Prediction of Trailing Edge Noise from Two-Point Velocity Correlations

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(ABSTRACT)

This thesis presents the implementation and validation of a new methodology developed by Glegg et al. (2004) for solving the trailing edge noise problem. This method is based on the premises that the noise produced by a surface can be computed by the integral of the cross product between the velocity and vorticity fields, of the boundary layer and shed vorticity (Howe (1978)). To extract the source terms, proper orthogonal decomposition is applied to the velocity cross spectrum to extract modes of the unsteady velocity and vorticity.

The new formulation of the trailing edge noise problem by Glegg et al. (2004) is attractive because it applies to the high frequencies of interest but does not require an excessive computational effort. Also, the nature of the formulation permits the identification of the modes producing the noise and their associated velocity fluctuations as well as the regions of the boundary layer responsible for the noise production.

The source terms were obtained using the direct numerical simulation of a turbulent channel flow by Moser et al. (1998). Two-point velocity and vorticity statistics of this data set were obtained by averaging 34 instantaneous fields. For comparisons purposes, experimental boundary layer data by Adrian et al. (2000) was chosen. Statistical reduction of 50 velocity fields obtained by particle image velocimetry was performed and analysis of the two-point correlation function showed features similar to the DNS data case. Also, proper orthogonal decomposition revealed identical dominant
modes and eddy structures in the flow, therefore justifying considering the channel flow as an external boundary layer for noise calculations.

Comparison of noise predictions with experimental data from Brooks et al. (1989) showed realistic results with the largest discrepancies, on the order of 5 dB, occurring at the lowest frequencies. The DNS results are least applicable at these frequencies, since these correspond to the longest streamwise length scales, which are the most affected by the periodicity conditions used in the DNS and also are the least representative of the turbulence in an external boundary layer flow. Most of the noise was shown to be produced by low-frequency streamwise velocity modes in the bottom 10% of the boundary layer and locations closest to the wall. Only 6 modes were required to obtain noise levels within 1 dB of the total noise.

Finally, the method for predicting spatial velocity correlation from Reynolds stress data in wake flows, originally developed by Devenport et al. (1999, 2001) and Devenport and Glegg (2001), was adapted to boundary-layer type flows. This method, using Reynolds stresses and the prescription of a length scale to extrapolate the full two-point correlation, was shown to produce best results for a length scale prescribed as proportional to the turbulent macroscale \( \kappa^{-3/2} / \varepsilon \).

Noise predictions using modeled two-point statistics showed good agreement with the DNS inferred data in all but frequency magnitude, a probable consequence of the modeling of the correlation function in the streamwise direction. Other quantities associated to noise were seen to be similar to the ones obtained using the DNS.
Completing my graduate research and thesis has been a very rewarding and fulfilling experience in my life. Only with the support of many people have I been able to reach the end of this remarkable journey.

Foremost, I would like to thank my parents Jean-Claude and Christiane Spitz. They have always been there for me, supporting and encouraging me throughout this experience. Without them I would never have been able to grow in such a favorable learning environment or to afford the great education I have received. I owe you this success and thank you for your love and dedication. Also, I would like to thank my brothers Christophe and Arnaud, and my sister Marie-Aude. You have listened to me and helped me through difficult times and I want you to know how much I appreciate you being there for me.

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$a_i$</td>
<td>Fourier coefficients as given in the DNS of Moser et al. (1998)</td>
</tr>
<tr>
<td>$a_i^*$</td>
<td>Complex conjugate of $a_i$</td>
</tr>
<tr>
<td>$a_n$</td>
<td>Stochastic random variables</td>
</tr>
<tr>
<td>$b$</td>
<td>Local enthalpy</td>
</tr>
<tr>
<td>$b_i$</td>
<td>Fourier amplitudes of the vorticity field the same way $a_i$ defines the velocity field of the DNS of Moser et al. (1998)</td>
</tr>
<tr>
<td>$b_i^*$</td>
<td>Complex conjugate of $b_i$</td>
</tr>
<tr>
<td>$B$</td>
<td>Total enthalpy, $B = b + v_i^2 / 2$ or $B = B_0 - \partial \phi / \partial t$</td>
</tr>
<tr>
<td>$B_0$</td>
<td>Log-law constant, $B = 5.2$</td>
</tr>
<tr>
<td>$c_o$</td>
<td>Mean total enthalpy</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Speed of sound</td>
</tr>
<tr>
<td>$c_v$</td>
<td>Coefficient of thermal expansion at constant pressure</td>
</tr>
<tr>
<td>$c_v$</td>
<td>Coefficient of thermal expansion at constant volume</td>
</tr>
<tr>
<td>$C_{ij}$</td>
<td>Cross spectrum</td>
</tr>
<tr>
<td>$C_{ij}^{(1)}$</td>
<td>Spanwise averaged cross spectrum</td>
</tr>
<tr>
<td>$\tilde{C}_{ij}$</td>
<td>Normalized value of $C_{ij}^{(1)}$</td>
</tr>
<tr>
<td>$e_v$</td>
<td>Net viscous stress</td>
</tr>
<tr>
<td>$G$</td>
<td>Free field Green’s function</td>
</tr>
<tr>
<td>$H$</td>
<td>Shape factor, $H = \delta^* / \theta$</td>
</tr>
</tbody>
</table>
Imaginary number, \( i = \sqrt{-1} \)

Integrand of the pressure

Normalized integrand of the pressure, \( I \)

Imaginary number, \( j = \sqrt{-1} \)

Total kinetic energy, \( k \equiv 1/2 \tau_{kk} \)

Magnitude of wavenumber, \(|k|\)

Acoustic wavenumber, \( k_o = \omega/c_o \)

Wavenumber vector

Half the wetted span

Half the domain length in the streamwise direction

Half the domain length in the spanwise direction

Mach number of the flow outside the boundary layer, \( M_o = U/c_o \)

Component of \( M_o \) in the direction of the observer

Mach number of the local flow, \( M_v = V(y_2)/c_o \)

Component of \( M_v \) in the direction of the observer

Mach number associated to the wake convection speed, \( M_w = W/c_o \)

Component of \( M_w \) in the direction of the observer

Number of points on the \( i \) direction

Origin of the coordinate system used for statistics in the boundary layer/channel

Pressure

Fluctuating pressure

Pressure stress tensor, \( p_{ij} = p \delta_{ij} - \tau_{ij} \)

Source term

Normalized source term, \( \hat{Q} \)

Propagation distance from source to observer

Reynolds number, \( \text{Re}_o = U_o \delta / \nu \)
Re$_{\theta}$: Reynolds number, $Re_{\theta} = U_{c} \theta / \nu$

Re$_{\tau}$: Reynolds number based on friction velocity, $Re_{\tau} = U_{\tau} \delta / \nu$

$R_{ij}$: Velocity cross correlation function

$s$: Local entropy

$s_i$: Spanwise averaged velocity

$s_{i}^{(n)}$: Eigenvector corresponding to the $n^{th}$ mode of $s_i$

$s_o$: Mean local entropy

$s'$: Fluctuating local entropy

$S_{pp}$: Far field sound spectrum

$t$: Time of the observer

$T$: Half the averaging time

$T'$: Mean temperature

$T$: Fluctuating temperature

$T_{ij}$: Lighthill’s stress tensor, $T_{ij} = p_{ij} + \rho v_i v_j - c_s^2 \rho \delta_{ij}$

$u, u_i$: Fluctuating velocity

$u_i^*$: Complex conjugate of $u_i$

$u^{(L)}$: Fluctuating velocity minus velocity potential $u^{(L)} = u - \nabla \varphi$

$u^+$: Mean velocity normalized on the friction velocity

$u_{\tau}$: Friction velocity

$U, U_i$: Mean velocity

$U_{c}$: Mean convection velocity in the boundary layer

$U_e$: Edge mean velocity

$v, v_i$: Velocity $v = U + u$ or $v = U + \nabla \varphi + u^{(L)}$

$V$: Volume of integration

$V$: Time mean convection velocity in the boundary layer

$\hat{V}$: Normalized mean convection velocity, $\hat{V} = V / u_{\tau}$
$W$ \hspace{1cm} Wake convection speed

$W_{ij}$ \hspace{1cm} Vorticity cross correlation function

$x, x_i$ \hspace{1cm} Position vector of observer

$y, y_i$ \hspace{1cm} Position vector of source

$y_i^+$ \hspace{1cm} Distance normalized by the viscous lengthscale, $\delta_v$

**Greek**

$\alpha$ \hspace{1cm} Angle formed between $x_3$ axis and origin-observer direction

$\delta$ \hspace{1cm} Boundary layer thickness

$\delta_{ij}$ \hspace{1cm} Kronecker delta

$\delta_v$ \hspace{1cm} Viscous lengthscale

$\delta^*$ \hspace{1cm} Displacement thickness

$\delta^+$ \hspace{1cm} Boundary layer thickness in wall units

$\varepsilon$ \hspace{1cm} Rate of dissipation of turbulent kinetic energy

$\theta$ \hspace{1cm} Angle formed between $x_i$ axis and origin-observer direction

$\Theta_{ij}$ \hspace{1cm} Frequency-wavenumber cross spectrum

$\theta$ \hspace{1cm} Momentum thickness

$\kappa$ \hspace{1cm} Von Karman constant, $\kappa = 0.41$

$\lambda$ \hspace{1cm} Eigenvalue

$\lambda^{(n)}_x$ \hspace{1cm} Eigenvalue corresponding to the $n$th mode

$\tilde{\lambda}^{(n)}_x$ \hspace{1cm} Normalized value of $\lambda^{(n)}_x$

$\nu$ \hspace{1cm} Kinematic viscosity, $\nu = \mu / \rho$

$\rho$ \hspace{1cm} Density

$\rho_o$ \hspace{1cm} Mean density
\( \rho_{ij} \) Correlation coefficient function

\( \rho' \) Fluctuating density

\( \sigma \) \[ \sigma = W/V(y) \]

\( \tau \) Time of the source

\( \tau_{ij} \) Viscous stress

\( \phi_i \) Eigenfunction

\( \varphi \) Velocity potential

\( .\varphi \) Partial derivative of \( \varphi \) with respect to time

\( \Phi_{ij} \) Wavenumber cross spectrum

\( \omega \) Angular frequency

\( \omega \) Vorticity

\( \tilde{\omega} \) Normalized angular velocity, \( \tilde{\omega} = \omega \delta / U_c \)

\( \omega, \omega_i \) Vorticity vector, \( \omega = \nabla \times v \)

\( \omega^{(L)} \) Fluctuating vorticity as \( \omega^{(L)} = \nabla \times u^{(L)} \)

\( \Omega, \Omega_i \) Mean vorticity \( \Omega = \nabla \times U \)

\( \tilde{\Omega} \) Normalized mean vorticity, \( \tilde{\Omega} = \Omega \delta / u_c \)

Symbols

\( \nabla \) Gradient operator

\( \nabla. \) Divergence operator

\( \nabla \times \) Curl operator

\( \nabla^2 \) Laplacian operator

\( \times \) Vector cross product
### Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>AIAA</td>
<td>American institute of aeronautics and astronautics</td>
</tr>
<tr>
<td>AST</td>
<td>Advanced subsonic technology</td>
</tr>
<tr>
<td>CES</td>
<td>Compact eddy structure</td>
</tr>
<tr>
<td>CFD</td>
<td>Computational fluid dynamics</td>
</tr>
<tr>
<td>DNS</td>
<td>Direct numerical simulation</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier transform</td>
</tr>
<tr>
<td>ICAO</td>
<td>International civil aviation organization</td>
</tr>
<tr>
<td>LDV</td>
<td>Laser Doppler velocimetry</td>
</tr>
<tr>
<td>LES</td>
<td>Large eddy simulation</td>
</tr>
<tr>
<td>LSE</td>
<td>Linear stochastic estimation</td>
</tr>
<tr>
<td>NACA</td>
<td>National advisory committee for aeronautics</td>
</tr>
<tr>
<td>NASA</td>
<td>National aeronautics and space administration</td>
</tr>
<tr>
<td>PIV</td>
<td>Particle image velocimetry</td>
</tr>
<tr>
<td>POD</td>
<td>Proper orthogonal decomposition</td>
</tr>
<tr>
<td>RANS</td>
<td>Reynolds-averaged Navier-Stokes</td>
</tr>
<tr>
<td>r.m.s.</td>
<td>Root mean square</td>
</tr>
<tr>
<td>SPL</td>
<td>Sound pressure level</td>
</tr>
</tbody>
</table>
CHAPTER ONE

Introduction

The development of aeronautical traffic since the middle of the twentieth century has led to important problems affecting people living in the vicinity of airports. Multiple issues such as health concerns due to air pollution, the fear of air accidents or the smell of aviation fuel are dominated by the noise disturbance associated with aircraft. Health status, annoyance and stress are among the consequences usually endured by the growing neighboring community (Job (1996)). As described by Thomas et al. (2004), a growing opposition arises not only from the increase in traffic frequency but also from a change in the notion of “acceptable” disturbance.

Economic consequences can be very important. Community concerns are responsible for restricting airports expansion, like in Strasbourg, where a project of expansion to night freight had to be declined. Also, the commercial viability of supersonic flight has been seen to be jeopardized by the lack of means to understand and control noise generation.

Some objectives such as confining the noise within airport are stated in the Office of Aerospace Technology third objective. The International Civil Aviation Organization (ICAO) is defining precise objectives to reduce noise production, but, in some cases, these guidelines have been judged insufficient by the public and led to stricter operational regimes (Thomas et al. (2004)).

In response to the augmentation in air traffic and in population living near airports, the Advanced Subsonic Technology (AST) program was initiated in 1992 (NASA facts, 2003). The three areas of noise reduction concern the engine, the nacelle and the airframe. In the case of turbofan engines, studies (Owens (1979)) showed that the fan noise was
governing the engine total noise in takeoff and approach stages. Noise reduction during these operations is the most relevant since they directly affect people on the ground.

With high bypass ratio engines, the importance of broadband noise from the fan is now considered at least as important as the fan tones (Glegg and Jochault (1998)). Broadband noise can be divided into two distinct mechanisms: leading edge and trailing edge noise. Trailing edge noise can be found dominant and is produced as turbulence in the boundary layer convects past the trailing edge of a blade (Ffowcs Williams and Hall (1970)). This turbulence creates a strong near field resulting in pressure fluctuations on the blade surface but does not propagate directly to the farfield since it is convected subsonically. However, a viscous wake will be generated. The waves propagating to the farfield are then the result of the interaction of the sound generated in the boundary layer and the wake with the trailing edge (Ffowcs Williams and Hall (1970), Howe (1978), Chase (1972), Amiet (1976)). Consequently, the noise produced by a surface can be computed by the integral of the cross product between the velocity and vorticity fields, of the boundary layer and shed vorticity (Howe (1978)). In theory, the details of vorticity and velocity fluctuations are therefore sufficient for trailing edge noise predictions.

Some of the attempts that have been made to predict the trailing edge noise are presented in the following section and illustrate the difficulties associated to relating the fields of aerodynamics and acoustics.

1.1 Review of Noise Prediction Schemes

Different approaches have been implemented to determine trailing edge noise. The early developments of acoustic predictions were mostly done for the idealized case of a semi-infinite rigid surface and low Mach-numbers. These assumptions imply a small acoustic wavelength compared with the chord of the plate.

Early methods that were implemented are based on the Lighthill acoustic analogy. Ffowcs Williams and Hall (1970) implemented a model using Lighthill’s formulation for trailing edge noise predictions. The assumptions included a medium at rest and an
isentropic flow at infinite Reynolds number. The farfield mean square sound pressure is being produced by compact turbulence of finite volume near the trailing edge. The turbulent Reynolds stress is assumed to be known and a Green function is used to solve for the sound pressure. The results imply a sound generation scaling of $U^5$ where $U$ is a characteristic flow velocity. This method only requires simple information about the flow and is accordingly very approximate. Also, problems associated to the infinite Reynolds number can be found close to the trailing edge. The back reaction of the plate is never taken into account, so no vortex shedding nor Kutta conditions are prescribed for these predictions.

Many other prediction schemes based on Lighthill’s analogy were developed. Crighton and Leppington (1970), Crighton (1972), Levine (1975) worked on the early implementations of the acoustic analogy. Howe (1975) also used Lighthill (1952) theory of aerodynamic sound. He reformulated the acoustic analogy and used the stagnation enthalpy as the fundamental acoustic variable.

More recently, radiated noise has been computed using integral solution to the Lighthill solution (e.g. Manoha et al. (2000), Wang and Moin (2000)). Another example is Oberai et al. (2002) that aims at solving the Navier-Stokes equations to obtain the turbulence information prescribing the Lighthill tensor. In this case, large eddy simulation (LES) is used to obtain velocity fluctuations and thus infer the corresponding pressure fluctuations. However, formulations of this kind are impractical in the turbofan engine application. The computational power required for such complex flows at very high Reynolds number does not permit efficient noise predictions.

Another way to approach the problem of trailing edge noise is based on solutions approximated by the linearized hydroacoustic equations. Major developments were made by Chase (1972 and 1975). Contrary to methods based on the acoustic analogy described previously, the radiation is determined from the pressure fluctuations on the edge of the plate. The wavevector-frequency spectral density of hydrodynamic pressure is obtained from measurable properties of the boundary layer. Then, using an evanescent wave model, the pressure spectra are related to the nearfield and farfield sound.
Howe (1978) generalized this method and obtained an analytical solution for the case of an incident hydrodynamic field scattered by the edge of the idealized plate. This solution was tested by Brooks and Hodgson (1981) against measurements of cross-spectra data obtained very close to the trailing edge of an airfoil. Scaling using the method of Ffowcs Williams and Hall showed some success in trailing edge noise spectra predictions.

Amiet (1976) also based his sound prediction in terms of surface pressure. In this case, the formulation includes also the effects of mean flow velocity. The turbulence is assumed stationary as it moves past the trailing edge of the semi-infinite plate. The convected pressure disturbance determines the acoustic field. Essentially, the farfield spectra levels are related to the spectrum of the convected boundary layer fluctuations and their spanwise correlation lengthscale.

The advantage of this method is that it can be inverted to obtain the components of the surface pressure spectrum from farfield measurements. Consequently, the availability of a database such as the one obtained by Brooks et al. (1989) makes it convenient to obtain the farfield spectra. For example, this scheme was implemented by Glegg and Jochault (1998) after reformulation of the problem.

Another approach is to use directly the farfield noise spectra measurements for noise predictions. Using empirical correlations Brooks et al. (1989) obtained predictions quantitatively representative of the farfield noise. The parameters entering the predictions include the measured pressure spectra associated with appropriate scaling based on boundary layer parameters and curve fits.

Although these methods based on surface pressure spectra or empirical spectral information show successful results validated by experiments, they suffer from their aerodynamic simplicity. Indeed, the absence of turbulence information or the detailed mean velocity usually available from computational fluid dynamics (CFD) solutions combined with the simplicity of the boundary layer modeling prevents from any insight
about the particular features that contribute to the noise. Thus, these techniques cannot be used as a basis for manipulation of the boundary layer to identify and control that noise.

In contrast, Glegg et al. (2004) developed a method using vorticity and velocity fluctuations in the boundary layer to predict trailing edge noise. The explicit formulation of these terms permits therefore identification of noise mechanisms and thus control of the noise production. Glegg et al. (2004) aims at predicting trailing edge noise flows for non uniform flows. For trailing edge noise, where high frequencies are important, it is important to find a model that does not require an excessive computational effort. Also, it is important to understand and identify the mechanisms associated to noise production. For this purpose, Glegg proposed a trailing edge noise prediction method based on Howe’s (1978) initial formulation.

Howe (1978) reformulated Lighthill’s acoustic analogy so that the pressure field could be related to the vorticity in the boundary layer. However, the stochastic nature of turbulent boundary layers prevents the use of analytical models to define the vorticity. However, measurements of the vorticity fluctuations are very complex and available spectra models are not applicable to vorticity so that methodologies as discussed earlier are usually preferred.

In contrast, Glegg’s theory uses the vorticity but with a technique only recently applied to aeroacoustics: proper orthogonal decomposition (POD). POD identifies the motions of a turbulent that contain the most energy on average (Pope 2000). The flow is defined by modes that represents actually a superposition of appearing eddy types (Devenport et al. (2001)) and thus the main eddies in the flow. Glegg shows that the vorticity source terms and pressure fluctuations can be theoretically calculated in terms of these velocity modes and summed to the total trailing edge noise. Also, the nature of the formulation permits to identify the modes and their associated velocities as well as the layers of the boundary layer responsible for the noise production. Details of the exact formulation of this theory are explained in Chapter 2.

One problem with Glegg’s method is that the two-point velocity correlation function must be estimated so the modes can be calculated, and there are no standard techniques for doing this in an airfoil boundary layer. It is generally not possible to obtain the two-point statistics of an external turbulent boundary layer at high Reynolds number
in a practical manner. Measurements of the two-point correlation are very difficult and time consuming. Computational fluid dynamics can at best provide single point information and time resolved turbulence calculations such as LES or direct numerical simulation (DNS) are impractical for fan applications as mentioned earlier. Another approach used to model the two-point correlation function is to assume a homogeneous isotropic behavior of the turbulence. With this technique the turbulence can be described as a sum of Fourier modes. However, the assumptions made lead to obvious accuracy problems due to its sweeping nature. Moreover, the physical aspect of the turbulence is lost so that this representation cannot permit the identification the inhomogeneous features of the flow contributing to the noise.

However, in their work on plane wakes, Devenport et al. (2001) found that Reynolds stress information and lengthscale information could be sufficient for two-point velocity correlation predictions. The method successfully reproduced the two-point velocity correlation measured in an airfoil wake. These predictions, which will be reviewed in Chapter 5, successfully reproduced the space-time two-point correlation function and therefore tend to imply that Reynolds stresses contain sufficient information to describe turbulences, at least in a wake type of flow. Consequently, it is reasonable to think that this method can be applied to other turbulent flows and maybe show comparable results. The applicability of this method to turbulent boundary layers would reinforce the assumption that enough information is available from Reynolds stresses and some lengthscale to reproduce the full two-point correlation function. Also, this method would be a great opportunity for rapid noise predictions on any lifting surface.

1.2 Motivation and Purpose: Validation of Glegg et al. (2004)’s Trailing Edge Noise Model and Two-Point Turbulence Modeling

The objectives of this study are:
To validate the noise prediction method of Glegg et al. (2004) using two-point correlation data obtained from a numerical simulation of a turbulent boundary layer

To show the information available from two-point statistics and the information that can be extracted using proper orthogonal decomposition for a boundary layer

To attempt to describe the mechanisms and part of the boundary layer responsible for noise production.

To demonstrate the calculation of two-point correlations in a boundary layer flow using the method of Devenport et al. (2001), based on Reynolds stress data

To attempt noise predictions from Reynolds stress data by combining the above two methods

The following chapter reviews in details Glegg et al. (2004)’s method. The developments of Glegg et al. (2004) assume that the two-point velocity statistics can be used as a basis to trailing edge noise predictions. Mathematical descriptions of the modal implementation and noise spectra are presented and represent the basis of the present research.

Chapter 3 defines and presents the two-point correlation function. This section presents the information contained in the two-point correlation function and aims at revealing the turbulent structure and properties of wall bounded flows. For this purpose, two different data sets are used: measurements of boundary layer fluctuations by Adrian et al. (2000) using Particle Image Velocimetry (PIV) and the DNS of Moser et al. (1998).

Chapter 4 presents the application of the method of Glegg et al. (2004) described in Chapter 2. Numerical implementation and noise calculations are performed in this section. However, no databases describing the complete statistics of a turbulent boundary layer close to the trailing edge of a plate exist. Thus, to test and validate this method a direct numerical simulation (DNS) is used to extract the two-point velocity statistics and
other relevant characteristics such as the mean velocity necessary to implement Glegg’s theory. This simulation is actually a solution obtained for a fully developed two-dimensional channel flow at a Reynolds number, based on the friction velocity and channel half-height, of 590. This type of flow behaves as two boundary layers back to back and it is therefore possible to consider one half of the channel to be representative of a turbulent boundary layer with a boundary layer thickness equal to half the channel height. The complete description of the channel DNS by Moser et al. (1998) is presented in Chapter 3.

To test and validate the method, a benchmark calculation is realized using a test case obtained from Brooks et al. (1989). The success of this implementation should reveal insights about the boundary layer mechanisms inherent to noise production.

Chapter 5 is concerned with two-point statistics modeling for noise prediction. In order to cope with the absence of methods to obtain the turbulence statistics needed for these trailing edge noise calculations, the model developed by Devenport and Glegg (2001) is tested for boundary layer types of flows. The intention is in a first time to test the original model and then to adapt it to a turbulent boundary layer flow, obviously more complex, while keeping the attractiveness of its inherent simplicity. For comparison purposes, the two data sets introduced in Chapter 3 are utilized. Finally, the modeled correlation function is tested for noise predictions using Glegg et al. (2004)’s method. The DNS two-point correlation function reproduced by the model, using single-point information is obtained and utilized for noise predictions. Comparisons using the test case obtained from Brooks et al. (1989) are performed as in Chapter 4.

Chapter 6 concludes this thesis. This section summarizes findings and conclusions from all the previous chapters.

Finally, appendices can be found at the end. Appendix A describes general definitions of the Fourier transform, cross spectrum and correlation functions. Appendix B describes the discrete relationship between the cross spectrum and the correlation function.

The recent developments of Devenport et al. (1999) and Glegg and Devenport (2001) in the application of proper orthogonal decomposition to aeroacoustics form the inspiration of Glegg et al.’s (2004) method for trailing edge noise prediction, which is based on Lighthill’s analogy and the initial work of Howe (1978). Specifically, the turbulence is prescribed as a set of statistically independent velocity modes. These modes are prescribed using proper orthogonal decomposition and this is particularly important for the trailing edge noise problem because only a few modes are necessary to represent the whole flow. Consequently, this method is suitable for the development of a method for trailing edge noise predictions, not limited by an unacceptably large computational effort. Also, this method permits in theory to relate the noise production to aerodynamic quantities such as velocity and vorticity. Noise sources are therefore identifiable and can provide better understanding of the noise mechanisms.
CHAPTER TWO

This section illustrates in details the method developed by Glegg et al. (2004), starting from the early development of Ffowcs Williams and Hall (1970), and also the later development by Howe (1978), all based on the acoustic analogy by Lighthill (1952).

2.1 Development of Trailing Edge Theory by Ffowcs Williams and Hall (1970)

The starting point of the trailing edge noise theory by Glegg et al. (2004) comes from the original development of Ffowcs Williams and Hall (1970). Their analysis was based on Lighthill’s (1952) acoustic analogy for sound generation by flows. Lighthill’s acoustic analogy is derived from the mass conservation and momentum equations, and is outlined here.

Conservation of mass relates the change in mass per unit volume to the inflow of mass represented by the mass-flux vector, \( \rho \mathbf{v} \) according to

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho \mathbf{v}_i) = 0
\]  

(2.1)

where \( \rho \) is the density, \( t \) the time, \( v_i \) and \( x_i \) the velocity and position components of respectively the velocity and position vectors \( \mathbf{v} \) and \( \mathbf{x} \) in the \( i \) directions. Lighthill (1952) proposed to take the derivative of this expression with respect to time leading to

\[
\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial t} (\rho v_i) \right) = 0
\]  

(2.2)

Conservation of momentum depends on the rate of change of momentum due to inflow of momentum and also the rate of change of momentum due to the body and surface forces acting on the fluid. So, considering the forces to be represented by the stress field \( p_{ij} \), the conservation of momentum can be expressed as
\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (p_{ij} + \rho v_i v_j) = 0 \]  

(2.3)

where \( p_{ij} = \rho \delta_{ij} - \tau_{ij} \) and \( \tau_{ij} \) is the viscous stress. The prime symbol represents here the perturbation associated to the quantity. Lighthill suggested taking the spatial differentiation of the momentum equation:

\[ \frac{\partial^2}{\partial t \partial x_i} (\rho v_i) + \frac{\partial^2}{\partial x_j \partial x_i} (p_{ij} + \rho v_i v_j) = 0 \]  

(2.4)

Subtracting equation (2.4) from equation (2.2) resulted then in

\[ \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho'}{\partial t^2} = \frac{\partial^2}{\partial x_j \partial x_i} (p_{ij} + \rho v_i v_j) \]  

(2.5)

So the acoustic variable was now defined as the perturbation density \( \rho' \). Subtracting from both sides

\[ c_o^2 \nabla^2 \rho' = c_o^2 \frac{\partial^2 \rho'}{\partial x_i \partial x_j} \]  

(2.6)

where \( c_o \) is the speed of sound and \( \delta_{ij} \) the Kronecker delta, the wave equation became

\[ \frac{\partial^2 \rho'}{\partial t^2} - c_o^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2}{\partial x_j \partial x_i} (p_{ij} + \rho v_i v_j - c_o^2 \rho \delta_{ij}) \]  

(2.7)

The source terms were given by the right hand side of this equation and the double divergence structure of these terms revealed a quadrupole source distribution. Then, introducing the definition of Lighthill’s stress tensor, \( T_{ij} = p_{ij} + \rho v_i v_j - c_o^2 \rho \delta_{ij} \), the wave equation became finally

\[ \frac{\partial^2 \rho'}{\partial t^2} - c_o^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_j \partial x_i} \]  

(2.8)

Using the free field Green’s function, \( G \), the solution to equation (2.8) in an unbounded medium can be expressed as
\[ \rho'(x,t)c_0^2 = \int_{-\tau}^{T} \int_{V} T_y(y,\tau) \frac{\partial^2}{\partial x_i \partial x_j} G(x,t|y,\tau) dV d\tau \]  

(2.9)

where \( y \) and \( x \) are respectively the position vectors of the source and of the observer, \( t \) and \( \tau \) the time respectively associated to the observer and the source. The integration is to range over all \( y \) in the volume \( V \) and \( 2T \) represents here the averaging time.

Ffowcs William and Hall (1970) applied Lighthill’s analogy to the case of a turbulent flow on an infinite rigid half-plane and used the solution shown in equation (2.9). The problem with Lighthill’s acoustic analogy is that the left hand side is given for a medium at rest. However, in the source region, sound waves can be refracted because of flow and temperature gradients which can obviously not be assimilated as sound generation mechanisms.

### 2.2 Development of Trailing Edge Theory by Howe (1978)

Responding to the problems expressed in the previous section concerning the use of Lighthill’s acoustic analogy, reformulations for the propagation of sound through a non-uniform medium were proposed by Howe (1978) and Goldstein (1978). The acoustic variable chosen by Howe to reformulate the acoustic analogy was the stagnation enthalpy, \( B \). The acoustic analogy was again based on the continuity and momentum equations.

The continuity equation was first modified from its original form described in equation (2.1). Glegg (2005) showed the main steps to obtain the general form of Goldstein’s continuity equation. Taking the time derivative of equation (2.1) led to

\[ \frac{\partial^2 \rho}{\partial t^2} + \nabla \left( \frac{\partial \rho}{\partial t} \mathbf{v} \right) + \nabla \left( \rho \frac{\partial \mathbf{v}}{\partial t} \right) = 0 \]

\[ \Leftrightarrow \frac{\partial^2 \rho}{\partial t^2} + \mathbf{v} \cdot \nabla \left( \frac{\partial \rho}{\partial t} \right) + \frac{\partial \rho}{\partial t} \nabla \mathbf{v} + \nabla \left( \rho \frac{\partial \mathbf{v}}{\partial t} \right) = 0 \]  

(2.10)

\[ \Leftrightarrow \frac{D}{Dt} \left( \frac{\partial \rho}{\partial t} \right) + \frac{\partial \rho}{\partial t} \nabla \mathbf{v} + \nabla \left( \rho \frac{\partial \mathbf{v}}{\partial t} \right) = 0 \]
The continuity equation, equation (2.1), can also be written as

\[ \frac{D \rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \quad (2.11) \]

So, using equation (2.11), equation (2.10) can be rewritten as

\[ \frac{D}{Dt} \left( \frac{\partial \rho}{\partial t} \right) - \frac{1}{\rho} \frac{\partial \rho}{\partial t} \frac{D \rho}{Dt} + \nabla \left( \rho \frac{\partial \mathbf{v}}{\partial t} \right) = 0 \]

\[ \Leftrightarrow \rho \left[ \frac{1}{\rho} \frac{D}{Dt} \left( \frac{\partial \rho}{\partial t} \right) + \frac{\partial \rho}{\partial t} \frac{D}{Dt} \left( \frac{1}{\rho} \right) \right] + \nabla \left( \rho \frac{\partial \mathbf{v}}{\partial t} \right) = 0 \quad (2.12) \]

and combining the terms in the bracket led to the general form of Goldstein’s continuity equation:

\[ \frac{D}{Dt} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right) + \frac{1}{\rho} \nabla \left( \rho \frac{\partial \mathbf{v}}{\partial t} \right) = 0 \quad (2.13) \]

The enthalpy variable was introduced from basic thermodynamic relationships. The stagnation enthalpy can be defined as:

\[ B = b + \frac{v_i^2}{2} \quad (2.14) \]

where \( b \) is the local enthalpy.

And also,

\[ db = T ds + \frac{dp}{\rho} \quad (2.15) \]

where \( T \) is the temperature and \( s \) the local entropy.

According to Glegg (2005), the total derivative of the stagnation enthalpy becomes then
CHAPTER TWO

\[
\begin{align*}
\frac{DB}{Dt} &= \frac{Db}{Dt} + \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} \\
\frac{DB}{Dt} &= T \frac{Ds}{Dt} + \frac{1}{\rho} \frac{Dp}{Dt} - \mathbf{v} \cdot \nabla p + \mathbf{v} \cdot \mathbf{e}_v \\
\Rightarrow \quad \frac{1}{\rho} \frac{\partial p}{\partial t} &= \frac{DB}{Dt} - T \frac{Ds}{Dt} - \mathbf{v} \cdot \mathbf{e}_v
\end{align*}
\]

(2.16)

where \( \mathbf{e}_v \) is the net viscous stress.

The entropy variable, \( s \), can be defined from the pressure and the density as

\[
ds = c_v \frac{dp}{\rho} - c_p \frac{d\rho}{\rho}
\]

(2.17)

where \( c_v \) and \( c_p \) correspond respectively to the coefficient of thermal expansion at constant volume and at constant pressure.

Note that taking the time derivative of equation (2.17)

\[
\frac{\partial s}{\partial t} = c_v \frac{\partial p}{\rho \partial t} - c_p \frac{\partial \rho}{\rho \partial t}
\]

(2.18)

\[
\Leftrightarrow \quad \frac{1}{\rho} \frac{\partial p}{\partial t} = \frac{c_v}{pc_p} \frac{\partial p}{\partial t} - \frac{1}{c_p} \frac{\partial s}{\partial t}
\]

and from the ideal gas law, Glegg (2005) showed

\[
\frac{1}{\rho} \frac{\partial \rho}{\partial t} = \frac{1}{pc_p} \frac{\partial p}{\partial t} - \frac{1}{c_p} \frac{\partial s}{\partial t}
\]

(2.19)

Replacing equation (2.16) in (2.19), Glegg obtained

\[
\frac{1}{\rho} \frac{\partial \rho}{\partial t} = \frac{1}{c_p^2} \left[ \frac{DB}{Dt} - T \frac{Ds}{Dt} - \mathbf{v} \cdot \mathbf{e}_v \right] - \frac{1}{c_p} \frac{\partial s}{\partial t}
\]

(2.20)

So, using equation (2.20), the continuity equation as presented in equation (2.13), became
\( \frac{D}{Dt}\left( \frac{1}{c_o^2} \left[ \frac{DB}{Dt} - T \frac{Ds}{Dt} - \mathbf{v} \cdot \mathbf{e}_v \right] - \frac{1}{c_p} \frac{\partial s}{\partial t} \right) + \frac{1}{\rho} \nabla \left( \rho \frac{\partial \mathbf{v}}{\partial t} \right) = 0 \) \hspace{1cm} (2.21)

The last equation needed to obtain Howe’s formulation of the acoustic analogy is the momentum equation. The momentum equation can be expressed in Crocco’s form as

\[ \frac{\partial \mathbf{v}}{\partial t} = -\nabla \mathbf{B} - \mathbf{\omega} \times \mathbf{v} + \mathbf{e}_v + T \nabla s \] \hspace{1cm} (2.22)

where \( \mathbf{\omega} \) is the vorticity vector.

Substituting equation (2.22) into equation (2.21) for \( \frac{\partial \mathbf{v}}{\partial t} \) leads to

\[ \frac{D}{Dt}\left( \frac{1}{c_o^2} \left[ \frac{DB}{Dt} - T \frac{Ds}{Dt} - \mathbf{v} \cdot \mathbf{e}_v \right] - \frac{1}{c_p} \frac{\partial s}{\partial t} \right) + \frac{1}{\rho} \nabla \left[ \rho (-\nabla \mathbf{B} - \mathbf{\omega} \times \mathbf{v} + \mathbf{e}_v + T \nabla s) \right] = 0 \] \hspace{1cm} (2.23)

Rearranging the terms, the final equation takes the form presented by Glegg (2005), i.e.

\[ \frac{D}{Dt}\left( \frac{1}{c_o^2} \left[ \frac{DB}{Dt} - T \frac{Ds}{Dt} - \mathbf{v} \cdot \mathbf{e}_v \right] - \frac{1}{c_p} \frac{\partial s}{\partial t} \right) + \frac{1}{\rho} \nabla \left( \rho \mathbf{\omega} \times \mathbf{v} - \rho T \nabla s - \rho \mathbf{e}_v \right) + \frac{D}{Dt}\left( \frac{1}{c_p} \frac{\partial s}{\partial t} + \frac{1}{c_o^2} \left[ T \frac{Ds}{Dt} + \mathbf{v} \cdot \mathbf{e}_v \right] \right) \] \hspace{1cm} (2.24)

In this expression, quadrupole terms are not immediately apparent. However, Glegg et al. (2004) showed that using the vector expansion

\[ (\rho \mathbf{\omega} \times \mathbf{v})_i = \rho v_j \frac{\partial v_i}{\partial x_j} - \rho \frac{\partial (v^2)}{\partial x_i} \] \hspace{1cm} (2.25)

and adding \( v_i \) times the continuity equation gave

\[ (\rho \mathbf{\omega} \times \mathbf{v})_i = \frac{\partial}{\partial x_j} (\rho v_j) + v_i \frac{\partial \rho}{\partial t} - \rho \frac{\partial (v^2)}{\partial x_i} \] \hspace{1cm} (2.26)

revealing terms similar to the ones contained in Lighthill’s equation.
Glegg proposed to linearize equation (2.24) for small perturbation about a mean time invariant mean flow with velocity $U$. Associating the subscript $o$ to mean variables, a prime to the perturbations, the variables from equation (2.24) were defined as:

$$B = B_o - \frac{\partial \phi}{\partial t} \quad \mathbf{v} = U + \nabla \phi + u^{(L)} \quad \mathbf{\Omega} = \nabla \times U \quad \mathbf{\omega} = \nabla \times \mathbf{v}$$

(2.27)

where $\phi$ is a velocity potential, $U$ the mean velocity, $u^{(L)}$ the velocity perturbation without the velocity potential and $\mathbf{\Omega}$ the mean vorticity.

For an inviscid, isentropic flow,

$$\frac{Ds}{Dt} = 0 \quad \frac{DB_o}{Dt} = 0 \quad \mathbf{e} = 0$$

(2.28)

Thus, setting these terms to zero reduced equation (2.24) to,

$$\frac{D}{Dt} \left( \frac{1}{c_o^2} \frac{DB}{Dt} \right) - \frac{1}{\rho} \nabla (\rho \nabla B) = \frac{1}{\rho} \nabla (\rho \mathbf{\omega} \times \mathbf{v} - \rho T \nabla s)$$

(2.29)

Howe’s wave equation could then be separated into 2 equations according to Glegg et al. (2004): one giving the mean flow and the second one giving the first order perturbation about the mean flow.

Expanding the terms of equation (2.29) gives

$$\frac{D}{Dt} \left( \frac{1}{c_o^2} \frac{D(B_o - \phi)}{Dt} \right) - \frac{1}{\rho} \nabla (\rho \nabla (B_o - \phi)) = \frac{1}{\rho} \nabla (\rho \nabla \times (U + \nabla \phi + u^{(L)}) \times (U + \nabla \phi + u^{(L)}) - \rho T \nabla s)$$

(2.30)

Noting that

$$T = T_o + T' \quad \nabla s = \nabla s_o + \nabla s' \quad \rho = \rho_o$$

(2.31)

The previous expression can be rewritten as
Neglecting second order term, setting to zero derivatives of mean values with respect to time and also cross products between two identical vectors, this equation reduces to:

\[
\frac{D_o}{Dt} \left( \frac{1}{c_o^2} \frac{D(B_o)}{Dt} \right) + \frac{D_o}{Dt} \left( \frac{1}{c_o^2} \frac{D(-\varphi)}{Dt} \right) - \frac{1}{\rho_o} \nabla(\rho_o \nabla(-\varphi)) - \frac{1}{\rho_o} \nabla(\rho_o \nabla(B_o)) = \frac{1}{\rho_o} \nabla \left[ \left( \rho_o \nabla \times U + \rho_o \nabla \times \varphi U + \rho_o \nabla \times U \times u^{(L)} \right) \right] + \left( \rho_o \nabla \times \nabla \varphi \times U + \rho_o \nabla \times \varphi \times \varphi \times U + \rho_o \nabla \times \varphi \times U \times u^{(L)} \right) + \left( \rho_o \nabla \times u^{(L)} \times U + \rho_o \nabla \times u^{(L)} \times \varphi + \rho_o \nabla \times u^{(L)} \times u^{(L)} \right)
- \left( \rho_o \Sigma \nabla s_o + \rho_o \Sigma \nabla s^\prime_o + \rho_o \Sigma \nabla s_o \right) \right]
\]

(2.32)

where \( \omega^{(L)} = \nabla \times u^{(L)} \)

Taking a time average of this equation eliminates the fluctuating term, and leads to the mean flow equation presented by Glegg et al. (2004):

\[
\nabla(\rho_o \nabla(B_o)) + \nabla \left[ \rho_o \Omega \times U - \rho_o \Sigma \nabla s_o \right] = 0
\]

(2.34)

Now, subtracting the mean flow equation from equation (2.33), and linearizing Crocco’s equation to define the velocity perturbations as

\[
\frac{\partial u^{(L)}}{\partial t} = -\left( \omega^{(L)} \times U \right) - \left( \Omega \times u^{(L)} \right) + T \nabla s_o + T \nabla s^\prime_o
\]

(2.35)

equation (2.33) reduces to the first order perturbation about the mean flow described by Glegg:
\[
\frac{D_o}{Dt} \left( \frac{1}{c_o^2} \frac{D_o(\phi)}{Dt} \right) - \frac{1}{\rho_o} \nabla(\rho_o \nabla(\phi)) + \frac{1}{\rho_o} \nabla(\rho_o \Omega \times \nabla \phi) = \frac{1}{\rho_o} \nabla \rho_o \frac{\partial \hat{u}^{(L)}}{\partial t} \tag{2.36}
\]

Note that the pressure perturbation appears implicitly in this equation as
\[
\frac{\partial p}{\partial t} = -\rho_o \frac{D_o \phi}{Dt},
\]
as demonstrated by Glegg et al. (2004).

### 2.3 Application to the Trailing Edge Noise Problem: Source Term Modeling

Assuming the velocity fluctuations \( u_i^{(L)} \) to be a statistically stationary process, Glegg defined the space time cross correlation function as

\[
R_{ij}(y, y', \tau) \equiv Ex\left[ u_i^{(L)}(y, t)u_j^{(L)}(y', t') \right] \tag{2.43}
\]

Similarly, Glegg defined the cross spectrum from the Fourier transform of the cross correlation function with respect to time as

\[
C_{ij}(y, y', \omega) = \frac{\pi}{T} Ex\left[ u_i^{(L)}(y, \omega)u_j^{(L)^*}(y', \omega) \right] \tag{2.44}
\]

where \( T \) is half the averaging time used in the statistical estimate and \( Ex[f] \) is the expected value of the statistics.

Using proper orthogonal decomposition, Glegg expressed the aforementioned averaged velocity, \( s_i(y_2, \omega) \), as a sequence of modes, solutions to the eigenvalue problem

\[
\frac{1}{R} \int_0^R C_{ij}^{(L)}(y_2, y_2', \omega)s_j^{(a)}(y_2', \omega)dy_2' = \frac{\pi}{T} \lambda^{(a)} s_i^{(a)}(y_2, \omega) \tag{2.45}
\]

with
where L is half the wetted span, $\lambda_i^{(n)}$ is the eigenvalue corresponding to the n$^{th}$ mode and $s_i^{(n)}$ the corresponding eigenvector representing the spanwise averaged velocity fluctuations.

The proper orthogonal decomposition problem, equation (2.45), was simplified by Glegg using the assumption of homogeneity of the flow in the spanwise direction for the formulation of the velocity cross spectrum.

From Glegg et al. (2004) the spanwise averaged velocity fluctuations were then expanded in the series

$$s_i(y_2, \omega) = \sum_{n=1}^{\infty} a_n(\omega) s_i^{(n)}(y_2, \omega)$$

(2.47)

where the coefficients $a_n$ are a set of stochastic random variables with zero mean and whose variance is equal to $\lambda_n^{(n)}$.

Accordingly, the acoustic pressure, equation (2.36), became

$$p(x, \omega) = \sum_{n=1}^{\infty} a_n(\omega) p^{(n)}(x, \omega)$$

(2.48)

where the pressure associated to each mode was given by

$$p^{(n)}(x, \omega) = \left( \rho_o \sin(\theta/2) e^{i\omega(t-y_1)/c_e} \right) \left[ \frac{\sqrt{M_y (1-\sigma)} Q^{(n)}(k, y_2)}{\sqrt{2(1-M_{sr})(1-M_y)^{1/2}}} \right] dy_2$$

(2.49)

and the source term by

$$Q^{(n)}(k, y_2) = \frac{1}{2\pi} \left[ \Omega_3 (i |\omega| s_1^{(n)} + \omega s_2^{(n)}) + i |\omega| \left( i\omega s_2^{(n)} - V(y_2) \frac{\partial s_1^{(n)}}{\partial y_2} \right) \right]_{y_1=d}$$

(2.50)
Finally, the far field spectrum could be obtained as a sum over all the modes, that is

\[
S_{pp}(x, \omega) = \frac{\pi}{T} \text{Ex}\left[p(x, \omega)\right]^2 = \frac{\pi}{T} \sum_{n=1}^{\infty} \lambda_{\omega}^{(n)} |p^{(n)}(x, \omega)|^2
\]  

(2.51)

The method by Glegg et al. (2004) shows two interesting features. First, the noise generation can be identified with specific modes from the use of proper orthogonal decomposition. Second, the pressure corresponds to an integral normal to the surface (cf. equation (2.49)). Thus, this expression permits not only to predict trailing edge noise but also to identify the regions of the flow that are responsible for the noise production.
Two Point Turbulence Data

The stochastic random nature of turbulent boundary layers requires approaches beyond simple analytical treatments. It is therefore usual to employ statistical analysis. Turbulent flows can be specified in terms of their time-averaged statistics. Some common forms of these statistics are the time-averaged turbulent stresses, the cross-correlation function and the cross-spectrum. The cross-correlation function applied to velocity fluctuations is defined in this chapter and general definitions of the cross-correlation and cross-spectrum functions are given in Appendix A.

In addition, two data sets representative of turbulent boundary layer types of flows are presented in this section. These data sets are chosen to illustrate the information inherent to two-point statistics. Also, these data serve as test cases for the application of the two-point correlation prediction method based on the Reynolds stresses developed in Chapter 5 and to validate the noise prediction method of Glegg et al. (2004) presented in Chapter 4. First, the channel flow DNS solution of Moser et al. (1998) was selected based on the completeness of its data set and initial availability of some explicit correlation results. Then, an experimental data set of a turbulent boundary layer was chosen for
comparison with a real flow. This data set from Adrian et al. (2000) is based on PIV measurements of a flat plate zero pressure gradient boundary layer. The data from Adrian et al. (2000), representing an external boundary layer, are presented to show that the DNS of Moser et al. (1998) can be used for trailing edge noise calculations since the characteristics of both flows are very similar. Main differences are only expected in the outer part of the boundary layer and for wavenumbers that imply scales of the order of the boundary layer thickness or larger.

Also, a less conventional way to specify turbulent flows known as compact eddy structures (CES) is described in this chapter. This method, based on proper orthogonal decomposition (POD) and on linear stochastic estimation (LSE), is presented in section 3.4. These techniques, POD and LSE, use two-point velocity correlations to describe the turbulence as a sum of modes and are the basis of Glegg’s trailing edge noise prediction scheme.

3.1 Two-Point Correlation Tensor

The space-time velocity correlation tensor $R_{ij}$, can be defined as

$$R_{ij}(y, y', \tau) \equiv \text{Ex}[u_i(y, t)u_j(y', t')]$$

see Appendix A, where the $i$ and $j$ subscripts can take the value 1, 2 or 3 corresponding respectively to the streamwise, normal to the wall and spanwise directions. $\text{Ex}[\ ]$ correspond to the expected value or ensemble average, $y$ and $y'$, to the positions of the two points with respect to the origin of the coordinate system, and $t$ and $t'$ the time corresponding to the two velocity fluctuations with $\tau = t' - t$. The velocity fluctuations, $u_i$, are assumed to be a statistically stationary process. Note that the correlation function 3.1 has the symmetry property

$$R_{ij}(y, y', \tau) = R_{ji}(y', y, -\tau)$$

(3.1a)
Figure 3.1 shows the coordinate system for the correlation function in a boundary layer. The correlation function is originally a 7 dimensional function, that is

\[ R_y(y_1, y_2, y_3, y_1', y_2', y_3', \tau) = Ex\left[u_i(y_1, y_2, y_3, t)u_j(y_1', y_2', y_3', t')\right] \]  
\[(3.1b)\]

If the streamwise and spanwise directions can be assumed homogeneous in the boundary layer, the absolute streamwise and spanwise locations are not important. Only the separation between the two points is sufficient to characterize these directions. Also, if the time correlation can be estimated from the spatial correlation (such as using Taylor’s hypothesis) then the time dependence can be dropped. From these assumptions, the correlation tensor reduces to a four dimensional function:

\[ R_y(\Delta y_1, y_2, y_2', \Delta y_3) = Ex\left[u_i(0, y_2, 0)u_j(\Delta y_1, y_2', \Delta y_3)\right] \]  
\[(3.1c)\]

The correlation has 9 components corresponding to the different combinations of \(i\) and \(j\). However, for a two-dimensional flow, the cross correlation components, \(R_{13}\) and \(R_{23}\) have to be zero for \(\Delta y_3 = 0\). Indeed, the symmetry in the correlation function requires \(R_{13}(\Delta y_3) = -R_{13}(-\Delta y_3)\), so that for \(\Delta y_3 = 0\) this cross correlation term has to vanish and similarly for \(R_{23}\). So usually, only the normal components and the cross term \(R_{12}\) are of most interest for zero streamwise and spanwise separation. The other cross term, \(R_{21}\), is in theory obtainable from \(R_{12}\) since from equation (3.1a), \(Ex[u_i(y)u_2(y')] = Ex[u_2(y')u_i(y)]\).

Another element related to the two point correlation function is the Reynolds stress tensor. This function corresponds to the two-point velocity correlation function for zero spanwise and streamwise separation, that is

\[ \tau_{ij}(y_2) \equiv Ex\left[u_i(y_2)u_j(y_2)\right] \]  
\[(3.2)\]

The definition of the two-point velocity correlation tensor can also be extended to a two-point vorticity correlation tensor.
\[ W_g(\Delta y_1, y_2, y'_2, \Delta y_3) = \text{Ex} \left[ \omega_i(0, y_2, 0) \omega_j(\Delta y_1, y'_2, \Delta y_3) \right] \]  
(3.3)

where \( \omega \) corresponds to the vorticity fluctuations.

In addition, other forms of the correlation function are useful. The cross-spectrum function can be obtained using Fourier transform of the correlation function. The Fourier transform can be applied on the time and spatial dimensions. As described in Appendix A, performing a Fourier transform on the time dimension, the time cross spectrum can be defined as

\[ C_g(y, y', \omega) = \frac{\pi}{T} \text{Ex} \left[ u^*_i(y, \omega) u_j(y', \omega) \right] \]  
(3.4)

where \( T \) corresponds to half the averaging time.

In homogeneous directions, the two-point correlation function is independent of position. The information contained in \( R_g \) can then be re-expressed in terms of the wavenumber spectrum in the homogeneous directions. So, following Appendix A, the wavenumber cross-spectrum can be defined as

\[ \Phi_g(k_1, y_2, y'_2, k_3) = \frac{1}{(2\pi)^2} \int_{-L_2}^{L_2} \int_{-L_3}^{L_3} R_g(\Delta y_1, y_2, y'_2, \Delta y_3) \exp \left( -j(k_1 \Delta y_1 + k_3 \Delta y_3) \right) d\Delta y_3 d\Delta y_1 \]

\[ = \frac{\pi}{L_1 L_3} \text{Ex} \left[ u^*_i(k_1, y_2, k_3) u_j(k_1, y'_2, k_3) \right] \]  
(3.5a)

where \( L_2 \) and \( L_3 \) correspond to half the domain lengths in the streamwise and spanwise directions. Similarly, the two-point correlation function can be obtained from the inverse Fourier transform of the wavenumber cross spectrum:

\[ R_g(\Delta y_1, y_2, y'_2, \Delta y_3) = \int \int \Phi_g(k_1, y_2, y'_2, k_3) \exp \left( j(k_1 \Delta y_1 + k_3 \Delta y_3) \right) dk_3 dk_1 \]  
(3.5b)

Both two-point velocity and vorticity correlation functions are presented in the following sections. The vorticity correlations are only presented for the channel flow DNS though. Indeed, it is very difficult to obtain instantaneous vorticity measurements experimentally.
3.2 Numerical Simulation of a Turbulent Channel Flow at $Re_\tau=590$

Moser et al. (1998) made direct numerical simulations of a turbulent channel flow up to $Re_\tau = 590$. The simulations were performed using a variant of the DNS channel code previously developed by Kim et al. (1987) for low Reynolds number.

Figure 3.2 shows a sketch of the channel and the coordinate system used for the simulation. As described by Moser et al. (1998) a Chebychev-tau formulation in the normal to wall direction and a Fourier representation in the spanwise and streamwise directions were used for this simulation. Also, a third-order Runge-Kutta discretization was used for the non-linear terms. Periodic boundary conditions were applied in the streamwise and spanwise directions. The periodic domain was chosen so that the two-point correlation in these streamwise and spanwise directions would be essentially zero at half the domain size separation.

For the purpose of this thesis, the simulation at a Reynolds number of $Re_\tau = 590$ is used. This Reynolds number is defined by the following relation:

$$Re_\tau = u_\tau \delta / \nu$$

(3.6)

where $u_\tau$ is the friction velocity, $\delta$ the channel half-width and $\nu$ the viscosity. This set of data was preferred to the sets of lower Reynolds numbers in order to avoid low-Reynolds number effects such as a short log layer or a large log law intercept. Also, only high Reynolds numbers are of interest for application to trailing edge noise.

The computation was conducted with 37,896,192 grid points ($384 \times 257 \times 384$, in $y_1$, $y_2$, $y_3$) for an actual Reynolds number of $Re_\tau=587.19$. The streamwise and spanwise computational periods were set respectively to $2\pi \delta$ and $\pi \delta$. For this domain, the grid spacing in the streamwise and spanwise directions were $\Delta y_1^+ = 9.7$ and $\Delta y_3^+ = 4.8$ in wall...
UNIT and the vertical resolution in the center of the channel was chosen as $\Delta y_{z}^* = 7.2$
where the distance measured in viscous lengths or wall units is defined as

$$y_i^+ = \frac{y_i}{\nu} = \frac{u_i y_i}{\nu}$$  

where $\delta_v$ is the viscous lengthscale.

Turbulent boundary layers and turbulent channel flow have particular mean velocity profiles where different mechanisms dominate. First, close to the wall, there is a very thin layer in which the mean velocity profile is determined by the viscous scales: the viscous sublayer. The viscous sublayer ranges from the wall to $y_2^* = 5$. Figure 3.3 shows the mean velocity profile close to the wall. For positions close to the wall, up to $y_2^* = 5$ or $y_2 = 0.008 / \delta$, the data from the DNS respect the law of the wall:

$$u^+ = y_2^*$$  

where $u^+$ is the mean velocity normalized on the friction velocity.

The next layer corresponds to the buffer layer, a region between the viscous sublayer and the log-law region. This is a transition region between the viscosity dominated and the turbulence dominated parts of the flow, that is for locations $5 < y_2^* < 30$ or also in this case $0.008 < y_2 / \delta < 0.05$.

For positions $30 < y_2^* < 150-180$ or $0.05 < y_2 / \delta < 0.25-0.30$ the influence of viscosity on the mean flow is negligible compared to the influence of turbulence. As shown in Figure 3.4, the mean velocity follows the logarithmic law of the wall due to von Karman (1930) for these positions, i.e.

$$u^+ = \frac{1}{\kappa} \ln y_2^* + B$$  

(3.9)
where $\kappa$ is the von Karman constant and $B$ is a constant. In the semi-log plot, Figure 3.4, these constant were chosen as $\kappa = 0.41$ and $B = 5.2$ and the resulting line compares well with the DNS data in the given range.

For locations higher than $y_2 / \delta \approx 0.30$ the data deviates slightly from the log law. This region is defined as the core.

### 3.2.1 Data Analysis and Reduction

The statistics extracted from the simulation are available at the URL http://www.tam.uiuc.edu/Faculty/Moser/channel. These data include the two-point correlations in the streamwise and spanwise directions for selected positions and also 34 complete instantaneous velocity fields. The Reynolds stresses are also available and were used for the two-point modeling described in Chapter 5.

In order to obtain a complete representation of the two-point correlation of the flow, data for the thirty four time instants (each 0.6 gigabytes) were processed. The data in these files were in terms of coefficients, $a_{i,j}$, of the Fourier decomposition of the velocity in the $y_1$ and $y_3$ directions, at each of the Chebychev Gauss-Lobatto collocation points in $y_2$. The collocation points were located at

$$\frac{y_{2i}}{\delta} = \cos\left(\pi \frac{i_z - 1}{n_z - 1}\right), \quad i_z = 1,...,n_z$$

(3.10)

where $i_z$ is the index corresponding to the $y_2$ locations and $n_z$ the number of points in the wall-normal direction (257). At each $y_2$ location, the three velocity components, $u_i$, are related to their coefficients by a truncated Fourier expression:

$$\frac{u_i(y_1, y_{2i}, y_3)}{u_r} = \sum_{k_1} \sum_{k_3} a_{i}(k_1, y_{2i}, k_3) \exp(jk_1 y_1 + jk_3 y_3)$$

(3.11)

where $k_1$ and $k_3$ correspond to the wavenumbers in the streamwise and spanwise directions and $j$ is the square root of -1. Here, the wavenumbers have the values
where $i_i$ and $i_3$ are the indices corresponding to the $y_i$ and $y_3$ locations, and $n_i$ and $n_3$ the number of points in these directions. The channel half height is represented by $\delta$ and the Fourier coefficients given by the DNS by $a_i$. Note that only the positive streamwise wavenumbers were given by Moser et al. (1998). Since $u_i$ is real, the complex number, $a_i$, must satisfy

$$a_i(-k_1, y_{2i}, -k_3) = a_i(k_1, y_{2i}, k_3)$$

and therefore the Fourier coefficients for negative $k_1$ are redundant.

The raw Fourier coefficient data from the instantaneous velocity fields were first used to compute the velocity wavenumber cross-spectrum function, equation 3.5a. In terms of the coefficients $a_i$ this is

$$\Phi_{ij}(k_1, y_2, y_2', k_3) = \frac{L_1 L_2 u_i^2}{\pi^2} \text{Ex}[a_i^*(k_1, y_2, k_3) a_j(k_1, y_2', k_3)]$$

The large number of points in each instantaneous velocity field made it impractical to determine this function at the full wall-normal resolution and using a full grid of streamwise and spanwise wavenumbers. Consequently, only every 4th wall-normal point was used for the calculations, reducing the full resolution of 129 points across the half height of the channel, to 33 $y_2$-locations. Then, the resulting function was interpolated to a set of exponentially spaced wavenumbers ranging between $-143/\delta$ and $143/\delta$ for $k_3$, and 0 and $143/\delta$ for $k_1$. This reduced the number of wavenumber components to 19 in $k_1$ and 37 in $k_3$.

Now averaging all these arrays leads to a single matrix: the averaged wavenumber function $\Phi_{ij}$. This was stored in normalized form as a 5 dimensional matrix,
\( \Phi_y(6,33,33,19,37) \) where the first index corresponds to the component number, i.e. 1 \( \Leftrightarrow \Phi_{11} \), 2 \( \Leftrightarrow \Phi_{12} \), 3 \( \Leftrightarrow \Phi_{13} \), 4 \( \Leftrightarrow \Phi_{22} \), 5 \( \Leftrightarrow \Phi_{23} \), 6 \( \Leftrightarrow \Phi_{33} \).

This matrix, obtained using Matlab, was about 200 megabytes in size.

The cross-spectrum matrix was then inverse Fourier transformed according to equation (3.5b) in order to obtain the spatial correlation function. From Appendix B, the discrete form of the relationship between the wavenumber cross spectrum, \( \Phi_y \), and the two-point correlation function, \( R_y \), is:

\[
R_y(y_2, y'_2, \Delta y_1, \Delta y_3) = \sum_{k_1} \sum_{k'_1} \frac{\pi^2}{L_y L_z \mu_r} \Phi_y(y_2, y'_2, k_1, k'_1) \exp(jk_1 \Delta y_1 + jk_3 \Delta y_3)
\]  

This inverse Fourier transform was performed using the IFFT2 function of Matlab. Note that this operation required reinterpolating the \( k_1 \) and \( k_3 \) wavenumbers of the spectrum tensors to a regular grid of 286 normalized wavenumbers in both directions ranging from -143 to 142. Then the resulting correlation function was then interpolated to a stretched grid of streamwise and spanwise separations. The stretched grid is similar to the one used for wavenumbers, using an exponential spacing ranging from \(-\pi/\delta\) to \(\pi/\delta\) in \( y_1 \) and from \(-\pi/(2\delta)\) to \(\pi/(2\delta)\) in \( y_3 \).

Reynolds stress profiles and some streamwise and spanwise correlations were compared with Moser et al. (1998)’s reduction in order to check the calculation. Note that Moser et al. (1998) used a larger set of instantaneous velocity fields to compute their statistics than were available in the present study. The Reynolds stress profiles, \( \tau_y \), are compared in Figure 3.5. For \( \tau_{11} \), \( \tau_{22} \), \( \tau_{33} \) and \( \tau_{12} \) the curves compare very well with less than 5% difference. For the streamwise normal stress, \( \tau_{11} \), the largest differences appear in the core, i.e. for \( y_2/\delta > 0.3 \). The other components, \( \tau_{22} \), \( \tau_{33} \) and \( \tau_{12} \), show differences for lower locations corresponding to the log-law region. The two cross components, \( \tau_{13} \) and \( \tau_{23} \), should be zero according to the definition of the correlation tensor. The number of fields required to zero these terms was however not sufficient. It can be seen that with more velocity fields, which Moser had for his statistical average, the terms are closer to zero than with the 34 fields used in the data reduction. However,
the differences in results is not very important since the absolute value of these terms is very low.

In general, normal stresses all show a similar behavior. Starting from zero at the wall, the stresses increase very rapidly in the viscous sublayer until the log-law region where it starts slowly decreasing toward the center of the channel. The dominating normal stress is the streamwise normal stress, $\tau_{11}$, followed by the spanwise and normal to wall normal stresses, $\tau_{33}$ and $\tau_{22}$ respectively. The shear stress, $\tau_{12}$, shows a similar behavior but reaches zero at the middle of the channel, to satisfy symmetry. Reynolds stresses are interesting because they are representative of the turbulence levels in a flow. They are related to production, kinetic energy and enclose information about the two-point correlation function, an assumption developed in details in Chapter 5.

Figure 3.6 shows profiles of the Reynolds stresses and of the kinetic energy, obtained from the data reduction, in the viscous wall region of the channel flow, i.e. for $y^_/\delta < 50$. The most vigorous turbulent activity is found in this region. The maximum turbulent kinetic energy is found in the buffer layer at $y^_/\delta = 20$. Also, other quantities such as anisotropy, production and dissipation are known to achieve their peaks in this region. From Figure 3.6, it can be seen that $\tau_{22}$ has a zero slope at the origin, that is

$$\left.\frac{\partial u_2}{\partial y_2}\right|_{y_2=0} = 0$$

(3.16)

This result is expected from continuity since at the wall $u_1$ and $u_3$ are zero for all $y_1$ and $y_3$ and consequently, the derivatives $\partial u_1 / \partial y_1$ and $\partial u_3 / \partial y_3$ at the wall are also zero.

Form continuity, $\partial u_i / \partial y_i = 0$ so that equation (3.16) must be satisfied.

Figure 3.7 shows the Reynolds stress profiles normalized by the kinetic energy across the channel half-height. This plot shows that the Reynolds stresses are anisotropic everywhere in the channel flow and especially in the log-law region. Also, in the log-law
region, there is self-similarity. The normalized Reynolds stresses are essentially uniform up to locations close to the centerline.

Figure 3.8 and Figure 3.9 compare the streamwise and spanwise one-dimensional correlation functions at $y_2 = 0.1685\delta$, that is $R_y(y_2, y_2, \Delta y_1)\bigg|_{y_2 = 0.1685}$ and $R_y(y_2, y_2, \Delta y_3)\bigg|_{y_2 = 0.1685}$. Note that here again, small differences appear because the data reduction was performed with only 34 instantaneous velocity fields whereas Moser had access to a larger set of instantaneous fields for his statistical averaging. More detailed descriptions of these one-dimensional correlations are given in the next section using correlation coefficients.

Estimates of the two-point vorticity correlation function for the channel flow were obtained in a similar way. The only difference in computing vorticity is that the Fourier coefficients used for velocity correlations have to undergo further processing before being assembled to form the vorticity cross spectrum function. Recalling equation (3.11) defining the Fourier decomposition of the velocity components, taking partial derivatives with respect to space gives

$$
\frac{\partial u_i(y_1, y_2, y_3)}{\partial y_i} = \sum_{k_1} \sum_{k_2} a_i(k_1, y_{2i}, k_3) jk_1 \exp(jk_1y_1 + jk_3y_3)
$$

$$
\frac{\partial u_i(y_1, y_2, y_3)}{\partial y_3} = \sum_{k_1} \sum_{k_2} a_i(k_1, y_{2i}, k_3) jk_3 \exp(jk_1y_1 + jk_3y_3)
$$

$$
\frac{\partial u_i(y_1, y_2, y_3)}{\partial y_2} = \sum_{k_1} \sum_{k_2} \frac{\partial a_i(k_1, y_{2i}, k_3)}{\partial y_2} \exp(jk_1y_1 + jk_3y_3)
$$

Also, from the definition of the collocation points given in equation (3.10)

$$
\frac{\partial y_{2i}}{\partial \tau_2} = -\sin \left(\pi \frac{i_2 - 1}{n_2 - 1}\right) \frac{\pi}{n_2 - 1}
$$

So, using the chain rule
\[
\frac{\partial a_i(k_1, y_{2i_2}, k_3)}{\partial y_2} = \frac{\partial a_i(k_1, y_{2i_2}, k_3)}{\partial y_2} - \frac{\partial a_i(k_1, y_{2i_2}, k_3)}{\partial y_2} (n_2 - 1) \pi \sin \left( \frac{\pi(i_2 - 1)}{n_2 - 1} \right) \tag{3.19}
\]

Now, the vorticity fluctuation vector, \( \omega \), is given by

\[
\omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \omega_3 \bar{e}_3 = \begin{vmatrix}
\bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\
\partial/\partial y_1 & \partial/\partial y_2 & \partial/\partial y_3 \\
u_1 & u_2 & u_3
\end{vmatrix}
\tag{3.20}
\]

where \( \bar{e}_1 \), \( \bar{e}_2 \) and \( \bar{e}_3 \) are unit vectors in the streamwise, wall-normal and spanwise directions.

Thus,

\[
\omega_1 = \frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3}, \\
\omega_2 = \frac{\partial u_1}{\partial y_3} - \frac{\partial u_3}{\partial y_1}, \\
\omega_3 = \frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2}
\tag{3.21}
\]

Therefore, if \( b_i \) represents the normalized Fourier amplitudes of the vorticity field the same way \( a_i \) defines the velocity field, then

\[
b_1(k_1, y_{2i_2}, k_3) = \frac{\partial a_i(k_1, y_{2i_2}, k_3)}{\partial y_2} - a_i(k_1, y_{2i_2}, k_3)2i_3j, \\
b_2(k_1, y_{2i_2}, k_3) = a_i(k_1, y_{2i_2}, k_3)2i_3j - a_i(k_1, y_{2i_2}, k_3)2i_3j, \\
b_3(k_1, y_{2i_2}, k_3) = a_i(k_1, y_{2i_2}, k_3)2i_3j - \frac{\partial a_i(k_1, y_{2i_2}, k_3)}{\partial y_2}
\tag{3.22}
\]

The vorticity cross spectrum and correlation functions can be obtained using these coefficients, the following equations, and the same interpolation and undersampling schemes used with the velocity correlation.
CHAPTER THREE

\[ W(y_2, y_2, \Delta y_1, \Delta y_3) = \sum_{y_i} \sum_{y_j} E[x(b'(k_i, y_2, k_j); b(k_i, y_2, k_j))] \exp(jk_i \Delta y_1 + jk_j \Delta y_3) \] (3.23)

To check the accuracy of these calculations, the mean square vorticity profiles were compared with the statistics given by Moser et al. (1998). Results are shown in figure 3.4. The cross terms of the mean square vorticity show differences with the statistics given by Moser. However, these terms are low in magnitude and should essentially be zero. Differences appear here again because of the difference in the number of fields utilized to perform the statistical average. The normal components show better agreement. The curves for \( W_{22} \) and \( W_{33} \) obtained from the data reduction match exactly the statistics given by Moser. Differences appear in \( W_{11} \) where the peak around \( y_2 / \delta = 0.05 \) is not well reproduced by the subset of the data reduced in the present study.

These profiles show as expected maxima in the viscous wall region. The vorticity is most intense at small scales and is connected to dissipation, which occurs mainly near the wall where viscous effects are important. Vorticity and dissipation can be mathematically related (cf. Durbin and Pettersson Reif (2001) for example). The dissipation can be expressed as

\[ \varepsilon = \nu \text{Ex} \left[ \frac{\partial u_k}{\partial y_i} \frac{\partial u_k}{\partial y_i} \right] \] (3.24)

For incompressible flows, the mean square magnitude of the vorticity can be expressed as

\[ \nu \text{Ex}[\omega^2] = \varepsilon - \nu \frac{\partial^2 \text{Ex}[u_k u_j]}{\partial y_j \partial y_k} \] (3.25)

In homogeneous turbulence, the last term of the right hand side of equation (3.25) vanishes. In this case the relationship between vorticity and dissipation becomes simply

\[ \nu \text{Ex}[\omega^2] = \varepsilon \] (3.26)

So vorticity is to be found where dissipation is the most important, i.e. in the viscous wall region.
3.2.2 Two-Point Correlation Results for the Channel Flow

Since the two-point velocity and vorticity correlation functions are 4-dimensional multi-component functions, they cannot be shown on a simple plot. Instead, some cross-sectional views and cuts through these functions are presented in this section with the goal of revealing their important physical features.

The first cross-section represented is the two-dimensional velocity correlation map for zero $y_1$ and $y_3$ separation, $R_{ij}(y_2, y'_2, \Delta y_3, \tau) = R_{ij}(y_2, y'_2, 0, 0)$. The nine components of the correlation function, normalized on $u^2$, are presented in Figure 3.11. The diagonals of these $y_2$-$y'_2$ correlation maps represent the Reynolds stress profiles and, because of the symmetry property of equation (3.1a), $R_{ij}(y_2, y'_2)$ must be equal to $R_{ji}(y'_2, y_2)$. This feature is visually observable in the symmetry of the nine component correlation map about the bottom left to top right diagonal.

For this type of flow, the largest correlations are the autocorrelations, $R_{11}$, $R_{22}$ and $R_{33}$. The correlations are most important on the diagonal, corresponding to the Reynolds normal stresses, and close to the wall. Also, the total kinetic energy is defined to be half the trace of the Reynolds stress tensor, i.e. $k \equiv 1/2 \tau_{kk}$. So, from Figure 3.8, most of the kinetic energy is seen to be associated with the streamwise autocorrelation. The correlations appear eye-shaped for the $R_{22}$ and $R_{33}$ components and the main correlations seem to appear in the log-law region. The correlation decays quickly as the separation between the two normal to wall location increase. For positions in the core, the correlations usually appear more extended away from the diagonal $y_2 = y'_2$. It seems that the farther the location from the wall, the more extended the correlations, probably a result of the presence of larger structures in the core of the channel. However, for the streamwise autocorrelation, velocities in the viscous wall region appear to be correlated with velocities close to the centerline of the channel, indicating maybe large streamwise
structures in the bottom half of the channel. Also, for $R_{y_1}$, the maximum appears to be located closer to the call, somewhere in the buffer layer.

Other cuts in the four-dimensional correlation function can be obtained in the streamwise and spanwise directions. In these directions, it is interesting to introduce the correlation coefficient function, which corresponds to a normalization of the correlation function. The correlation coefficient function can be defined as:

$$\rho_y(y_2, \Delta y_1, \Delta y_3) = \frac{R_y(y_2, \Delta y_1, \Delta y_3)}{\sqrt{\tau_y(y_2)}}$$  \hspace{1cm} (3.20)

(No summation implied)

Figure 3.8 represents the correlations in the $y_1 - y_3$ plane for different $y_2$ locations, i.e. $\rho_y(y_2, \Delta y_1, \Delta y_3)$ where $y_2$ is a fixed normal to wall position. These cuts show the behavior of the correlation coefficient for separations in the streamwise and spanwise directions corresponding to the limits of the computational domain. In the $y_1$ direction the domain limits are equal to $\pm \pi/\delta$ and in the $y_3$ direction to $\pm \pi/2\delta$. The planes are presented for different $y_2$ positions located in the different regions of the channel flow defined in Figure 3.4, i.e. the viscous sublayer with $y_2/\delta = 0.005$, the buffer layer with $y_2/\delta = 0.030$, the log-law region with $y_2/\delta = 0.169$ and finally the core with $y_2/\delta = 0.404$ and $y_2/\delta = 0.853$. In addition to these planar cuts, one-dimensional cuts are performed in these planes according to Figure 3.13. The one-dimensional correlation coefficient function is presented for positive streamwise separation and zero spanwise separation in Figure 3.14 and for positive spanwise separation and zero streamwise separation in Figure 3.15, that is $\rho_y(y_2, \Delta y_1, 0)$ and $\rho_y(y_2, 0, \Delta y_3)$ respectively.

The extent of the correlation coefficients appears very large in the streamwise compared to the spanwise direction especially in the viscous sublayer and in the buffer layer. This behavior is probably the result of streaks originating on the wall and creating
structures elongated in the streamwise direction. These structures are well described in the literature (cf. Pope (2000), Moin and Moser (1988) for example). The correlations are particularly extended for $\rho_{11}$ and $\rho_{12}$ at all the $y_2$-positions depicted. Moreover, the correlations do not decay to zero by the end of the computational domain as would be needed if the DNS solution were to be completely independent of the size of the computational box. Also, one would expect the DNS to show these correlations to be symmetric with respect to $y_1 = 0$ and $y_3 = 0$ since these directions are homogeneous. However, Figure 3.12 shows slightly asymmetric correlations. This result is actually a consequence of the uncertainties arising from the reduced number of velocity fields available. The consequences of this “shortage” might not affect the statistical determination of the smallest structures but the largest structures present in the flow, since the latter are the last to reach a stationary state.

Now, Figure 3.14 and Figure 3.15 reveal the same features as Figure 3.12 in terms of one-dimensional cuts. The correlation coefficients appear to decay faster in the spanwise direction (Figure 3.15) than in the streamwise direction (Figure 3.14). Generally, they decrease quickly as the streamwise or spanwise separation increases. The autocorrelation coefficients $\rho_{22}$ and $\rho_{33}$ decay faster than the coefficient $\rho_{11}$, though. For example, looking at streamwise separation, the coefficients get close to zero in less than a channel half-height of separation whereas $\rho_{11}$ takes usually one to two channel half-heights. Also, the damping of the correlation function becomes less important as the position normal to the wall increases. This is a result of the variation of scales with normal position. The largest scale are present in the outer layer and therefore are correlated over greater distances than the smaller scales present in the viscous wall region.

The cross correlation coefficient, $\rho_{12}$, is from its definition negative. It starts at a value of about -0.4 near the wall and becomes null after half to one channel half-height. The cross correlation coefficient lowers in magnitude as the normal position increases. Finally, at the center of the channel $\rho_{12}$ is zero for every streamwise and spanwise separation. This is a result of the symmetries inherent to the definition of the correlation function.
Here again, the size of the computational can be find to be too small and the correlations often do not asymptote a zero value. This result is likely to affect the largest structures.

Vorticity correlations, $W_y(\Delta y_1, y_2', y_2', \Delta y_3')$, representative of the flow are presented in Figure 3.16 and Figure 3.17. Figure 3.16 shows the two-point vorticity correlation map for zero streamwise and spanwise separations, $W_y(0, y_2, y_2', 0)$. As for the velocity correlations, the correlations dominate on the main diagonal and are symmetric with respect to it. In this case, however, the major correlations in magnitude appear in the spanwise autocorrelation followed by the streamwise and normal to wall autocorrelations. These correlations do not show much extent from the main diagonal corresponding to the mean square vorticity correlations and represented in Figure 3.10. This result is probably related to the behavior observed for the velocity correlations shown in Figure 3.11. Since the major velocity correlations were present in the streamwise and normal to wall autocorrelations, the spanwise vorticity correlation has to be important. Also, this feature might be physically explained by the formation of spanwise eddies in the bottom of the boundary layer. An example of the spanwise vorticity in the bottom of the boundary layer can be associated to the legs of the well recognized hairpin vortices. In addition, the vorticity correlations are logically found very important in magnitude near the wall since they are related to the dissipation of kinetic energy which occurs mainly near the wall.

The correlation maps for fixed normal positions, $W_y(\Delta y_1, 0, 0, \Delta y_3')$, are presented in Figure 3.17 for the same normal to wall locations used in Figure 3.12 to present $R_y(\Delta y_1, 0, 0, \Delta y_3')$. Figure 3.17 shows dominant streamwise extent for the correlations in the viscous sublayer and the buffer layer. This result is expected since most of the dissipation happens in these regions and since the structures in this region are usually elongated in the streamwise direction. For locations in the log-law region and in the core, the normal to wall and spanwise correlations also appear extended in the streamwise direction. However, the streamwise extent is not as important. Also, for these higher locations, the vorticity correlations decay very fast with distance both in the streamwise and spanwise directions, and also with increasing $y_2'$-positions. As mentioned previously,
these observations are related to dissipation. Dissipation decreases with increasing normal to wall positions, and is also related to the smallest structures. Since the smallest structures are mostly located in the viscous wall region, the vorticity correlations decay very fast in the streamwise and spanwise directions in the core.

Similar aspects of the statistics extracted for the DNS are now studied in the next section for an external boundary layer. Since this study involved a trailing edge noise calculation being carried using the DNS for source terms, it is useful to know how the simulated channel flow compares to an external boundary layer.

### 3.3 PIV Measurements of a Boundary layer at \( \text{Re}_\theta = 7705 \)

Features similar to those seen in some of Moser’s channel flow results can also be found in an external boundary layer such as in the experiment of Adrian et al. (2000). This data set was chosen to show if important differences existed between the channel flow DNS and the boundary layer measurement. Indeed, noise calculations are made from a boundary layer near the trailing edge of an airfoil. No complete statistical set were available for such a flow though, so that the DNS of a channel flow was preferred to carry the trailing edge noise calculations.

Adrian et al. (2000) made PIV measurements on a flat plate for Reynolds numbers up to \( \text{Re}_\theta = 7705 \) or \( \text{Re}_r = 2230 \). The turbulent boundary layer was grown on a flat plate 100 mm above the floor of a wind tunnel test section and a 4.7 mm diameter wire strip was used for to produce spanwise uniform transition of the boundary layer as well as for the stabilization of the downstream location of the transition. A sketch of a flat plate boundary layer coordinate system is given in Figure 3.18.

Measurements were performed in an Eiffel-type low-turbulence boundary layer wind tunnel with a test section 914 mm wide, 457 mm high and 6090 mm long. The root-mean square intensity turbulence was less than 0.2 % in the free stream. The parameters
of the flow for the set of measurements gathered at the Reynolds number of interest are listed in Table 3.1.

Table 3.1: Parameters of the boundary layer of Adrian et al. (2000) at \( \text{Re}_b = 7705 \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Re}<em>b = \theta \text{ U}</em>\infty / \nu )</td>
<td>7705</td>
</tr>
<tr>
<td>( \text{Re}<em>b = \delta \text{ U}</em>\infty / \nu )</td>
<td>61860</td>
</tr>
<tr>
<td>( \text{U}_\infty ) (m/s)</td>
<td>11.39</td>
</tr>
<tr>
<td>( \delta ) (mm)</td>
<td>83.1</td>
</tr>
<tr>
<td>( \theta ) (mm)</td>
<td>10.35</td>
</tr>
<tr>
<td>( \delta^* ) (mm)</td>
<td>14.4</td>
</tr>
<tr>
<td>( u_t ) (m/s)</td>
<td>0.41</td>
</tr>
<tr>
<td>( \delta^* )</td>
<td>2216</td>
</tr>
<tr>
<td>( H = \delta^* \theta )</td>
<td>1.39</td>
</tr>
<tr>
<td>( \delta / \theta )</td>
<td>0.125</td>
</tr>
</tbody>
</table>

For the PIV measurements, 0.5-2 \( \mu \text{m} \) diameter oil particles were suspended in the flow and illuminated by a 0.25 mm thick light sheet. The double pulsed images were photographed using a side viewing photographic camera. The schematic of the system used is represented in Figure 3.19.

The data obtained from these measurements can be downloaded at the following URL: http://ltcf.tam.uiuc.edu/Downloads/Data/BL/index.html. To obtain the statistics required by the model, 50 files containing two-dimensional instantaneous velocity fields were downloaded. Each of the 50 files showed a streamwise cross section through the flow and contained data for \( y_1, y_2, u_1 \) and \( u_2 \). First, mean velocities were extracted to subtract from the instantaneous velocity records before computing the correlation function. This operation ensures that the round off error does not build up in the calculations. The mean velocity profile is compared to the channel flow mean velocity profile and to the log-law of the wall due to von Karman (equation (3.9)). The closest data point to the wall was located at \( y_2^* = 30 \), which corresponds to the lower end of the log-
CHAPTER THREE

law region. No data points were available in the viscous wall region probably because near wall measurements in PIV are very difficult and characterized by large uncertainties due to difficulties encountered with scattering of light from wall boundaries and lack of seed particles near the wall. Also, these uncertainties might explain why the first data point seems to be off the line representing the log-law of the wall. For higher locations the points follow the relationship implied by the log-law. Note that for the boundary layer measurements, the log-law region is more extended than in the channel flow case and so is the outer region, the equivalent of the core, consistent with the higher Reynolds number of this flow.

Then the correlation function was computed from the instantaneous velocity fields provided by multiplying the velocity profiles to form \( y_2 - y'_2 \) planes for the different streamwise separations. These planes are then averaged for similar streamwise separations and finally for the 50 fields. The resulting velocity correlation function is a function of \( y_2, y'_2 \) and \( \Delta y_1 \), that is \( R_y(y_2, y'_2, \Delta y_1) \).

The spanwise Reynolds stress profile was not measured in the Adrian data set. This element is however essential for the two-point velocity prediction model presented in Chapter 5. This information was therefore provided from measurements of a turbulent boundary layer on a smooth wall, made by George and Simpson (2001) using laser Doppler velocimetry (LDV), at almost the same conditions. The parameters associated to this flow are represented in Table 3.2. The Reynolds number based on the friction velocity, \( Re_{\tau} \), were similar for both flows: 2310 for George and Simpson (2001), and 2230 for Adrian et al. (2000). The comparison of both flows show that the boundary layer studied with PIV was thicker and was obtained for a lower freestream velocity. However, the Reynolds number being very similar, it is possible to assume that both data sets should show similar statistics.
The Reynolds stress fields of both boundary layer experimental data are shown in Figure 3.21. As mentioned before, the spanwise Reynolds stress profile is not represented for Adrian et al. (2000)’s data set. The Reynolds stresses of the boundary layer appear fairly similar to the profiles given by the channel flow DNS, reinforcing the justification of considering the channel flow DNS as a boundary layer flow. Major differences appear however for \( \tau_{22} \) and \( \tau_{33} \). The slopes in both the log-law region and the outer region/core appear similar but show lower values for the DNS. Figure 3.21 shows differences of up to 70% in \( \tau_{22} \). Similarly, the outer region appears slightly different in \( \tau_{11} \) and \( \tau_{12} \). Many of these differences can be attributed to the difference in Reynolds numbers between the two flows, i.e. 590 for the channel flow and 2230 for the boundary layer.

However, not only the comparison between the boundary layer and the DNS shows differences. Comparison between the two experimental data sets shows important differences too. For similar Reynolds number, it is interesting to notice that the boundary layer data sets show large variations, especially for \( \tau_{11} \) and \( \tau_{22} \) at low normal to wall positions. These discrepancies show reveal the difficulty of making repeatable turbulence measurements and generating repeatable boundary layers in different facilities, and also set a standard for how close one should expect predictions to come to experimental data of the second order statistics.

Figure 3.22 shows the velocity correlation maps of \( R_{11} \), \( R_{22} \) and \( R_{12} \) inferred from the Adrian data set for zero streamwise and spanwise separation. These are the only correlation components available from the measurements by Adrian et al. (2000). As in

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_e )</td>
<td>27.0 m/s</td>
</tr>
<tr>
<td>( u_\tau )</td>
<td>0.98 m/s</td>
</tr>
<tr>
<td>( \delta )</td>
<td>39.0 mm</td>
</tr>
<tr>
<td>( \delta^* )</td>
<td>6.09 mm</td>
</tr>
<tr>
<td>( \theta )</td>
<td>4.47 mm</td>
</tr>
<tr>
<td>( \text{Re}_{\theta} )</td>
<td>( \sim ) 7300</td>
</tr>
</tbody>
</table>
the channel flow case, the correlations appear eye-shaped and the strongest correlations correspond to the diagonals: the Reynolds stresses. The main differences appear in the extent in the levels near the diagonals, an aspect already observed in Figure 3.21 with the Reynolds stresses. Also, the $R_{11}$ correlation associated to the boundary layer does not show the correlations between the viscous wall region and the core/outer region that were present in the channel flow case. A final difference can be seen in the contours which appear idealized for the DNS calculations and not for the experimental data set.

Figure 3.23 represents the streamwise correlation coefficients at different $y_+$-positions in the flow. The velocity fluctuations generally appear similar to the ones obtained from the DNS. The correlation coefficient, $\rho_{22}$, shows a similar rate of decay as in the DNS case. The correlation coefficient $\rho_{11}$ appears very similar for both data sets, at least in the log-law region. For higher locations, the decay rate appears to be faster for the boundary layer than for the channel flow. Also, differences appear for $\rho_{12}$. The magnitude of $\rho_{12}$ at the wall is for the boundary layer.

The channel flow DNS and the boundary layer experimental data sets both show the same principal features of the raw two-point correlation function. The only major differences that could be established between the channel flow and the boundary layer appeared to be corresponding to phenomena associated to the largest structures. Correlation and correlation coefficient functions appeared very similar in both flows. The similarity in the two-point statistics provides therefore some justification for considering the channel flow DNS as an external boundary layer and for using the DNS data to extract the source terms used in the trailing edge noise prediction scheme described in Chapter 2.

A last comparison that can be made lies in the structures present in these flows. It can be tempting to use the plots shown previously in this section to infer the structures responsible for the correlation function. However, correlation maps are only two-dimensional cuts in a four-dimensional function and thus not obvious representations of the eddies. An alternative and more comprehensive approach is to use characteristic eddy decomposition. Characteristic eddy decomposition does not aim to finding the
CHAPTER THREE

instantaneous structures present in the flow, but at finding averaged structures representative of the turbulence. Here, proper orthogonal decomposition and linear stochastic estimation are used to extract the eddy-like structures.

3.4 Proper Orthogonal Decomposition and Compact Eddy Structures

The recent developments of the application of proper orthogonal decomposition to aeroacoustics are the basis of Glegg’s method for trailing edge noise prediction. As mentioned before, the turbulence can be prescribed as a set of statistically independent velocity modes through POD. As the name of this procedure implies, the optimum modes are orthogonal and independent (Lumley 1968). Moreover, Lumley showed that the modes were eigenfunctions of the two-point correlation tensor and the corresponding eigenvalues their spectrum. The magnitudes of the eigenvalues correspond to the proportion of kinetic energy produced by the corresponding eigenfunction or mode. Thus, a limited number of modes are required to determine the main eddies, at least in the case of a fan-wake type of flow as described by Devenport et al. (2001). For these wake measurements, the first two modes were counting for about 27% of the total kinetic energy in the wake.

The one-dimensional eigenvalues \( \lambda \) and eigenfunctions \( \phi_i \) of a boundary layer velocity correlation function can be obtained by solving the following integral equation:

\[
\lambda \phi_i(y_2) = \int_{-\infty}^{\infty} R_{y_i}(y_2, y'_2) \phi_j(y'_2) dy'_2
\]  

(3.21)

From the discretization of this expression, normalization of the eigenvalues and eigenfunctions is required. Consider the discretization of equation (3.21)

\[
\sum R_{y_i}(y_2, y'_2) \phi_j(y'_2) \Delta y_2 = \lambda \phi_i(\Delta y_2)
\]  

(3.22)
Now by definition, at each frequency, the average energy contained in a velocity mode is equal to one, so, for a unit thickness in the $y_2$ direction,

$$\int_0^{L_2} \phi_j^2(y_2') dy_2' = 1 \quad (3.23)$$

or in discrete form,

$$\sum_{y_2} \phi_j^2(y_2') \Delta y_2' = 1 \quad (3.24)$$

The way Matlab handles the eigenvectors is by setting the r.m.s. to 1, or

$$\sum_{y_2} \phi_{\text{Matlab}}^2(y_2') = 1 \quad (3.25)$$

From equations (3.24) and (3.25) follows the eigenfunction normalization

$$\phi_j(y_2') = \frac{\phi_{\text{Matlab}}(y_2')}{\sqrt{\Delta y_2'}} \quad (3.26)$$

Substitution of this expression in equation (3.22) leads to

$$\sum R_{ij}(y_2, y_2') \phi_{\text{Matlab}}(y_2') \Delta y_2 = \lambda \frac{\phi_{\text{Matlab}}(y_2)}{\sqrt{\Delta y_2}} \quad (3.27)$$

So,

$$\sum R_{ij}(y_2, y_2') \phi_{\text{Matlab}}(y_2') = \frac{\lambda}{\Delta y_2} \phi_{\text{Matlab}}(y_2) \quad (3.28)$$

and finally

$$\lambda_{\text{Matlab}} = \frac{\lambda}{\Delta y_2} \quad (3.29)$$
The eigenfunctions obtained through POD represent the most probable instantaneous velocity profiles, and thus can be thought of as representing the most probable eddy structures.

However, the use of POD is only relevant in the inhomogeneous directions. In the homogeneous directions, POD reduces to Fourier decomposition and requires a large number of wavenumber components, and consequently of modes, to describe completely the flow (cf. Glegg and Devenport (2001)). Furthermore, these Fourier modes are clearly not realistic eddy velocity fields.

Another way to extract probable eddy velocity fields is therefore used for the homogeneous directions. Linear stochastic estimation (LSE) gives the best linear estimate of the velocity field and is one way to do this. The best linear estimate of the instantaneous velocity field based on the velocity value at a specific point \( u_i(y_2) \) is given by the following equation.

\[
\left. u_j(y_2, \Delta y_3, \Delta y_1) \right|_{LSE} = \frac{u_i(y_2)}{\text{Ex}[u_i(y_2)^2]} \cdot R_{y} (y_2, y_2, \Delta y_3, \Delta y_1) \tag{3.30}
\]

Glegg and Devenport (2000) used this method to obtain the best linear estimate of the instantaneous velocity fields associated with each of the one dimensional proper orthogonal modes (the most probable profiles). This method, called compact eddy structures (CES), uses POD to calculate the modes in the inhomogeneous direction while LSE describes the homogeneous direction to obtain the complete three-dimensional velocity field.

For the \( n^{th} \) mode, using CES, the velocity fluctuations associated to the \( n^{th} \) mode become

\[
\left. u_j^{(n)}(y_2, \Delta y_3, \Delta y_1) \right|_{CES} = \frac{1}{\lambda^{(n)}} \int \phi^{(n)}(y_2) R_{y} (y_2, y_2, \Delta y_3, \Delta y_1) dy_2 \tag{3.31}
\]

This method has the advantage to describe the whole flow in a few numbers of modes, that have eddy-like forms and serve as source terms in certain acoustic calculations, see Glegg and Devenport (2001).
3.4.1 Characteristic Eddy Decomposition Applied to the DNS Results

As mentioned previously, eigenvalues and associated eigenvectors, representing the velocity modes, can be extracted from the two-point velocity correlation function. The eigenvalue spectrum for the channel flow and the relative energy of the eigenvalues are plotted in Figure 3.24 for the first 100 modes for the proper orthogonal decomposition in the inhomogeneous normal-to-wall direction. The eigenvalue spectrum shows that the first two modes contain considerably more energy than the other modes. They account for almost 37% of the total turbulence kinetic energy in the channel. Also, only four modes are necessary to account for 50% of the total energy.

The corresponding modal profiles are presented in Figure 3.25. As with other typical eigenfunctions, the number of zero-crossings increases with the order of the eigenfunctions. In this case, the first two modes are dominated by \( u_1 \) and \( u_2 \). In these modes \( u_3 \) is negligible. For the modes 3 and 4, the motion shows more complex motions with all three velocity components. Also, for all these modes \( u_1 \) shows high gradients near the walls. It is interesting to note that for all these modes the streamwise, \( u_1 \), and vertical, \( u_2 \), components have opposite signs throughout the domain. They therefore make a positive contribution to Reynolds shear stress. Figure 3.26 shows a grid independence study to examine the resolution necessary in the normal to wall direction to obtain quantitatively correct modes. Eigenvalues and modal profiles are presented for 51, 33 and 26 points in Figure 3.26a, Figure 3.26b and Figure 3.26c respectively. The eigenvalue spectra look similar for the first modes at every resolution. Down to 33 points, the first four modes are exactly identical. For 26 points, the modes look very similar, but the fourth mode starts showing discrepancies. Thus the 33 \( y_2 \)-locations used to represent the DNS data appear adequate.

The compact eddy structures that are obtained using LSE with these profiles are shown in Figure 3.27. The first mode shows a velocity field with almost no rotation. In
the other modes, spanwise roller-type structures appear dominantly in the flow. They appear singly in the second and third modes, and in pairs in the fourth mode. In the second mode, the eddy appears to be centered around $0.25 y_z / \delta$ and in mode 3 around $0.5 y_z / \delta$. Mode 4 shows counter rotating eddies, one in the upper half of the half channel and one in the lower part.

The same analysis is applied to the boundary layer flow data set. It is interesting to see if the same velocity structures present in the channel flow also dominate the truly external boundary layer.

### 3.4.2 Characteristic Eddy Decomposition Applied to the Boundary Layer

The eigenvalue spectrum for the Adrian boundary layer was extracted through proper orthogonal decomposition and is plotted with the relative eigenvalues energy in Figure 3.28 for the first 100 modes. As in the channel flow case the first few modes contain most of the energy. The first two modes account for 47% of the total turbulent kinetic energy, and the first four modes sum up to over 60% of the kinetic energy. The decrease in energy per mode is faster than in the channel flow case so that even fewer modes are necessary to describe this flow.

Figure 3.29 represents the modal profiles associated to the first four modes. The first modes show some simple velocity motions of $u_1$ and $u_2$. The spanwise fluctuations do not appear in the modes since the two-point function extracted from the files only contained $u_1$ and $u_2$ information. It is interesting to notice that the first four modes compare to the ones obtained for the channel flow (Figure 3.25). The same motions of $u_1$ and $u_2$ appear at similar locations on the profiles.

A grid independence study is also performed for the modal profiles of the present data set. The dependence on the number of points in the normal to wall direction for the use of proper orthogonal decomposition is represented in Figure 3.30a, Figure 3.30b and Figure 3.29c for 40, 20 and 10 points respectively. The resolution seems to have a very
limited influence. Only 20 points appear to be sufficient to represent quantitatively the first velocity modes. Even for 10 points, the profiles corresponding to the first two modes appear similar to the profiles obtained for 86 points. For higher modes, the qualitative representation of the velocity profiles is respected but differences appear in the velocity fluctuation magnitudes.

Finally, the compact eddy structures associated to the first four modes are plotted in Figure 3.31. Spanwise roller-type structures appear dominantly in the flow. They appear singly in the second mode and in pairs in the third and fourth mode. Again, they look similar to the ones obtained for the DNS, suggesting maybe that the dominant averaged eddy structures are the same for turbulent boundary layers types of flows.

### 3.5 Conclusion

Different characteristics of the two-point correlation function have been illustrated for a channel flow DNS and a boundary layer. It was felt wise to compare the channel flow DNS with a truly external boundary layer to determine the extent to which it might provide suitable source terms for a trailing edge noise calculation. The DNS was chosen for the trailing edge noise calculations based on the completeness of this data set and initial availability of some explicit correlation results. Both data sets showed very similar features. The decay of the correlation coefficient function in the streamwise direction as well as the two-dimensional correlation maps were similar for the channel flow and for the boundary layer. Furthermore, exploitation of the two-point velocity function through proper orthogonal decomposition revealed identical dominant modes and eddy structures in the flow. These similarities definitely provide some justification for using the channel flow DNS data to compute the source terms necessary for the implementation of Glegg et al. (2004)’s noise prediction method. The only discrepancies that might arise from the use of the channel flow DNS to represent a truly boundary layer are expected for the largest structures and therefore the lowest frequencies and other associated features. The application of the trailing edge prediction method to the turbulent channel flow DNS is described in the next section.
Figure 3.1. Definition of the two-point correlation coordinate system. The two points are positioned at a distance $y$ and $y'$ from the origin $O$, and are separated by the distance $s$.

Figure 3.2. Sketch of the channel with the coordinate system and the flow geometry.
Figure 3.3. Near wall profile of mean velocity from the DNS of Moser et al. (1998): symbols represent the DNS data; line represents the law of the wall, $u^+ = y_2^+$.

Figure 3.4. Mean velocity profile across the channel from the DNS of Moser et al. (1998): symbols represent the DNS data; line represents the log-law of the wall due to von Karman, equation (3.9).
Figure 3.5. Comparison between the Reynolds stress profiles given by Moser et al. (1998), represented by the symbols, and the Reynolds stress profiles obtained from the two-point correlation resulting from the data analysis/reduction, represented by the lines. The Reynolds stresses are normalized on $\frac{u_r^2}{\delta}$. 
Figure 3.6. Profiles of Reynolds stresses and kinetic energy normalized by the friction velocity squared in the viscous wall region of the turbulent channel flow obtained from the reduction of the DNS data of Moser et al. (1998).

Figure 3.7. Profiles of Reynolds stresses normalized by the turbulent kinetic energy from the reduction of the DNS data of Moser et al. (1998).
Figure 3.8. Comparison between the streamwise velocity correlation given by Moser et al. (1998), represented by the symbols, and the streamwise velocity correlation obtained from the two-point correlation resulting from the data analysis/reduction, represented by the lines. Correlations normalized on $u_c^2$ at $y_2 = 0.1685$, $\Delta y_3 = 0$
Figure 3.9. Comparison between the spanwise velocity correlation given by Moser et al. (1998), represented by the symbols, and the spanwise velocity correlation obtained from the two-point correlation resulting from the data analysis/reduction, represented by the lines. Correlations normalized on $u_c^2$ at $y_2 = 0.1685$, $\Delta y_1 = 0$
Figure 3.10. Comparison between the mean square vorticity correlation profiles given by Moser et al. (1998) (represented by the symbols), and the mean square vorticity correlation profiles obtained from the two-point vorticity correlation resulting from the data analysis/reduction, (represented by the lines). These profiles are given for zero streamwise and spanwise separations and are normalized on $Re_{\tau}$.
Figure 3.11. Two-point velocity correlation maps for zero streamwise and spanwise separations as a function of $y_2'$ locations of the two points. Velocity correlations normalized on $u'_2^2$. 
Figure 3.12. Two-point velocity correlation coefficients maps at $y_2 = y_2$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.12. Two-point velocity correlation coefficients maps at $y_2 = 0.030$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.12. Two-point velocity correlation coefficients maps at $y_2 = y_2$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.12. Two-point velocity correlation coefficients maps at $y_2 = y_2^*$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.12. Two-point velocity correlation coefficients maps at $y_2 = y_2^*$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.13. Definition of the one-dimensional cuts in the streamwise (top left) and spanwise (top right) directions performed in the correlation coefficient plane (bottom left).
Figure 3.14. Streamwise velocity correlation coefficients given by the DNS solution at different $y_2$-locations and $\Delta y_3 = 0$. 

$y_2/\delta = 0.005$

$\rho_{11}$

$\rho_{22}$

$\rho_{33}$

$\rho_{12}$

$y_2/\delta = 0.030$

$\rho_{11}$

$\rho_{22}$

$\rho_{33}$

$\rho_{12}$

$\Delta y_1/\delta$
Figure 3.14. Streamwise velocity correlation coefficients given by the DNS solution at different $y_2/\delta$-locations and $\Delta y_3 = 0$. 

$y_2/\delta = 0.169$

$\rho_{11}$

$\rho_{22}$

$\rho_{33}$

$\rho_{12}$

$y_2/\delta = 0.404$

$\rho_{11}$

$\rho_{22}$

$\rho_{33}$

$\rho_{12}$

$\Delta y_1/\delta$
Figure 3.14. Streamwise velocity correlation coefficients given by the DNS solution at different $y_2$-locations and $\Delta y_3 = 0$.
Figure 3.15. Spanwise velocity correlation coefficients given by the DNS solution at different $y_2/\delta$ locations and $\Delta y_1 = 0$. 
$y_2/\delta = 0.169$

![Graphs showing $\rho_{11}$, $\rho_{22}$, $\rho_{33}$, and $\rho_{12}$ for $y_2/\delta = 0.169$.](image)

$y_2/\delta = 0.404$

![Graphs showing $\rho_{11}$, $\rho_{22}$, $\rho_{33}$, and $\rho_{12}$ for $y_2/\delta = 0.404$.](image)

$\Delta y_2/\delta$

Figure 3.15. Spanwise velocity correlation coefficients given by the DNS solution at different $y_2$-locations and $\Delta y_1 = 0$.  

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$y_2/\delta = 0.853$

Figure 3.15. Spanwise velocity correlation coefficients given by the DNS solution at different $y_2$ - locations and $\Delta y_1 = 0$. 
Figure 3.16. Two-point vorticity correlation, $W_{ij}$, maps for zero streamwise and spanwise separation as a function of locations of the two points, $y_2'$. Vorticity correlations normalized on $Re^2$. 
Figure 3.17. Two-point velocity correlation coefficients maps at $y_2 = y_2'$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.17. Two-point vorticity correlation coefficients maps at $y_2 = y_2'$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.17. Two-point vorticity correlation coefficients maps at $y_2 = y_2$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.17. Two-point vorticity correlation coefficients maps at $y_2 = y_2^*$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.17. Two-point vorticity correlation coefficients maps at $y_2 = y_2^*$, as a function of the streamwise and spanwise separations of the two points.
Figure 3.18. Sketch of a flat-plate boundary layer with the coordinate system and the flow geometry.

Figure 3.19. Schematic of PIV photographic recording system used by Adrian et al. (2000).
Figure 3.20. Comparison between the mean velocity profile across the boundary layer from the PIV measurements of Adrian et al. (2000) and the mean velocity profile across the channel from the DNS of Moser et al. (1998): \(\times\) represents the DNS data; \(\circ\) represents the PIV measurements; \(-\) represents the log-law of the wall due to von Karman, equation (3.9).
Figure 3.21. Comparison of the Reynolds stress profiles given for the boundary layers and the channel flow; − represents Adrian et al. (2000)'s PIV measurements; ⃝ represents George and Simpson (2001)'s LDV measurements; + represents the DNS data from Moser et al. (1998). Profiles normalized on $u^2$. 
Figure 3.22. Comparison of the two-point velocity correlation function for zero streamwise and spanwise separation given by the PIV measurements of Adrian et al. (2000) on the left, and given by the DNS of Moser et al. (1998) on the right.
Figure 3.23. Comparison between the Streamwise velocity correlation coefficients from Adrian et al.’s measurements and the DNS data at different $y_2$-locations in the flow; — DNS data; — boundary layer measurements.
$y_x/\delta = 0.404$

$y_x/\delta = 0.853$

Figure 3.23. Comparison between the Streamwise velocity correlation coefficients from Adrian et al.’s measurements and the DNS data at different $y_x$-locations in the flow; DNS data; boundary layer measurements.
Figure 3.24. Eigenvalue spectrum of the DNS data set from the one-dimensional proper orthogonal decomposition in the $y_2$-direction (left) and relative energy of the eigenvalues (right).

Figure 3.25. Modal profiles for the first four modes of the DNS obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction.
Figure 3.26a. Eigenvalue spectrum and modal profiles for the first four modes of the DNS obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction. Lines represent the solution obtained for 101 points, symbols for 51 points.
Figure 3.26b. Eigenvalue spectrum and modal profiles for the first four modes of the DNS obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction. Lines represent the solution obtained for 101 points, symbols for 33 points.
Figure 3.26c. Eigenvalue spectrum and modal profiles for the first four modes of the DNS obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction. Lines represent the solution obtained for 101 points, symbols for 26 points.
Figure 3.27. Compact eddy structures for the first four modes deduced from the DNS data set.
Figure 3.28. Eigenvalue spectrum of the boundary layer data set from the one-dimensional proper orthogonal decomposition in the $\gamma_2$-direction (left) and relative energy of the eigenvalues (right).

Figure 3.29. Modal profiles for the first four modes of the boundary layer obtained from the one-dimensional proper orthogonal decomposition in the $\gamma_2$-direction.
Figure 3.30a. Eigenvalue spectrum and modal profiles for the first four modes of the boundary layer obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction. Lines represent the solution obtained for 86 points, symbols for 40 points.
Figure 3.30b. Eigenvalue spectrum and modal profiles for the first four modes of the boundary layer obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction. Lines represent the solution obtained for 86 points, symbols for 20 points.
Figure 3.30c. Eigenvalue spectrum and modal profiles for the first four modes of the boundary layer obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction. Lines represent the solution obtained for 86 points, symbols for 10 points.
Figure 3.31. Compact eddy structures for the first four modes deduced from the boundary layer data.
The theory of Glegg et al. (2004) described in Chapter 2 is now validated in this chapter using for source term the DNS data set of a turbulent channel flow calculated by Moser et al. (1998). This section aims at showing the applicability of the new method and also what operations and transformations have to be applied on the two-point correlation function for the implementation of this method. The DNS of a turbulent channel flow by Moser et al. (1998) was described in the previous chapter and was shown to possess similar characteristics to an external boundary layer. Cuts through the four dimensional
two-point correlation function and proper orthogonal decomposition showed that a channel flow could be considered as two turbulent boundary layers back to back, where the boundary layer thickness $\delta$ corresponds to the half channel height and the edge velocity to the channel centerline velocity. As mentioned in the previous chapter, the largest differences with a truly external boundary layer flow are expected in the outer part of the layer and for frequencies implying scales of the boundary layer or larger.

This section first describes the relationship between the data given by Moser et al. (1998) and the quantities used in the method of Glegg et al. (2004). The DNS calculations from Moser et al. (1998) are used here to provide the cross-spectrum needed to obtain the source terms on which noise predictions are based. The Matlab implementation of this noise calculation is fully described in this chapter as are comparisons with experimental noise measurements from a test case from the study of Brooks et al. (1989). Specifically run 295 of this study, involving a tripped NACA 0012 airfoil at zero angle of attack, was chosen to serve as a basis for comparison. This run was chosen since the trailing edge boundary layers were presumably fully turbulent and symmetric. Also, the trailing edge boundary layer thickness Reynolds number, $\text{Re}_e = U_e \delta / \nu$, for this particular case was about 15000, matching well the DNS for which the equivalent Reynolds number was close to 13000.

4.1 Major Steps in the Trailing Edge Noise Calculations

The method of Glegg et al. (2004) provides the acoustic spectrum through a series of major steps ranging between (1) the two-point velocity statistics and (6) the final acoustic spectrum as illustrated in Figure 4.1.

The final step in the method implementation corresponds to step (6): obtaining the acoustic spectrum. The acoustic spectrum is given by a sum of eigenvalues obtained from the two-point velocity statistics and pressure modes according to

$$S_{pp}(x, \omega) = \frac{\pi}{T} \sum_{n=1}^{\infty} \lambda_{\omega n} \left| p_{\omega n}(x, \omega) \right|^2 \quad (4.1)$$
These pressure modes were inferred from the integrand of the pressure modes and diverse information relative to the observer modes following step (5).

\[
p^{(n)}(x, \omega) = \left( \frac{\rho_0 \sin(\theta/2)e^{i\omega r_x(1-M_\infty)/(c_a)}}{i\omega r_c(1+M_{\infty})(1-M_{\infty})} \right) \int_0^\infty I^{(n)}(\omega)dy_2
\]  

(4.2)

The integrand of the pressure modes corresponds to the product between the source term and an exponential damping term before integration in the normal to wall direction (step (4)). The exact expression is given by

\[
I^{(n)}(\omega) = \sqrt{M_\infty}(1-\sigma)Q^{(n)}(k, y_2)e^{i|y_2-V(y_2)}
\]

\[
\sqrt{2(1-M_{\infty})(1-M_\infty)^{1/2}}
\]

(4.3)

where the source term \( Q^{(n)} \) was defined as (step 3))

\[
Q^{(n)}(k, y_2) = \frac{1}{2\pi} \left[ \Omega_k(i|\omega|s^{(n)}_1 + \omega s^{(n)}_2) + i|\omega|\left( i\omega s^{(n)}_2 - V(y_2)\frac{\partial s^{(n)}_1}{\partial y_2} \right) \right]_{y_2=d}
\]

(4.4)

As discussed in Chapter 2, the source term is only function of the mean velocity and the velocity fluctuations extracted from the spanwise averaged cross-spectrum. The velocity were extracted using proper orthogonal decomposition (step 2) according to equation (4.5).

\[
1 \int_0^R \frac{C^{(l)}_{ij}(y_2, y'_2, \omega)}{R} s^{(n)}_i(y_2, \omega)dy'_2 = \frac{\pi}{T} \lambda^{(n)}_{s_i}(y_2, \omega)
\]

(4.5)

Obtaining this spanwise averaged cross-spectrum constitute the first step. This operation is realized from two-point velocity statistics of the channel flow DNS and is presented in the next section. Subsequent sections will also provide complete information about the implementation of the other steps, mainly concerning normalization of the intermediary quantities.
4.2 Spanwise Averaged Velocity Cross Spectrum: Step (1)

The first step in the implementation of the trailing edge noise prediction method of Glegg et al. (2004) is to relate the velocity cross-spectrum of the DNS to the spanwise averaged velocity cross-spectrum required to extract the velocity fluctuations forming the source term.

Recalling Chapter 2, the spanwise averaged velocity cross-spectrum needed in the noise calculation is given by equation (2.46) as

\[
C_{ij}^{(1)}(y_2, y_2', \omega) = \frac{T}{L} \int_{-L}^{L} \mathcal{E}[u_i^{(L)}(y_2, 0, \omega)^* u_j^{(L)}(y_2', \Delta y_3, \omega)] d(\Delta y_3)
\]  

The goal is therefore to relate \(C_{ij}^{(1)}(y_2, y_2', \omega)\) to the cross-spectrum, \(\mathcal{E}[u_i^*(k_1, y_2, k_3) a_j(k_1, y_2', k_3)]\), obtained from the DNS data reduction presented in Chapter 3.

To find the relationship between the two cross-spectra, use is made of some results developed in Appendix A and the different definitions of cross-spectra.

Consider the two random stochastic variables to be the two velocity components \(u_i\) and \(u_j\) respectively at \(y_2\) and \(y_2'\). Keeping \(y_2\) and \(y_2'\) implicit in the derivations, the frequency-wavenumber cross-spectrum as defined in equation (A.7) becomes

\[
\Theta_{ij}^{(1)}(\omega, k_3) = \frac{1}{2\pi} \int_{-L_3}^{L_3} \mathcal{E}[u_i^*(\omega, y_3) u_j(\omega, y_3 + \Delta y_3)] e^{-jk_3\Delta y_3} d\Delta y_3
\]

For zero \(k_3\) this expression takes the form of an average over \(2L_3\),

\[
\Theta_{ij}(\omega, 0) = \frac{1}{2T} \int_{-L_3}^{L_3} \mathcal{E}[u_i^*(\omega, y_3) u_j(\omega, y_3 + \Delta y_3)] d\Delta y_3
\]

or for \(y_3 = 0\), since the spanwise direction is homogeneous and only depends on separation,
\[ \Theta_y(\omega,0) = \frac{1}{2T} \int_{-L}^{L} \text{Ex}[u_i^*(\omega,0)u_j(\omega,\Delta y_i)]d\Delta y_i \] (4.9)

At the limits of the computational domain, the term \( \text{Ex}[u_i^*(\omega,0)u_j(\omega,\Delta y_i)] \) decays to zero. Consequently, the integral over the width of the computational domain is equivalent to the integral over the wetted span, \( 2L \). Equation (4.9) can then be rewritten as

\[ \Theta_y(\omega,0) = \frac{1}{2T} \int_{-L}^{L} \text{Ex}[u_i^*(\omega,0)u_j(\omega,\Delta y_i)]d\Delta y_i \] (4.10)

The cross-spectrum obtained in equation (4.10) can then be related to the spanwise averaged cross-spectrum, \( C_{ij}^{(1)} \). Comparison with equation (4.6) lead to

\[ \Theta_y(\omega,k_3)|_{k_3=0} = \frac{L}{2\pi} C_{ij}^{(1)} \] (4.11)

And from equation (A.7)

\[ \Theta_y(\omega,0) = \frac{\pi^2}{TL_3} \text{Ex}[u_i^*(\omega,k_3)u_j(\omega,k_3)]|_{k_3=0} \] (4.12)

So that

\[ C_{ij}^{(1)} = \frac{2\pi^3}{TL_3 L} \text{Ex}[u_i^*(\omega,k_3)u_j(\omega,k_3)]|_{k_3=0} \] (4.13)

The frequency, \( \omega \), is here inferred from the time delay in the spectrum, \( \tau \). This information is however not available from the data set. Another way to obtain the frequency is then to relate it to the streamwise wavenumber, \( k_i \). This operation is possible using Taylor’s hypothesis.

Assuming homogeneity in the streamwise direction, Taylor’s hypothesis relates streamwise location and time such that
where $U_c$ is the convection velocity in the streamwise direction.

So, considering the velocity fluctuation $u$, function of time or space, the definition of Fourier transform (cf. Appendix A) gives

$$u(\omega) = \frac{1}{2\pi} \int_{-T}^{T} u(t) e^{-j\omega t} \, dt$$  \hspace{1cm} (4.15)$$

and

$$u(k_1) = \frac{1}{2\pi} \int_{-L_i}^{L_i} u(y_1) e^{-j k_1 y_1} \, dy_1$$  \hspace{1cm} (4.16)$$

Now, according to Taylor’s hypothesis, the following relations must held

$$T = -\frac{L_i}{U_c}, \quad -k_1 U_c = \omega, \quad dy_1 = -U_c \, dt$$  \hspace{1cm} (4.17)$$

Thus,

$$u(\omega) = \frac{1}{2\pi} \int_{-L_i/U_c}^{-L_i/U_c} u \left( -\frac{y_1}{U_c} \right) e^{-j k_1 y_1} \, dy_1 \quad \frac{dy_1}{-U_c}$$

$$\Rightarrow \quad u(\omega) = \frac{1}{2\pi} \int_{L_i}^{L_i} u(y_1) e^{-j k_1 y_1} \, dy_1$$  \hspace{1cm} (4.18)$$

$$\Rightarrow \quad u(\omega) = \frac{1}{2\pi U_c} \int_{-L_i}^{L_i} u(y_1) e^{-j k_1 y_1} \, dy_1$$

So the velocity fluctuation can be transformed from wavenumber to frequency domain according to the relationship

$$u(\omega) = \frac{u(k_1)}{U_c}$$  \hspace{1cm} (4.19)$$

where the index between $\omega$ and $k_1$ is reversed. And consequently
\[ \text{Ex}[u_i'(\omega)u_j'(\omega)] = \frac{\text{Ex}[u_i'^*(k_1)u_j'(k_1)]}{U_e^2} \quad (4.20) \]

So, applying Taylor’s hypothesis, the spanwise averaged cross-spectrum, \( C_{ij}^{(1)} \), can be expressed as

\[
C_{ij}^{(1)} = \frac{2\pi^3}{L_1U_e} \left( \frac{1}{U_e^2} \right) \text{Ex}[u_i'^*(k_1,k_3)u_j(k_1,k_3)]_{k_3=0}
\]
\[
= \frac{2\pi^3}{L_1L_2L_3U_e} \text{Ex}[u_i'^*(k_1,k_3)u_j(k_1,k_3)]_{k_3=0}
\]

\[(4.21)\]

The final step in relating this cross-spectrum to the data reduction lies in the appropriate scaling of the velocity fluctuations. According to Chapter 3,

\[ \text{Ex}[a_i'(k_1,y_2,k_3)a_j'(k_1,y_2,k_3)] = \frac{\pi^3}{L_1^2L_3^2U_e^2} \text{Ex}[u_i'^*(k_1,y_2,k_3)u_j'(k_1,y_2,k_3)] \quad (4.22) \]

with \( 2L_1 = 2\pi \delta \) and \( 2L_3 = \pi \delta \), so that,

\[
\text{Ex}[a_i'(k_1,y_2,k_3)a_j'(k_1,y_2,k_3)] = \frac{4}{\delta u_r^2} \text{Ex}[u_i'^*(k_1,y_2,k_3)u_j'(k_1,y_2,k_3)]
\]

\[(4.23)\]

And using the relation developed in equation (4.21),

\[
C_{ij}^{(1)} = \frac{2\pi^3}{\pi \delta \frac{\pi \delta}{2} L U_e} \frac{\delta u_r^2}{4} \text{Ex}[a_i'^*(k_1,y_2,k_3)a_j'(k_1,y_2,k_3)]_{k_3=0}
\]
\[
= \frac{\pi \delta^2 u_r^2}{L U_e} \text{Ex}[a_i'^*(k_1,y_2,k_3)a_j'(k_1,y_2,k_3)]_{k_3=0}
\]
\[(4.24)\]

or introducing the normalized spanwise averaged cross-spectrum, \( \tilde{C}_{ij} \),

\[
\tilde{C}_{ij} = \frac{L U_e}{\pi \delta^2 u_r^2} C_{ij}^{(1)}
\]
\[(4.25)\]
Plots and descriptions of this cross-spectrum function for the channel-flow boundary layer and quantities determined from it to obtain pressure modes and noise spectra follow.

The first quantity used in the noise calculation was the spanwise averaged cross-spectrum stored as $\tilde{C}_{y}$, its normalized value. Figure 4.2 shows two-dimensional cuts of this cross-spectrum, $\tilde{C}_{y}(y_2, y_2')$, at different frequencies $\tilde{\omega} = \omega \delta / U_c$. These frequencies span a wide range of frequencies available from the data reduction. The cross-spectra are represented for every other two exponentially stored frequency ($0 \leq \tilde{\omega} \leq 143$).

In general, the magnitude of the cross-spectra decrease as the frequency increases. This result is not surprising since the lowest frequencies are associated to the largest structures and therefore contain more energy. Figure 4.2 shows that the most energy at the lowest frequencies is associated mainly with $\tilde{C}_{11}$ followed by $\tilde{C}_{33}$. The other terms of the cross spectrum appear negligible compared to these two. As the frequency increases, energy associated to $\tilde{C}_{11}$ decreases in magnitude, and energy starts dominating in $\tilde{C}_{33}$ followed by $\tilde{C}_{22}$. Also, the cross-spectra weaken, show less and less extend and start localizing exclusively near the wall. This result is expected since the smaller structures, which are associated to high frequencies, are usually confined in the near wall region.

For the cross-spectrum at zero frequency, all the $u_2$ terms are zero due to continuity. Zero frequency corresponds to zero streamwise and spanwise wavenumber for this particular spectrum. Consequently, from the definition of the Fourier transform (cf. Appendix A), taking the Fourier component for $k_1 = k_3 = 0$ is equivalent to averaging $u_2$ in the streamwise and spanwise directions at a particular $y_2$ location. The $u_2$ terms have therefore to be zero or mass would be created between the wall and this $y_2$ position. Also, the cross term $\text{Ex}[u_1^* u_2]$ is not zero for zero $y_2$-separation as one would expect from the symmetry in the two-point correlation function. As mentioned before, zero spanwise wavenumber corresponds to a spanwise average of the correlation function. The symmetry in the correlation function requires: $R_{13}(y_2, y_2, \Delta y_3) = -R_{13}(y_2, y_2, -\Delta y_3)$. So,
an average of the correlation function over the span should theoretically be zero. The reason for the discrepancies lies probably in the residual uncertainty associated with the fact that the DNS is not fully statistically converged.

4.3 Proper Orthogonal Modes: Step (2)

Using proper orthogonal decomposition on this spanwise averaged cross-spectrum, \( C_{ij}^{(1)} \), it is now possible to obtain the proper orthogonal modes representing the spanwise averaged velocity fluctuations and associated eigenspectrum forming the source term.

The proper orthogonal modes of \( C_{ij}^{(1)} \) are computed solving the eigenvalue problem addressed in equation (4.1). Discretization of this expression leads to

\[
\frac{1}{L_2} \sum_{y_2} C_{ij}^{(1)} (y_2, y'_2, \omega) s_j^{(n)}(y_2, \omega) dy'_2 = \frac{\pi}{T} \lambda_i^{(n)} s_i^{(n)}(y_2, \omega) \tag{4.26}
\]

To solve this eigenvalue problem, the eig function of Matlab was used on \( C_{ij}^{(1)} \), which was calculated for 101 points evenly spaced across the boundary layer. This subroutine provides directly the eigenvalues and associated eigenmodes necessary to build the source terms.

Now by definition, at each frequency, the average energy contained in a velocity mode is equal to one, so

\[
\frac{1}{L_2} \int_0^{L_2} \left| s_j^{(n)}(y'_2) \right|^2 dy'_2 = 1 \tag{4.27}
\]

or in discretized form

\[
\frac{1}{L_2} \sum_{y_2} \left| s_j^{(n)}(y'_2) \right|^2 \Delta y'_2 = 1 \tag{4.28}
\]
The discrete eigenmodes, \( \tilde{s}^{(n)}_j(y'_2) \) are computed normalized such that

\[
\sum_{j_2} \left| \tilde{s}^{(n)}_j(y'_2) \right|^2 = 1
\]  

(4.29)

So,

\[
\frac{1}{L_2} \sum_{j_2} \left| s^{(n)}_j(y'_2) \right|^2 \Delta y' = \sum_{j_2} \left| \tilde{s}^{(n)}_j(y'_2) \right|^2
\]

\[
\Rightarrow \sum_{j_2} \left| \frac{s^{(n)}_j(y'_2) \sqrt{\Delta y'_2}}{\sqrt{L_2}} \right|^2 = \sum_{j_2} \left| \tilde{s}^{(n)}_j(y'_2) \right|^2
\]

(4.30)

\[
\Rightarrow s^{(n)}_j(y'_2) = \tilde{s}^{(n)}_j(y'_2) \frac{L_2}{\Delta y'_2}
\]

where \( \tilde{s}^{(n)}_j \) is the stored spanwise averaged velocity mode. Substituting into equation (4.26) and using Taylor’s hypothesis leads to

\[
\frac{1}{\delta} \sum \tilde{C}_{ij} \frac{\pi \delta^2 u_c^2}{L U_c} \tilde{s}^{(n)}_j(y'_2) \sqrt{\frac{\delta}{\Delta y'_2} \Delta y'_2} = \frac{\pi U_c}{\delta^2} \lambda^{(n)}_i \tilde{s}^{(n)}_j(y'_2, \omega) \sqrt{\frac{\delta}{\Delta y'_2}}
\]  

(4.31)

So,

\[
\sum \tilde{C}_{ij} \tilde{s}^{(n)}_j(y'_2) = \frac{U_c^2 L}{\pi \delta^2 u_c^2 \Delta y'_2} \lambda^{(n)}_i \tilde{s}^{(n)}_j(y'_2, \omega)
\]  

(4.32)

And consequently, the stored eigenvalue, \( \tilde{\lambda}^{(n)}_i \), relates to the eigenvalue, \( \lambda^{(n)}_i \), according to

\[
\tilde{\lambda}^{(n)}_i = \frac{U_c^2 L}{\pi \delta^2 u_c^2 \Delta y'_2} \lambda^{(n)}_i
\]

(4.33)

The eigenspectra are shown in Figure 4.3 for the same frequencies \( \tilde{\omega} \) presented in Figure 4.1. The matrix \( \tilde{C}_{ij}(y'_2, y'_1) \) is of order (303, 303) for every frequency leading therefore to 303 eigenvalues with 303 associated eigenmodes. For the lowest frequencies, the first few modes contain considerably more energy than the others. After 30 modes or
so, the eigenspectra disappears since the eigenvalues become very small in magnitude. As the frequency increases, the modes appear to contain less energy and the decrease in energy for increasing mode number becomes smaller. As noticed for the cross-spectra, Figure 4.4, the eigenvalues decrease in intensity as the frequency increases because they are associated to the smallest structures which contain less energy.

The velocity mode shapes, both imaginary and real parts, corresponding to the first four eigenvalues at the set of frequencies used in the previous figures are displayed in Figure 4.4. For zero frequency, only $u_1$ and $u_3$ fluctuations are present. The $u_2$ fluctuations are zero since the spatial average at a fixed $y_z$-location must be zero even instantaneously, otherwise continuity would not be satisfied.

In general, several features of the velocity modes are interesting. First, at low frequencies, i.e. up to $\tilde{\omega} = 34.5$, the velocity profiles span the entire channel half height, indicating large eddy structures. As the frequency increases, the extent of the modes diminishes and they become localized in the near wall region, just as in the cross-spectrum case. For example, at $\tilde{\omega} = 34.5$ the velocity fluctuations vanish for $y_z/\delta > 0.5$; at $\tilde{\omega} = 143$ the velocity fluctuations reach zero by $y_z/\delta = 0.25$. Comparison with the cross-spectra shows other similarities. For low frequencies, the cross-spectra were associated mainly with $u_1$ and $u_3$ terms. Similarly, the eigenmodes show dominant $u_1$ and $u_3$ fluctuations up to $\tilde{\omega} = 7.5$. As the frequency increases, the cross-spectra showed a switch from the $u_1$ to $u_2$ terms. The same feature can be observed in the modes for $\tilde{\omega} > 21$ where the amplitude of $u_1$ fluctuations starts to become smaller and the amplitude of $u_2$ fluctuations larger or even dominant like in the second mode at $\tilde{\omega} = 34.5$.

Finally, it is interesting to notice that the $u_3$ fluctuations in the modes show a change of sign in the vicinity of the wall at almost every frequency. This change of sign probably indicates the presence of streamwise eddies near the wall.

The numerical resolution of 101 points across the channel was tested against a 201 points resolution. Figure 4.5 shows a comparison of the eigenvalues and eigenmodes
obtained using the two different resolutions. Both eigenspectra and eigenmodes show similar results justifying the resolution chosen in the calculations.

The spanwise averaged velocity fluctuations can now be used to define the source term used in noise predictions.

### 4.4 Source Term: Step (3)

The source term for the \( n^{th} \) mode was defined in chapter 2 as

\[
Q^{(n)}(k, y_2) = \frac{L}{2\pi} \left[ \Omega_3 (i\omega \hat{s}_1^{(n)} + \omega s_2^{(n)}) + i\omega \left( i\omega s_2^{(n)} - V(y_2) \frac{\partial s_1^{(n)}}{\partial y_2} \right) \right]_{y_2 = d}
\]  

(4.34)

In addition to the previous normalizations for the velocity fluctuation, additional normalized quantities can be substituted for the stored quantities using the relationships

\[
\tilde{V}u_r = V \\
\frac{\tilde{\omega}U_c}{\delta} = \omega \\
\tilde{\Omega}u_r = \Omega
\]

(4.35)

Also, noting that in the case of a two-dimensional boundary layer the mean vorticity can be expressed as

\[
\Omega = \Omega_3 = -\frac{\partial V}{\partial y_2}
\]

(4.36)

the source term can then be defined as

\[
Q^{(n)}(k, y_2) = \frac{L}{2\pi} \left[ \frac{\tilde{\Omega}u_r}{\delta} \left( i\frac{\tilde{\omega}U_c}{\delta} \hat{s}_1^{(n)} (y_2') \sqrt{\frac{L_2}{\Delta y_2}} + \frac{\tilde{\omega}U_c}{\delta} \hat{s}_2^{(n)} (y_2') \sqrt{\frac{L_2}{\Delta y_2}} \right) - \left( \frac{\tilde{\omega}U_c}{\delta} \right)^2 \hat{s}_2^{(n)} (y_2') \sqrt{\frac{L_2}{\Delta y_2}} - i\frac{\tilde{\omega}U_c}{\delta} \tilde{V}u_r \frac{\partial s_1^{(n)}}{\partial y_2} \sqrt{\frac{L_2}{\Delta y_2}} \right]_{y_2 = d}
\]

(4.37)
So,

\[
Q^{(n)}(k, y_2) = \frac{u_c U_c}{\pi \delta^2} \sqrt{\frac{L_2}{\Delta y_2}} \frac{1}{2} \left[ \hat{\omega} \hat{i} \hat{\epsilon} \hat{s}^{(n)}_1 + \hat{s}^{(n)}_2 \right] - \frac{U_c}{u} \hat{\omega} \hat{s}^{(n)}_2 - i \hat{\omega} \hat{V} \frac{\hat{s}^{(n)}_1}{\delta} \right] \tag{4.38}
\]

Or,

\[
\hat{Q}^{(n)}(k, y_2) = \frac{\pi \delta^2}{Lu u_c} \sqrt{\frac{\Delta y_2}{\delta}} Q^{(n)}(k, y_2) = \frac{1}{2} \left[ \hat{\omega} \hat{i} \hat{\epsilon} \hat{s}^{(n)}_1 + \hat{s}^{(n)}_2 \right] - \frac{U_c}{u} \hat{\omega} \hat{s}^{(n)}_2 - i \hat{\omega} \hat{V} \frac{\hat{s}^{(n)}_1}{\delta} \right] \tag{4.39}
\]

where \( \hat{Q}^{(n)}(k, y_2) \) is the stored value of the source term \( Q^{(n)}(k, y_2) \).

The source term modes \( \hat{Q}^{(n)}(k, y_2) \) are plotted in Figure 4.6 for similar frequencies as the previous quantities. The source term modes appear highly oscillatory and span the entire half channel at low frequencies. For frequencies between 21 and 90, the source term appears to be confined in the lower 40% of the half channel height. Higher frequencies show the source term to lie in the lower 20% of the boundary layer. The physical aspect behind this is probably the same as for the previous quantities. Lower frequency structures extend over the whole boundary layer while higher frequency structures confine in the bottom of the channel.

The source term so defined can then be integrated to obtain the acoustic spectrum. Before obtaining the spectrum though, an integrand of the pressure mode can be defined and is normalized in the following section.

### 4.5 Integrand of the Pressure Mode: Step (4)

Defining the integrand of the pressure mode from equation (4.3) as
$I^{(n)} = \sqrt{M_v (1-\sigma)} Q^{(n)}(k, y_2) e^{-\frac{M_2 y_2}{\nu}} \left(1 - \frac{1}{1 - M_{\nu}} \right)^{1/2}$ \hspace{1cm} (4.40)

For low Mach numbers, this expression reduces to

$I^{(n)} = \sqrt{M_v (1-\sigma)} Q^{(n)}(k, y_2) e^{-\frac{M_2 y_2}{\nu}} \left(1 - \frac{1}{1 - M_{\nu}} \right)^{1/2}$ \hspace{1cm} (4.41)

Now, the term $\sigma$, which determines the effect of the Kutta condition, is expressed as $\sigma = \frac{U_c}{V(y_2)}$. So here, the wake convection speed is assumed to be similar to the convection speed used inside the boundary layer.

Then,

$I^{(n)} = \sqrt{U_c \sqrt{c_o} \left(1 - \frac{U_c}{V} \right)} \tilde{Q}^{(n)}(k, y_2) \frac{L u_c U_c}{\pi \delta^2} \frac{M_2}{\Delta y_2} e^{-\frac{M_2 y_2}{\nu}} \left(1 - \frac{1}{1 - M_{\nu}} \right)^{1/2}$ \hspace{1cm} (4.42)

And

$\tilde{I}^{(n)} = \frac{\pi \delta^2}{L u_c U_c} \sqrt{\frac{\Delta y_2}{\delta \nu u_c I^{(n)}}}$ \hspace{1cm} (4.43)

or

$\tilde{I}^{(n)} = \sqrt{\frac{U_c}{V} \left(1 - \frac{U_c}{V} \right)} \tilde{Q}^{(n)}(k, y_2) e^{-\frac{M_2 y_2}{\nu}} \left(1 - \frac{1}{1 - M_{\nu}} \right)^{1/2}$ \hspace{1cm} (4.44)

This integrand, $\tilde{I}^{(n)}$, is plotted in Figure 4.7 at the same frequencies shown in earlier figures and shows the same general characteristics as the source term. The main differences can be seen in the extent over the half channel. Mathematically, this result can be explained by the exponential term damping the source term in equation (4.44). As the frequency and $y_2$ increase, the integrand lowers exponentially in magnitude. For high frequencies, the integrand shows mainly spikes very close to the wall.

All the elements necessary to obtain pressure modes and the acoustic spectrum are now defined and are to be used in the next section.
4.6 Pressure Modes and Acoustic Spectrum: Step (5) and (6)

Pressure modes and acoustic spectrum are the ultimate quantities wanted in the noise calculation. For low Mach number, the pressure for the \( n^{th} \) mode, given by equation (4.2) becomes

\[
|p^{(n)}(x, \omega)| = \left( \frac{\rho_o \sin(\theta/2)}{\sqrt{2\omega r_c}} \right) \int_0^\infty \frac{I dy_2}{\Delta y_2}
\]

Discretizing this equation and using the previous results leads to

\[
|p^{(n)}(x, \omega)| = \left( \frac{\rho_o \sin(\theta/2)}{\sqrt{2\omega r_c}} \right) \frac{L u_c U_c}{\pi \delta^2} \sqrt{\frac{\partial}{\Delta y_2}} \sqrt{\frac{u_c}{c_o}} \left| \sum \tilde{I} \Delta y_2 \right|
\]

\[
\Rightarrow |p^{(n)}(x, \omega)| = \left( \frac{\rho_o \sin(\theta/2)}{\sqrt{2\omega r_c}} \right) \frac{\delta}{U_c \tilde{\omega}} \frac{L u_c U_c}{\pi \delta^2} \sqrt{\frac{u_c}{c_o}} \left| \sum \tilde{I} \right|
\]

\[
\Rightarrow |p^{(n)}(x, \omega)| = \left( \frac{\rho_o \sin(\theta/2)}{\sqrt{2r_c}} \right) \sqrt{\frac{L^2 u_c^3 \Delta y_2}{\pi^2 c_o \delta}} \sum i
\]

And the corresponding acoustic spectrum is then

\[
S_{pp}(x, \omega) = \frac{\pi}{T} \mathbb{E} \left[ |p(x, \omega)|^2 \right] = \frac{\pi}{T} \sum_{n=1}^\infty \lambda_n^{(n)} \left| p^{(n)}(x, \omega) \right|^2
\]

So,

\[
S_{pp}(x, \omega) = \frac{\pi}{T} \sum_{n=1}^\infty \left( \frac{\pi \delta^2 u_c^2 \Delta y_2}{U_c^2 L} \right) \tilde{\lambda}_n^{(n)} \left( \frac{\rho_o \sin(\theta/2)}{\sqrt{2r_c}} \right)^2 \frac{L^2 u_c^3 \Delta y_2}{\pi^2 c_o \delta} \left| \sum \tilde{I} \right|^2
\]

\[
S_{pp}(x, \omega) = \left( \frac{\rho_o \sin(\theta/2)}{r_c} \right)^2 \frac{L u_c^3 \Delta y_2}{2\pi U_c c_o} \sum \tilde{\lambda}_n^{(n)} \sum \tilde{I}^2
\]
The so defined quantities are now related to experimental results using the normalizations developed in this section.

### 4.7 Comparison with Experiment

To scale the results from the DNS and compare with experimental results, run 295 from Brooks et al. (1989) is chosen. This particular test case is interesting since the Reynolds number based on the boundary layer thickness, \( U_e \delta / \nu \), is close to 15000. Indeed, the equivalent Reynolds number for the DNS is about 13000.

The model tested in this case was a NACA 0012 airfoil with a chord length of 10.16 cm and a 45.7 cm span. The tunnel used had a 30.48 by 45.72 cm rectangular exit at a zero angle of attack. The freestream velocity was of 39.6 m/s.

For measurement purposes, eight 1.27 cm diameter free-field response microphones were distributed in the plane perpendicular to the airfoil midspan at a distance of 1.22 m. One microphone was offset from this plane.

The far field noise spectra from Brooks et al. were obtained for an observer located perpendicular to the midspan, \( \theta = 90^\circ \) and \( \alpha = 90^\circ \), at a distance \( r_e = 1.22 \) m. The density and sound speed were respectively taken as 1.1 kg/m\(^3\) and 340 m/s.

For the calculations, the convection velocities utilized for the wake, \( W \), and for Taylor’s hypothesis in the boundary layer, \( U_c \), were both chosen as \( 0.6 U_e \).

Also, for this symmetric airfoil, predictions were doubled to account for the boundary layers on both sides of the trailing edge.

The noise predictions based on the source terms of the DNS are compared with Brooks et al.’s measured spectra and associated curve fit in Figure 4.8. The results are presented as single-sided one third octave band sound pressure level (SPL) spectra. From the previous derivations, it is important to see that no empirical information connecting the boundary layer properties to the farfield or to the unsteady pressure around the
trailing edge was involved in the predictions. The calculations rely purely on the two-point velocity statistics and the flow mean-velocity provided by the DNS.

Interestingly, the predictions come very close to the measurements and curve fit. For medium to high frequencies, 3 kHz and above, the results lie within 3 dB of the measurements. Also, the calculated spectrum seems to predict the measured change in slope of the measured spectrum observed around 5 kHz.

The main discrepancies appear for low frequencies, below 3 kHz, where most calculations underestimate the measurements by about 5 dB. The disagreements for these frequencies are not surprising though. The DNS data used were generated for a channel flow and not an external boundary layer. DNS for channel flows are available at higher Reynolds number than for external boundary layers because periodicity conditions can be applied in the streamwise direction. So, the largest differences with an external boundary layer is expected in the outer part of the boundary layer and at frequencies or wavenumbers implying scales of the order of the boundary layer thickness or larger. The lowest frequencies correspond to the longest streamwise length scales and the four points corresponding to this range of frequencies have wavelengths of $2\pi/\delta$, $\pi/\delta$, $2\pi/3\delta$ and $\pi/2\delta$. Consequently, this portion of the spectra is the most affected by the periodicity conditions and can suffer from significant errors.

In the implementation of the noise prediction method, assumptions were made concerning convection velocities. In the boundary layer and the application of Taylor’s hypothesis the convection velocity was arbitrarily chosen as $0.6U_e$. Also, in the application of the Kutta conditions, the term $\sigma = \frac{W}{V}$ was introduced. The convection speed of the wake was also set to $0.6U_e$. These assumptions represent therefore potential sources of error. However, Figure 4.9 clearly shows that the importance of the convection speed is minimal. The comparison of noise spectra for convection speeds of $0.4U_e$, $0.6U_e$ and $0.8U_e$ do not show significant differences. The level and shape of the spectrum are very similar. The only major differences that arise from the change in convection speed are expected and concern the frequencies associated with the DNS source terms.
The agreements between the predictions and the measurements of Brooks et al. suggest that the present method is appropriate for noise calculations and can therefore be used to reveal the features and parts of the boundary layer contributing to the far field noise.

4.8 Noise Contributions

The source term in equation (4.34) can be decomposed into four terms. These terms are the explicit representation of the velocity fluctuation contribution.

\[
\begin{align*}
\text{Term 1} & = i\omega s_1^{(n)} + \Omega_3 \omega s_2^{(n)} - \omega \omega_2^{(n)} - i\omega V(y_2) \frac{\partial^2 s_1^{(n)}}{\partial y_2^2} \\
\text{Term 2} & = i\omega s_1^{(n)} + \Omega_3 \omega s_2^{(n)} - \omega \omega_2^{(n)} - i\omega V(y_2) \frac{\partial^2 s_1^{(n)}}{\partial y_2^2} \\
\text{Term 3} & = i\omega s_1^{(n)} + \Omega_3 \omega s_2^{(n)} - \omega \omega_2^{(n)} - i\omega V(y_2) \frac{\partial^2 s_1^{(n)}}{\partial y_2^2} \\
\text{Term 4} & = i\omega s_1^{(n)} + \Omega_3 \omega s_2^{(n)} - \omega \omega_2^{(n)} - i\omega V(y_2) \frac{\partial^2 s_1^{(n)}}{\partial y_2^2}
\end{align*}
\]

Terms 1 and 2 represent the contributions from the streamwise and normal to wall velocity fluctuations whereas terms 3 and 4 represent the contributions of these velocities to the spanwise vorticity fluctuations. It is therefore possible to compute the individual contribution of these terms to the overall noise. The individual contributions are presented in Figure 4.10 and do not add up to the total noise since the squaring in equation (4.47) generates cross terms that are not included for these calculations.

The contribution of the normal to the wall velocity term 2 appears negligible. Also, the other term associated to the normal to the wall velocity, the vorticity term 3, is a minor contributor to the overall noise. The streamwise velocity terms however are the dominant contributor to the noise spectrum, especially term 4.

The noise can also be decomposed in a modal form. From the velocity modes, it is possible to find the source term modes, integrand modes and finally pressure modes contributing to the overall noise.
A major motivation in the use of modes to represent the turbulences and compute the noise is that only a limited number of modes are needed to obtain an accurate representation of the flow. Figure 4.11 shows the cumulative contributions to the overall sound pressure level from the modes. Only the first five or six dominant modes at each frequency are needed to obtain an overall sound pressure level within 1 dB of the total predicted level.

The first five modes contributions to the noise at each frequency are represented in Figure 4.12. These modes are ordered by the magnitude of their contributions to the noise and consequently guarantee the decay with increase in mode number. However, these modes are not much different than that based on the eigenvalues of the cross spectrum $C_{ij}^{(l)}$ as shown in Figure 4.13. So, the most energized modes of the cross spectrum function tend to be dominating the noise production too.

Finally, the formulation this noise prediction scheme can be utilized to compute the different part of the boundary layer responsible of the noise production. Figures 4.14 through 4.16 show the different contributions divided up by location in the boundary layer. Figure 4.14 represents the noise spectra predicted when the integration of the source term in equation (4.45) is carried out over slices one tenth of the boundary layer thickness in size. The sum is not equal to the total noise since the cross products arising from the squaring in equation (4.47) are not included. This plot shows clearly a predominance of the source terms in the bottom 10% of the boundary layer. This result can be explained mathematically by the exponential weighting function in the integrand, equation (4.40). Physically, the results show that the largest scales at low frequency motion in the near wall region are the major noise producer. This result is reinforced when the same calculations are carried out in the bottom 10% of the boundary layer. Figure 4.15 shows the predicted noise spectrum for the ten points calculated in the bottom of the boundary layer. Here again, the lowest points of this part of the boundary layer are the major contributor to the overall trailing edge noise.

The integration of the source term can also be cumulatively carried over the boundary layer. Figure 4.16 shows that actually not every part of the boundary layer
described previously contribute to the noise. The figure shows that the external part of the boundary layer \((y_2/\delta \geq 0.7)\) are reducing the overall noise. For the largest structures, physical explanation can be found in the phase difference between the near wall and outer region of the boundary layer. Source terms in the outer region are then canceling the production of the lower parts. This might be an effect of the conditions imposed by the boundaries of the channel flow and therefore the use of a complete data set for an external boundary layer would be interesting to get more insight about the reality of this cancellation mechanism.

Finally, the numerical resolution used for the calculations was tested again by performing the same noise calculations but using 201 points across the channel half-height. Figure 4.17 shows a comparison of the SPLs obtained using 101 and 201 points. Once again, the results are similar and the 101 points can be assumed sufficient to perform the calculations.

4.9 Conclusions

The application of the trailing edge noise model developed by Glegg et al. (2004) using source terms extracted from the DNS showed some realistic results. The realism of these predictions provided some justification in the use of the modes of the turbulent velocity field near the trailing edge for noise prediction. Although the predictive capability of this model has been shown, this method suffers from the lack of available two-point velocity information. The following section describes how the turbulent velocity field might be extrapolated from basic information in order to obtain the source terms required by Glegg’s method.
Figure 4.1. Schematic of the major steps in the calculation of the trailing edge noise acoustic spectrum from two-point velocity statistics and mean velocity.
Figure 4.2. Magnitude of the normalized nine components of the spanwise averaged cross-spectrum, $\tilde{C}_{ij}(y_2', y_2')$, for different normalized frequencies $\tilde{\omega}$.
Figure 4.2. Magnitude of the normalized nine components of the spanwise averaged cross-spectrum, $\tilde{C}_{ij}(y_2,y'_2)$ for different normalized frequencies $\tilde{\omega}$. 
Figure 4.3. Eigenvalue spectra of the cross-spectrum for the first 50 modes obtained from the one-dimensional proper orthogonal in the $y_z$-direction for different normalized frequencies $\omega$. 
Figure 4.3. Eigenvalue spectra of the cross-spectrum for the first 50 modes obtained from the one-dimensional proper orthogonal in the $y_2$-direction for different normalized frequencies $\tilde{\omega}$. 
Figure 4.4. Modal profiles for the first four modes of the spanwise averaged cross-spectrum obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction for different normalized frequencies $\tilde{\omega}$. 
Figure 4.4. Modal profiles for the first four modes of the spanwise averaged cross-spectrum obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction for different normalized frequencies $\tilde{\omega}$. 

$\omega = 7.5$

$\omega = 13$

$\omega = 21$
Figure 4.4. Modal profiles for the first four modes of the spanwise averaged cross-spectrum obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction for different normalized frequencies $\bar{\omega}$. 
Figure 4.4. Modal profiles for the first four modes of the spanwise averaged cross-spectrum obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction for different normalized frequencies $\bar{\omega}$. 

Mode 1

Mode 2

Mode 3

Mode 4

$\frac{y_2}{\delta}$
Figure 4.5. Comparison of the eigenvalue spectra (top) and modal profiles (bottom) for the first four modes of the spanwise averaged cross-spectrum obtained using a resolution of 101 and of 201 points across the channel half-height. Symbols in the modal profiles represent the 201 points resolution; lines represent the 101 points resolution.
Figure 4.6. Source term modal profiles for different normalized frequencies $\tilde{\omega}$. 

$\frac{y}{\delta}$
Figure 4.6. Source term modal profiles for different normalized frequencies $\tilde{\omega}$. 

$\omega = 7.5$

$\omega = 13$

$\omega = 21$
Figure 4.6. Source term modal profiles for different normalized frequencies $\tilde{\omega}$. 

$y_2/\delta$
Figure 4.6. Source term modal profiles for different normalized frequencies $\tilde{\omega}$. 
Figure 4.7. Integrand of the modal profiles for different normalized frequencies $\tilde{\omega}$. 

$y_2/\delta$
Figure 4.7. Integrand of the modal profiles for different normalized frequencies $\tilde{\omega}$. 

Mode 1 

$\omega = 7.5$

Mode 2 

$\omega = 13$

Mode 3 

$\omega = 21$

$y_2/\delta$
Figure 4.7. Integrand of the modal profiles for different normalized frequencies $\tilde{\omega}$. 

Integrand vs. $y_2/\delta$
\[ y_2/\delta \]

Figure 4.7. Integrand of the modal profiles for different normalized frequencies \( \tilde{\omega} \).
Figure 4.8. Far field trailing edge noise spectra from Brooks et al., run 295, compared with predictions made using the current method with source terms computed from a DNS solution for $U_c = W = 0.6U_e$. The two plots show different frequency ranges.
Figure 4.9. Effect of convection speed on the computed noise level. The convection speed, $U_c$, is assumed equal to the wake convection velocity.

Figure 4.10. Contribution to the total noise from the four terms forming the source term in equation (4.50).
Figure 4.11. Cumulative contribution to the predicted total noise as a function of mode number.
Figure 4.12. Contribution to the predicted noise spectrum from the five most dominant modes.
Figure 4.13. Comparison of the contributions to the noise from the modes ordered as the eigenvalues of the cross spectrum $C_{ij}^{(1)}$ and ordered by the magnitude of their contribution to the noise.
Figure 4.14. Contributions to the predicted noise spectrum from different parts of the boundary layer in slices 1/10th of the boundary layer thickness in size.
Figure 4.15. Contributions to the predicted noise spectrum from different parts of the boundary layer bottom 10% of the boundary layer.
Figure 4.16. Cumulative contributions to the predicted noise spectrum from different parts of the boundary layer in slices 1/10th of the boundary layer thickness in size.
Figure 4.17. Far field trailing edge noise spectra from Brooks et al., run 295, compared with predictions made using the current method with source terms computed from a DNS solution with different numerical resolutions.
Prediction of the Two-Point Velocity Correlation Function for Noise Calculations

The basis of the modeling of the two-point correlation function for a turbulent boundary layer type flow comes from the development of Devenport et al. (1999, 2001) and Devenport and Glegg (2001) for the prediction of velocity correlations in wake type of flows. This method is described in the subsequent section and is subject to modifications in later sections of this chapter to account for the evident differences between wakes and boundary layers.

The present chapter develops in a first part the direct application of the original model of Devenport et al. (2001) using a single lengthscale to describe the entire flow. Then, the model is subject to modifications, using a mixing lengthscale and a macroscale lengthscale. Also, some of the numerical differentiation of the original method is replaced
by analytical differentiation, and an improvement to the von Karman spectrum model, a necessary condition for vorticity correlation prediction, is made. Moreover, since CFD does not always provide Reynolds stress statistics, a simple turbulence model, the algebraic stress model as defined by Rodi (1984), is presented to extrapolate this information from turbulence kinetic energy and dissipation results. Finally, noise predictions using the modeled two-point correlation of the channel flow are performed and compared to the predictions of the previous chapter.

For comparison purposes, the main data set utilized is the channel flow DNS, due to its completeness. Comparisons with the boundary layer experimental data set are also presented to observe how general and realistic the predictions can be.

5.1 Description of the Original Model

5.1.1 Modeling

This section describes the basis of the theoretical method developed for plane wakes by Devenport et al. (1999, 2001) and Devenport and Glegg (2001). This method is based on the hypothesis that sufficient information is included in the Reynolds stress field which coupled with the continuity constraints should be enough to determine the inhomogenous aspects of the two-point correlations anywhere in the flow.

The two-point correlation tensor is defined by:

$$R_{ij}(\mathbf{y}, \mathbf{y}') = \text{Ex}[u_i(y)u_j(y')]$$

where $y = (y_1, y_2, y_3)$ and $y' = (y'_1, y'_2, y'_3)$ denote the two points, and $u_i$ ($i = 1,2,3$) denotes the velocity components according to Figure 3.1.

To satisfy the condition of incompressibility of the flow, the spatial correlation tensor, $R_{ij}$, must be divergence free with respect to both positions. This condition can be guaranteed by expressing this tensor as the double curl of the vector-potential correlation, $q_{ij}$ (Chandrasekhar, 1950).
\[ R_{ij}(\mathbf{y}, \mathbf{y}') = \varepsilon_{ikl} \varepsilon_{jmn} \frac{\partial^2 a_{in}(\mathbf{y}, \mathbf{y}') h(s)}{\partial x_m \partial x_k} \] (5.2)

For homogeneous turbulence it is straightforward to show that the vector-potential is simply

\[ q_{ij}(\mathbf{r}, \mathbf{r}') = -\frac{1}{2} u^2 h(s) \delta_{ij} \] (5.3)

where \( u \) is the turbulence intensity, \( s \) is the scalar distance between \( \mathbf{y} \) and \( \mathbf{y}' \), and \( h(s) \) is the first moment of the longitudinal correlation coefficient function as defined by Hinze (1973), i.e.

\[ h(s) = \int_0^s f(s') ds' \] (5.4)

For a von Karman turbulence spectrum the first moment of the correlation function can be expressed as

\[ h(s) = 2^{2/5} \left[ \frac{1}{\sqrt{2\Gamma(\frac{4}{5})}} - (k_s)_{4/5} K_{4/5}(k_s) \right] / \Gamma(\frac{4}{5})/k_s^2 \] (5.5)

where \( k_s \approx 0.75/L \) and \( L \) is the longitudinal integral scale of turbulence.

Devenport et al. (1999) proposed an expression analogous to equation (5.3) for inhomogeneous turbulence. This new formulation separates \( q_{ij} \) into the product of the homogeneous function \( h(s) \), that controls the manner in which the correlation falls off with separation between points, and a scaling function \( a_{ij}(\mathbf{y}, \mathbf{y}') \), i.e.

\[ q_{ij}(\mathbf{y}, \mathbf{y}') = a_{ij}(\mathbf{y}, \mathbf{y}') h(s) \] (5.6)

Substituting equation (5.6) into (5.2) leads to

\[ R_{ij}(\mathbf{y}, \mathbf{y}') = \varepsilon_{ikl} \varepsilon_{jmn} \left[ \frac{\partial^2 a_{in}(\mathbf{y}, \mathbf{y}') h(s)}{\partial y_m \partial y_k} + \frac{\partial a_{in}(\mathbf{y}, \mathbf{y}') \partial h(s)}{\partial y_m \partial y_k} + \frac{\partial a_{in}(\mathbf{y}, \mathbf{y}') \partial h(s)}{\partial \mathbf{y}'_m \partial \mathbf{y}'_k} \right] \] (5.7)
The two-point correlation function must be consistent with the prescribed Reynolds stress field of the flow. So, when \( y = y' \) or \( s = 0 \), \( R_{ij} \) must be equal to the Reynolds stress tensor \( \tau_{ij} \). Now, as the first moment of a correlation coefficient function, 

\[ -h(s) \] and its first derivative tend to zero as \( s \) tend to zero

-the second derivative, \( \frac{\partial^2 h(s)}{\partial x_m \partial x_k} \), tends to \(-\delta_{mk}\) as \( s \) tends to zero

So, for zero separation (\( s = 0 \))

\[
R_{ij} (y, y') = \tau_{ij} (y) = -\varepsilon_{ikl} \varepsilon_{jmn} a_{ln} (y, y') \delta_{mk} \\
= a_{ij} (y, y') - \delta_{ij} a_{oo} (y, y')
\]

(5.8)

To satisfy equation (5.8) the scaling function must then be equal to:

\[
a_{ij} (y, y') = \tau_{ij} (y) - \frac{1}{2} \delta_{ij} \tau_{pp} (y)
\]

(5.9)

This relationship shows that it is simple to prescribe a vector potential correlation function consistent with any Reynolds stress field. This result is unexpected because of the otherwise second-order differential relationship between the vector potential and velocity correlation functions. The simplicity of this expression is what makes this method practical and possible to implement.

Devenport and Glegg (2001) proposed an expression for \( a_{ij} (y, y') \) consistent with equation (5.9). The scaling function is expressed in terms of the arithmetic average of the Reynolds stress tensor at \( y \) and \( y' \), specifically

\[
a_{ij} (y, y') = \frac{1}{2} \left[ \tau_{ij} (y) - \frac{1}{2} \delta_{ij} \tau_{pp} (y) + \tau_{ij} (y') - \frac{1}{2} \delta_{ij} \tau_{pp} (y') \right]
\]

(5.10)

which substituted into equation (5.7) gives,
\[ R_{ij}(y,y') = \epsilon_{ikl} \epsilon_{jmn} \frac{\partial^2 (b_{ln}(y) + b_{ln}(y')) h(s)}{\partial x_m \partial x_k} \]

\[ = \epsilon_{ikl} \epsilon_{jmn} \left[ \frac{\partial b_{ln}(y')}{\partial x_m} \frac{\partial h(s)}{\partial x_k} + \frac{\partial b_{ln}(y)}{\partial x_m} \frac{\partial h(s)}{\partial x_k} + (b_{ln}(y) + b_{ln}(y')) \frac{\partial^2 h(s)}{\partial x_m \partial x_k} \right] \]  

(5.11)

This formulation is attractive because of its simplicity and because it makes \( R_{ij} \) dependent only on the derivatives of \( h(s) \) and not its actual value. This property guarantees decay of the resulting correlation at large separations (cf. Devenport and Glegg (2001)).

To apply this model to a flow requires as input only the Reynolds stress field \( \tau_{ij}(y) \) and the lengthscale \( L \), used to scale the decay function in equation (5.5). The inhomogeneous flow is treated as homogeneous turbulence. However, the formulation of the model implies constraints as to be consistent with the prescribed Reynolds stress field. The modifications are applied on \( q_{ln} \) and therefore the result is not a trivial product of the von Karman spectrum and the turbulence stress field. The expression implied by the double curl of the vector-potential correlation is a much more complex function. The lengthscales and form of this function vary spatially in a manner implied by continuity and inhomogeneity, as well as the von Karman form (cf. Devenport and Glegg (2001)).

Devenport et al. (1999, 2001) and Devenport and Glegg (2001) tested this model by using it to predict the 4-dimensional 2-point correlation tensor of a plane wake, specifically \( R_{ij}(y,y',z,\tau) \). The model used the measured Reynolds stress profile as input, as well as a lengthscale \( L \) set to be three-quarters of the half-wake width. They found the model provided a satisfactory prediction of the entire two-point velocity correlation of the wake. Also, proper orthogonal modes and compact eddy structures were found to be well reproduced by the model. These modes are needed for rigorous aeroacoustic noise calculations (such as leading edge noise produced by a blade impacting on the wake). These predictions are presented in the next section. The results obtained using the model of Devenport et al. (2001) were reproduced to ensure the reproducibility of the results and also have a basis for applying this method to boundary layers. Indeed, the first step in predicting the two-point correlation function in a boundary
layer was to adapt the original model used for wakes but using the appropriate input for lengthscale and Reynolds stress fields.

5.1.2 Measurements of Devenport et al. (2001)

The measurements were made in the Virginia Tech Low-Speed Wind Tunnel (Figure 5.1). The empty test-section is 914mm by 610mm and produces a low-turbulence (<0.3%). The airfoil tested was a 203mm section NACA 0012 airfoil with a distributed roughness trip in order to increase the Reynolds number of its boundary layers and consequently of the wake.

Measurements were taken using hot-wire probes capable of simultaneous three-component measurement from a 0.5 mm³ measurement volume. These measurements were made at a chord Reynolds number of $3.28 \times 10^5$ and at a $x/c = 8.33$, $x/\theta = 900$, position where the flow is fully developed.

The half-wake width $L$ (distance from the centerline to the location where the mean velocity deficit is $0.5U_w$) at that position was 17.4 mm and the centerline mean velocity deficit, $U_w$, 5.6% the freestream velocity $U_e$.

The smallest separation between the probes was 2.5mm, which corresponds to 16 times the expected Kolmogorov lengthscale. The first probe was placed at 17 positions on the $y$-axis and for each of these positions the second probe was placed at some 400 positions at different $z$ locations. 3072 samples were taken for each point and 50 records recorded by the eight sensors. The data are available from the following URL: http://www.aoe.vt.edu/flowdata.

The velocity fluctuations presented in this section are normalized on the mean velocity deficit, $U_w$. 
5.1.3 Results and Discussion for the Results of Devenport et al. (2001)

Reynolds stress field

The first basis in the development of the model is the incorporation of the Reynolds stress tensors obtained from the measurements described in the previous section. The Reynolds stress profiles used in equation (5.10) to calculate the two-point correlations were first smoothed and are plotted in Figure 5.2 using the model. The Reynolds stresses represents the correlations for \( y = y', \) or \( R_{y}(y', y',0,0)/U_{w}^{2}. \)

The results obtained are consistent with the requirements and expectations. The Reynolds normal stress tensors \( \overline{u_{1}^{2}}, \overline{u_{2}^{2}}, \) and \( \overline{u_{3}^{2}} \) tend to zero near the edges of the wake and are symmetric with respect to the centerline. The Reynolds shear stress tensor, \( u_{1}u_{2} \) at zero separation, tends also to zero near the edges and is symmetrically opposed with respect to the centerline. The presence of negative correlations in \( u_{1}u_{2} \) comes from the symmetry of the velocities in the \( y_{2} \)-direction with respect to the centerline of the wake \( (y_{2} = 0). \)

Lengthscale:

The model developed by Devenport et al. (2001) is based on a single integral lengthscale that appears implicitly in the longitudinal correlation coefficient function (equation 5.5).

The lengthscale is determined from the experimental records. The streamwise correlation coefficient for \( u_{2} \) fluctuations at the centerline of the channel, \( \rho_{22}(0,0,0,\tau) \) is used to find the single lengthscale (Figure 5.3). Note that the time-delay, \(-\tau U_{c}\), and the streamwise separation are assumed to be interchangeable because turbulence intensities and mean velocity variations across the wake are small (\( \leq 5\% \)).

The model fits the experimental values for short streamwise correlations but does not reproduce the quasi-periodicity that can be observed experimentally. The absence of
oscillatory behavior was explained by Devenport et al. (2001) by the non-incorporation of physics into the model and the use of the non-oscillatory von Karman decay function. Through a trial and error method to fit the curve the lengthscale for the model was found to be equal to 0.73L.

Correlations

In order to test the model, two-dimensional velocity correlation maps comparing measured and modeled correlations are plotted in Figure 5.4. These plots are made for zero streamwise and spanwise separation and actually only represent a cut through the four-dimensional correlation function.

For the nine components of the correlation function plotted the lines where $y_2 = y'_2$ represent the Reynolds stress profiles, that is $R_{ij}(y_2, y_2, 0, 0)$. As mentioned in Chapter 2, $R_{ij}(y_2, y_2)$ must be equal to $R_{ji}(y'_2, y'_2)$ to be consistent with tensor properties. This feature is visually observable in the correlation map by symmetry about the bottom left to top right diagonal. The measurements show quite good results concerning this symmetry.

The model reproduces the main results obtained from the measurement. However some differences are noticeable. The main difference appears on the $R_{12}$ and $R_{21}$ correlations where the asymmetry about the Reynolds stress profile lines does not appear. Another difference can be noticed in the extent away from the Reynolds stress profile line in $R_{22}$, shorter for the model. Finally, the $R_{11}$ and $R_{33}$ correlations obtained from the model show higher correlations near the boundaries of the wake and differences in the size of the “wings”.

Characteristic Eddy Decomposition

Based on the proper orthogonal decomposition technique developed in Chapter 3, the eigenvalues from the experimental data set and from the modeled two-point correlation function are extracted and compared in Figure 5.5. The eigenvalue spectra are obtained from the solution of the Fredholm integral
∫ \int R_{ij}(y, y', 0, 0) \phi_j(y') dy' = \lambda \phi_i(y) \quad (5.12)

Figure 5.5 shows the eigenvalues for the first twenty modes. An analysis of the measured eigenvalue spectrum reveals that the first two modes account for 27% of the total turbulence kinetic energy and the first four modes to about 37%.

The modal profiles are plotted in Figure 5.6 for the first four modes of the eigenvalue spectra. The modal profiles corresponding to the first two modes combine symmetric and antisymmetric $u_1$ and $u_2$. They are consequently associated with the generation of the Reynolds shear stress $R_{12}$. For the two other modes the motions appear to be more complexly determined. For all these modes the spanwise velocity $u_3$ appears negligible.

Now, using LSE, the velocities are extracted in the homogeneous direction to form the compact eddy structures, as described in Chapter 2.

Figure 5.7 shows two-dimensional cuts in the three-dimensional compact eddy structures for the first four modes deduced from the measured correlation functions and for the first four modes deduced from the model correlation functions. These plane cuts are done at $y_3 = 0$. Both model and measurements deducted plots show the same structures. Every mode shows some predominant spanwise roller-type structures. For the measurements, the structures appear either singly as in modes 1 and 3 or in symmetric pairs as in modes 1, 2 and 4. The model reproduces the dominant features of all the characteristic eddy structures.

**Conclusion**

In the study of a wake flow Devenport et al. (2001) arrived to some interesting conclusions concerning the two-point correlation model developed. This model reproduced many of the features that were obtained from the measurements. The dominant proper orthogonal modes as well as their spectrum and their associated compact eddy structures were reproduced.
Interestingly, only a single lengthscale and information concerning the Reynolds stress field were required to successfully predict the main characteristics of a turbulent wake. No other physics were used in the computations, leading to an interrogation about the amount of information enclosed in the Reynolds stress field and the lengthscale. A simplification of the two-point correlation problem can thus be expected since this information is available through CFD calculations.

The main problem with such a simplistic approach is to see how well the model can be applied to other flows such as boundary layers. This subject is developed in the following section.

5.2 Adaptation of the Model of Devenport et al. (1999, 2001) to a Turbulent Boundary Layer Type of Flow

5.2.1 Modeling the Two-Point Correlation Function

In order to adapt the model developed for wakes several approaches are possible. The first approach would be to use the exact same model. The other approach is to include some of the physics associated to boundary layers. Using this knowledge about boundary layers mechanisms can be useful in order to avoid errors that obviously arise from oversimplistic approaches.

Application of the original two-point model for wakes requires the specification of the Reynolds stress field (for equation (5.10)) and the single constant lengthscale used in equation (5.5). The lengthscale for the model is determined by a curve-fit on the time delay correlation coefficient for $u_2$ fluctuations in the streamwise direction at some normal to wall location. Using the same procedure, it is therefore possible to compute the two-point correlation function for the channel flow of Moser et al. (1998) and for the boundary layer of Adrian et al. (2000).
5.2.1.1 **Modeling the Correlation Decay Function**

Figure 5.8 shows the curve fits for the channel flow and for the boundary layer. The lengthscale was chosen by a trial and error method aiming at fitting the time-delay correlation coefficient function for $u_2$ fluctuations at the center of the channel for the DNS, and at a normal to wall position of $0.5049\delta$ for the boundary layer, i.e. $\rho_{22}(\Delta y_i)$. Both data sets show a best fit for a lengthscale $L = 0.28\delta$.

For the channel flow curve-fit, the main differences appear for short time-delay where the model underestimates slightly the measurements and for longer time-delay where the measurements oscillate about the model curve. For the boundary layer curve fit, the results diverge for large streamwise separation. However, the main cause of these differences does not come from the model but from the uncertainties in the measurements described in Chapter 3.

It is interesting to notice that the reference lengthscale is exactly the same for the application to both data sets, suggesting maybe that $0.28\delta$ could be a general reference lengthscale for boundary layer flows with high Reynolds number. Testing against more experimental data sets would be required though to support this hypothesis.

A single lengthscale might seem unrealistic to characterize the correlation behaviors in the whole flow. In contrast to wakes, boundary layers show some discontinuity in the $y_2$-direction because of the presence of the wall. Boundary layer flows are therefore flows containing regions of different scales. The integral lengthscale at the wall has to be zero and varies with normal to wall positions. Allowing a variable lengthscale to be prescribed in the model could improve predictions. Prescribing such a lengthscale is still consistent with the ultimate application of the model, i.e. as a post processor for CFD results that provides 2-point correlation estimates, since almost all standard turbulence models used in CFD codes give as output some macroscopic lengthscale as a function of position throughout the flow.

Devenport and Glegg (2001) and Devenport *et al.* (2001) discussed how a variable lengthscale and anisotropic decay function could be accommodated within the
model framework. They argue that such variations of this type can be incorporated into the model by replacing equation (5.11) with,

$$R_y(y,y') = \varepsilon_{ijkl} e_{jmn} \frac{\partial^2 (h(s, e_s, y)b_{ln}(y) + h(s, e_s, y')b_{ln}(y'))}{\partial x_m' \partial x_k}$$  \hspace{1cm} (5.13)

where the dependence of $h()$ on $e_s$, the vector direction of separation, allows for the prescription of an anisotropic decay function, and the dependence on $y$ allows for point to point changes in the prescription of the decay function form, or its lengthscale. If only concerned with varying the lengthscale, equation (5.13) can be written as,

$$R_y(y,y') = \varepsilon_{ijkl} e_{jmn} \frac{\partial^2 (h(s, L(y))b_{ln}(y) + h(s, L(y'))b_{ln}(y'))}{\partial x_m' \partial x_k}$$  \hspace{1cm} (5.14)

with $h()$ still given by equation (5.5). As discussed by Devenport and Glegg (2001) this type of manipulation of the decay function does not alter the simple relationship between the scaling function $b()$ and the prescribed Reynolds stress field. The only requirement lies in the first term of the Maclaurin series expansion of $h()$ with respect to $s$. The first term has to be simply $s^2/2$ for the relationship to hold. This condition is satisfied as long as $h()$ can be expressed as a first moment in $s$ of a correlation coefficient-type function such as equation (5.4) where the longitudinal correlation coefficient function, $f()$, is unity and differentiable when $s = 0$.

Two major formulations were used for prescribing the lengthscale $L(y)$ in the present work:

- a formulation using a lengthscale prescribed as proportional to a standard mixing length distribution which is a single fixed function
- a formulation using the turbulent macroscale as basis for prescribing the model lengthscale which is a function based on local conditions and is known from homogeneous turbulence studies to have a formal relationship with the correlation lengthscale.
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The first formulation prescribes the lengthscale, $L$, as proportional to a standard mixing length distribution. The van Driest (1956) mixing length formulation prescribes a length that increases with normal distance to the wall, starting from zero at the wall until it reaches a value of $0.09\delta$. From this point on the mixing length is prescribed as constant at $0.09\delta$. The variation in the near-wall region is given by the equation

$$
\ell_m = \delta_v \kappa y^+_2 \left[ 1 - \exp\left(-y^+_2/A^*\right) \right]
$$

(5.15)

where $\delta_v$ is the viscous lengthscale, $\kappa = 0.41$ and $A^*$, the van Driest constant, equals to 26. For Moser et al.’s (1999) channel flow this reaches a value of $0.09\delta$ at $y_2/\delta = 0.225$. The constant of proportionality between $L$ and $\ell_m$ was chosen so that the value of $L$ at the channel center would be the same as in the constant lengthscale calculations. Specifically, $L = 3.11\ell_m$.

The second formulation is based on a feature commonly used in turbulence models: the association of turbulent kinetic energy, $k$, and of the dissipation of kinetic energy, $\varepsilon$. These quantities are usually given by CFD and permit to define the turbulent macroscale, $k^{3/2}/\varepsilon$, to prescribe the model lengthscale. The turbulent macroscale is calculated by widely used turbulence models such as the $k-\varepsilon$ model based on local conditions. The lengthscale is assumed to be proportional to the turbulent macroscale, i.e.

$$
L \sim \frac{k^{3/2}}{\varepsilon}
$$

(5.16)

For the two-point correlation prediction model, the lengthscale is then scaled so that its maximum, at $y_2/\delta = 0.5$, is equal to the lengthscale used for the constant lengthscale model, that is $L\big|_{y_2/\delta=0.5} = 0.28$.

Several variations on this turbulent macroscale approach may be defined where the turbulent macroscale is calculated using the individual Reynolds normal stress components rather than the complete turbulence kinetic energy. In particular, using wall normal stress $\tau_{22}$ results in a qualitatively different behavior in the near wall region.
Comparison of the different lengthscales used in the model for the channel flow is presented in Figure 5.9a. Also, Figure 5.9b shows the behavior of the lengthscales for positions near the wall. In the viscous wall region, the rate of increase of the lengthscale is the lowest for the macroscale model using $\tau_{22}$, followed by the mixing length and the macroscale model using the total kinetic energy. The mixing lengthscale is the first to reach its maximum at $y_2/\delta = 0.5$. The macroscale models are normalized so that they reach a value of 0.28 at $y_2/\delta = 0.5$. All the models start decreasing around this value except for the $\tau_{22}$ model which increases steadily up to a value of 0.35. The behavior of the lengthscale is important since it determines the rate of decay of the correlation function with separation.

The other input used in the model is the Reynolds stress field. Here no modification in the implementation is necessary. The difference with the wake modeling is that now the Reynolds stresses from the boundary layer data sets are used. The Reynolds stress fields were shown in Figure 3.21 for the channel flow and the boundary layer. However, the Reynolds stress field, required as an input to the model, is not usually available from most computational fluid dynamics (CFD) calculations. It is therefore useful to have a turbulence model associated to the two-point model in order to obtain the Reynolds stress information.

5.2.1.2  **Modeling the Reynolds Stress Field**

Most commonly used CFD calculations provide predictions only of turbulence kinetic energy, $k$, and of dissipation rate, $\varepsilon$. Consequently, in order to use the present method to create the full two-point correlation function, a means is needed to first extrapolate the Reynolds stress field from the kinetic energy and the dissipation rate.
A way to obtain the Reynolds stress field from these parameters is to use a turbulence model. An algebraic stress turbulence model, defined by Rodi (1984), was chosen for its simplicity of implementation.

The closure expression for this model provides a set of linear simultaneous equations. Given mean velocity, the turbulence kinetic energy and the dissipation rate, the six components of the Reynolds stress tensor can then be inferred using these equations.

Neglecting buoyancy, Rodi’s expression for the Reynolds stress becomes

\[
\tau_{ij} = k \left[ \frac{2}{3} \delta_{ij} + \frac{(1 - \gamma) \left( \frac{\mathcal{P}}{\varepsilon} - \frac{2}{3} \delta_{ij} \frac{\mathcal{P}}{\varepsilon} \right)}{c_1 + \frac{\mathcal{P}}{\varepsilon} - 1} \right]
\]

(5.17)

where \(\mathcal{P}\) is the rate of production of turbulent kinetic energy, \(\mathcal{P}_{ij}\) the rate of production of Reynolds stress, \(c_1 = 1.8\) and \(\gamma = 0.6\).

To simplify the problem the flow is assumed to be a simple shear flow. The only significant mean velocity gradient is given by \(\frac{\partial U}{\partial y_2}\). Also, the normal-stress productions are given by:

\[
\begin{align*}
\mathcal{P}_{11} &= 2\mathcal{P} = -2\rho_{12} \frac{\partial U}{\partial y_2} \\
\mathcal{P}_{22} &= 0 \\
\mathcal{P}_{33} &= 0
\end{align*}
\]

(5.18)

Now, equation (5.17) can be obtained for the different components of interest:

\[
\tau_{11} = k \left[ \frac{2}{3} + \frac{(1 - \gamma) \left( -4\frac{\tau_{12}}{3\varepsilon} \frac{\partial U}{\partial y_2} \right)}{c_1 + \frac{\tau_{12}}{\varepsilon} \frac{\partial U}{\partial y_2} - 1} \right]
\]

(5.19)
\[
\tau_{22} = k \left[ \frac{2}{3} + \left(1 - \gamma \right) \frac{2\tau_{12} \frac{\partial U}{\partial y_2}}{3\varepsilon \frac{\partial U}{\partial y_2}} \right] \tag{5.20} \\
\tau_{33} = \tau_{22} \tag{5.21} \\
\tau_{12} = k \left[ \frac{2}{3} + \left(1 - \gamma \right) \frac{-\tau_{22} \frac{\partial U}{\partial y_2}}{3\varepsilon \frac{\partial U}{\partial y_2}} \right] \tag{5.22}
\]

Substituting equation (5.20) into (5.22) gives a third order polynomial that can be solved for \( \tau_{12} \). The constraints in solving this equation are to have positive normal stresses and a negative shear stress \( \tau_{12} \).

Figure 5.10 shows a comparison between the Reynolds stress profiles provided with DNS data and the Reynolds stress profiles inferred from the turbulence model using \( k \), \( \varepsilon \) and the mean velocity gradient. The turbulence model is very simplistic and leads obviously to differences with the original DNS Reynolds stresses. It is interesting to notice however that \( \tau_{11} \) is well modeled except for the maximum near the wall. It was shown in Chapter 3 that measurements in the viscous wall region are difficult and showed differences between different data sets. Also, \( \tau_{33} \) is quite well reproduced, slightly underestimated in the core. Major differences appear for \( \tau_{12} \) and \( \tau_{22} \) in the viscous wall region where the turbulence model overestimates the Reynolds stresses, a probable consequence of the assumptions used in the modeling, e.g. \( \tau_{22} = \tau_{33} \). The differences shown by the turbulence model might however not be crucial since it is not known how much accuracy is needed to model the full two-point correlation.
5.2.2 Results Obtained from the Different Models

Similarly to Chapter 3, some cuts in the correlation function are presented here to show how accurately the different models can interpolate the full two-point correlation function. The normalizations are analogous to the ones used in Chapter 3: $u_\tau$ for velocities, $u_\tau^2$ for the correlation function and $\delta$ for distances.

The results were obtained from Matlab codes using standard double precision. Double differentiation implied by the double curl of the potential correlation function was calculated using central difference method twice with a differentiation step size of $dr = 10^{-6}\delta$.

The first correlations presented correspond to $y - y'$ planes. Figure 5.11, (a) through (f), show the nine components of $R_{ij}(y, y')$ obtained using the constant lengthscale model, the mixing length model, and the $k$ based, $\tau_{11}$ based, $\tau_{22}$ and $\tau_{33}$ based macroscale models respectively.

Using the constant lengthscale model, the correlation map shows, as expected strong correlations along the diagonals where they represent the Reynolds stress profiles. Furthermore they show the modeled correlation functions to be symmetric about this line, as they must be. However, the $R_{11}$ and $R_{33}$ correlation functions show very unrealistic behavior close to the $y$ and $y'$ axes, indicating that velocity fluctuations near the wall are closely correlated with fluctuations across the entire channel, whereas those near the channel center are not. This unlikely behavior turns out to be a consequence of the constant lengthscale assumption, as demonstrated below.

Figure 5.11, (b), shows the same correlation data plotted for the constant lengthscale, recomputed with the lengthscale varying according to the mixing length prescription. The introduction of the variable lengthscale appears to eliminate the sudden increase in the extent of the $R_{11}$ and $R_{33}$ correlations close to the walls.
The macroscale model based on $\tau_{11}$, Figure 5.11 (d), shows a similar behavior as the constant lengthscale model did. Again, strong correlation near the axes appear in the $R_{11}$ and $R_{33}$ correlations.

The other models based on the macroscale, i.e. using $k$, $\tau_{22}$ and $\tau_{33}$ show similar behavior. The correlations look pretty similar to the one obtained from the DNS, Figure 3.11. Main differences appear in the extent from the diagonals and also in the cross-terms that do not show the asymmetry observed for the DNS.

In addition, the correlation function obtained using the Reynolds stresses inferred from the algebraic stress turbulence model is presented in Figure 5.11 (g) for the $k$ based macroscale model. The modeled correlation functions are very similar. The main diagonal reflects the differences in Reynolds stress fields. Also, the cross components, $R_{11}$, show more extent from the diagonal than the original model. The general agreement provides justification for later use of this simple turbulence model for cases where Reynolds profiles are not available. This argument is also reinforced from the previous discussion in section 5.2.1.2.

The $y_2 - y'_2$ correlations are important, since they represent the correlations in the inhomogeneous direction. However, it is also important to obtain good predictions in the homogeneous directions. For example, the streamwise direction will have an impact on frequency when it comes to noise calculations.

Figure 5.12, (a) through (g), show the correlations maps $R_{yy}(y_1, y_2)$ for the DNS and the different models at a normal to wall position located in the log-law region, at $y_2 = y'_2 = 0.118$. Figure 5.13, (a) through (g), shows the same correlations but in the core at $y_2 = y'_2 = 0.404$.

Figure 5.13 shows similar results for all the models, a consequence of the quasi constant value of the lengthscale in the core. Comparison with the DNS data shows differences, principally for $R_{11}$ where the model seriously underpredicts the extent of the correlation in the streamwise direction. Also, the other components of the correlation
function show a principal direction of extent in the spanwise direction for the different models whereas the DNS shows more extent in the streamwise direction.

For locations closer to the wall, like in Figure 5.12, the different models show different results in the streamwise and spanwise directions. Surprisingly, the best results for $R_{11}$ are obtained with the constant lengthscale model. The streamwise extent is still underestimated but less than with the other models. However, the other components of the correlation function are overestimated by the constant lengthscale model. The other models show similar results. Again, the major extent for the modeled $R_{22}$ and $R_{33}$ appears in the spanwise direction and not in the streamwise direction as in the DNS case.

From the diverse cuts in the correlation function, the macroscale models appear to show the best prediction abilities. The $\tau_{22}$ based model showed predictions equivalent to the total kinetic energy based model but the latter is preferred since more commonly used. So, more comparisons with the DNS are provided now using the $k$ based macroscale model for proper orthogonal modes and compact eddy structures.

Figure 5.14 and Figure 5.15 show a comparison between the proper orthogonal modes obtained using the DNS two-point correlation data and the two-point correlation function extrapolated from the Reynolds stresses using the $k$ based macroscale model. Figure 5.14 shows a comparison of the eigenvalue spectra and of the relative energy of these eigenvalues. The first four eigenvalues obtained from the DNS and from the model are very similar. The first modes contain also the most energy but the rate of decrease in energy for higher mode numbers is inferior for the model.

Not only the eigenvalues, but also the modes inferred from the model are very similar. Figure 5.15 shows a comparison of the first four modes. The first three modes are essentially associated to $u_1$ and $u_2$ fluctuations. These modes are very well reproduced by the model. However, the model does not seem to reproduce the $u_3$ fluctuations. A consequence of this behavior results less agreement in the fourth mode. The model still recreates the $u_1$ and $u_2$ fluctuations behavior but shows differences in their amplitude and no agreement for $u_3$ fluctuations.
The compact eddy structures obtained from the modeled two-point correlation function are also similar to the ones obtained using the original DNS data reduction. The first mode shows a velocity field with almost no rotation. In the other modes, spanwise roller-type structures appear dominantly in the flow. They appear singly in the second and third modes, and in pairs in the fourth mode. In the second mode, the eddy appears to be centered around $0.25 \frac{y}{\delta}$ and in mode 3 around $0.5 \frac{y}{\delta}$. Mode 4 shows counter rotating eddies, one in the upper half of the half channel and one in the lower part.

The velocity correlation function is therefore well reproduced by the $k$ based macroscale model. It is consequently interesting to see if the model can be used to interpolate the vorticity correlation function and how well this function, usually not available experimentally, can be reproduced.

### 5.3 Extension to Vorticity Calculations

#### 5.3.1 Numerical Implementation

The vorticity is not directly used in the trailing edge method developed by Glegg et al. (2004) but appears in the source term of the wave equation derived by Howe (1978). Since vorticity enters mechanisms associated to noise production, it might be of interest to be able to extrapolate the vorticity correlation tensor. In theory, the vorticity could be computed by simply taking the double curl of the velocity tensor, i.e. the curl with respect to each position. So, basing the analysis on the vector-potential correlation tensor, $\mathbf{q}$, the vorticity correlation tensor, $\mathbf{W}$, can be expressed as:

$$\mathbf{W} = \nabla \times \nabla' \times \nabla \times \nabla' \times \mathbf{q}$$

(5.23)

where $\nabla$ corresponds to the curl with respect to $\mathbf{y}$ and $\nabla'$ corresponds to the curl with respect to $\mathbf{y}'$. This expression can also be rewritten as

$$W_{ab} = \nabla \times \nabla' \times \varepsilon_{ijn} \varepsilon_{jm\nu} \frac{\partial^2 q_{ln}}{\partial y'_m \partial y'_k}$$

(5.24)
So,

\[ W_{ab} = \varepsilon_{ikl} \varepsilon_{jmn} \nabla \times \left( e_o \frac{\partial}{\partial y_o} \times \frac{\partial^2 q_{ln}}{\partial y_m \partial y_k} \right) \]  

(5.25)

and \( e_o \times e_i = \varepsilon_{aol} e_a \),

\[ W_{ab} = \varepsilon_{aol} \varepsilon_{ikl} \varepsilon_{jmn} \nabla \times \left( \frac{\partial^3 q_{ln}}{\partial y_m \partial y_k \partial y_o} \right) \]

\[ \Rightarrow W_{ab} = \varepsilon_{aol} \varepsilon_{ikl} \varepsilon_{jmn} \left( e_o \frac{\partial}{\partial y_o} \times \frac{\partial^3 q_{ln}}{\partial y_m \partial y_k \partial y_o} \right) \]  

(5.26)

and \( e'_p \times e'_j = \varepsilon_{tno} e'_t \) leading to

\[ W_{ab} = \varepsilon_{tno} \varepsilon_{aol} \varepsilon_{ikl} \varepsilon_{jmn} \left( e'_p \frac{\partial}{\partial y'_o} \times \frac{\partial^3 q_{ln}}{\partial y'_m \partial y'_k \partial y'_o} \right) \]  

(5.27)

Another useful way to express the vorticity correlation tensor is to reformulate equation (5.23) as

\[ W = \nabla' \times \nabla' \times (\nabla \times \nabla \times q) \]  

(5.28)

Now, let

\[ \nabla \times \nabla \times q = U \]  

(5.29)

and

\[ \nabla' \times \nabla' \times U = W \]  

(5.30)

Recalling the double curl identity,

\[ \nabla \times \nabla \times q = \nabla \left( \nabla' \cdot q \right) - \nabla'^2 q \]  

(5.31)

equation 5.25 can be rewritten as,
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\[
U_{in} = \frac{\partial}{\partial x_i} \frac{\partial q_{in}}{\partial j} = \frac{\partial^2 q_{in}}{\partial x_k \partial x_k}
\]

\[\Rightarrow \quad U_{in} = \frac{\partial^2 q_{in}}{\partial x_k \partial x_k} - \frac{\partial^2 q_{in}}{\partial x_k \partial x_k} \quad (5.32)\]

Similarly, equation (5.26), can be rewritten as,

\[
W_{ma} = \frac{\partial^2 U_{mb}}{\partial x_a \partial x_b} - \frac{\partial^2 U_{ma}}{\partial x_c \partial x_c} \quad (5.33)
\]

where

\[
U_{mb} = \frac{\partial^2 q_{jb}}{\partial x_m \partial x_j} - \frac{\partial^2 q_{mb}}{\partial x_k \partial x_k}
\]

and

\[
U_{ma} = \frac{\partial^2 q_{ja}}{\partial x_m \partial x_j} - \frac{\partial^2 q_{ma}}{\partial x_k \partial x_k} \quad (5.34)
\]

So that the two-point vorticity tensor becomes

\[
W_{ma} = \frac{\partial^4 q_{jb}}{\partial x_m \partial x_j \partial x_a \partial x_b} - \frac{\partial^4 q_{mb}}{\partial x_k \partial x_k \partial x_a \partial x_b} - \frac{\partial^4 q_{ja}}{\partial x_m \partial x_j \partial x_c \partial x_c} + \frac{\partial^4 q_{ma}}{\partial x_k \partial x_k \partial x_c \partial x_c} \quad (5.35)
\]

According to equation (5.35), the implementation of the two-point vorticity correlation function could then be obtained by simple modifications of the original code used for two-point velocity predictions. However, some problems arise from the first moment of the longitudinal correlation function, \(h()\), used in the predictions. The formulation of this function implies a von Karman spectrum that decays as \(k^{-5/3}\) at high wave numbers. The corresponding vorticity spectrum, obtained through a double spatial derivative described previously, is consequently multiplied twice by the wavenumber and asymptotes to a \(k^{1/3}\) variation. This von Karman spectrum, which is an infinite Reynolds number approximation that implies no dissipation, results therefore in infinite mean-square vorticity.

As a consequence, the vorticity statistics, and thus the associated modes and eigenvalues, are determined by the manner in which the spectrum rolls off at
wavenumbers beyond the inertial subrange. The source terms involving vorticity modes are consequently determined by the dissipation rate.

Dissipation is usually available from DNS or from turbulence models and it is possible to extend the current method to account for it. A way to account for dissipation is to modify the first moment decay function, \( h() \). It is possible to change \( h() \) explicitly so as to produce a quadratic variation at small separations consistent with a prescribed dissipation, and then proceed to the numerical evaluation of the vorticity correlation tensor. This approach has a very practical aspect since it requires minimal modifications to the model. However, this implementation suffers from some accuracy problems relative to the fourth derivative implied by the relationship between the vector potential and the vorticity. The fourth derivative cannot be computed by difference, at least using standard double precision arithmetic. Also, it can be difficult to find a modification to the decay function that implies a realistic spectrum.

### 5.3.2 Validation of the Analytical Implementation

First of all, the new decay function, which accounts both for the macroscale of the turbulence and the microscale responsible for dissipation, was validated by the recalculation of the two-point spatial velocity correlations of Moser et al. (1998) with the dissipation rate effects incorporated into the decay function. Figure 5.18 shows the \( y_2 - y_2' \) correlation map for zero streamwise and spanwise separation. Comparison with Figure 5.11 (c), which uses the numerical model with the macroscale based on total kinetic energy, shows good agreement. Differences appear for small separation. These differences are expected since the microscale is going to have some effects on the correlation function for small separation.

The analytical implementation itself was validated by comparison of velocity two-point correlations calculated for homogeneous turbulence and for the channel flow using the numerical formulation, the new analytic formulation and the formulation from Hinze (1975) described below.
Hinze gives an expression of the two-point velocity correlation based on the separation between the two points, a longitudinal correlation coefficient function \( f() \) and a lateral correlation coefficient function \( g() \).

The two-point velocity correlation using Hinze’s formulation is given as

$$ R_y = u^2 \left[ \frac{f(s) - g(s)}{s^2} \xi_i \xi_j + g(s) \delta_{ij} \right] $$

where \( s \) is the distance between the two points, \( \xi \) corresponds to the separation in the \( y \) direction and \( u^2 \) is the turbulence intensity.

The correlation coefficient functions are defined as

$$ f(s) = \exp \left( -\frac{s^2}{8\nu t} \right) $$

where \( \nu \) is the kinematic viscosity and

$$ g(s) = f(s) + \frac{s}{2} \frac{\partial f(s)}{\partial s} $$

In this case the lateral correlation function becomes then

$$ g(s) = \left( 1 - \frac{s^2}{8\nu t} \right) \exp \left( -\frac{s^2}{8\nu t} \right) $$

So that the two-point correlation tensor can be defined as

$$ R_y = u^2 \left[ \frac{1}{8\nu t} \xi_i \xi_j + \left( 1 - \frac{s^2}{8\nu t} \right) \delta_{ij} \right] \exp \left( -\frac{s^2}{8\nu t} \right) $$

The analytical model is tested against Hinze’s formulation (cf. equation (5.58)). Figure 5.19 shows a comparison of the \( y_2 - y_2 \) correlation maps for the different component of \( R_y \) at zero streamwise and spanwise separation.

The two different implementations show perfectly similar results. As expected from the formulation, the correlations are only function of separation and not position.
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This result is visible in Figure (5.19) where the contours form diagonal stripes decreasing in magnitude as separation increases. Also, the cross terms are identically zero. Taking the double curl of the velocity expression, the vorticity correlation can also be extrapolated for the formulation derived by Hinze (1975).

Figure 5.20 shows a comparison of the two-point vorticity functions obtained from the analytical model and from Hinze (1975)’s formulation. Plotted are the correlation maps, \( W_{\phi}(y_2, y_2') \), for zero streamwise and spanwise separation. The vorticity shows similar behavior as the velocity, i.e. a function of separation decreasing in magnitude with increasing separation. Analytical prediction method and Hinze’s formulation show exact same results.

This implementation was successful for homogeneous turbulence but showed inaccuracies for boundary layer calculations. The accurate implementation is left for future work.

5.4 Trailing Edge Noise Predictions Using the Modeled Two-Point Correlation and the Theory of Glegg et al. (2004)

This section presents trailing edge noise predictions using the modeled two-point correlation function to extract the source terms. The model used here for two-point prediction prescribes the lengthscale as proportional to the turbulent macroscale \( k^{3/2}/\varepsilon \). The results presented here are similar to the results shown in Chapter 4. The difference is of course the use of the model instead of the DNS data to extract the source terms utilized in the method of Glegg et al. (2004).

As in the previous chapter, scaling and comparison are made with run 295 from Brooks et al. (1989). The single-sided one third octave band SPL spectra are compared in Figure 5.21. The spectrum using the modeled two-point correlation shows good agreement in all but frequency magnitude. The peak spectral level and shape are respected. However, the frequencies appear about two to four times too large, a probable consequence of the features observed in the modeled spatial two-point correlations in Figure 5.12 and Figure 5.13. The streamwise extent in these correlations appeared to be
about two to four times too short explaining the shift observed from the spectrum inferred by the modeled two-point correlation.

The noise contributions can also be identified as in Chapter 4. Figure 5.22 shows the contributions of the different terms forming to the source terms as in equation (4.46):

\[ i\Omega_3 |\omega| s_1^{(n)} + \Omega_3 \omega s_2^{(n)} - |\omega| \omega s_2^{(n)} - i |\omega| V(y_2) \frac{\partial s_1^{(n)}}{\partial y_2} \]  

where terms 1 and 2 represent the contributions from the streamwise and normal to wall velocity fluctuations, and terms 3 and 4 represent the contributions of these velocities to the spanwise vorticity fluctuations. As in the DNS based calculations, the major noise contribution at low and mid-frequencies is associated to the streamwise vorticity term. Main differences appear for higher frequencies where the modeled correlation leads to a source term dominated by the streamwise velocity term first and then the normal to wall vorticity term.

Contributions to the predicted noise spectrum from different parts of the boundary layer in slices 1/10th of the boundary layer thickness in size are presented in Figure 5.23. The results are similar to the ones obtained from the use of the DNS. Again, the plot shows a predominance of the source terms in the bottom 10% of the boundary layer. Also, the lowest points in the bottom 10% of the boundary layer appear to be the major noise contributor (see Figure 5.24). These results are identical to the ones obtained using the DNS.

5.5 Conclusion

The method for predicting spatial velocity correlation from Reynolds stress data in wake flows, originally developed by Devenport et al. (1999, 2001) and Devenport and Glegg (2001), has been adapted to boundary-layer type flows, and tested on DNS and experimental data sets.
The original method, using the prescription of a single lengthscale to describe the entire flow, has been found deficient. In consequence, several new methods, where the lengthscale is prescribed as proportional to a standard mixing length or to the turbulent macroscale, have been developed. The use of the turbulent macroscale, $k^{3/2}/\varepsilon$, was found to produce the most realistic when tested against the DNS data set.

The dominant proper orthogonal modes, their spectra and associated compact eddy structures implied by the results of the new method have been extracted and reproduced the main features found in experimental data.

The method has been extended to the calculation of vorticity correlations and modes. Such calculations require that the effects of dissipation be included, and methods for doing this were proposed.

Also, an algebraic stress turbulence model, as defined by Rodi (1984), was chosen to model the Reynolds stress field. Most CFD solutions do not provide Reynolds stress statistics but only provide kinetic energy and its dissipation. Consequently, a turbulence model appears of use in order to utilize the two-point correlation prediction method. The two-point correlation function obtained using the Reynolds stresses inferred from the turbulence model showed good agreement with the correlation function obtained from the original stresses.

The two-point correlation prediction model was also used with the trailing edge prediction method of Glegg et al. (2004). The results showed good agreement in all but frequency magnitude. The reasons appeared to lie in the modeling of the two-point correlation function in the streamwise direction. A different prescription of the lengthscale might be useful to develop in the future for accurate noise prediction using RANS methods to extrapolate the source term.
Figure 5.1. Schematic of the airfoil and test section showing the coordinate system and location of two-point measurements by Devenport et al. (2001). Dimensions in millimeters.

Figure 5.2. Reynolds stress profiles in the wake obtained from the model. Stresses normalized on $U_e^2$. 

$u_1 u_2$

$u_2 u_2$

$u_3 u_3$

$x/c = 7.5$: Wake profiles fully developed

$x/c = 8.33$: Two point measurements

$NACA 0012$ $U_e = 27.5 m/s$
Figure 5.3. Time-delay/x-wise correlation coefficient function for \( u_2 \) fluctuations on the wake centerline (\( y=0, z=0 \)).

Figure 5.4. Measured (left) and modeled (right) two-point velocity correlation maps for zero \( x \) and \( z \) separation \( R_{yy}(y, y', 0, 0)/U_w^2 \).
Figure 5.5. Eigenvalue spectra obtained for the wake measurements and the modeled correlation function from the one-dimensional proper orthogonal decomposition in the $y_2$-direction.

Figure 5.6. Modal profiles for the first four modes of the wake obtained from the one-dimensional proper orthogonal decomposition in the $y_2$-direction: symbols show measurements, and lines show model results.
Figure 5.7. Comparison between the compact eddy structures for the first four modes of the wake obtained from the model (right) and experimental data (left).
$y_2/\delta = 0.5049$

Figure 5.8. Time-delay/streamwise correlation coefficient for $u_2$ fluctuations in the boundary layer (top) and in the channel (bottom). — Model; ○ - data from the measurements/DNS.
Figure 5.9a. Lengthscale used in the two-point prediction model as a function of the normal to wall position for the different models tested.

Figure 5.9b. Lengthscale used in the two-point prediction model as a function of the normal to wall position for the different models tested for locations near the wall.
Figure 5.10. Comparison of the Reynolds stress profiles given by the DNS data and inferred from the algebraic stress turbulence model.
Figure 5.11. Two-point velocity correlation maps for zero streamwise and spanwise separations as a function of $y'_2$-locations of the two points. Velocity correlations normalized on $u'_r^2$. 
Figure 5.11. Two-point velocity correlation maps for zero streamwise and spanwise separations as a function of $y_2$ locations of the two points. Velocity correlations normalized on $u_τ^2$. 

(c) $k$ based macroscale lengthscale

(d) $τ_{11}$ based macroscale lengthscale
Figure 5.11. Two-point velocity correlation maps for zero streamwise and spanwise separations as a function of $y_2$ locations of the two points. Velocity correlations normalized on $u_r^2$. 

(e) $\tau_{22}$ based macroscale lengthscale

(f) $\tau_{33}$ based macroscale lengthscale
Figure 5.11. Two-point velocity correlation maps for zero streamwise and spanwise separations as a function of $y_2$ locations of the two points. Velocity correlations normalized on $u_+^2$. Reynolds stress profiles utilized inferred from algebraic stress turbulence model.
Figure 5.12. Two-point velocity correlation maps $R_{ij}(\Delta y_i, \Delta y_j)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.12. Two-point velocity correlation maps $R_y(\Delta y_1, \Delta y_3)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.12. Two-point velocity correlation maps $R_{ij}(\Delta y_i, \Delta y_j)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.12. Two-point velocity correlation maps $R_{ij}(\Delta y_i, \Delta y_j)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.12. Two-point velocity correlation maps $R_\gamma(\Delta y_i, \Delta y_j)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.12. Two-point velocity correlation maps $R_{ij}(\Delta y_i, \Delta y_j)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.12. Two-point velocity correlation maps $R_{ij}(\Delta y_i, \Delta y_j)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.13. Two-point velocity correlation maps $R_{ij}(\Delta y_1, \Delta y_3)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.13. Two-point velocity correlation maps $R_{ij}(\Delta y_1, \Delta y_3)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.13. Two-point velocity correlation maps $R_{ij}(\Delta y_1, \Delta y_3)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.13. Two-point velocity correlation maps $R_{ij}(\Delta y_i, \Delta y_j)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.13. Two-point velocity correlation maps $R_{ij}(\Delta y_1, \Delta y_3)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.13. Two-point velocity correlation maps $R_{ij}(\Delta y_1, \Delta y_3)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.13. Two-point velocity correlation maps $R_{ij}(\Delta y_1, \Delta y_3)$, as a function of the streamwise and spanwise separations of the two points.
Figure 5.14. Comparison of the eigenvalue spectra of the o- DNS data set and of the + -modeled correlation function from the one-dimensional proper orthogonal decomposition in the $y_2$-direction (top) and relative energy of the eigenvalues (bottom).
Figure 5.15. Comparison of the modal profiles for the first four modes of the DNS data set (symbols) and of the modeled correlation function (lines) obtained from the one-dimensional proper orthogonal decomposition in the $y_2/\delta$ direction.
Figure 5.16. Compact eddy structures for the first four modes deduced from the modeled two-point correlation function of the channel flow.
Figure 5.17. One-dimensional spectrum implied by equations (5.36), (5.37) and (5.38) (denoted as ‘Modified’ in the legend) for different $\lambda_g k_e$ compared with the von Karman form, and that implied by the three-dimensional energy spectrum function of Pope (2000) for the same dissipation rate.
Figure 5.18. Two-point velocity correlation maps for zero streamwise and spanwise separations as a function of $y_2$-locations of the two points. Velocity correlations normalized on $u_r^2$. Correlations obtained using the analytical formulation which accounts both for the macroscale of the turbulence and the microscale.
Figure 5.19. Comparison of two-point velocity correlation maps for zero streamwise and spanwise separations as a function of $y_2$-locations of the two points. Correlations obtained for homogeneous turbulence using the formulation from Hinze (1975) (top) and the analytical implementation of the model (bottom).
Figure 5.20. Comparison of two-point vorticity correlation maps for zero streamwise and spanwise separations as a function of $y_2$ locations of the two points. Correlations obtained for homogeneous turbulence using the formulation from Hinze (1975) (top) and the analytical implementation of the model (bottom).
Figure 5.21. Far field trailing edge noise spectra from Brooks et al., run 295, compared with predictions made using the method of Glegg et al. with source terms computed from a DNS solution and from the two-point model for $U_c = W = 0.6U_e$.

Figure 5.22. Contribution to the total noise from the four terms forming the source term in equation (5.59).
Figure 5.23. Contributions to the predicted noise spectrum from different parts of the boundary layer in slices 1/10\textsuperscript{th} of the boundary layer thickness in size.

Figure 5.24. Contributions to the predicted noise spectrum from different parts of the boundary layer bottom 10 \% of the boundary layer.
Conclusion

The objectives of this study were multiple. First of all, mathematical derivations of the theory of Glegg et al. (2004) for trailing edge noise prediction were reproduced in Chapter 2. The application of proper orthogonal decomposition for the trailing edge noise problem was presented. This method, which needs only a few modes to represent the whole flow was shown suitable for the present problem, not limited by unacceptably large computational efforts.

The method of Glegg et al. (2004) was shown to rely on the two-point correlation function to extract the source terms used in trailing edge noise prediction. The information available from two-point statistics was presented in Chapter 3 for the DNS of a channel flow and for PIV measurements of a boundary layer over a flat plate. Both data sets showed similar features in the form of two-dimensional cuts or one-dimensional correlation coefficient functions. Also, exploitation of the two-point correlation function through proper orthogonal decomposition revealed identical dominant modes and eddy structures in the flow, therefore justifying the use of the DNS as an external boundary layer for noise calculations. These similarities provided some justification for using the
channel flow DNS data to compute the source terms necessary for the implementation of Glegg et al. (2004)'s noise prediction method, developed in Chapter 4.

The trailing edge noise prediction method of Glegg et al. (2004) was implemented and validated in Chapter 4 using two-point correlation data obtained from a numerical simulation of a turbulent boundary layer. The prediction method was applied to the calculation of the noise produced by an isolated airfoil using the source term extracted from the DNS of a turbulent channel flow.

Comparison with experimental data from Brooks et al. (1989) showed realistic results with the largest discrepancies, on the order of 5 dB, occurring at the lowest frequencies. At these frequencies, the DNS results are least applicable since these correspond to the longest streamwise lengthscales, which are the most affected by the periodicity conditions used in the DNS and also are the least representative of the turbulence in an external boundary layer flow.

The accuracy and realism of the predictions provided some justification in the analysis of the different noise contributors in the boundary layer. Most of the noise was shown to be produced by low-frequency streamwise velocity mode in the bottom 10% of the boundary layer and locations closest to the wall. Few modes were shown to be needed. Only 6 modes were required to obtain noise levels within 1 dB.

Finally, the method for predicting spatial velocity correlation from Reynolds stress data in wake flows, originally developed by Devenport et al. (1999, 2001) and Devenport and Glegg (2001), was adapted to boundary-layer type flows, and tested on both DNS and experimental data sets in Chapter 5.

The original prescription of the lengthscale was shown to be not applicable to boundary layers, because of the dramatic changes in lengthscales between the wall and outer regions. Prescription of the lengthscale as proportional to the turbulent macroscale, $k^{3/2}/\varepsilon$, was shown to produce the most accurate results. In addition, the dominant proper orthogonal modes, their spectra and associated compact eddy structures implied by the results of the new method have been extracted and reproduced the main features found in experimental data.
In cases where the Reynolds stress statistics are not available, an algebraic stress turbulence model based on kinetic energy and dissipation was developed (cf. Rodi (1984)). The results obtained from this model are imperfect but showed reasonable results.

Extension of the two-point correlation method to vorticity correlation predictions was also developed. Such calculations require that the effects of dissipation be included, and methods for doing this were proposed.

Lastly, the two-point correlation prediction model was used to extract the source terms used in the trailing edge noise calculations. The results showed good agreement in all but frequency magnitude. The reasons appeared to lie in the modeling of the two-point correlation function in the streamwise direction. A different prescription of the lengthscale might therefore be useful to develop in the future for accurate noise prediction using RANS methods to extrapolate the source term.

In conclusion, a method to calculate trailing edge noise, not limited by unacceptably large computational efforts, was demonstrated. It was shown that, in theory, using only information about kinetic energy and dissipation of the kinetic energy, noise predictions can be obtained. Discrepancies in the predictions appeared to be caused from the modeling of the correlations in the streamwise direction. Future work should then focus on the modeling of the decay function in the homogeneous direction so that trailing edge noise can be predicted directly from CFD aerodynamic information and using affordable computers.
To analyze stationary random processes, time-averaged functions are usually utilized. This appendix describes the general definition of correlation and cross spectra functions, and also derives more useful forms of the cross spectrum used along this thesis. More information can be found in “Time Series Analysis” by Bendat and Piersol (1986) or “Theoretical Acoustics” by Morse and Ingard (1968).

Let the two random stochastic variables $x$ and $y$. The definition of the Fourier transform by Bendat and Piersol (1986) gives

$$\frac{1}{2T} \overline{x^*(f)y(f)} = \int_{-T}^{T} R_{xy}(\tau) e^{-j2\pi f \tau} d\tau$$  \hspace{1cm} (A.1)

where the cross-correlation function is defined by $R_{xy}(\tau) = \overline{x(t)y(t+\tau)}$ and $f$ corresponds to the frequency in Hz. $T$ corresponds to half the averaging time and $\tau$ the
time difference between the functions $x$ and $y$. The superscript * indicate the complex conjugate of the function it is associated to.

In terms of angular frequency, $\omega$, equation A.1 becomes

$$
\frac{(2\pi)^2}{2T} x^*(\omega)y(\omega) = \int_{-T}^{T} R_{xy}(\tau)e^{-j\omega\tau} d\tau
$$

(A.2)

Also, the normal definition of cross-spectrum is given by

$$
S(\omega) = \frac{2\pi}{2T} x^*(\omega)y(\omega)
$$

(A.3)

And from equation (A.2),

$$
S(\omega) = \frac{\pi}{T} x^*(\omega)y(\omega) = \frac{1}{2\pi} \int_{-T}^{T} R_{xy}(\tau)e^{-j\omega\tau} d\tau
$$

(A.4)

Analogously,

$$
S'(k_m) = \frac{\pi}{L_m} x^*(k_m)y'(k_m) = \frac{1}{2\pi} \int_{-L_m}^{L_m} R_{xy}'(\Delta y_m)e^{-jk_m\Delta y_m} d\Delta y_m
$$

(A.5)

For the problem described in Chapter 4, assumptions are made about the homogeneity of the flow in some directions. So, assuming the functions $x$ and $y$ to be additionally dependent on separation $\Delta y_3$, it is possible to define a new cross-spectrum for this flow such that

$$
\Theta(\omega,k_3) = \frac{1}{(2\pi)^2} \int_{-L_3}^{L_3} \int_{-\tau}^{\tau} R_{xy}(\tau,\Delta y_3)e^{-j\omega\tau-jk_3\Delta y_3} d\tau d\Delta y_3
$$

(A.6)

where $L_3$ is the half width of the computational domain.

Substituting successively equations (A.4) and (A.5), for $m=3$, into this last expression, the cross-spectrum becomes
Similarly, the two-wavenumber spectrum can be defined as

\[ \Phi(k_1, k_3) = \frac{1}{(2\pi)^2} \int_{-L_3}^{L_3} \int_{-L_3}^{L_3} R_{y_3}(\Delta y_1, \Delta y_3) e^{-jk_1\Delta y_1 - jk_3\Delta y_3} d\Delta y_1 d\Delta y_3 \quad (A.8) \]

where \( L_1 \) is the half length of the computational domain.

Substituting equation (A.5), successively for \( m=1 \) and \( m=3 \), gives

\[ \Phi(k_1, k_3) = \frac{1}{2\pi} \int_{-L_3}^{L_3} x^*(k_1, y_3) y(k_1, y_3 + \Delta y_3) e^{-jk_1\Delta y_1} d\Delta y_3 \left( \frac{\pi}{L_1} \right) \]

\[ = \frac{\pi}{L_1 L_3} x^*(k_1, k_3) y(k_1, k_3) \quad (53) \]

\[ = \frac{\pi^2}{L_1 L_3} x^*(k_1, k_3) y(k_1, k_3) \]

The different expressions described in this appendix are to be used in Chapter 4 to relate velocity fluctuations to the cross spectrum function. The particular cross spectra utilized in the method of Glegg et al. (2004) are used to obtain velocity modes defining the source term entering noise prediction calculations.
It is important to derive relationships between the inverse fast Fourier transform of the averaged cross spectrum and the two point correlation using the Fourier decomposition of the flow field given by Moser et al. (1998).

This appendix uses the definitions of the coordinate system and elements associated to the DNS of Moser et al. (1998) given in Chapter 3 to relate the fast Fourier transform (FFT) of the data of the DNS and the resulting FFT of the cross spectrum. According to the information provided by Moser, the three velocity components, were given by the truncated Fourier expression

\[ u_i(y_1, y_2, y_3) = \sum_{k_1} \sum_{k_3} a_i(k_1, k_3) \exp(jk_1y_1 + jk_3y_3) \]  

(B.1)

with the streamwise and spanwise wavenumbers are defined as
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\[ k_1 = \frac{2i_1 \pi}{\pi \delta} \quad \Rightarrow \quad k_1 \delta = i_1 \quad i_1 = -\frac{n_1}{2}, \ldots, \frac{n_1}{2} - 1 \]  \hspace{1cm} (B.2)

\[ k_3 = \frac{2i_3 \pi}{\pi \delta} \quad \Rightarrow \quad k_3 \delta = 2i_3 \quad i_3 = -\frac{n_3}{2}, \ldots, \frac{n_3}{2} - 1 \]

and

\[ \Delta k_1 = \frac{1}{\delta} \quad \Delta k_3 = \frac{2}{\delta} \]  \hspace{1cm} (B.3)

So,

\[ \Delta y_1 = \frac{2\pi}{n_1 \Delta k_1} = \frac{2\pi h}{n_1} \quad \Rightarrow \quad \frac{\Delta y_1}{\delta} = \frac{2\pi}{n_1} \]  \hspace{1cm} (B.4)

\[ \Delta y_3 = \frac{2\pi}{n_3 \Delta k_3} = \frac{\pi h}{n_3} \quad \Rightarrow \quad \frac{\Delta y_3}{\delta} = \frac{\pi}{n_3} \]

And the streamwise and spanwise locations become

\[ y_1 = \frac{2\pi j_1 \delta}{n_1} \quad \text{for} \quad j_1 = 0, \ldots, n_1 - 1 \]  \hspace{1cm} (B.5)

\[ y_3 = \frac{\pi j_3 \delta}{n_3} \quad \text{for} \quad j_3 = 0, \ldots, n_3 - 1 \]

So, equation (B.1) can be discretized as

\[
u_i(y_{1i}, y_{2i}, y_{3i}) = \sum_{i_1=\frac{n_1}{2}}^{\frac{n_1}{2}} \sum_{i_2=\frac{n_2}{2}}^{\frac{n_2}{2}} a_i(i_1, y_{2i}, i_3) \exp \left( j \frac{i_1}{h} \frac{2\pi j_1 \delta}{n_1} + j \frac{i_3}{n_3} \right) \]

\[ \Rightarrow \quad \nu_i(j_1, y_{2i}, j_3) = \sum_{i_1=\frac{n_1}{2}}^{\frac{n_1}{2}} \sum_{i_2=\frac{n_2}{2}}^{\frac{n_2}{2}} a_i(i_1, y_{2i}, i_3) \exp \left( j \frac{2\pi j_1 j_1}{n_1} + j \frac{2\pi j_3 j_3}{n_3} \right) \]  \hspace{1cm} (B.6)

Now from equation (B.1):

\[
\sum_{y_1} \sum_{y_3} \nu_i(y_{1i}, y_{2i}, y_{3i}) \exp (-jk_{1i}y_1 - jk_{3i}y_3) \\
= \sum_{y_1} \sum_{y_3} \sum_{k_1} \sum_{k_3} a_i(k_1, y_{2i}, k_3) \exp \left( j(k_1 - k'_{1i})y_1 + j(k_3 - k'_{3i})y_3 \right) \\
= \sum_{i_1=\frac{n_1}{2}}^{\frac{n_1}{2}} \sum_{i_2=\frac{n_2}{2}}^{\frac{n_2}{2}} \sum_{i_3=\frac{n_3}{2}}^{\frac{n_3}{2}} a_i(i_1, y_{2i}, i_3) \exp \left( j \frac{2\pi j_1 j_1}{n_1} (i_1 - i'_{1i}) + j \frac{2\pi j_3 j_3}{n_3} (i_3 - i'_{3i}) \right) 
\]  \hspace{1cm} (B.7)
So, summing over $j_1$, representing a period, the term $\exp\left( j \frac{2\pi j_1}{n_1} (i_1 - i'_1) \right)$ sums to zero except when $i_1 - i'_1$, in which case it sums to $n_1$, and similarly for $j_3$.

Consequently, equation (B.7) reduces to

$$a_j(k_1,k_3) = \frac{1}{n_1} \frac{1}{n_3} \sum_{y_1} \sum_{y_3} u_j(y_1,y_3) \exp(-jk_1y_1 - jk_3y_3)$$

(B.8)

and the cross spectrum becomes

$$Ex[a_j^*(k_1,k_3) a_j(k_1,k_3)] =$$

$$\frac{1}{n_1^2} \frac{1}{n_3^2} \sum_{y_1} \sum_{y_3} \sum_{y_1'} \sum_{y_3'} Ex[u_j(y_1,y_3)u_j(y_1',y_3')] \exp(jk_1y_1 + jk_3y_3 - jk_1y_1' - jk_3y_3')$$

(B.9)

Equivalently,

$$Ex[a_j^*(k_1,k_3) a_j(k_1,k_3)] =$$

$$\frac{1}{n_1^2} \frac{1}{n_3^2} \sum_{y_1} \sum_{y_3} \sum_{y_1'} \sum_{y_3'} \sum_{y_1} \sum_{y_3} R_{y_j}(y_1 - y_1', y_3 - y_3') \exp(jk_1(y_1 - y_1') + jk_3(y_3 - y_3'))$$

(B.10)

Now, operating a change of coordinate, the summation can be decomposed as

$$\sum_{y_1} \sum_{y_3} = \sum_{j_1 = -n_1/2}^{n_1/2-1} \sum_{j_1 = -n_1/2}^{n_1/2-1} = \sum_{j_1 = -n_1/2}^{n_1/2-1} \sum_{j_1 = -n_1/2}^{n_1/2-1}$$

(B.11)

with $f_j = j_1 - j_1$. This operation is similar for $j_3$.

Then, $\quad Ex[a_j^*(k_1,k_3) a_j(k_1,k_3)] =$

$$\frac{1}{n_1^2} \frac{1}{n_3^2} \sum_{j_1 = -n_1/2}^{n_1/2-1} \sum_{j_3 = -n_3/2}^{n_3/2-1} \sum_{j_1 = -n_1/2}^{n_1/2-1} \sum_{j_3 = -n_3/2}^{n_3/2-1} R_{y_j}(y_1 - y_1', y_3 - y_3') \exp(j \frac{2\pi \delta}{n_1} (j_1 - j_1') + j \frac{2\pi \delta}{n_3} (j_3 - j_3'))$$

(B.12)

or
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\[
\begin{align*}
\text{Ex}[a^*_i(k_1, k_3) a_j(k_1, k_3)] &= \\
&= \frac{1}{n_1} \frac{1}{n_3} \sum_{j_1=-n_1/2}^{n_1/2-1} \sum_{j_1=-n_3/2}^{n_3/2-1} R_{ij}(y_1-y_1^*, y_3-y_3^*) \exp(-j \frac{2\pi i_1}{n_1} j_1^* - j \frac{2\pi i_3}{n_3} j_3^*) \\
&= (B.13)
\end{align*}
\]

And since the summation over \( j_1^* \) and \( j_3^* \) run over a whole period of the exponential term, the starting and ending point is irrelevant and \( j_1 \) and \( j_3 \) can be set to zero. Also, from equation B(.13) it can be seen that the expression inside the summation is now independent of \( j_1 \) and \( j_3 \). Consequently, the summations over these indices become simply multiplications by \( n_1 \) and \( n_3 \). As a result, equation (B.13) becomes

\[
\begin{align*}
\text{Ex}[a^*_i(k_1, k_3) a_j(k_1, k_3)] &= \\
&= \frac{1}{n_1} \frac{1}{n_3} \sum_{j_1=-n_1/2}^{n_1/2-1} \sum_{j_1=-n_3/2}^{n_3/2-1} R_{ij}(y_1-y_1^*, y_3-y_3^*) \exp(-j \frac{2\pi i_1}{n_1} j_1^* - j \frac{2\pi i_3}{n_3} j_3^*) \\
&= (B.14)
\end{align*}
\]

which is the same as

\[
\begin{align*}
\text{Ex}[a^*_i(k_1, k_3) a_j(k_1, k_3)] &= \frac{1}{n_1} \frac{1}{n_3} \sum_{j_1=-n_1/2}^{n_1/2-1} \sum_{j_1=-n_3/2}^{n_3/2-1} R_{ij}(\Delta y_1, \Delta y_3) \exp(-jk_1\Delta y_1 - jk_3\Delta y_3) \\
&= (B.15)
\end{align*}
\]

In a similar way, it is possible to obtain the inverse of this relationship. Details are not shown here, but the expression relating the two quantities is simply:

\[
\begin{align*}
R_{ij}(y_2, y_2^*, \Delta y_1, \Delta y_3) &= \sum_{k_1} \sum_{k_3} \text{Ex}[a^*_i(k_1, k_3) a_j(k_1, k_3)] \exp(jk_1\Delta y_1 + jk_3\Delta y_3) \\
&= (B.16)
\end{align*}
\]

or in terms of the wavenumber cross spectrum

\[
\begin{align*}
R_{ij}(y_2, y_2^*, \Delta y_1, \Delta y_3) &= \sum_{k_1} \sum_{k_3} \frac{\pi^2}{L_1 L_3 u_r^2} \Phi_{ij}(y_2, y_2^*, k_1, k_3) \exp(jk_1\Delta y_1 + jk_3\Delta y_3) \\
&= (B.17)
\end{align*}
\]


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REFERENCES


REFERENCES


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