\[ y10'[t] = (c-b) \left(-\frac{2 y10[t]}{(c+b t-ct)} \right) - \frac{y9[t]}{(c+b t-ct)^2} + \frac{y8[t]}{(c+b t-ct)^3 - p^-} \}
\[ \left(c-b\right) \left(1-v^2\right) \left(\frac{\nu (y8[t]) y12[t] + y11[t] y9[t] + 0.5 (y8[t])^3}{(c+b t-ct)} \right) + \frac{y8[t] y11[t]}{(c+b t-ct)^2} + \]
\[ \frac{1}{2} (y8[t])^2 y9[t] + y9[t] y12[t] \right), \quad y11'[t] = (b-c) y12[t], \]
\[ y12'[t] = (b-c) \left(-\frac{y12[t] + 0.5 (v-1) (y8[t])^2}{(c+b t-ct)} + \frac{y11[t]}{(c+b t-ct)^2} - y8[t] y9[t] \right) \}
\[ \text{leftBC[b_, c_, y2_, y3_, y4_, y5_, y6_, y8_, y9_, y10_, y11_, y12_] :=} \]
\[ \{y1[0] == 0, y2[0] == y02, y3[0] == y03, y4[0] == y04, y5[0] == y05, y6[0] == y06, \]
\[ y7[0] == 0, y8[0] == -y02, y9[0] == -y03, y10[0] == -y04, y11[0] == -y05, y12[0] == y06 \}
\[ \text{soln :=} \]
\[ \text{NDSolve[Flatten[Append[de[b, c, y2, y3, y4, y5, y6, y8, y9, y10, y11, y12], \}
\[ \quad \text{leftBC[b, c, y02, y03, y04, y05, y06]]}, \]
\[ \{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12], \{t, 0, 1\}, \text{MaxSteps} \to 3000] \}
\]
endpt[b_, c_, y02_, y03_, y04_, y05_, y06_] :=
{y1[t], y2[t], y3[t], y4[t], y5[t], y6[t], y7[t], y8[t], y9[t], y10[t],
y11[t], y12[t]}/.
Last[
NDSolve[Flatten[Append[de[b, c, y2, y3, y4, y5, y6],
    leftBC[b, c, y02, y03, y04, y05, y06]]],
{y1[t], y2[t], y3[t], y4[t], y5[t], y6[t], y7[t], y8[t], y9[t], y10[t],
y11[t], y12[t]}, {t, 0, 1}, MaxSteps -> 3000] /. t -> 1;
endpt[gb, gc, gy02, gy03, gy04, gy05, gy06]
Clear[b, c, y02, y03, y04, y05, y06]

rts := FindRoot[{endpt[b, c, y02, y03, y04, y05, y06][1] = h - \[Alpha],
    endpt[b, c, y02, y03, y04, y05, y06][2] = 0,
    endpt[b, c, y02, y03, y04, y05, y06][5] = 0,
    endpt[b, c, y02, y03, y04, y05, y06][7] = \[Alpha],
    endpt[b, c, y02, y03, y04, y05, y06][8] = 0,
    endpt[b, c, y02, y03, y04, y05, y06][9] = 0,
    endpt[b, c, y02, y03, y04, y05, y06][12] -
    endpt[b, c, y02, y03, y04, y05, y06][11] = 0}, {b, {gb, 0.99 gb}},
{c, {gc, 0.99 gc}}, {y02, {gy02, 0.99 gy02}}, {y03, {gy03, 0.99 gy03}},
{y04, {gy04, 0.99 gy04}}, {y05, {gy05, 0.99 gy05}}, {y06, {gy06, 0.99 gy06}},
AccuracyGoal -> 7, MaxIterations -> 200]
endpt[b/. rts, c/. rts, y02/. rts, y03/. rts, y04/. rts, y05/. rts,
y06/. rts]
b = b/. rts
c = c/. rts
y02 = y02/. rts
y03 = y03/. rts
y04 = y04/. rts
y05 = y05/. rts
y06 = y06/. rts
Vita

Sally Anne White was born in Jackson, Tennessee, on January 1, 1976 to Roy L. and Laurel White. She graduated from Jackson Central-Merry High School in 1994 and began studying engineering at Tennessee Technological University the same year. While an undergraduate, she spent one year in the cooperative learning program working for the Corps of Engineers, Nashville District, and graduated with a Bachelor of Science degree in Civil Engineering in May 1999. She went on to pursue a Master of Science in Civil Engineering at Virginia Polytechnic Institute and State University, and graduated in May 2001.