Chapter 3

Polarization Analysis

3.1 Initial Steps in the Analysis

This chapter addresses the problems associated with measuring the linear polarization of starlight, as well as extended Hα sources. In particular, the problem of bias removal needs to be addressed so that a proper report of the polarization is made. The method used to estimate the true polarization from the measured value which is biased is discussed in the first section. These procedures are of the utmost importance since they are the basis of assessing whether and to what extent a source is polarized. These procedures are followed throughout the analysis of the polarized light in this entire project.

Since a single image obtained with SLIC covers a large portion of the sky (10°), care must be taken to correctly report polarization position angles. As mentioned in Chapter 1, the angle that the vibration of the electric vector makes with respect to a line of constant right ascension in the equatorial coordinate system is the reported polarization position angle for linearly polarized light. Due to the alignment of the polarizer with respect to the CCD chip, polarization position angles are measured with respect to the $y$ axis of the chip. However, lines of constant right ascension are generally not parallel the $y$ axis of the CCD chip, and so a correction must be applied in order to properly report the polarization position angle. In section 3.3, the method for making this correction is discussed.
Section 3.4 discusses the method of photometry used in this project. Since one goal is to measure the linear polarization of starlight, care must be taken in order to perform the photometry accurately. In addition to this, the method used to determine the Stoke's parameters $q$ and $u$ from the photometric measurements is mentioned. In particular, the use of trimmed means is shown to be a useful method of handling outliers in the data and another way of determining the parameters $q$ and $u$. This method of stellar photometry is discussed in section 3.4.1 and trimmed means are covered in section 3.4.2. Section 3.4.3 discusses instrumental polarization which occurs far outside the useable $10^\circ$ extent of the H$\alpha$ and no-filter images. A comparison of previous measurement of stellar polarization in the areas of interest to measurements obtained in this project is the focus of section 3.4.4. In particular, it is shown in this section that the use of trimmed means as a way of determining $q$ and $u$ is justified. In those cases where previous polarization measurements of stellar sources exist but none detected in this project, an explanation of the possible reasons why it was not detected is also covered.

Finally, section 3.5 covers the making of the H$\alpha$ polarization maps. Before the maps are made, the individual images must be registered as discussed in section 2.4.3 in Chapter 2. In order to obtain a higher signal to noise ratio, the maps are smoothed. Section 3.5.1 explains the smoothing procedure as well as the effect it has on the data itself. Since interest is focused on the extended H$\alpha$ sources, stars are not a major concern when making the maps. As such, their removal before the map making procedure is discussed in section 3.5.2. The final maps are presented in section 3.5.3.

3.2 Statistics Applied to Polarization

The linear polarization of radiation can be expressed by the three Stoke's parameters $I$, $Q$, and $U$ which were defined in Chapter 1. The normalized Stoke's parameters are expressed by $q = Q/I$ and $u = U/I$. The degree of polarization $p$ and the position angle $\theta$ were shown to be given by
However, measured values of \( q \) and \( u \) do not give the true value of polarization since noise is usually present. The true degree of polarization \( p_0 \) would be given by equation (3.1) using the true values of the normalized Stoke's parameters \( q_0, u_0 \) instead of the measured values \( q, u \). If the mean values of \( ar{q} \) and \( ar{u} \) from a series of measurements of \( q \) and \( u \) were used in equation (3.1), the degree of polarization will be overestimated. This can be seen to occur when both \( ar{q} \) and \( ar{u} \) are zero. In such a case the experimental noise on \( ar{q} \) and \( ar{u} \) will be non-zero, and equation (3.1) will result in a non-zero value for the degree of polarization. Therefore, when the degree of polarization is computed using mean values of measured normalized Stoke's parameters \( q, u \), a bias will result. It has been shown by various authors (Wardle and Kronberg 1974, Simmons and Stewart 1985) that the bias becomes increasingly pronounced when the signal-to-noise ratio is small. Therefore, a method of removing this bias is needed. Estimators of the true degree of polarization are discussed in section 3.3.1.

Another concern in this project is the reporting of confidence intervals on the estimated degree of polarization and position angle. Since the goal of this thesis is to map magnetic fields, confidence intervals on the polarization position angle determines the precision with which the magnetic field direction can be measured. Confidence intervals are also needed on the degree of polarization as well. Methods of determining confidence intervals for the degree of polarization and position angle are given in sections 3.3.2 and 3.3.3 respectively.
3.2.1 Bias Removal from Percent Polarization

It has long been known that experimental noise will have a bias effect on polarization measurements, particularly at low levels of polarization or measurements with low signal-to-noise ratios. It is known that experimental measurements of the degree of polarization $p$ are distributed about the true value $p_0$, and that the average value of the experimentally determined degree of polarization $\overline{p}$ is not equal to the true value $p_0$. In this section I discuss the methods used to produce estimators of the true value of the degree of polarization.

As pointed out by Clarke et al. (1983), measurements of $q$ and $u$ will not be normally distributed when low photon counts used to measure $q$ and $u$ are obtained, however, differences from normality are low when photon counts are high. For a CCD, the number of photons recorded in a pixel $C_\gamma$ is directly related to the number of analog digital units (ADU), or counts, $C_{ADU}$ in that pixel by $C_\gamma = g \cdot C_{ADU}$, where $g$ is the gain of the CCD. Since the gain of SLIC was set at 2.53, the actual number of photons recorded is about $2^{1/2}$ times the number of counts in an image pixel. Most stellar and nebula measurements obtained for this thesis are on the order of a few thousand counts, so the number of photons recorded is quite high. Therefore, I assume that photon counting statistics do not introduce significant bias.

In deriving the estimators of the true value of polarization $p_0$, it is assumed that the standard deviations of both $q$ and $u$ are given by a single population standard deviation $\sigma$, which, is also taken to be the standard deviation on the measured polarization $p$. Therefore, letting $\sigma$ be the standard deviation of the measured polarization $p$, the probability distribution function for $p$ is given by the Rice distribution function

$$ F(p, p_0) = \frac{p}{\sigma^2} e^{-\frac{p^2 + \sigma^2}{2\sigma^2}} I_0\left(\frac{pp_0}{\sigma^2}\right) $$

(3.3)

where $I_0$ is the modified Bessel function of order zero. This distribution function is shown graphically in Figure 3.1 for three values of $\frac{pp_0}{\sigma} = 0, 1, \text{ and } 2$. It can be seen that even when the true value of polarization is zero ($\frac{pp_0}{\sigma} = 0$), values of $p$ are distributed asymmetrically around $\frac{p}{\sigma} > 0$. 
Various estimators of the true polarization have been derived. Serkowski (1962) constructed an estimator \( \hat{p}_o \) which was defined as the value of \( p_o \) for which an observed value of \( p \) is equal to the mean of the Rice distribution. It has been shown that a threshold value exists for this estimator which is easily derived as follows.

![Figure 3.1](image)

**Figure 3.1** The Rice distribution function plotted as a function of \( p = \frac{P}{\sigma} \) for three values of \( p_o = \frac{P_o}{\sigma} \) equal to 0, 1, and 2. It is clearly seen that even if the true value of polarization is zero, the measured value is not distributed about zero.

Suppose a source is known to have no polarization. In this case \( p_o = 0 \). If the Serkowski estimator is used to estimate this true value of polarization from a measurement of \( p > 0 \), the estimator \( \hat{p}_o \) should equal zero. However, the expectation value for the measured polarization when the true polarization is
zero is

\[ <p> = \int_0^\infty \frac{p}{\sigma} F(p, 0) \, dp = \int_0^\infty \frac{p^2}{\sigma^2} e^{-\frac{2p^2}{\sigma^2}} \, dp = \sigma \sqrt{\frac{\pi}{2}} = 1.25\sigma. \] (3.4)

Therefore, any measured value of \( p \) less than 1.25\( \sigma \) must be set equal to zero (that is \( \hat{p}_s = 0 \) when \( p < 1.25\sigma \)). If the measured value of polarization \( p \) is greater than or equal to 1.25\( \sigma \) then the estimated polarization \( \hat{p}_s \) can be determined from

\[ \int_0^\infty \frac{\sigma}{\sigma} F(p', \hat{p}_s) \, dp' = \frac{p}{\sigma}. \] (3.5)

Another method to estimate the true value of polarization used by radio astronomers was proposed by Wardle and Kronberg (1974). Their estimator \( \hat{p}_W \) is defined as the value of the true polarization \( p_o \) for which the observed value is a maximum of the Rice distribution. Therefore, by differentiating the Rice distribution function and setting this result equal to zero, the estimator of the true polarization can then be found from

\[ \frac{\partial F(p, p_o)}{\partial p} \bigg|_{p_o=\hat{p}_W} = I_o\left(\frac{\hat{p}_W}{\sigma^2}\right)(1 - \hat{p}_W^2) + \frac{\hat{p}_W}{\sigma^2} I_1\left(\frac{\hat{p}_W}{\sigma^2}\right) = 0. \] (3.6)

Again, there is a threshold value for this estimator. This threshold value \( K_W \) was shown by Simmons and Stewart (1985) to be given by

\[ \frac{\partial F(p, 0)}{\partial p} \bigg|_{p=K_W} = e^{-\frac{K_W^2}{\sigma^2}} \left(1 - \frac{K_W^2}{\sigma^2}\right) = 0 \] (3.7)

which yields \( K_W = \sigma \). Therefore, the estimated value of the true polarization \( p_o \) for a measured polarization \( p \) which is less than \( K_W \) must equal zero (i.e., \( \hat{p}_W = 0 \) when \( p < K_W \)). Any other estimated value of true polarization \( p_o \) for which \( p \) is greater than \( K_W \) can be estimated by solving for \( \hat{p}_W \) in equation (3.6).

Two other estimators have been critiqued by Simmons and Stewart (1985) along with the Serkowski and Wardle-Kronberg estimator mentioned above. These two other estimators are the maximum likelihood estimator and median estimator. They define the median estimator as the value of \( p_o \) for which the measured value \( p \) is equal to the median of the Rice distribution. They derive a threshold value of this estimator \( K_{MD} \) from
\[
\int_0^{K_{MD}} F(p', 0) \, dp' = \int_0^{K_{MD}} \frac{p'}{\sigma} e^{-\frac{p'^2}{2\sigma^2}} \, dp' = 1 - e^{-\frac{K_{MD}^2}{2\sigma^2}} = \frac{1}{2}
\]

which gives a threshold value of \( K_{MD} = \sqrt{\ln(4)} \sigma = 1.18\sigma \). If \( p \) measured is greater than \( K_{MD} \) the estimator \( \hat{p}_{MD} \) is determined from

\[
\int_0^p F(p', \hat{p}_{MD}) \, dp' = \frac{1}{2}.
\]

The maximum likelihood estimator of the true polarization \( p_0 \) discussed by Simmons and Stewart (1985) is defined as the value of \( p_0 \) which maximizes the Rice distribution function. For a measured polarization \( p \), \( p_0 \) can be estimated by solving

\[
\frac{\partial F(p, p_0)}{\partial p} = p_0 I_0 \left( \frac{pp_0}{\sigma^2} \right) + ip I_1 \left( \frac{pp_0}{\sigma^2} \right) = 0
\]

where \( i = \sqrt{-1} \). Like the previous estimators, a threshold value for this equation exists as well. Unlike the other thresholds however, this one is obtained by numerically evaluating equation (3.10). This has been done by Simmons and Stewart (1985) where they find the threshold value \( K_M = 1.41\sigma \).

Therefore, any observed value of polarization \( p \) which is less than \( K_M \) must have maximum likelihood estimator \( \hat{p}_M = 0 \). For values of observed polarization greater than \( K_M \), the estimated polarization can be obtained by solving for \( \hat{p}_M \) in the equation

\[
p_0 I_0 \left( \frac{\hat{p}_M}{\sigma^2} \right) + ip I_1 \left( \frac{\hat{p}_M}{\sigma^2} \right) = 0.
\]

Solving for the estimator \( \hat{p}_M \) in equation (3.11) is a difficult task. It is simplified by making the assumption that the observed polarizations are small. With this assumption, the estimator in equation (3.11) can be approximated by

\[
\hat{p}_M = \sqrt{p^2 - (1.41\sigma)^2}
\]

when \( p \geq K_M \). In fact all of the above estimators can be approximated in a form similar to equation (3.12). In the case of the Wardle-Kronberg estimator,
Chapter 3 Polarization Analysis

the approximate expression for the estimator when \( p \geq K_W \) is

\[
\tilde{p}_w = \sqrt{p^2 - \sigma^2}.
\]

Simmons and Stewart (1985) have critiqued the use of the four estimators mentioned above and have shown that each one leaves behind a differing amount of residual bias. By their analysis, they have shown that the Serkowski and median estimators should not be used to estimate the true polarization from the measured polarization. They show that the range over which these two estimators are best is very small. They also demonstrate that the estimator that best minimizes the square error, \( \langle (\tilde{p}_o - p_o)^2 \rangle \), is the maximum likelihood estimator when the signal-to-noise ratio of the estimated true polarization is less than or on the order of \( 0.7 \) (i.e., \( p_o/\sigma \leq 0.7 \)). When \( p_o/\sigma \geq 0.7 \), they find the Warble-Kronberg estimator to be better. When either of these cases hold, the Serkowski estimator is shown to have a larger square error than either the Wardle-Kronberg or maximum likelihood estimators. It is for these reasons that only the Wardle-Kronberg and maximum likelihood estimators are used in this thesis to estimate the true polarization.

The choice to either use the Wardle-Kronberg estimator \( \tilde{p}_W \) or the maximum likelihood estimator \( \tilde{p}_M \) is decided upon a signal-to-noise argument. In their critique of the various estimator methods, Simmons and Stewart (1985) made the assumption that the \( p_o/\sigma \) is known \textit{a priori}. However, neither \( p_o \) nor \( \sigma \) is known before a measurement is made. This being the case, both the Wardle and maximum likelihood estimators are used to estimate the true polarization. If the degree of polarization is high, then the Wardle estimator is reported. If on the other hand the polarization is small, the maximum likelihood estimator is reported.

It should be noted that experience with calculating degrees of polarization in this project where \( p > 1\% \) has shown that the two methods yield similar results. Usually the difference obtained between the two different estimators is on the order of a few tenths of a percent. Indeed, as pointed out by Simmons and Stewart (1985) all of the various estimators agree asymptotically, and when \( p/\sigma > 4 \), the true polarization can be estimated by \( \sqrt{p^2 - \sigma^2} \).

From the discussion of polarization estimators above, an estimate of the true polarization using either \( \tilde{p}_m \) or \( \tilde{p}_w \) can be made, provided the source has a
measured polarization greater than $1.41\sigma$. When the source has a measured polarization only slightly greater than $1.41\sigma$ (or $\sigma$), the estimated polarization will be close to zero. Another way of testing whether or not a source can be considered polarized is discussed in section 3.3.4 after the discussion of confidence intervals.

Throughout the discussion of estimators it has been assumed that the standard deviation of the measured polarization $p$ was given by a population standard deviation $\sigma$, and that $\sigma_q$ and $\sigma_u$ were themselves samples used to estimate $\sigma$. It has been pointed out by Clarke and Stewart (1986) that, in many polarization experiments which measure $q$ and $u$, the above situation holds. I have found that values of $\sigma_q$ are similar to those of $\sigma_u$ for all measurements of the normalized Stoke's parameters which have been obtained so far. As a numerical check on this assumption I use the F-statistic which is a method of comparing variances from two different distributions.

The use of the F-statistic here is a straightforward two tailed test. The null hypothesis is $H_0 : \sigma_q^2 = \sigma_u^2$, alternative hypothesis $H_1 : \sigma_q^2 \neq \sigma_u^2$, and the significance level set at 10%. For example, there are 20 $q$ and $u$ images for the Cygnus region. Therefore the numerator and denominator degrees of freedom are each 19 in the $F$ distribution. The upper critical value obtained from a table of $F$ distributions for $f_{0.05}(19,19) = 2.174$ and the lower critical value obtained from $f_{0.95}(19,19) = 2.174^{-1} = 0.459$. If the ratio of the two observed variances $F = \frac{\sigma_q^2}{\sigma_u^2}$ is between these two critical values, then the null hypothesis is accepted. This indicates that $\sigma_q^2$ and $\sigma_u^2$ can be considered to be estimates of a population variance $\sigma^2$. If the test is successful, then $p_0$ is estimated. If however, $F$ is outside these critical values, then the null hypothesis is rejected in favor of the alternative hypothesis.

In most cases where polarizations have been measured, the value of $F$ usually lies within the $f_{0.05}$ and $f_{0.95}$ range. A few sources have values of $F$ which are just slightly above $f_{0.95}$. When these sources have a measurable polarization, the null hypothesis is accepted anyway if the value of $F$ is only slightly above $f_{0.95}$. In cases where $F$ is considerably larger than $f_{0.05}$ the source is usually found to be unpolarized, at least in the sense that any measurable polarization can not be achieved by this experiment. In these cases, the source is noted as not having any measurable polarization.
In determining the noise on the measured signal \( p \), the standard deviation \( \sigma \) is computed from the method of propagation of errors as

\[
\sigma^2 = \left( \frac{\partial p}{\partial q} \right)^2 \sigma_q^2 + \left( \frac{\partial p}{\partial u} \right)^2 \sigma_u^2 + 2 \frac{\partial p}{\partial q} \frac{\partial p}{\partial u} \sigma_{qu} \tag{3.14}
\]

where \( p \) is given by equation (3.1) and the covariance \( \sigma_{qu} \) by

\[
\sigma_{qu} = \frac{1}{N} \sum_{i=1}^{N} (q_i - \bar{q})(u_i - \bar{u}) \tag{3.15}
\]

In the above equation, \( \bar{q} \) and \( \bar{u} \) are the averages value of the normalized Stoke's parameters of \( q \) and \( u \) respectively. When \( p \) from equation (3.1) is substituted into equation (3.14), the resulting expression for the standard deviation of \( p \) is

\[
\sigma = \sqrt{\bar{q}^2 \sigma_q^2 + \bar{u}^2 \sigma_u^2 + 2 \bar{u} \bar{q} \sigma_{qu}} \tag{3.16}
\]

Obtaining the normalized Stoke's parameters in this project is done in such a way that the \( q \) and \( u \) are independent. Therefore, for a large number of measurements, \( \sigma_{qu} \) tends toward zero. However, I still use equation (3.15) and (3.16) to compute \( \sigma \) since equation (3.14) holds regardless of whether or not measurements of \( q \) and \( u \) are independent.

In order to estimate the polarization for stars from the images obtained using the RPD in front of the SLIC, I have written a FORTRAN (stokes.f) program which uses as inputs the results from the appropriate photometry package from IRAF. (See the appendix for a listing of the code for this program.) In this program, the normalized Stoke's parameters and their descriptive statistics are computed. The measured polarization \( p \) is obtained using equation (3.1) with \( q \) and \( u \) given by \( \bar{q} \) and \( \bar{u} \), and equation (3.16) is used to compute the standard deviation of \( p \). Once the signal-to-noise ratio is computed, equations (3.12) and (3.13) are then used to estimate the degree of polarization and both results reported. The program also computes the position angle by using \( \bar{q} \) and \( \bar{u} \) in equation (3.2), however, a note about the bias in the position angle is appropriate at this point.
Like the bias in the degree of polarization, there also exists a bias in the polarization position angle. However, unlike the degree of polarization, there does not exist a simple method of estimating the true polarization position angle. Serkowski (1962) has shown that the distribution function for the position angle $\theta$ is given by

$$G(\theta, \theta_o, p_o) = \frac{1}{\sqrt{\pi}} \left\{ \frac{1}{\sqrt{\pi}} - \eta_o e^{\eta_o^2} \left[ 1 - \text{erf}(\eta_o) \right] \right\} e^{-\frac{\theta^2}{2\sigma^2}}$$  \hspace{1cm} (3.17)

where

$$\eta_o = \frac{1}{\sqrt{2}} \left( \frac{p_o}{\sigma} \right) \cos 2(\theta - \theta_o).$$  \hspace{1cm} (3.18)

It has been pointed out by Clarke and Stewart (1986) that this distribution function is symmetric and that an unbiased value of $p_o$ would yield an unbiased estimate of $\theta_o$. It may be possible to derive an estimator like those for the degree of polarization from this distribution function based on the maximum likelihood estimator method for example. However, such a daunting task has not been undertaken by this author. Therefore, any value of position angle has not been corrected for bias in this project.

Estimators of the degree of polarization are close to the value of the average measured polarization in this project. I assume such is the case for the polarization position angle as well. A comparison of measured position angles for a few stellar sources reveal that the amount of bias in position angle is very small. Considering that the initial alignment of the polarizer may be off by approximately 1 degree, the agreement between sources with known polarization position angles and their position angles measured in this project is very good.

The stokes.f program also computes confidence intervals on the degree of polarization and the position angle. In the next section I discuss the method used to construct confidence intervals on the estimated degree of polarization. In section 3.2.3 the method of constructing confidence intervals on the position angle is discussed.
3.2.2 Confidence Intervals (Degree of Polarization)

An estimation of the degree of polarization is complete when a confidence interval is also reported. A procedure for determining exact confidence intervals on the degree of polarization was developed by Simmons and Stewart (1985) which is based on a general procedure for developing confidence intervals by Mood and Graybill (1974). I will summarize the general method for determining confidence intervals by Mood and Graybill by using the Rice distribution function. I will then show how Simmons and Stewart modified this method to obtain confidence intervals on the degree of polarization.

The goal here is to determine the upper and lower confidence interval limits on the estimator of the true polarization $\hat{p}_o$ for any value of measured polarization $p$. Using the Rice distribution as the density of the estimator, (i.e. $F(p, \hat{p}_o)$) limits for a $1 - \alpha$ confidence interval can be determined. For any value of measured polarization $p'$ which is substituted for $p$ in $F(p, \hat{p}_o)$, the distribution of $\hat{p}_o$ is completely specified. Therefore, two numbers $p_i$ and $p_s$ can be found such that the probability of $\hat{p}_o$ is less than $p_i$ is

$$P [\hat{p}_o < p_i] = \int_0^{p_i} F(\hat{p}_o, p') d\hat{p}_o = \frac{\alpha}{2}$$

(3.19)

and the probability that $\hat{p}_o$ is greater than $p_s$ is

$$P [\hat{p}_o > p_s] = \int_{p_s}^{\infty} F(\hat{p}_o, p') d\hat{p}_o = \frac{\alpha}{2}$$

(3.20)

The numbers $p_i$ and $p_s$ will depend on the value of $p'$ in $F(\hat{p}_o, p')$. Therefore, $p_i$ and $p_s$ can be written as functions of $p'$ (i.e. $p_i = p_i(p')$ and $p_s = p_s(p')$). Now the area under the $F(\hat{p}_o, p)$ curve is equal to one which can be written mathematically as

$$\int_0^{\infty} F(\hat{p}_o, p') d\hat{p}_o = 1$$

But the area under the curve is also equal to
The first and last terms on the left hand side are both equal to $0.5$ from equations (3.19) and (3.20) above. Upon substitution, the above equation becomes

$$\int_0^{p_1(p')} F(\hat{p}_b, p') \, d\hat{p}_b + \int_{p_2(p')}^{\infty} F(\hat{p}_b, p') \, d\hat{p}_b + \int_{p_3(p')}^{p_2(p')} F(\hat{p}_b, p') \, d\hat{p}_b = 1.$$ 

which is the probability of finding $p_1(p') < \hat{p}_b < p_2(p')$. Therefore, for a measured value of $p$ given by $p'$, there exist two numbers given by $p_1(p')$ and $p_2(p')$ such that the above integral (3.21) is satisfied. If the functions $p_1(p)$ and $p_2(p)$ are graphed as a function of $p$ (where $p$ can have any value $p'$), for a particular value of $\alpha$, graphs such as those shown in Figure 3.2 are obtained.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3_2.png}
\caption{Two confidence interval curves $p_1(p)$ and $p_2(p)$ for a particular value of $\alpha$. A confidence interval for the estimated degree of polarization $\hat{p}_b$ is given by $[p_l, p_u]$. See text for details.}
\end{figure}
Chapter 3  Polarization Analysis  

Confidence intervals for \( p_0 \) are then easily found by drawing a horizontal line at the observed value of \( p \) (given by \( p' \)) and then projecting down to the \( p_0 \) axis. The points of intersection on the \( p_0 \) axis are the two limits between which \( \tilde{p}_o \) will occur with a \( (1 - \alpha) \) probability. From Figure 3.2 it can be seen that if the value of \( p' \) is larger than \( p_0 \), the horizontal line will intersect both the \( p_0(p') \) and \( p_0 \) curves and the confidence interval will be given by \([p_1, p_2]\) as shown. If the value of \( p' \) is such that \( p_1 < p' < p_2 \) as seen in Figure 3.2, then the confidence interval is given by \([0, p_2]\). Finally, if \( p' < p \) then no confidence interval will exist even though \( p' > 0 \). To remedy this, Simmons and Stewart (1985) draw tangent lines from the origin to the upper curve, in this case the \( p_0 \) curve.

Simmons and Stewart have created plots like those given in Figure 3.2 for the 68%, 95%, and 99% confidence intervals (see Simmons and Stewart (1985), Figure 6.) However, the graphs obtained by Simmons and Stewart have axes which are in units of \( p/\sigma \) and \( p_0/\sigma \) instead of \( p \) and \( \tilde{p}_o \) as have been derived here. For this project I have reproduced their curves which are shown in Figure 3.3. Since these are not easily obtained numerically from equations (3.19) and (3.20), and since I wish to use them in a computer program, instead of drawing lines by hand, I have determined equations which approximate their graphs in the following manner.

First, the figure was enlarged and a square grid overlaid. Pairs of numbers \((p_0, p)\) were then measured everywhere the grid intercepted the \( p_0(p_0) \) and \( \tilde{p}(p_0) \) curves. (In order to make an accurate copy, the grid spacing was very narrow.) Finally, for each set of \( p_0(p_0) \) and \( \tilde{p}(p_0) \) curves corresponding to a \((1 - \alpha)\) confidence interval, the pairs of numbers were put into the TableCurve software package and numerical approximation for the curve obtained. The results are shown in Figure 3.3 are very close to Fig. 6 of Simmons and Stewart (1985). The equations used to create the graphs of Figure 3.3, as well as mathematically determine the 68%, 95%, and 99% confidence intervals, are given by equations (3.22a)...(3.22f).

Using the graphs derived by Simmons and Stewart, if the value of \( p/\sigma \) is greater than 1.5 (that is, if the signal-to-noise ratio is above the lowest value of the 68% confidence curve), then the horizontal line will intersect at least two of the confidence interval curves, and the resulting projection down to the \( p_0/\sigma \) axis will result in a confidence interval given by \([p_1/\sigma, p_2/\sigma]\). It is then a trivial task to convert these into intervals of \( p_0 \) by simply multiplying the upper and
Figure 3.3. The above are plots of the curves used to determine the 68%, 95%, and 99% confidence intervals on the degree of polarization. For any measured value of $p/\sigma$ the confidence interval is constructed by drawing a horizontal line across the graph at this value and then projecting down to the $p_b/\sigma$ axis from the intersection of the appropriate percentage curve. As an example, for a measured value of $p/\sigma = 3.5$, the method of determining the 68% confidence intervals is shown above.
\[
\frac{p_o}{\sigma} \text{ [68 \%]}_{\text{upper}} = \begin{cases} 
\left( \frac{p}{0.514} \right)^{\frac{1}{2.56}} & 0 < \frac{p}{\sigma} \leq 2.4 \\
\frac{p}{0.925} + 0.955 & \frac{p}{\sigma} > 2.4 
\end{cases} (3.22a)
\]

\[
\frac{p_o}{\sigma} \text{ [68 \%]}_{\text{lower}} = \begin{cases} 
\left( \frac{p - 1.455}{0.49} \right)^{\frac{1}{2.56}} & 1.5 \leq \frac{p}{\sigma} \leq 1.7 \\
\frac{p - 1.1}{0.992} & \frac{p}{\sigma} > 1.7 
\end{cases} (3.22b)
\]

\[
\frac{p_o}{\sigma} \text{ [95 \%]}_{\text{upper}} = \begin{cases} 
\left( \frac{p}{0.203} \right)^{\frac{1}{2.76}} & 0 < \frac{p}{\sigma} \leq 5.1 \\
\frac{p + 1.64}{0.979} & \frac{p}{\sigma} > 5.1 
\end{cases} (3.22c)
\]

\[
\frac{p_o}{\sigma} \text{ [95 \%]}_{\text{lower}} = \begin{cases} 
\left( \frac{p - 2.495}{0.517} \right)^{\frac{1}{2.76}} & 2.5 \leq \frac{p}{\sigma} \leq 3.1 \\
\frac{p - 1.95}{1.039} & \frac{p}{\sigma} > 3.1 
\end{cases} (3.22d)
\]

\[
\frac{p_o}{\sigma} \text{ [99 \%]}_{\text{upper}} = \begin{cases} 
\left( \frac{p}{0.0376} \right)^{\frac{1}{2.77}} & 0 < \frac{p}{\sigma} \leq 1.2 \\
\frac{p}{\sigma} + 2.29 & \frac{p}{\sigma} > 1.2 
\end{cases} (3.22e)
\]

\[
\frac{p_o}{\sigma} \text{ [99 \%]}_{\text{lower}} = \begin{cases} 
\left( \frac{p - 3.0}{0.584} \right)^{\frac{1}{2.65}} & 3.0 \leq \frac{p}{\sigma} \leq 3.6 \\
\frac{p - 2.55}{1.0435} & \frac{p}{\sigma} > 3.6 
\end{cases} (3.22f)
\]
lower limit by $\sigma$. Any value of $p/\sigma$ such that $0 < p/\sigma < 1.5$ will have a confidence interval of $[0, p/\sigma]$. This projection technique would be used if the confidence intervals were to be determined by hand. However, equations (3.22a)...(3.22f) can be used to determine the confidence intervals on the degree of polarization fairly accurately. That is, using the biased measured value of the degree of polarization $p$ and its standard deviation $\sigma$ in equations (3.22a)...(3.22f) the 68%, 95%, and 99% confidence intervals on the estimated true degree of polarization $p_0$ can be easily obtained. The stokes.f program makes these calculations and prints out all three confidence intervals for the estimated degree of polarization. Now all that remains is to determine the confidence intervals on the polarization position angle.

### 3.2.3 Confidence Intervals (Polarization Position Angle)

In the last section it was noted that reporting the estimated degree of polarization was not complete unless a confidence interval was also reported. Likewise, a confidence interval must also be reported along with the polarization position angle $\theta$ as well. The procedure for determining the confidence intervals follows the method developed by Clarke and Stewart (1986). I shall outline their procedure and note how it is applied in this project.

In this experiment, repeated measurements of $q$ and $u$ are obtained, which have mean values $\bar{q}$ and $\bar{u}$ and standard mean errors $\sigma_\pi = \sigma_q / \sqrt{n}$ and $\sigma_\Pi = \sigma_u / \sqrt{n}$. These mean values are sample estimates of the true values for $q$ and $u$ denoted $q_o$ and $u_o$. As pointed out by Stewart (1985) and Clarke and Stewart (1986), if $q$ and $u$ are not correlated, then confidence intervals for $q_o$ and $u_o$ can be constructed by noting that

$$\frac{(\bar{q} - q_o)^2}{\sigma_q^2} + \frac{(\bar{u} - u_o)^2}{\sigma_u^2} \sim \chi^2_n$$

is distributed in the form of chi-squared $\chi_2$. Since the left hand side has two independent square terms, there are 2 degrees of freedom. Equation (3.26) is an equation for an ellipse. In order to create angular confidence intervals from this equation, Clarke and Stewart (1986) first draw an ellipse centered on $\bar{q}$ and $\bar{u}$ with the semi-major and/or semi-minor axes $\sqrt{\frac{\sigma_q^2 \chi_2}{\bar{q}}} \text{ and } \sqrt{\frac{\sigma_u^2 \chi_2}{\bar{u}}}$ respectively.
Both $\sigma_q$ and $\sigma_u$ are determined from repeated measurements of $q$ and $u$ and, for a 68%, 95%, or 99% confidence interval, the corresponding values of $\chi^2$ with two degrees of freedom are 2.29, 5.99, and 9.21 respectively. Tangent lines are then drawn from the origin to either side of the ellipse as shown in Figure 3.4.

The first angle to notice in Figure 3.4 is $\phi$ which is simply twice the polarization position angle given by equation (3.2) where $u = \bar{u}$ and $q = \bar{q}$. That is, $\theta = \frac{1}{2} \phi = \frac{1}{2} \tan^{-1}(\bar{u}/\bar{q})$. The lower and upper confidence intervals are determined from $\phi_{\text{lower}}$ and $\phi_{\text{upper}}$ by the same argument. Therefore, $\theta_{\text{lower}} = \frac{1}{2} \phi_{\text{lower}}$ and $\theta_{\text{upper}} = \frac{1}{2} \phi_{\text{upper}}$, creating the confidence interval $[\theta_{\text{lower}}, \theta_{\text{upper}}]$.

To determine $\phi_{\text{lower}}$ and $\phi_{\text{upper}}$ it is necessary to know where the tangent lines intersect the ellipse. The equation for a line tangent to an ellipse at a point $q_o, u_o$ on the ellipse is obtained by implicit differentiation of equation (3.26) and, after a little algebra, it can be shown to be

$$\frac{(q_o - \bar{q})(q_1 - \bar{q})}{\sigma_q^2 \chi_2} + \frac{(u_o - \bar{u})(u_1 - \bar{u})}{\sigma_u^2 \chi_2} = 1 \quad (3.27)$$

where $q_1$ and $u_1$ are any other point outside the ellipse in the $q-u$ plane. By making this outside point equal to the origin (i.e. $q_1 = u_1 = 0$) the above equation becomes

$$\frac{\bar{q}(q_o - \bar{q})}{\sigma_q^2 \chi_2} + \frac{\bar{u}(u_o - \bar{u})}{\sigma_u^2 \chi_2} = -1. \quad (3.28)$$

Solving the above equation for $u_o$ and substituting into equation (3.26) we obtain an expression solely in terms of $q_o$

$$\frac{(q_o - \bar{q})^2}{\sigma_q^2 \chi_2} + \left(1 + \frac{\bar{q}(q_o - \bar{q})}{\sigma_q^2 \chi_2}\right)^2 \frac{\sigma_u^2 \chi_2}{u^2} = 1 \quad (3.29)$$

Likewise, by solving equation (3.28) for $q_o$ and substituting into equation (3.26) we obtain an expression solely in terms of $u_o$

$$\left(1 + \frac{\bar{u}(u_o - \bar{u})}{\sigma_u^2 \chi_2}\right)^2 \frac{\sigma_q^2 \chi_2}{q^2} + \frac{(u_o - \bar{u})^2}{\sigma_u^2 \chi_2} = 1. \quad (3.30)$$
Figure 3.4 The geometrical construction of the confidence intervals for the position angle. To obtain the angular confidence intervals, $\phi_{\text{lower}}$ and $\phi_{\text{upper}}$ are each divided by 2 since the position angles are $\theta = \frac{1}{2} \phi$.

Solving equation (3.29) for $q_o$ and (3.30) for $u_o$ and making the quotient $u_o/q_o$ we get that

$$\frac{u_o}{q_o} = \frac{\bar{u} \left( \sigma_u^2 \bar{q}^2 \chi_2 + \sigma_q^2 \bar{u}^2 \chi_2 - \sigma_q^2 \sigma_u^2 \chi_2 \right) \pm \sigma_u^2 \bar{q} \chi_2 \sqrt{\sigma_u^2 \bar{q}^2 \chi_2 + \sigma_q^2 \bar{u}^2 \chi_2 - \sigma_q^2 \sigma_u^2 \chi_2}}{\bar{q} \left( \sigma_q^2 \bar{q}^2 \chi_2 + \sigma_q^2 \bar{u}^2 \chi_2 - \sigma_q^2 \sigma_u^2 \chi_2 \right) \pm \sigma_q^2 \bar{u} \chi_2 \sqrt{\sigma_q^2 \bar{q}^2 \chi_2 + \sigma_q^2 \bar{u}^2 \chi_2 - \sigma_q^2 \sigma_u^2 \chi_2}}$$

which can be rewritten as
\[
\frac{u_o}{q_o} = \frac{u \sigma_\theta^2 \sigma_\pi^2 \chi_2^2 \left( \frac{\chi^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1 \right) \pm \sigma_\pi \sigma_\theta \chi_2 \chi_2^2 \sqrt{\frac{\chi^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1}}{q \sigma_\theta^2 \sigma_\pi^2 \chi_2^2 \left( \frac{\chi^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1 \right) \pm \sigma_\pi \sigma_\theta \chi_2 \chi_2^2 \sqrt{\frac{\chi^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1}}
\]
and simplified to
\[
\frac{u_o}{q_o} = \frac{\sigma_\pi}{\sigma_\theta} \cdot \frac{\pm \frac{\eta}{\sqrt{\sigma_\theta^2 \chi_2}} + \frac{\pi}{\sqrt{\sigma_\pi^2 \chi_2}} \sqrt{\frac{\eta^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1}}{\pm \frac{\eta}{\sqrt{\sigma_\theta^2 \chi_2}} + \frac{\eta}{\sqrt{\sigma_\pi^2 \chi_2}} \sqrt{\frac{\eta^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1}}. \tag{3.31}
\]

Equation (3.31) above gives the slope of a line from the origin to a point tangent to an ellipse in the \(q_o-u_o\) plane. Using this equation then, the angles \(\phi_{upper}\) and \(\phi_{lower}\) are thus given by
\[
\phi_{upper} = \tan^{-1} \left[ \frac{\sigma_\pi}{\sigma_\theta} \cdot \frac{\pm \frac{\eta}{\sqrt{\sigma_\theta^2 \chi_2}} + \frac{\pi}{\sqrt{\sigma_\pi^2 \chi_2}} \sqrt{\frac{\eta^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1}}{\pm \frac{\eta}{\sqrt{\sigma_\theta^2 \chi_2}} + \frac{\eta}{\sqrt{\sigma_\pi^2 \chi_2}} \sqrt{\frac{\eta^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1}} \right] \tag{3.32}
\]
\[
\phi_{lower} = \tan^{-1} \left[ \frac{\sigma_\pi}{\sigma_\theta} \cdot \frac{-\frac{\eta}{\sqrt{\sigma_\theta^2 \chi_2}} + \frac{\pi}{\sqrt{\sigma_\pi^2 \chi_2}} \sqrt{\frac{\eta^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1}}{-\frac{\eta}{\sqrt{\sigma_\theta^2 \chi_2}} + \frac{\eta}{\sqrt{\sigma_\pi^2 \chi_2}} \sqrt{\frac{\eta^2}{\sigma_\theta^2 \chi_2} + \frac{\pi^2}{\sigma_\pi^2 \chi_2} - 1}} \right] \tag{3.33}
\]

When the tangent points have been determined from the above equations, it is now a simple matter of computing the angle \(\theta = \frac{1}{2} \phi\). The subroutines in the \texttt{stokes.f} program use the procedure described above to determine the lower and upper 68\%, 95\%, and 99\% confidence intervals. Once this is done the program reports the polarization position angle and all three confidence intervals.

One problem still remains regarding the position angle. Since the images have a usable 10\' diameter, position angles not at the very center of the image need to be adjusted for this large image size. The method for doing this is discussed in the next section.
3.3 Correcting Position Angle for Large Image Size

The images obtained with the spectral line imaging camera using the Hα filter have a usable 10° angular field of view. This large field of view causes a slight complication when reporting polarization position angles. Since the polarizer has its transmission axis aligned with one axis of the CCD (call it the y axis), all polarization position angles are referred to this axis. However, polarization position angles are referred to lines of constant right ascension. On either side of the central y axis of the CCD, the curved lines of constant right ascension are generally not parallel with the y axis of the chip.* There exists an angle $\gamma$ between lines of constant $x$ and constant right ascension as shown in Figure 3.5. This angle depends not only on right ascension but also on declination since $\gamma$ increases with increasing declination. Therefore, it becomes important to know the magnitude of $\gamma$ so that a correct position angle is reported.

The line of constant $x$ and as well as constant right ascension are approximately collinear along the central y axis of the chip. As was shown in chapter 2, there exists a very small rotation between the x, y coordinate system of the CCD and the equatorial coordinate system at the center of the chip. The amount of rotation was determined to be consistently on the order of 1/4°, counter-clockwise with respect to the line of constant right ascension at the center of the CCD. This small amount of rotation is removed from measurements of the position angle as well. Determining $\gamma$ is relatively straightforward.

Consider a polarized source located at some point other than the central y axis in the x, y coordinate system of the CCD. Let the line of constant right ascension which runs through this point be given by the value $\alpha_s$. When the polarized source is located to the right of the central y axis, the line of constant right ascension will have a negative "slope" as shown in Figure 3.6.

*Only when the x axis of the CCD is aligned with the celestial equator will lines of constant right ascension exist which are perpendicular to the y axis and only along the x axis.
Figure 3.5 A portion of the celestial sphere imaged onto the square CCD chip. The central y axis ($x = 256$) of the chip is parallel with a line of constant right ascension. To either side of the central y axis, lines of constant right ascension are curved and so an angle $\gamma$ exist between the two.

With a knowledge of the right ascension and declination ($\alpha_0$, $\delta_0$) at the mechanical center of the chip ($x = 256, y = 256$), the line of constant right ascension $\alpha_s$ is given by

$$\alpha_s = \alpha_0 + \tan^{-1}\left\{\frac{-X}{\cos(\delta_0) - Y \sin(\delta_0)}\right\} \quad (3.31)$$

where $X, Y$ are the standard coordinates for any point along the line of constant right ascension. Recall from Chapter 2 that the standard coordinates $X$ and $Y$ were related to the $x$ and $y$ position on the chip by $x/F = X$ and $y/F = Y$ where $F$ is the focal length of the camera lens in pixel units. Therefore, the formula for $\gamma$ can be derived in either $x, y$ or $X, Y$. Taking $\alpha = \alpha(X, Y)$, the differential $d\alpha$ is...
In order to determine the angle $\gamma$ makes with respect to a line of constant right ascension $\alpha_x$ and a line parallel to the $y$ axis ($x_0$), the slope of the tangent line at the point $(x_0, y_0)$ is determined and $\gamma$ given by $\gamma = \tan^{-1}(dx/dy)$.

$$d\alpha = \frac{\partial \alpha}{\partial X} dX + \frac{\partial \alpha}{\partial Y} dY = 0$$

which is equal to zero since $\alpha$ is a constant. To get the slope of a tangent line at any point on the curve, the terms in the equation above are rearranged as

$$\frac{dY}{dX} = -\left( \frac{\partial \alpha}{\partial X} \right) / \left( \frac{\partial \alpha}{\partial Y} \right).$$

Differentiating equation (3.31) above with respect to $X$ and $Y$ and substituting into the equation above, the slope of the line at any point $X, Y$ along the curve $\alpha_x$ is
It can be seen from Figure 3.6 that the angle $\gamma$ is related to $d_x/dy$ by the tangent of $\gamma$. Since I want $\gamma$ to always be positive, the absolute value of $d_x/dy$ is used. Therefore, $\gamma$ is given by

$$\gamma = \tan^{-1}\left| \frac{d_x}{d_y} \right| = \tan^{-1}\left| \frac{x \sin(\delta_o)}{F \cos(\delta_o) + y \sin(\delta_o)} \right|. \quad (3.32)$$

Therefore, given any point $(x, y)$ which is referenced to the center of the image, the angle $\gamma$ between a line parallel to the $y$ axis and a line of constant right ascension running through that point is given by equation (3.32).

So far the angle $\gamma$ has been derived for a point located to the right of the central $y$ axis. However, the above procedure applies when the point of interest is to the left of the central $y$ axis as well. Since equations (3.31) and (3.32) are referred to the center of the CCD chip, and, as long as the $x, y$ coordinates of any point is referred to the center of the image as well, the above procedure works to determine the angle $\gamma$ anywhere on the chip.

Now that the angle $\gamma$ has been determined, correcting the position angle is an easy task. There are two cases to consider; either the point of interest is to the left or to the right of the central $y$ axis. The computed polarization angle $\theta$ referred to the $y$ axis can either be between the line of constant right ascension and the $y$ axis or located outside of it, as shown in Figures 3.7 and 3.8. Each of these cases must be considered separately.

Suppose the point of interest is to the left of the central $y$ axis and the computed polarization position angle $\theta$ is between the line of constant right ascension $\alpha_s$ and a line of constant $x$ ($x_o$) as shown in Figure 3.7 (a). The correctly reported polarization position angle $\theta'$ is measured counter clockwise with respect to the line of constant right ascension $\alpha_s$ in the image.
By looking at Figure 3.7 (a), and from simple geometry, it is easily seen that \( \gamma, \theta, \) and \( \theta' \) are related by \((\gamma - \theta') + \theta = 180^\circ\). Therefore, the correctly reported polarization position angle is

\[
\theta' = \theta + \gamma - 180^\circ. \tag{3.34}
\]

If, on the other hand, the polarization position angle lies outside the angle \( \gamma \) as shown in Figure 3.7 (b), the correctly reported polarization position angle is given by the easily seen geometrical relationship

\[
\theta' = \theta + \gamma. \tag{3.35}
\]
Figure 3.8 Configurations between the angle $\gamma$, the computed polarization angle $\theta$, and the correctly reported polarization position angle $\theta'$ when the polarization source is to the right of the central $y$ axis.

Now consider the case when the polarization source is to the right of the central $y$ axis. If the polarization position angle is between the angle of constant right ascension $\alpha_s$ and the line of constant $x (x_o)$ as shown in Figure 3.8 (a), then the relationship between $\gamma$, $\theta$, and $\theta'$ is $(\gamma - \theta) + \theta' = 180^\circ$. This makes the correctly reported polarization position angle

$$\theta' = \theta - \gamma + 180^\circ. \quad (3.36)$$

However, if the polarization position angle is outside the angle $\gamma$ as shown in Figure 3.8 (b), then the correctly reported polarization position angle is

$$\theta' = \theta - \gamma. \quad (3.37)$$

Therefore, by using the formulae derived in this section, the calculated polarization position angle for a source anywhere on the CCD can be adjusted so that it is reported with respect to the line of right ascension which runs through the source. These formulae are used to not only adjust the polarization position...
angle but also the limits on the polarization position angle confidence intervals. In the case of stellar data, I have written subroutines in the *stokes.f* program which perform all the necessary calculations that were derived in this section. The only required user input for the program is the position of the star. The rest of the decisions and calculations are done automatically by the subroutines.

In the case of the Hα maps, the adjustment is across the usable 10° extent of the image. The procedure is automated in the *mapmaker.f* program since each pixel is adjusted appropriately. A discussion of the *mapmaker.f* program is given in section 3.6 and the program code is given in the Appendix.

Until now, no mention has been made of any possible instrumental polarization. Now that some of the tools needed to correctly analyze the polarization (e.g. photometry, bias removal, corrections for image size) have been sharpened, the analysis of stellar polarization can now begin. But before this is done, and before any Hα maps are made, the question of whether or not there is instrumental polarization has to be answered.
3.4 Photometry

Section 3.4.1 introduces the method of aperture photometry used to determine the brightness of a stellar source in the CCD images obtained with SLIC. In order to accurately determine the Stoke's parameters $q$ and $u$, care must be exercised performing aperture photometry. The first section below discusses some of the problems and solutions to doing accurate stellar photometry.

In section 3.4.2 the method of determining $\bar{q}$ and $\bar{u}$ by trimmed means is introduced. The reason for using trimmed means is to handle the problem of outliers in measurements of $q$ and $u$. By removing these outliers, the central tendency of $q$ and $u$ is more accurately represented when averaging. In section 3.4.3 instrumental polarization is briefly mentioned. Any significant amount of instrumental polarization occurs in the corner regions of the images where the image is distorted by the optics of the system. Since no reliable data are obtained in these regions anyway, any instrumental polarization in these regions is of no concern in this project.

Finally, section 3.4.4 compares stellar polarization measurements obtained in this project with previously published values. In some cases the agreement is very good, however in others it is not. In the cases where no measurement was obtained or significant disagreement with the published value exists, possible explanations of why are given.

3.4.1 Stellar Photometry

Using aperture photometry to determine the brightness of a star on the CCD chip would at first seem like a relatively straightforward task. First one would need to determine how large an aperture to use which contains all the light from the star. Then, centering this aperture on the star, one sums up all the light (pixel values) in this aperture. The sum obtained contains not only light contributed by the star but also the background as well. An estimate of the background contribution is obtained by measuring nearby pixels. Once the background value has been determined and appropriately scaled to the area of
the aperture centered on the star, it is subtracted from the value obtained in the aperture centered on the star. This leaves only the contribution from the star.

However, life is not that simple, and obtaining accurate photometric results is fraught with many difficulties. Among the concerns is how large the aperture should be in order to measure all the light from the star? What is the best way to determine the contribution from the background? Or even, simply, how to center the aperture on the star?

Luckily, a lot of these questions have already been answered by other astronomers. In this section I will explain how aperture photometry is done in this project and the answers to the questions above. The method used here follows the general aperture photometry practices outlined by Da Costa (1992) and follows closely the procedures described by Davis (1989) and Massey and Davis (1992). In this project aperture photometry is performed using the PHOT task in IRAF package APPHOT.

The first and easiest problem to address is how to accurately center the photometric aperture on a star. The method used to determine the centers of stars for aperture photometry in this project is based on calculating the centroid of the light distribution inside a box which contains the light from the star. The box is set to a width large enough to encompass the bright pixels which represent the star but not so large as to contain a lot of sky background. An initial guess of the center of the star is obtained using the IMEXAMINE task in IRAF and is usually accurate to within one decimal place. The initial star center is then written to a file which is read by the CENTER task via the CENTERPARS parameter set.

When calculating the centroid of the star, IRAF computes the marginal $x$ and $y$ distributions which are extracted from the pixels in the box initially centered on the star. The intensity weighted centroid in $x$ and $y$ is then computed via

$$x_c = \frac{\sum (I(x_i) - \bar{T}(x)) x_i}{\sum (I(x_i) - \bar{T}(x))}$$

$$y_c = \frac{\sum (I(y_j) - \bar{T}(y)) y_j}{\sum (I(y_j) - \bar{T}(y))}$$

where summation is over those pixels for which the marginal sums $I(x_i)$ and $I(y_i)$ are greater than the mean intensity of each marginal $\bar{T}(x)$ and $\bar{T}(y)$.
respectively, Davis (1987). Here the marginal sums are given by

\[ I(x_i) = \sum_j I_{ij} \quad I(y_j) = \sum_i I_{ij} \]

where the summation is over the pixels in the box centered on the star. The mean intensity of each marginal sum is simply then the average of the marginal sums above.

The next problem to consider is how to determine the background around the star. More explicitly, what needs to be estimated is the value of the background region where the star would be if it were not there. It is important to obtain a good background estimate and a means of achieving an accurate background determination is possible using the sky fitting algorithms in IRAF. In particular, the FITSKY task used by the photometry task PHOT provides powerful algorithms which determine accurate background values.

Firstly, an annulus whose inner radius and width are set by the user is centered on the star. By using an annulus, any gradient in the background is, to first order, removed, Da Costa (1992). The inner radius is set far enough from the center of the star so that any light from the star contributes only a negligible amount to the background value. The width is set to a large value so that the number of background pixels is on the order of hundreds in order to obtain good statistics.

The FITSKY task then computes a histogram of the sky values by first sorting and then computing the mean value of the background distribution as well as the standard deviation with respect to the mean background value. Any pixels which have saturated values (greater than 64,000) and which are negative are rejected before computing the background statistics. The remaining values of the background are then binned, and the histogram computed. A Gaussian is fit to the histogram using non-linear least squares techniques, Davis (1987).

If the background annulus is large enough, it may contain bright sources such as H\(\alpha\) nebulosity or even other stars. In these cases, the FITSKY task has a number of options which allow the user to decide when, and how many, pixels should be rejected from the annulus. If the annulus contains bright sources, then these pixels are rejected in two ways. First, a certain percentage of the total number of pixels in the background are clipped off from the high end of the...
sorted array of background pixels. The background statistics are then computed using the remaining sky pixels values. This has the effect of reducing any bias due to a small number of large pixels values which are not truly representative of the background in the region around the star.

The other way of rejecting these bright sources is to remove any strongly deviant pixels from the histogram itself. If the computed standard deviation using the background pixels is \( \sigma_{\text{sky}} \), then by rejecting any pixels which are \( k \cdot \sigma_{\text{sky}} \) above (or even below) the mode of the computed histogram and then refitting the Gaussian removes the bias caused by any strongly deviant pixels. Usually the value of \( k \) is chosen to be around 2.5 or 3 as suggested by Da Costa (1992). The above procedure of removing pixels above (and below) \( k \cdot \sigma_{\text{sky}} \) and then refitting the Gaussian work best when performed iteratively. This is exactly what is done in the FITSKY task. The procedure is done iteratively until either no more pixels are rejected or until the maximum number of iteration cycles are reached, Davis (1987).

In cases where no strongly deviant pixels are noticeable, the \( k \cdot \sigma_{\text{sky}} \) rejection algorithm is still employed anyway with \( k \) set to a value of 3. This insures that any deviant pixels are rejected even though their appearance in the image is not immediately noticeable. In areas with a lot of stars or H\( \alpha \) nebulosity, both method mentioned above are used simultaneously. In addition, regions where small stars are in the background annulus are noticeable; pixels inside a radius \( r \) centered on these stars are removed. Removal of many pixels is another reason for choosing an annulus large enough to contain hundreds of pixels.

All the necessary parameters are contained in the FITSKYPARS parameter set which is used by the FITSKY task which implements the chosen algorithm and preset parameters to estimate the background value. This algorithm provides a good estimate of the background. The final background value that is used is the mode of the fitted histogram. This, however, represents the value in just one pixel, and so this value must be scaled to the area of the aperture used to measure the flux of the star. Knowing the area of the aperture used to measure the flux of the star in square pixels, the modal value is multiplied by this area and is used as an estimate of the background for the star.

The last thing that must be considered when doing aperture photometry is how large to make the aperture centered on the star which measures the
brightness of the star. The apertures used are circular in order reduce the sampling error and partial pixel values are included when measuring the light from the star.

The aperture should be large enough to contain all the light from the star. However, the aperture size will naturally depend on the brightness of the star. Da Costa (1992) has shown that, in magnitudes, the error in a photometric observation is given by

$$
\delta m = \frac{1.09}{g C_{ADU}} \sqrt{g C_{ADU} + n_{pix}(r^2 + g S_{ADU})}
$$

where $g$ is the gain of the CCD in electrons per ADU, $C_{ADU}$ is the number of counts (ADU) attributed to the star in the aperture centered on the star, $S_{ADU}$ is the number of background counts in the aperture, $n_{pix}$ is the number of pixels in the aperture, and $r$ is the readout noise in electrons. From this equation, Da Costa considers two limiting cases. For bright stars, $gC_{ADU} \gg n_{pix}(r^2 + g S_{ADU})$ and faint stars for which $gC_{ADU}$ does not dominate $n_{pix}(r^2 + g S_{ADU})$

For bright stars, large apertures centered on the stars are acceptable. This is because the number of photoelectrons ($g C_{ADU}$) is very large, in which case equation (3.38) is approximated by $\delta m \sim (g C_{ADU})^{-1/2}$. Therefore, for bright stars, the photometric error is dominated by Poisson statistics. Choosing how large the aperture should be is determined by the method of growth curves (Howell, 1989).

The growth curve is created in the following manner. First, the flux through a series of different radii apertures centered on the star is obtained as well as estimates of the background value by means of the methods discussed above. Next the background value is scaled to the area of the aperture centered on the star and subtracted from the summed values in that aperture which measures the flux from the star alone. Next, the difference in stellar flux between successive apertures is plotted as a function of radius. If the star is fairly isolated, the curve of growth will almost always have an appearance as shown in the top graph of Figure 3.9.

Notice that for this star, the difference in stellar flux asymptotically approaches zero. This indicates that after a certain radius, the difference in flux between two successive apertures is extremely small. Therefore, choosing
between an aperture of 3 or 4 pixels in which to measure the stellar flux is not critically important. This fact is also illustrated by the middle graph of Figure 3.9 which is a plot of the stellar flux as a function of radius. For this isolated bright star the stellar flux is seen to reach an asymptotic value, and choosing an aperture radius of 3 or 4 pixels does not substantially alter the recorded flux. The two apertures will, for all practical purposes, yield similar results.

The bottom graph in Figure 3.9 is a plot of the signal-to-noise using the CCD equation versus aperture radius. The CCD signal-to-noise equation used here is given by

\[ S/N = \frac{N_{\text{star}}}{\sqrt{N_{\text{star}} + n_{\text{pix}}(N_{\text{sky}} + N_r^2)}} \]  

(3.39)

where \( N_{\text{star}} \) is the background subtracted sum in the aperture centered on the star, \( N_{\text{sky}} \) is the single pixel background value determined by the mode of a Gaussian fitted to the histogram of background pixels, \( n_{\text{pix}} \) is the number of pixels in the aperture centered on the star, and \( N_r \) is the readout noise in the CCD (6.9 e\(^-\) pixel\(^{-1}\) for SLIC). Other versions of the CCD equation can be found in the literature (e.g. Newberry 1991, Howell 1992), however, the CCD equation used above is adequate for purposes of this project.

The bottom plot in Figure 3.9 shows that the signal-to-noise reaches a maximum value at a radius of about 2 pixels. However, there is not a substantial decrease in \( S/N \) at a radius of 3 pixels where the measured flux starts to level off as seen in the middle graph. At 3 pixels the growth curve indicates that the difference in flux through two successive apertures is practically the same. It is these criteria that determine the aperture radius. That is, the radius of the aperture in which to measure the stellar flux is always chosen to be the radius at which the growth curve indicates a negligible difference in flux from the star. This corresponds to the aperture radius at which the stellar flux starts to level off. As a matter of consistency, all of the apertures chosen are done so using the above mentioned criteria.

The left graphs in Figure 3.9 were for an isolated star. Many of the stars in the H\(\alpha\) images are somewhat crowded, especially for stars of fainter magnitude. A bright star of similar magnitude as the one just discussed but which is located very near another star is shown in the right hand graphs of Figure 3.9. In the case where the bright star has a faint nearby companion, the growth curve (top
right curve in Figure 3.9) shows a deviation away from an asymptotic approach to zero difference in flux. This results because the fainter star is included in the large aperture used to measure the stellar flux of the bright star. The deviation of the change in flux is seen to start to occur at an aperture radius of 5 pixels.

In this case, the growth curve is extremely helpful in deciding the aperture radius. From the top right growth curve in Figure 3.9, an aperture radius of about 4 pixels is large enough to measure the stellar flux of the bright star while at the same time minimizing the flux due to the fainter nearby star. A look at the flux versus radius (middle right curve of Figure 3.9) confirms that an aperture radius of 4 pixels is an appropriate choice. Beyond 4 pixels, the flux is seen to increase as more of the nearby star's flux enters the larger apertures. The \( S/N \) as a function of radius plot (bottom right curve in Figure 3.9) indicates that there is no substantial decrease in \( S/N \).

Therefore, for a bright star, choosing an aperture in which to measure the stellar flux is rather straightforward. By plotting growth curves as well as the stellar flux as a function of radius, the choice of aperture radius is easily determined from inspection of the resulting graphs. In the H\( \alpha \) images, bright stars which have a high number of photon counts, and for which the above procedure works well, usually have a magnitude of less than or equal to 6.

However, for faint stars, the choice of aperture has to be much smaller than those used for bright stars. For faint stars, \( gC_{ADU} \) in equation (3.38) is not the dominant term, but rather \( n_{pix}(r^2 + g S_{ADU}) \) dominates. If a large aperture were used to measure the flux for faint stars, a large fraction of the sum in aperture centered on the star would contain a significant amount of counts which are attributed to the background sky as well as the readout noise of CCD. In these cases, a small aperture which contains a larger proportion, but perhaps not all, of the light from the star is used.
Figure 3.9 Plots for two bright stars; one that is isolated (left) and one that has a nearby companion star (right). The top plots are the growth curves of the two stars, the middle plots are the stellar fluxes (background subtracted) of the two stars through apertures of increasing radius, and the bottom are plots of the signal-to-noise ratios as a function of aperture radius.
However, using such a small aperture does not place such a severe limitation on the photometry process. The limiting magnitude for stars in the H\(\alpha\) images is around magnitude 8. However, as mentioned in the next section, reliable polarimetric measurements are usually not attainable for these faint stars and so a photometric limiting magnitude of approximately 7 is adopted for stars in the H\(\alpha\) images. Stars around this magnitude have point spread functions which are on the order of few pixels. The size of the pixels in the CCD chip are 1.6\('\) square implies a point spread function on the order of 4\('\) in radius for these stars. However, as shown by King (1971), the majority of the light in a star falls inside the innermost core of a star's point spread function. For these magnitude stars an aperture whose radius is on the order of a few pixels is sufficient to measure most of the light from the star. For these stars an aperture radius on the order of 2 to 3 pixels is chosen based on the same methods used for brighter stars.

Choosing such a small radius in which to measure the light from the star is not a severe hindrance in determining the Stoke's parameters \(q\) and \(u\). The Stoke's parameters are derived from differences between photometric measurements of stellar images obtained for different polarizer position angles as discussed in Chapter 1. Polarimetric results obtained for faint stars in the H\(\alpha\) images (presented in the next section) indicates that it is sufficient to measure a large portion of the stellar flux in order to detect differences between photometric measurements.

One other problem needs to be addressed concerning aperture photometry performed in this project and that is how to deal with crowded fields. A good portion of the stars in the CCD images obtained are crowded next to other stars. Some of the stars are located next to, and even in, H\(\alpha\) nebulosity in these images as well. For these stars, aperture photometry is an extremely difficult task. Light from neighboring stars would contaminate a large aperture, and it is difficult to determine a reliable background estimate of stars in regions where H\(\alpha\) nebulosity is present. Aperture photometry is still performed, but measurements presented in the next section indicate that only highly polarized stars are directly measurable in these instances.

Now that the method of aperture photometry has been discussed, all that remains is to describe how the Stoke's parameters \(q\) and \(u\) are derived from the measured stellar fluxes. As mentioned in Chapter 1, the Stoke's parameters \(q\) and \(u\) are obtained by differences in measurements between a polarizer
positioned at different angles with respect to a fiducial setting. Here the measurements are with respect to \( y \) axis of the CCD chip and the polarizer is positioned at 45° increments from this axis. As stated in Chapter 1, the normalized Stoke's parameters are given by

\[
q = \frac{I(0°) - I(90°)}{I(0°) + I(90°)} = \frac{I(180°) - I(270°)}{I(180°) + I(270°)} \tag{3.40}
\]

\[
u = \frac{I(45°) - I(135°)}{I(45°) + I(135°)} = \frac{I(225°) - I(315°)}{I(225°) + I(315°)} \tag{3.41}
\]

where \( I(\theta) \) is the measured stellar intensity with the polarizer oriented at angle \( \theta \) with respect to the \( y \) axis of the CCD. Since the intensity is related to the flux, it suffices to measure only the stellar flux for \( I(\theta) \).

The measuring of the stellar flux is handled by the \textsc{phot} task in IRAF. All the necessary parameters discussed above are set in their proper parameter files (i.e., \textsc{fiskypars}, \textsc{photpars}, etc.) and the task is run non-interactively by noting the position of the star of interest in a coordinate file. The resulting output file is run through a cleaning program \textsc{photc.f} which extracts the pertinent data as shown in Figure 3.10. What results is a file which contains, for a specific polarizer position angle, the stellar flux (ON – OFF) of a particular star in each image (Image) which is the table headings in Figure 3.10. Also included in this output file are the background values around the star in each image (OFF), the sum in the aperture centered on the star (ON), the area of the aperture centered on the star (AREA), the computed instrumental magnitude (MAG), and the estimated instrumental magnitude error (MAGER). Located at the bottom of the table shown in Figure 3.10 are the average, standard deviation, and standard deviation of the mean for all of the columns except MAGER.

Since the transmission axis of the polarizer has the same orientation at 0° and 180° with respect to the CCD's \( y \)-axis, measurements at both 0° and 180° are included in the same photometry file as shown in Figure 3.10. Photometry is performed on all of the images as well. That is, a single photometry file is created for each of the 45° and 225°, 90° and 270°, and 135° and 315° images. These four files are then input into another program (\textsc{stokes.f}) which calculates the stokes parameters \( q \) and \( u \) using equations (3.40) and (3.41) above as well as their respective statistics.
Photometry Data for HD0
Note: OFF Flux is scaled to size of ON aperture

<table>
<thead>
<tr>
<th>Image</th>
<th>ON</th>
<th>OFF</th>
<th>ON-OFF</th>
<th>AREA</th>
<th>MAG</th>
<th>MAGER</th>
</tr>
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<td>21.39775</td>
<td>23.229</td>
<td>0.112</td>
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<td>23.202</td>
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</tr>
<tr>
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<td>21.45976</td>
<td>23.342</td>
<td>0.123</td>
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<tr>
<td>i97000t</td>
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<td>36.95</td>
<td>396.85</td>
<td>21.39424</td>
<td>23.196</td>
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<tr>
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<td>21.46414</td>
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<td>i121000t</td>
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<td>435.87</td>
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<td>21.46212</td>
<td>23.362</td>
<td>0.127</td>
</tr>
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<td>23.364</td>
<td>0.123</td>
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<table>
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<th>OFF</th>
<th>ON-OFF</th>
<th>AREA</th>
<th>MAG</th>
<th>MAGER</th>
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</thead>
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<td>Std</td>
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<td>0.46</td>
<td>32.82</td>
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<td>0.096</td>
<td></td>
<td></td>
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<tr>
<td>SDOM</td>
<td>7.98</td>
<td>0.10</td>
<td>7.34</td>
<td>0.00966</td>
<td>0.022</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 3.10** Photometric measurement file for a single star in the Hα images where the polarizer was oriented at 0° and 180° with respect to the y axis of the CCD chip. Individual measurements are listed with the average, standard deviation, and standard deviation of the mean for each column except MAGER listed at the bottom.
The *stokes.f* program also computes the degree of polarization and position angle for a single star. The statistics associated with the degree of polarization are also computed as discussed in section 3.2. The program then computes the estimated degree of polarization and the associated confidence intervals. The confidence intervals on the position angle are also computed and the appropriate corrections for large image size are applied to both the position angle and its confidence intervals as well according to the procedure laid out in section 3.3.

### 3.4.2 Instrumental Polarization

Before any results are presented and before any analysis of $\text{H}\alpha$ maps or stellar data is performed, it is necessary to check to what extent, if any, of the amount of polarization detected is due to the instrument itself. Any measurement which is determined to have instrumental polarization effects must either be corrected or not used depending on the situation. Therefore, a means is required of determining if any instrument polarization is present and to what extent.

A commonly used method of testing for instrumental polarization is to observe a star which is known to be unpolarized (or at least polarized to a very small degree) and observe if any measurable polarization results. Such stars are usually referred to as unpolarized standard stars. These stars are usually limited to A, F, G, and K-type stars and are usually located in the solar neighborhood to reduce the risk of interstellar polarization.

A listing of such stars is presented by Serkowski (1974). These nearby standard stars have been observed by using rotatable tube telescopes. The use of a rotatable tube telescope eliminates polarization caused by the telescope's optics, (Mathewson and Ford 1970). A list of standard stars presented by Serkowski (1974) all have polarizations on the order of hundredths of a percent. If the unpolarized star is observed to have a polarization much higher than its accepted value, and if all factors such as background polarization have been accounted for, then the excess polarization results from the instrument itself.
Figure 3.11 The location of $\beta$ Cassiopeiae in the series of H$\alpha$ images taken to assess the instrumental polarization of SLIC. The circle indicates the useable 10° extent in the H$\alpha$ images.
In this project, the following approach is used to determine if instrumental polarization is a factor in any polarimetric measurements. A standard star whose polarization is known to be exceedingly small is \( \beta \) Cassiopeiae, which is a nearby F2IV star. Its polarization is listed as \( 0.009 \pm 0.009\% \), Serkowski (1974). \( \beta \) Cassiopeiae has a high declination (59\( ^\circ \) 09\( ^\prime \)) and so was easily observable.

Polarimetric observations of this star were made on the night of November 25, 1997. As with all observations made for this project, data was taken only on clear, moonless nights in order eliminate any background polarization due to scattered moonlight. Exposure times for all observations were 30 seconds and the star did not saturate the detector. Since its visual magnitude is 2.3 it created a large (\( \sim 5 \) pixel radius) point spread function. The closest star was more than 10 pixels away and none were nearly as bright as \( \beta \) Cassiopeiae. Growth curves indicated that the change in flux was fairly constant out to a radius of about 9 pixels.

In order to test for the presence of instrumental polarization across the extent of the CCD, a series of observations of \( \beta \) Cassiopeiae were made so that its point spread function was located on different parts of the CCD chip. This was done in order to determine if instrumental polarization was a factor in any particular region due to the optics of the system. Figure 3.11 indicates the location of the image of \( \beta \) Cassiopeiae observed at various places on the CCD. The circle in Figure 3.11 indicates the location of the useable \( \pm 10^\circ \) extent of the H\( \alpha \) images. At each location, five images with the polarizer oriented at every \( 45^\circ \) were obtained resulting in a total of ten images each at \( 0^\circ, 45^\circ, 90^\circ, \) and \( 135^\circ \). For each star, photometry was performed in the manner discussed in the previous section, and polarimetric computations performed using the stokes.f program. The resulting polarimetric results are summarized in Table 3.1.

From Table 3.1, we see that only two measurements show any large measurable polarization as a result of the instrument itself. Four of the measurements are within \( 5^\circ \) of the mechanical center of the image. Location 6 is just outside the circle and has a radius of \( 5.4^\circ \) and is 54 pixels away from the edge of the CCD. Location 7 is in the corner of the image and has a very high degree of polarization.
Table 3.1
RESULTING POLARIMETRIC OBSERVATIONS OF β CASSIOPEIAE

<table>
<thead>
<tr>
<th>Location No.</th>
<th>Stellar Image Location (Pixels)</th>
<th>( p_\theta ) (%)</th>
<th>( \sigma ) (%)</th>
<th>( \tilde{p} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>259 256</td>
<td>0.8</td>
<td>4.7</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>271 458</td>
<td>0.6</td>
<td>2.7</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>413 270</td>
<td>0.6</td>
<td>1.8</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>342 143</td>
<td>0.8</td>
<td>0.9</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>117 251</td>
<td>0.5</td>
<td>0.7</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>54 259</td>
<td>1.6</td>
<td>1.1</td>
<td>1.1</td>
</tr>
<tr>
<td>7</td>
<td>55 50</td>
<td>3.1</td>
<td>1.3</td>
<td>2.7</td>
</tr>
</tbody>
</table>

In the H\(\alpha\) images the corners suffer the greatest optical aberrations. This is most likely the cause of the instrumental polarization. It is not evidently clear from visual inspection of the images that location 6 also suffers from these optical aberrations, but is probably the cause of the instrumental polarization in this region as well.

Since all other locations have no observable polarization it is reasonable to accept that polarimetric measurements in these regions are relatively free of any measurable instrumental polarization. Therefore, only stars which are inside the 5° radius are measured. Likewise for the H\(\alpha\) maps, only the emission which is recorded inside the 5° radius is taken as a reliable measurement of polarization. No attempt is made to adjust or remove the instrumental polarization for stars outside of the 5° radius, and, since the intensity of H\(\alpha\) drops off dramatically outside this region, there is no need to remove the instrumental polarization from these regions.

### 3.4.3 Measurement Compared to Previously Published Results

Now that the question of instrumental polarization has been addressed, and the criteria for where on the CCD polarimetric measurements are free of any measurable instrumental polarization, it is now time to present some results of stellar polarimetric measurements. These measurements are useful in determining to what extent polarization can be measured using the SLIC in conjunction with the RPD. The H\(\alpha\) images were obtained primarily for the
purpose of making polarization maps around regions which were recognized as supernova remnants and shell-like structures. However, the large number of stars seen in these images makes them a natural choice to try and measure any polarization that exists for these stars. But before any measurements are made of stars whose degree of polarization is unknown, a reasonable first step is to see how measurements obtained for stars which are known to be polarized compare with their published values.

The data presented in this section results from the measurement of stellar fluxes from a series of H\(_\alpha\) images. The dates of the observations as well as the field centers are listed in Table 3.2. Each series of observations was taken at or around new moon so that the presence of polarized scattered moonlight was not a contributing factor in the observations. On all but one of the nights (Monoceros I) observations were made when the transparency of the sky was very good. On that one bad night the transparency was quite variable. Therefore, an analysis of stars in the images of that night is not performed.

<table>
<thead>
<tr>
<th>Field</th>
<th>Date of Observation</th>
<th>Field Center</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cygnus 1</td>
<td>September 5, 1997</td>
<td>20° 21' 36° 52'</td>
</tr>
<tr>
<td>Cygnus 2</td>
<td>September 28, 1997</td>
<td>20° 47' 43° 10'</td>
</tr>
<tr>
<td>Cygnus 3</td>
<td>September 28, 1997</td>
<td>20° 02' 44° 11'</td>
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<tr>
<td>Cygnus 4</td>
<td>September 5, 1997</td>
<td>21° 02' 35° 14'</td>
</tr>
<tr>
<td>Monoceros I</td>
<td>February 2, 1998</td>
<td>6° 40' 06° 50'</td>
</tr>
<tr>
<td>Monoceros II</td>
<td>March 3, 1998</td>
<td>6° 40' 06° 50'</td>
</tr>
</tbody>
</table>

Existing published polarimetric data for stars in each of these regions were obtained from the catalogs of Mathewson et al. (1978) and Axon and Ellis (1976). Only stars that were within the 5° of the center of the image were used since these stars are more likely not to have any measurable instrumental polarization associated with them (see previous section). A large portion of these stars had magnitudes much greater than 8.0 and therefore were not measured. Some of the remaining stars were in areas of bright H\(_\alpha\) nebulosity and were indistinguishable from the nebula itself. These stars were obviously not measured. Finally, a large number of stars had degrees of polarization much
Polarimetric measurements were then made on the remaining stars. The results are shown in Table 3.3 below. The first column is the Henry Draper Catalog number for the star. The second is the star's visual magnitude obtained from the SIMBAD astronomical database. The third and fourth columns are the stars' biased measured polarization and standard deviation in percent. Column five is the estimated degree of polarization (in percent) using the Wardle-Kronberg estimator, and column six is the 68% confidence interval on the degree of polarization in percent. Column seven and eight are the stars' polarization position angle as referred to a line of constant right ascension and its corresponding 68% confidence interval. Column nine and ten are the published degree of polarization and position angle from the Mathewson et al. catalog in percent and degrees respectively. Stars which are marked with an asterisk are those where the use of the γ-trimmed mean method has been used in the calculations of the polarimetric parameters (see section 3.4.4 for an explanation of the γ-trimmed mean method). Stars which appear twice are those which have been observed in two separate fields and are within the 5° radius of the center of the image.

The first thing to notice about the data is that there is very good agreement between the measured and published values of both the degree of polarization and position angle as well. In some cases the agreement is quite striking. For example, HD 198478 is the Henry Draper designation for the star 55 Cygni which is a highly polarized standard star listed by Serkowski (1974). The list of highly polarized standard stars presented by Serkowski (1974) are useful to observe as a means of relating the instrumental polarization position angle to a standard frame, namely the equatorial system which is used in this project. They are also useful for checking polarizer modulation efficiency, (Clarke 1994). This standard star was used to check the initial alignment of the polarizers reference axis with respect to the north-south line. It was also used as a check to check the consistency of the degree of polarization obtained.
### Table 3.3

**Comparison of Polarimetric Results from Stars in the Hα Images with Stars of Known Published Polarimetric Values**

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<th>$P_\lambda$</th>
<th>$\sigma$</th>
<th>$\bar{P}$</th>
<th>68% C.I., $\theta$</th>
<th>68% C.I., $\theta$</th>
<th>$P_p$</th>
<th>$\theta_p$</th>
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<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
<td>(9)</td>
</tr>
<tr>
<td>45910*</td>
<td>6.8</td>
<td>1.8%</td>
<td>1.4%</td>
<td>1.1%</td>
<td>[0.0, 2.9]%</td>
<td>139°</td>
<td>[124, 155]°</td>
<td>1.02%</td>
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<tr>
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<td>6.3</td>
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<td>1.8</td>
<td>[0.7, 3.2]</td>
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<td>[0.0, 10.3]</td>
<td>136</td>
<td>[126, 144]</td>
<td>5.0</td>
</tr>
</tbody>
</table>

*Trimmed Means Reported

The spectral type of 55 Cygni is B3 Ia, which indicates that it is a supergiant. This star is known to have a variable spectrum. However, it is assumed to have a high, constant level of polarization caused by the interstellar medium. It is a standard polarimetric reference star listed by Serkowski (1974) as having a degree of polarization of 2.8% and position angle 3° with respect to the line of constant right ascension measured at the star's location. From observations by Serkowski (1975), it was shown that polarized starlight is wavelength dependent, that is, the observed degree of polarization and position angle depends on the wavelength at which it is observed. He found that the basic variation of polarization with wavelength varied little from star to star; as the wavelength increased, the polarization increased until reaching a maximum value $p_{max}$ at the wavelength $\lambda_{max}$, and then decreased. While variation exists in the value of $p_{max}$ and $\lambda_{max}$ from star to star, the wavelength dependence is virtually identical when normalized values of $p/p_{max}$ are plotted against $\lambda/\lambda_{max}$. The empirical expression describing the wavelength dependence of polarization when $p/p_{max}$ is plotted against $\lambda/\lambda_{max}$ is known as Serkowski's law and is
where \( K = -0.10 + 1.86\lambda_{\text{max}} \) (Evans 1993). The values of percent polarization and position angle for HD 198478 are reported at the maximum wavelength value of 0.53\( \mu \text{m} \).

Using equation (3.42), the percent polarization at the H\( \alpha \) wavelength of 656.3 nm is computed to be 2.7%. The degree of polarization determined for this star using the technique of aperture photometry yields a value of 2.4% with a 68% confidence interval of [1.7%, 3.3%]. The position angle is 6° with a 68% confidence interval of [3°, 9°]. Both the degree of polarization and position angle agree very well with the reference values. This value was calculated using the regular mean to determine the average of the Stoke's parameters \( q \) and \( u \), that is, the mean as computed in the standard way by summing the individual observations and then dividing by the total number of observations.

Stars with asterisks in Table 3.3 had polarization measurements that were determined using the \( \gamma \)-trimmed mean to determine the normalized Stoke's parameters \( \bar{q} \) and \( \bar{u} \). The reason for this is that originally, the stars showed no sign of polarization when the arithmetic mean was used to compute \( \bar{q} \) and \( \bar{u} \). However, three of the four stars have published values of polarization which are comparable to the degree of polarization of 55 Cygni, and therefore should be detectable. Likewise, polarization of HD 199478 which has a published degree of polarization of 1.8% was detected. Therefore, it was surprising that the polarization was estimated to be zero by the Wardle-Kronberg estimator.

Upon inspection, outliers were seen in the histograms of each of these four stars. These outliers had the effect of inflating the standard deviation. When this inflated value was used in the Wardle-Kronberg estimator, its value was larger than the measured polarization, hence, by definition, the estimate of the degree of polarization is zero. Therefore, a means of handling the outliers was found in the use of the \( \gamma \)-trimmed mean which is described in the next section. Upon applying this method, the four stars whose polarizations were previously set to zero by the Wardle-Kronberg estimator, yielded degree of polarization and position angles consistent with published values.
3.4.4 Use of the $\gamma$-Trimmed Means

Stars in Table 3.3 which are marked with an asterisk indicate that measurements were obtained using the method of $\gamma$-trimmed means. In some of the measurements of $q$ and $u$ outliers are present. In the presence of these outliers, the mean is no longer a robust measure of the population mean. These outliers tend to pull the mean in their direction, giving it a value different from what it would be if they were not present. Using the $\gamma$-trimmed mean removes the influence of these outliers from the mean giving more weight to the central portion of the distribution. The method of $\gamma$-trimmed mean is an established statistical technique for robust estimations of the mean of a population when the distribution is heavily tailed or outliers are present. I shall review the method as described by Wilcox (1997) and describe how it is applied here.

Firstly, measurements of the normalized Stoke's parameters are sorted in ascending order. The $\gamma$-trimmed mean truncates the ordered distribution at the $\gamma$ and $1-\gamma$ quantiles, where $0 \leq \gamma < 0.5$. The amount of trimming is left to the individual, however, according to Wilcox, empirical investigations from actual studies indicates that a value of $\gamma$ between 0 and 0.25 is an optimal amount of trimming in terms of minimizing the standard error. For this project, a value of $\gamma = 0.2$ is consistently used. Letting $g = \gamma n$ where $g$ is rounded down to the nearest integer, the largest and smallest $g$ measurements of the ordered $q$ and $u$, are truncated. The remaining ordered values of $q$ and $u$ are then averaged as

$$
\bar{q} = \frac{q_{(g+1)} + \cdots + q_{(g-1)}}{n-2g} \quad \bar{u} = \frac{u_{(g+1)} + \cdots + u_{(g-1)}}{n-2g}
$$

(3.43)

These mean values of $q$ and $u$ normalized Stoke's parameters are used in the calculations instead of the simple arithmetic mean which include all measurements of $q$ and $u$, including the outliers. In order to calculate the estimated true degree of polarization, as well as construct confidence intervals on the polarization position angle, the standard deviation and standard deviation of the $\gamma$-trimmed mean needs to be computed. The standard deviation of the mean can not be computed by simply computing the standard deviation of the untrimmed measurements and then dividing by $\sqrt{n-2g}$. In order to calculate the standard deviation, the Winsorized sample mean is first computed. The Winsorized sample mean for $q$ is given by
where

\[ Q_i = \begin{cases} 
q_{(g+1)}, & \text{if } q_i \leq q_{(g+1)} \\
q_i, & \text{if } q_{(g+1)} < q_i < q_{(n-g)}, \\
q_{(n-g)}, & \text{if } q_i \geq q_{(n-g)}
\end{cases} \]

The Winsorized mean of \( u \) is found in like manner. The Winsorized mean pulls in the \( g \) smallest values of \( q \) and sets them equal to \( q_{(g+1)} \) and pulls the \( g \) largest values of \( q \) down and sets them equal to \( q_{(n-g)} \). From this, the Winsorized variance for \( q \) is computed from

\[ \sigma^2_{w,q} = \frac{1}{n-1} \sum (q_i - \bar{q}) \]  

and a similar expression is used for \( \sigma^2_{w,u} \) as well. The standard deviation of the trimmed mean of \( \bar{q} \) is the square root \( \sigma_{w,q} \) and finally, the standard deviation of the mean for the trimmed mean of \( \bar{q} \) is given by

\[ \sigma_{\bar{q}} = \frac{\sigma_{w,q}}{(1 - 2\gamma) \sqrt{n}} \]  

where again a similar expression for \( \sigma_{\bar{u}} \) is used. With the trimmed means of \( q \) and \( u \), as well as the standard deviation and standard deviation of the mean of \( q \) and \( u \), the analysis of the estimated degree of polarization, polarization position angle, and related confidence intervals is performed.

In order to test the reliability of this method, the stars in Table 3.3 were analyzed using both a simple arithmetic mean and the method of trimmed means just discussed. The results are shown in Table 3.4 which compares the measurements using the two methods. In Table 3.4, four stars had appreciable standard deviations larger than their measured bias polarizations. In these cases the signal-to-noise ratio was less than 1 indicating a non-detection of polarization and according to the discussion of estimators in section 3.2.1, the estimated degree of polarization is set to zero. However, all four stars have published polarizations at or above 1\%.
### Table 3.4
Comparison of Arithmetic and Trimmed Mean Methods in Determining the Polarization Characteristics of the Stars Measured in This Project

<table>
<thead>
<tr>
<th>HD</th>
<th>$p_i$ (%)</th>
<th>$\sigma$ (%)</th>
<th>$\bar{p}$ (%)</th>
<th>68% C.I.</th>
<th>$\theta$</th>
<th>68% C.I.</th>
<th>Arithmetic Mean</th>
<th>Trimmed Mean</th>
<th>68% C.I.</th>
<th>$\theta$</th>
<th>68% C.I.</th>
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<tbody>
<tr>
<td>45910</td>
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<td>0</td>
<td>[0.0, 3.2]</td>
<td>143</td>
<td>[128, 155]</td>
<td>1.8</td>
<td>1.4</td>
<td>1.1</td>
<td>[0.0, 2.9]</td>
<td>139</td>
</tr>
<tr>
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<td>7.4</td>
<td>0</td>
<td>[0.0, 9.2]</td>
<td>136</td>
<td>[128, 144]</td>
<td>6.3</td>
<td>3.4</td>
<td>5.3</td>
<td>[2.6, 8.8]</td>
<td>136</td>
</tr>
<tr>
<td>195047</td>
<td>5.6</td>
<td>3.0</td>
<td>4.7</td>
<td>[2.4, 7.9]</td>
<td>71</td>
<td>[68, 75]</td>
<td>5.5</td>
<td>2.1</td>
<td>5.1</td>
<td>[3.2, 7.1]</td>
<td>73</td>
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<td>2.8</td>
<td>[1.1, 5.0]</td>
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<td>[49, 54]</td>
<td>3.4</td>
<td>1.7</td>
<td>2.9</td>
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<td>195593</td>
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<td>74</td>
<td>[68, 98]</td>
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<td>[13, 18]</td>
<td>4.7</td>
<td>1.2</td>
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<td>6.0</td>
<td>3.1</td>
<td>[0.0, 10.3]</td>
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<td>3.0</td>
<td>5.3</td>
<td>[2.8, 8.4]</td>
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</table>

Note: HD stands for Hipparcos Data.
All four of these stars had outliers. The effect the outliers have is to inflate the standard deviation of the measured polarization. Removal of the outliers with the trimmed mean method greatly reduced the standard deviations while at the same time only slightly altering their measured degree of polarization. With the trimmed mean method however, the value of the standard deviation of the measured value of polarization dropped, (significantly in one case) below the value of the measured polarization so that the signal-to-noise ratio was now greater than 1. This allowed an estimate of the true degree of polarization as well as its confidence intervals. For two of the stars (45910 and 195593), the agreement between the estimated degree of true polarization and their published values is very good. For the star 199356 the estimated degree of polarization is only 1% lower than its published value of polarization, however, for the star 193426 it is significantly higher. For the position angles of these stars, only one star (HD 199356) shows any significant difference between its value measured using the arithmetic mean and the \( \gamma \)-trimmed mean. Using the \( \gamma \)-trimmed mean significantly alters the position angle for HD 199356. Measurements of the position angle using the trimmed mean for all four stars are in fairly good agreement with their published values listed in Table 3.4.

One thing that should also be noticed about the measurement of the estimated true degree of polarization and position angle is the extent to which measurements which were possible using the arithmetic mean are altered when using the \( \gamma \)-trimmed mean. For all but two of the eight stars which had both arithmetic and trimmed means, the estimated true degree of polarization was only slightly altered, and in only one was it altered by 0.5%. Only two stars were significantly altered by the \( \gamma \)-trimmed mean method; HD 199356 by 0.9%, and HD 226868 by 2.2%. However, both of the trimmed mean values for the estimated true degree of polarization are in good agreement with the published values of degree of polarization. For the position angles for these eight stars, three are exactly the same for each method, and the other five are altered by at most 2°.

The above observations indicate that the \( \gamma \)-trimmed mean is a useful and practical way of removing outliers from the measurements of the normalized Stoke's parameters. Its effects on measurements which were attainable using the ordinary arithmetic mean are overall small, only slightly altering the measured position angle. As for its effect on the degree of polarization, it is capable of producing results which are in fair agreement with the published values.