Wideband Digital Filter-and-Sum Beamforming with Simultaneous Correction of Dispersive Cable and Antenna Effects

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Abstract

Optimum filter-and-sum beamforming is useful for array systems that suffer from spatially correlated noise and interference over large bandwidth. The set of finite impulse response (FIR) filter coefficients used to implement the optimum filter-and-sum beamformer are selected to optimize signal-to-noise ratio (SNR) and reduce interference from the certain directions. However, these array systems may also be vulnerable to dispersion caused by physical components such as antennas and cables, especially when the dispersion is unequal between sensors. The unequal responses can be equalized by using FIR filters. Although the problems of optimum-SNR beamforming, interference mitigation, and per-sensor dispersion have previously been individually investigated, their combined effects and strategies for mitigating their combined effects do not seem to have been considered.

In this dissertation, combination strategies for optimum filter-and-sum beamforming and sensor dispersion correction are investigated. Our objective is to simultaneously implement optimum filter-and-sum beamforming and per-sensor dispersion correction using a single FIR filter per sensor. A contribution is to reduce overall filter length, possibly also resulting in a significant reduction in implementation complexity, power consumption, and cost.

Expressions for optimum filter-and-sum beamforming weights and per-sensor dedispersion filter coefficients are derived. One solution is found via minimax optimization. To assess feasibility, the cost is analyzed in terms of filter length. These designs are considered in the context of LWA1, the first “station” of the Long Wavelength Array (LWA) radio telescope, consisting of 512 bowtie-type antennas and operating at frequencies between 10 MHz and 88 MHz. However, this work is applicable to a variety of systems which suffer from non-white spatial noise and directional interference and are vulnerable to sensor dispersion; e.g., sonar arrays, HF/VHF-band riometers, radar arrays, and other radio telescopes.
This dissertation is dedicated to

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Contents

1 Introduction

1.1 Application Example: LWA1 ......................................... 5
1.2 Problem Statement ...................................................... 8
1.3 Contributions ............................................................ 10
1.4 Organization of This Dissertation ............................... 12

2 Wideband Filter-and-Sum Beamforming ........................................ 13

2.1 Analytic Model in Bandpass Representation ....................... 14
  2.1.1 Signal Representation ............................................ 14
  2.1.2 Power at the Output of the Beamformer ....................... 17
2.2 Beamforming Techniques ............................................... 22
  2.2.1 Delay-and-Sum Beamforming .................................. 22
  2.2.2 Wideband Optimum-SNR Beamforming ....................... 24
2.2.3 Optimum-SNR Beamforming with Null Constraints ............................. 25

2.2.4 Comparison Between Beamforming Techniques ................................. 26

2.3 Implementation of Fine Delay FIR Filters ........................................... 32

2.3.1 Selection of Truncation Length in Reconstruction-based Delay Filter ....... 33

2.3.2 Alternative Designs for the Fractional Sample Period Delay FIR Filter .... 34

2.4 Summary ......................................................................................... 38

3 Cable Distortion and Correction 39

3.1 Transmission Line Theory ................................................................. 39

3.2 Characteristics of Coaxial Cables ..................................................... 41

3.2.1 Cable Loss and Cable Dispersion .................................................. 41

3.2.2 Equalization of Cable Loss and Cable Dispersion ........................... 43

3.2.3 Time Domain Filter for Correction of Cable Distortion ................. 49

3.3 Application to KSR200DB Cable ..................................................... 51

3.3.1 Characteristics of KSR200DB Cable .............................................. 51

3.3.2 Correction Filter for KSR200DB Cables ..................................... 53

3.3.3 Alternative Design Methods for the KSR200DB Cable Correction Filter ... 59

3.3.4 Demonstration ............................................................................... 62

3.4 Summary ......................................................................................... 63
4 Antenna Dispersion and Dedispersion

4.1 Relevant Antenna Theory ................................................................. 66

4.1.1 Equivalent Circuit Model ............................................................. 66

4.1.2 Example: A Thin Straight Dipole ................................................... 69

4.2 Design of a Per-Sensor Antenna Dedispersion FIR Filter for a Thin Straight Dipole . 72

4.3 Description of a Single LWA1 Stand ................................................ 80

4.3.1 Electromagnetic Model ................................................................. 80

4.3.2 Self-Impedance ........................................................................... 82

4.3.3 Effective Length ........................................................................... 84

4.3.4 Group delay .................................................................................. 86

4.4 Design of a Per-Sensor Antenna Dedispersion FIR Filter for a Single LWA1 Stand . 89

4.5 Design of a Per-Sensor Antenna Dedispersion FIR Filter for a LWA1 Antenna Stand in the Presence of a Second LWA1 Antenna Stand ................................................. 97

4.6 Summary ......................................................................................... 100

5 Combination Approaches ........................................................................ 102

5.1 Direct Method ................................................................................... 103

5.1.1 Combined Filter for Sensor Dedispersion ....................................... 103

5.1.2 Combined Filter Including Beamforming and Sensor Dedispersion ................................................. 109
5.2 Optimization Method ........................................ 115
  5.2.1 Combined Filter for Sensor Dedispersion ............. 116
  5.2.2 Combined Filter Including Beamforming and Sensor Dedispersion .... 117
5.3 Comparison .................................................. 119
5.4 Application to a Simplified Version of LWA1 ............. 120
5.5 Summary .................................................... 127

6 Application to LWA1 ........................................ 128
  6.1 SEFD for a LWA1 Standalone Stand ....................... 130
  6.2 LWA1 Array Manifold .................................... 135
  6.3 LWA1 Beamforming ........................................ 136
    6.3.1 Confirmation of Results from Previous Work ........ 136
    6.3.2 Effects of Unequal Cable Losses and Dispersive Delays .......... 138
    6.3.3 Correction of Unequal Cable Losses .................. 145
    6.3.4 Correction of Unequal Cable Losses and Dispersive Delays ...... 149
    6.3.5 Discussion ........................................... 152
  6.4 Summary .................................................. 154

7 Conclusions ................................................ 159
7.1 Findings ................................................................. 159

7.2 Future Work ............................................................ 161

A Taylor Series Expansion ............................................. 163

B Real-Valued Filter ....................................................... 167

C LWA1 Stand Positions and Cable Lengths ....................... 172

Bibliography .................................................................. 175
1.1 Model for characterizing effects of dispersion and unequal responses on delay-and-sum beamforming. .......................................................... 3

1.2 Block diagram of implementation schemes. ........................................... 4

1.3 Aerial view of LWA1. ........................................................................ 7

1.4 Close-up view of LWA1. ...................................................................... 7

2.1 Filter-and-sum beamforming process block diagram. .......................... 15

2.2 Coordinate system for array analysis. .................................................. 16

2.3 Block diagram of a digital delay-and-sum beamforming implementation. .. 23

2.4 Sensor arrangement in the LWA1 array. ................................................. 27

2.5 Impact of the filter length $M$ on the SNR improvement over that of a single sensor. 29

2.6 Impact of external noise correlation and filter length on the SNR improvement. . . 31

2.7 Impact of interference on SNR of the array systems. ............................ 33
2.8 Frequency response deviation of the fine delay FIR filter from the ideal for various $M$. 35

3.1 Equivalent circuit of a transmission line of infinitesimal length. 40

3.2 Typical composition of a coaxial cable. 42

3.3 Response of KSR200DB cables having different lengths, using Equation (3.5) with
$\zeta = 5.4 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{1/2}$ and $\kappa = 3.8 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{1/2}$. 53

3.4 Impact of cable dispersion on the array system. 54

3.5 Frequency response of a KSR200DB cable having length of 150 m. 55

3.6 Frequency response of the correction filter for a KSR200DB cable of length 150 m. 55

3.7 Close-up view of the $2^{17}$-tap impulse response of the correction filter obtained using
the numerical inverse Fourier transform method. 56

3.8 Frequency response of the convolution of $h_c$ and $h_{cd}$. 56

3.9 Frequency response errors in the cable correction filter using the closed-form formul-
lation described in Equation (3.38). 58

3.10 Close-up view of the impulse response of the correction filter from Equation (3.40). 58

3.11 Difference between the frequency responses of the filter $h_{cd}$ and $\tilde{h}_{cd}$. 59

3.12 Performance of KSR200DB cable correction filters of different filter lengths. 60

3.13 Performance of the cable correction FIR filter using different design methods. 61

3.14 SNR improvement at the output of the delay-and-sum beamformer accounting for
the effect of unequal cable losses and dispersive delays in different cases. 64

xii
4.1 Circuit model for an antenna in receive mode. .................................. 67
4.2 Calculation for the VEL of an arbitrarily-shaped wire antenna. ............ 69
4.3 TTG circuit model for $Z_A$ of a straight dipole antenna. ....................... 70
4.4 Calculated impedance of a straight dipole of length 3.7 m and radius 5 mm. . 71
4.5 Group delay due to antenna dispersion. .............................................. 72
4.6 Frequency response $H_a(\omega)$ of the dipole antenna, as described in Section 4.1.2. . . 74
4.7 Frequency response $H_{ad}(\omega)$ of the antenna dedispersion filter. .......... 74
4.8 Close-up view of the impulse response of the dipole antenna dedispersion FIR filter. 75
4.9 Verification of the dipole antenna dedispersion FIR filter in the time domain for
   different filter lengths $M_{ad}$. ..................................................... 76
4.10 Frequency response errors of the dipole antenna dedispersion FIR filters from the
   ideal for different filter lengths. ................................................. 77
4.11 Performance of the dipole antenna dedispersion FIR filters using different design
   approaches. ................................................................. 79
4.12 Wire grid model of a LWA1 stand. .................................................. 81
4.13 Geometry of wire grid used to model the dipole. ................................. 81
4.14 Calculated impedance of the antenna in a standalone stand in transmit mode. . 83
4.15 Calculated impedance of an antenna in a standalone stand in receive mode for various
   incident signals. ............................................................... 85
4.16 Comparison of the impedance of the antenna in a standalone stand calculated using
the transmit-mode and receive-mode methods. ................................. 85

4.17 Calculated effective length of the antenna in a standalone stand with the incident
direction of \( \theta = 0^\circ \) and \( \phi = 45^\circ \). ................................. 87

4.18 Calculated effective length of the antenna in a standalone stand with the incident
direction of \( \theta = 74^\circ \) and \( \phi = 45^\circ \). ................................. 87

4.19 Calculated effective length of the antenna in a standalone stand with the incident
direction of \( \theta = 74^\circ \) and \( \phi = 0^\circ \). ................................. 88

4.20 Group delay of an antenna in a standalone stand due to a signal incident from \( \theta = 0^\circ \)
and \( \phi = 45^\circ \). ................................. 88

4.21 Group delay of an antenna in a standalone stand due to a signal incident from \( \theta = 74^\circ \)
and \( \phi = 45^\circ \). ................................. 89

4.22 Frequency response of the antenna in a standalone stand. ................................. 91

4.23 Close-up view of the impulse response \( h_a \) of \( \tilde{H}_a(\omega) \). ................................. 91

4.24 Frequency response \( H_{ad}(\omega) \) of the antenna dedispersion filter for the antenna in a
standalone stand. ................................. 92

4.25 Close-up view of the impulse response \( h_{ad} \) of the antenna dedispersion filter for the
antenna in a standalone stand. ................................. 92

4.26 Verification of the antenna dedispersion FIR filter for the antenna in a standalone
stand in the time domain for different filter lengths \( M_{ad} \). ................................. 93
4.27 Frequency response error of the antenna dedispersion FIR filters for the antenna in a standalone stand using the prototype truncation method from the ideal for different \( M_{ad} \). ................................................................. 94

4.28 Performance of the antenna dedispersion FIR filters for the antenna in a standalone stand using windowing and minimax optimization. ................................. 96

4.29 Wire grid model of an array consisting of two identical stands. ...................... 98

4.30 Difference between the self-impedance of antenna in a two-stand pair and the same antenna in a standalone stand. ................................................................. 98

4.31 Difference between the effective length of an antenna in a two-stand pair and the same antenna in a standalone stand for the incident direction \( \theta = 0^\circ \) and \( \phi = 45^\circ \). . 99

4.32 Same as Figure 4.31, except for \( \theta = 74^\circ \). ................................................................. 100

4.33 Performance of antenna dedispersion FIR filters for an antenna in the two-stand pair using different design approaches. ................................................................. 101

5.1 Desired frequency response \( H_{sd}(\omega) \) of the ideal combined filter. ..................... 104

5.2 Close-up view of the impulse response of the combined filter \( h_{sd0} \). ......................... 105

5.3 Frequency response errors of \( h_{sd0} \) from \( H_{sd}(\omega) \) with different filter lengths. ......... 106

5.4 The 118-tap combined filter \( h_{sd} \) using convolution. ............................................... 107

5.5 The 175-tap combined filter \( \tilde{h}_{sd} \) using convolution. ........................................... 108

5.6 Frequency response errors of \( h_{sd} \) described in Figure 5.5 after truncation. .............. 108
5.7 Desired frequency response $H(\omega)$ of the ideal combined filter that implements 2.5 ns delay at the sample rate of 196 MSPS and sensor dedispersion. .......................... 110

5.8 The 109-tap combined filter $h_0$ obtained using the inverse Fourier transform of $H(\omega)$ described in Figure 5.7. ............................................................... 111

5.9 The 27-tap delay FIR filter $h_b$ using Kaiser windowing. ............................. 111

5.10 The 201-tap Combined filter $h_1$ using convolution. ................................. 112

5.11 Desired frequency response $H(\omega)$ of the ideal combined filter that implements 1.6 ns delay at the sample rate or 196 MSPS and sensor dedispersion. .......................... 113

5.12 The 206-tap combined filter $h_0$ obtained using the inverse Fourier transform of $H(\omega)$ described in Figure 5.11. ............................................................... 113

5.13 The 31-tap delay FIR filter $h_b$ using Kaiser windowing. ............................. 114

5.14 The 205-tap Combined filter $h_1$ using convolution. ................................. 114

5.15 The 102-tap combined filter using the optimization method. ...................... 118

5.16 The 99-tap combined filter using the optimization method. ...................... 118

5.17 The 101-tap combined filter using the optimization method. ...................... 119

5.18 Delay-and-sum beamforming performance for different methods in the condition described in the text of Section 5.4. ............................................................... 122

5.19 Same as Figure 5.18, except now relative to the “ideal” result. .................. 123

5.20 Same as Figure 5.18, except now relative to the “w/o correction” result. ......... 123
5.21 Optimum-SNR beamforming performance for different methods in the condition described in the text of Section 5.4. ................................. 125

5.22 Same as Figure 5.21, except now relative to the “ideal” result. ................................. 125

5.23 Same as Figure 5.21, except now relative to the “w/o correction” result. ........................ 126

6.1 Calculated SEFD for the \( x \)-aligned dipole of a LWA1 standalone stand in different planes. ................................................................. 133

6.2 Calculated SEFD for a LWA1 standalone stand in different planes. ............................... 134

6.3 SEFD for various beamforming techniques in the \( \phi = 0^\circ \) plane. ............................... 137

6.4 SNR improvement over that of a single stand by various beamforming techniques in the \( \phi = 0^\circ \) plane. ................................................................. 137

6.5 Same as Figure 6.4, except the external noise correlation zeroed. ............................... 139

6.6 Distribution of constant delays in LWA1 cable system. ............................................. 139

6.7 Distribution of cable losses and dispersive delays at 20 MHz. ...................................... 140

6.8 Distribution of cable losses and dispersive delays at 38 MHz. ...................................... 140

6.9 Distribution of cable losses and dispersive delays at 74 MHz. ...................................... 141

6.10 Same as Figure 6.4, except now accounting for effects of unequal cable losses and dispersive delays, but without correction. ................................. 142

6.11 Difference between Figures 6.4 and 6.10. ................................................................. 143
6.12 SNR improvement over that of a single stand by optimum-SNR beamforming neglecting external noise correlation, in the presence of unequal cable losses and dispersive delays, but without correction. ........................................... 143

6.13 Difference between the Figure 6.12 result and the optimum-SNR beamforming result in Figure 6.10. .......................................................... 144

6.14 Difference between the Figure 6.12 result and the phase-and-sum beamforming result in Figure 6.10. .......................................................... 144

6.15 Same as Figure 6.10, but now perfectly correcting the effects of unequal cable losses. 146

6.16 Difference between Figures 6.15 and 6.4. .................................................. 146

6.17 Difference between Figures 6.16 and 6.11. .................................................. 147

6.18 Same as Figure 6.12, but now perfectly correcting the effects of unequal cable losses. 147

6.19 Difference between the Figure 6.18 result and the optimum-SNR beamforming result in Figure 6.15. .................................................. 148

6.20 Difference between the Figure 6.18 result and the phase-and-sum beamforming result in Figure 6.15. .................................................. 148

6.21 Difference between Figures 6.19 and 6.13. .................................................. 149

6.22 Difference between Figures 6.20 and 6.14. .................................................. 150

6.23 Same as Figure 6.4, but now for optimum-SNR beamforming neglecting external noise correlation. .................................................. 151
6.24 Difference between the Figure 6.23 result and the optimum-SNR beamforming result in Figure 6.4. ......................................................... 151

6.25 SNR improvement by phase-and-sum beamforming repeated 100 times with random magnitude and phase errors. ............................................. 153

6.26 SNR improvement by optimum-SNR beamforming repeated 100 times with random magnitude and phase errors. ............................................. 153

6.27 SNR improvement by optimum-SNR beamforming neglecting external noise correlation repeated 100 times with random magnitude and phase errors. .............. 154

6.28 Differences between Figure 6.25 and the phase-and-sum beamforming results in Figure 6.4. ................................................................. 155

6.29 Differences between Figure 6.26 and the optimum-SNR beamforming results in Figure 6.4. ................................................................. 156

6.30 Differences between Figures 6.27 and 6.23. ......................................................... 157

A.1 The Taylor series for $f(x) = e^{g\sqrt{x}}$ with different terms, where $g$ is given in Equation (A.6). ................................................................. 165

A.2 Approximation errors of the Taylor series for $f(x) = e^{g\sqrt{x}}$ where $g$ is given in Equation (A.6). ................................................................. 165

B.1 Impulse response of the filter as shown in Figure 3.6. ................................. 171
B.2 Differences between the frequency response of the filter described in Figure B.1(a) and the Figure 3.6 result. ................................................................. 171
List of Tables

2.1 Description of interference signals. ................................................. 32

2.2 Design techniques for fractional sample period delay FIR filters. .......... 36

3.1 Summary of the performance of cable correction FIR filters using different design approaches. ................................................................. 63

4.1 Summary of performance of the dipole antenna dedispersion FIR filters for different filter lengths. ................................................................. 78

4.2 Minimum number of taps required for less than 1° phase error over 10 – 88 MHz for dipole antenna dedispersion FIR filters designed using different methods. ........ 79

4.3 Summary of performance of the antenna dedispersion FIR filters for the antenna in a standalone stand using prototype truncation for different $M_{ad}$. ................. 95

4.4 Summary of performance of the antenna dedispersion FIR filters for the antenna in a standalone stand. ................................................................. 97
4.5 Summary of performance of antenna dedispersion FIR filters for an antenna in the two-stand pair using different design approaches. ....................... 101

5.1 Summary of the performance of combined filters for sensor dedispersion versus different design methods. .......................................................... 120

5.2 Summary of the performance of combined filters for beamforming delay and sensor dedispersion versus different design methods. ....................... 121

6.1 Summary of SNR improvement by different beamforming schemes in the $\phi = 0^\circ$ half plane. ................................................................. 158

6.2 Variations due to the finite number of correction filter taps with respect to different beamforming schemes at three frequencies. ....................... 158

C.1 LWA1 stand positions and cable lengths. ........................................ 172
Chapter 1

Introduction

Array signal processing is limited to narrowband signals in many applications (e.g., radar and communication). However, for the systems of interest in this work and other cases (e.g., sonar), received signals are wideband. A common approach to wideband digital beamforming combines sensor outputs via a delay-and-sum operation. The fractional sample period delays can be accurately approximated using finite impulse response (FIR) filters as described in [1] and references therein. In most conditions, noise associated with different sensors is assumed to be uncorrelated [2–5]. Delay-and-sum beamforming weights based solely on sensor positions (i.e., geometrical delays) can optimize signal-to-noise ratio (SNR) in this case. However, the correlation of noise between sensors is significant in some systems of interest (e.g., antenna arrays operating at frequencies below 300 MHz [6]) and delay-and-sum beamforming cannot optimize SNR in the presence of noise correlation. A solution to compensate the correlation of noise between sensors was considered in [7], but this solution was applicable to narrowband signals. Furthermore, wideband systems are sensitive to interference and not all interference can be effectively filtered out in the analog domain.
Consequently, the wideband beamformer may be required to maximize SNR in the presence of spatially-correlated noise and also reduce interference from specified directions. These additional functions can be implemented using the available degrees of freedom of the delay FIR filters [8]. In this work, we describe optimum filter-and-sum beamforming in bandpass representation and derive beamforming weights to optimize SNR in the presence of spatially-correlated noise and interference.

Many of these systems are also vulnerable to dispersion and unequal responses from physical components such as cables and antennas, in the manner described in Figure 1.1. Unequal cable lengths and non-uniform coupling between array elements may lead to unequal dispersion between sensors. Thus the correction of dispersion (dedispersion) and equalization of unequal responses may be required before summing.

Cables exhibit frequency-dependent loss and delay. Previous works were mostly related with compensation for the frequency-dependent loss; e.g., line repeaters [9], reduction in the number of amplifiers in cascade [10], lumped and variable amplifiers [11], distributed amplifiers [12]. The phase characteristics of cables was considered in [13]; however, the analog equalizer was specified with high-speed cables having minimum-phase-like characteristics. The effect and equalization of cable loss and delay have been considered in [14], but the equalizer was designed by trial and error and the solution was only applicable to narrowband analog signals. The problem and equalization of cable distortion were also considered for the transmission of digital signals via cables [15–17], but the equalizers were developed to eliminate inter-symbol interference (ISI) and improve bit-error-rate (BER) due to signal reflection, excessive attenuation, or limited bandwidth. In digital communications systems, signals are usually known and training signals can be used to obtain the equalizer; however, we do not know the characteristics of signals in our applications. In this
work, we considered the problem of cable distortion due to imperfect conductivity of the center conductor and shield. We described the frequency response of cables and proposed a solution to the compensation for the frequency-dependent cable loss and delay.

The effect of unequal antenna dispersion on beamforming was considered in [7], but the correction scheme was constrained to narrowband signals. This work described the frequency response of an antenna based on an equivalent circuit model, and proposed a wideband solution to equalize the frequency-dependent group delay due to antenna dispersion.

Although some aspects of these problems are well-understood, optimal processing for combination of them has not been previously addressed. In this work, we considered the combined effect of these problems (correlation of noise between sensors and sensor dispersion) on array beamforming, and proposed the strategy for the correction of the combined effect. Figure 1.2 illustrates two candidate implementation schemes for an array consisting of $N$ sensors (e.g., antennas): “concatenation scheme” shown in Figure 1.2(a), and “combination scheme” shown in Figure 1.2(b). For the $n^{th}$
Figure 1.2: Block diagram of implementation schemes.
sensor, the combination approach uses a single FIR filter $H_n(\omega)$ to simultaneously perform the functions of $H_{cn}^{-1}(\omega)$, $H_{an}^{-1}(\omega)$ and $\tilde{H}_{bn}(\omega)$; where the filter $H_{cn}^{-1}(\omega)$ is used to correct the dispersion in cables, the filter $H_{an}^{-1}(\omega)$ is used to correct the dispersion by antennas, and the filter $\tilde{H}_{bn}(\omega)$ gives the weights for wideband optimum-SNR beamforming and deterministic nullforming. The combination scheme has the potential to yield a smaller overall filter length, possibly also resulting in a corresponding reduction in implementation complexity, power consumption, and cost. For these reasons, the combination approach is investigated in this dissertation.

This work is generally applicable to systems, which suffer from non-white spatial noise and interference, and are vulnerable to unequal dispersion of sensor responses, including sonar arrays [18], HF/VHF band riometers [19], radar arrays [20], and radio telescopes [6, 21–23].

This chapter is organized into four sections: In Section 1.1 (“Application Example: LWA1”), we describe the first “station” of the Long Wavelength Array (LWA) radio telescope [6] as a primary application example of this work. Section 1.2 (“Problem Statement”) frames the specific problems that this dissertation addresses, and Section 1.3 (“Contributions”) summarizes the research contributions of this dissertation. Finally, Section 1.4 (“Organization of This Dissertation”) provides an explanation of how the rest of this dissertation is organized.

1.1 Application Example: LWA1

In this dissertation, we use LWA1 as an application example. Figure 1.3 presents a full view of LWA1, which is composed of 256 dual-polarized “stands” arranged in pseudorandom fashion. Each stand consists of a pair of collocated orthogonally-polarized active dipole antenna elements shown in
Figure 1.4 within a 110 × 100 m elliptical footprint. LWA1 operates at frequencies between 10 MHz and 88 MHz using receivers having noise figure of about 2.7 dB. Additional detailed information on LWA is available in [6, 24] and online\(^1\).

In the design for the digital processing subsystem of LWA1, “simple” delay-and-sum beamforming (i.e., as shown in Figure 1.1) is used [25, 26]. This approach utilizes FIR filters for fractional sample period delay compensation. However, in the operating frequency band of LWA1, Galactic noise is dominant [24]. The correlation of this noise between sensors significantly desensitizes the array for beam pointings which are not close to the zenith [7]. Ellingson (2009) recently showed that optimum-SNR beamforming could significantly improve the SNR performance over delay-and-sum beamforming in this case [7].

In order to be able to detect weak signals from distant radio sources, it is often necessary to mitigate radio frequency interference (RFI), which is a growing problem for radio astronomy [27]. When the desired signal and interferers occupy the same frequency band but originate from different directions, it is anticipated that space-time nullforming will be desired to reduce such interferers. Data-adaptive nullforming algorithms may not be suitable for radio astronomy applications because the resulting beam patterns may exhibit considerable variability between updates; for example, as shown in [28]. For this reason, and also because interference typically comes from directions which are known \textit{a priori}, deterministic nullforming algorithms are preferred for LWA1.

LWA1 is vulnerable to antenna dispersion [29–31] and cable dispersion [32]. Because the antennas are arranged in a non-uniform geometry (see Figure 1.3), the way mutual coupling changes the impedance of each antenna is different. As a result, the dispersion caused by antennas varies

\(^1\)http://lwa.unm.edu
Figure 1.3: Aerial view of LWA1. LWA1 is located near the center of the Very Large Array (VLA) site, and parabolic VLA antennas appear in the background.

Figure 1.4: Close-up view of LWA1 (picture taken November 2009).
between antennas. Also, the cables are of different lengths, so the excess delay due to cable
dispersion is different between cables. Thus, the correction of these contributions to the sensor
dispersion may be desired before summing.

LWA1 is thus a good example of an application that would benefit from the approach depicted in
Figure 1.2(b). The problem statement with respect to LWA1 becomes: To what extent it is possible
to simultaneously implement wideband optimum-SNR beamforming, deterministic nullforming, and
sensor dispersion correction using a single digital delay FIR filter per sensor?

1.2 Problem Statement

In this dissertation, we consider the possibility of combining all of these functions (wideband
optimum-SNR beamforming, deterministic nullforming, and sensor dedispersion) in the same set
of filters, as shown in Figure 1.2(b), instead of simply concatenating the separate filters given in
Figure 1.2(a), thereby achieving a significant reduction in overall filter length and a corresponding
reduction in implementation complexity, power consumption, and cost. Specifically, this work is
composed of the following elements:

1. Characterization of cable distortion and equalization of cable distortion. Previous work con-
sidered this problem using narrowband analog solutions. We would like to develop a wideband
digital scheme to correct and equalize the frequency-dependent cable distortion. The primary
difficulty is the inverse Fourier transform of the frequency response of the cable. We use a
three-termed Taylor series approximation to bypass this problem and obtain a mathematical
expression of the prototype for the cable correction filter.
2. *Characterization of antenna dispersion and correction scheme for antenna dedispersion.* We describe antennas in receive mode using an equivalent circuit model. The group delay due to antenna dispersion is firstly considered in the case of standalone (in the absence of other sensors of the array) to show the dependence of frequency. We also consider the antenna in the case of two-pair sensor to show the effect of mutual coupling. This shows the significant effect of unequal antenna dispersion. So our next task is to design appropriate FIR filters to correct such dispersion over the interested frequency range.

3. *Strategy of simultaneous implementation of optimum filter-and-sum beamforming and sensor dispersion correction using a single FIR filter per sensor.* The combined effect of these problems (e.g., correlation of noise between sensors, and sensor dispersion) and the strategy for the correction of the combined effect has not been previously addressed. We would like to use the combination scheme (e.g., Figure 1.2(b)) to achieve this goal. One approach is to combine several functions in a single FIR filter by convolution of the desired impulse responses, followed by truncation to obtain a filter of the desired length. The final filter length is determined by specifications such as magnitude and phase errors over the bandwidth of interest. In order to achieve a significant reduction in overall filter length, we consider obtaining the combined filter via minimax optimization, as opposed to convolution of the desired impulse responses. We wish to find the strategy best-suited to obtain the combined filter by comparing these two methods in terms of the final filter length with respect to the same specifications (e.g., phase accuracy over the given frequency band). The computational burden for the calculation of combined filter coefficients is also considered.

4. *Application to LWA1.* As an application example of this work, we would like to predict the
performance (e.g., SNR performance over the frequency range of interest) of LWA1 as it is currently implemented, and figure out the improvement that might be achieved as a result of the designs proposed in this research.

1.3 Contributions

The objective of this research is to simultaneously implement wideband optimal-SNR beamforming with null constraints for interference mitigation, and sensor dispersion correction, using only one FIR filter per sensor, as shown in Figure 1.2(b). Specifically, the contributions of this research are composed of the following elements:

1. A rigorous description of the compensation for frequency-dependent cable distortion was derived (See Sections 3.2.2 and 3.2.3). This description characterized the cable correction filter using physical parameters of cables, and the solution was applicable to systems vulnerable to frequency-variant cable distortion.

2. The effect of unequal cable losses and dispersive delays on wideband beamforming was analyzed (See Section 3.3.1). The procedures are helpful in determining the impact of unequal cable distortion on array SNR performance.

3. A scheme for the implementation of sensor dedispersion by modification of the coefficients of the same FIR filters used to implement filter-and-sum beamforming (“combination scheme”) was developed (See Section 5.2). This method was shown to significantly improve array SNR performance without a large increase in the overall filter length.

4. A general description of filter-and-sum beamforming in bandpass representation, using a priori
information such as incident directions, does not seem to have been previously considered for long-wavelength radio astronomy applications. In this dissertation, the beamforming weights in \textit{bandpass representation} have been derived to serve this purpose (see Section 2.2). This description is particularly useful for wideband applications to digital signals in real-valued quantities.

5. A description of antenna dispersion based on a circuit model was developed (See Section 4.1.1). This description is helpful in characterizing the frequency-dependent dispersion by antennas. It is also useful in describing the effect of mutual coupling when the antenna is embedded in an array.

6. The designs are demonstrated using an application example, the LWA1 radio telescope. A consistent framework for predicting the performance (i.e., SNR improvement over that of a single sensor over the bandwidth of interest) of LWA1 (see Section 6.1) has been developed.

Note that the improvements in SNR provided by the proposed methods are fractional numbers in decibel scale in this dissertation. Although these may seem to be only minor improvements, these are significant in the context of a large array. For example, 0.3 dB improvement in SNR can be interpreted as 7% reduction in the number of sensors required. For LWA1, this corresponds to 17 fewer sensors. Since a LWA1 station costs about US$800,000 [24] and the cost is approximately linear in the number of sensors, this amounts to a savings of about US$53,000 plus associated installation, power, and maintenance costs. These minor SNR improvements can also be significant in radio science. The minimum detectable flux density is proportional to $1/\sqrt{\Delta \tau}$ , where $\Delta \tau$ denotes the integration time required to achieve a specified SNR. Therefore, 0.3 dB improvement in system sensitivity corresponds to a 13% reduction in the integration time required.
1.4 Organization of This Dissertation

The rest of this dissertation is arranged as follows. Chapter 2 (“Wideband Filter-and-Sum Beamforming”) describes the approach to the derivation of the filter-and-sum beamforming weights, which are used to optimize SNR and mitigate interference from the certain directions. In Chapter 3 (“Cable Distortion and Correction”), we analyze the impact of dispersion in cables on array systems and propose a solution to the correction of cable dispersion. In Chapter 4 (“Antenna Dispersion and Dedispersion”), we address the dispersion by antennas and present the method to the correction of antenna dispersion. Chapter 5 (“Combination Approaches”) summarizes two strategies to implement several functions using only one FIR filter per sensor and makes a comparison between them in terms of increased filter length with respect to the same specifications. Chapter 6 (“Application to LWA1”) provides the procedures to predict the performance of LWA1 as it is currently implemented and figure out the improvement that might be achieved as a result of the designs proposed in this research. Finally, Chapter 7 (“Conclusions”) summarizes the findings of this research and presents recommendations for future work.
Chapter 2

Wideband Filter-and-Sum Beamforming

This chapter describes several beamforming techniques and considers implementation issues for long-wavelength radio astronomy applications. Section 2.1 ("Analytic Model in Bandpass Representation") provides a theoretical description in bandpass representation and expressions for the real-valued desired signal, interference, and noise. In Section 2.2 ("Beamforming Techniques"), we present the calculation of beamforming weights for various beamforming techniques. Section 2.3 ("Implementation of Fine Delay FIR Filters") addresses the issues of beamforming implementation. Finally, Section 2.4 ("Summary") summarizes this chapter.
2.1 Analytic Model in Bandpass Representation

Bandpass representation here means signal representation using real-valued samples. We consider an array consisting of \( N \) sensors. Each sensor is followed by a \( M \)-tap FIR filter, as Figure 2.1 shows. In this dissertation, vectors are denoted by boldface lower case letters and matrices are denoted by boldface upper case letters.

2.1.1 Signal Representation

To characterize the incident direction, we consider a coordinate system in which the origin is arbitrarily selected, and in which Cartesian axes \( x \), \( y \) and \( z \) correspond to East, North, and the zenith, respectively, as shown in Figure 2.2. The associated spherical coordinate system is used to specify the direction of the incoming plane wave. The incident direction is represented as \( \{ \theta, \phi \} \), where \( \theta \) is the angle measured from the \( z \) axis, and \( \phi \) is the angle measured from the \( x \) axis toward the \( y \) axis. The direction of incidence \( \{ \theta, \phi \} \) is also indicated using the symbol \( \psi \). The delay of the signal incident from direction \( \psi \) at the \( n^{th} \) sensor is

\[
\tau_n(\psi) = -\frac{1}{c} (x_n \sin \theta \cos \phi + y_n \sin \theta \sin \phi + z_n \cos \theta),
\]

where \( c \) is the speed of light in free space, and \((x_n, y_n, z_n)\) are the coordinates of the \( n^{th} \) sensor.

Assume that the incident signal consists of a desired signal, \( J \) interference signals, and noise. The signal of interest is incident from \( \{ \theta_0, \phi_0 \} \), which is henceforth indicated as \( \psi_0 \); similarly, the \( j^{th} \) \((j = 1, \cdots, J)\) interference signal is incident from \( \{ \theta_j, \phi_j \} \), which is thus represented as \( \psi_j \). The
Figure 2.1: Filter-and-sum beamforming process block diagram.
Figure 2.2: Coordinate system for array analysis. \( \hat{r} \) is the unit vector pointing from the origin toward the direction of incidence. \( \mathbf{p}_n \) indicates the position of the \( n^{th} \) sensor.

The sum of signals at the position of the \( n^{th} \) sensor at sample \( k \) is then

\[
x_n[k] = s \left[ k - \frac{\tau_n(\psi_0)}{T_s} \right] + \sum_{j=1}^{J} \left\{ v_j \left[ k - \frac{\tau_n(\psi_j)}{T_s} \right] \right\} + z_n[k] + u_n[k],
\]

where: \( T_s \) is the sample period, \( s[k] \) is the signal of interest as received at the origin of the coordinate system; \( v_j[k] \) is the \( j^{th} \) interference signal as received at the origin of the coordinate system; \( z_n[k] \) is the contribution from noise external to the system; and \( u_n[k] \) is the contribution from noise internal to the system referred to the input of the sensor. Without loss of generality, we interpret these to be real-valued (bandpass representation) quantities.

Using the beamforming structure as described in Figure 2.1, the signal associated with the \( m^{th} \) tap of the \( n^{th} \) sensor is

\[
x_{nm}[k] = x_n[k - (m - 1)], \quad m = 1, \cdots, M.
\]
The beamformer output at sample $k$, $y[k]$, is formed from the weighted sum of tap signal samples such that

$$y[k] = \sum_{m=1}^{M} \sum_{n=1}^{N} w_{nm} x_{nm}[k],$$  \hspace{1cm} (2.4)$$

where $w_{nm}$ is the real-valued weight for the $m$th tap of the $n$th sensor.

### 2.1.2 Power at the Output of the Beamformer

The power at the output of the beamformer is

$$P_y = \left\langle |y[k]|^2 \right\rangle,$$  \hspace{1cm} (2.5)$$

where $\left\langle \cdot \right\rangle$ denotes time-domain averaging. Representing the tap signal samples and weights as column vectors, Equation (2.4) is rewritten as

$$y = w^T x,$$  \hspace{1cm} (2.6)$$

where the subscript “$T$” denotes transpose,

$$w = [w_{11} \cdots w_{N1} \hspace{0.1cm} w_{12} \cdots w_{N2} \hspace{0.1cm} \cdots \hspace{0.1cm} w_{1M} \cdots w_{NM}]^T, \text{ and}$$  \hspace{1cm} (2.7a)$$
$$x = [x_{11}[k] \hspace{0.1cm} \cdots \hspace{0.1cm} x_{N1}[k] \hspace{0.1cm} x_{12}[k] \hspace{0.1cm} \cdots \hspace{0.1cm} x_{N2}[k] \hspace{0.1cm} \cdots \hspace{0.1cm} x_{1M}[k] \hspace{0.1cm} \cdots \hspace{0.1cm} x_{NM}[k]]^T. \hspace{1cm} (2.7b)$$

The output power as described in Equation (2.5) is then

$$P_y = w^T R_x w,$$  \hspace{1cm} (2.8)$$
where $R_x$ is an $NM \times NM$ matrix given by

$$R_x = \langle xx^T \rangle . \quad (2.9)$$

In the process of expanding Equation (2.9) by the substitution of Equation (2.2), we assume that the signal of interest, interference, external noise, and internal noise are mutually uncorrelated for any $n$. The possibility that external noise $z_m(t)$ and $z_n(t)$ are correlated for $m \neq n$ is not precluded. Under these assumptions, Equation (2.9) can be rewritten as

$$R_x = R_s + \sum_{j=1}^{J} R_j + R_z + R_u , \quad (2.10)$$

where $R_s$, $R_j$, $R_z$ and $R_u$ are the $NM \times NM$ covariance matrices associated with the signal of interest, the $j^{th}$ interference signal, external noise, and internal noise, respectively. In the following, we will describe the calculations for these covariance matrices.

**Covariance matrices for desired signal and interference signals**

Let $E_\theta[k]$ and $E_\phi[k]$ be the $\theta$- and $\phi$-polarized components of the electric field of the signal of interest, having units of $V \cdot m^{-1} \cdot Hz^{-1/2}$. The signal received at the $n^{th}$ sensor, incident from the desired direction $\psi_0$, is then

$$s_n[k] = a_n^\theta(\psi_0)E_\theta[k] + a_n^\phi(\psi_0)E_\phi[k] , \quad (2.11)$$

where $a_n^\theta(\psi_0)$ and $a_n^\phi(\psi_0)$ are the “effective lengths”, having units of meters, of the $n^{th}$ sensor for the signal incident from $\psi_0$, associated with $\theta$ and $\phi$ polarizations, respectively. Since we have
not yet made any assumption about the correlation between $E_\theta(t)$ and $E_\phi(t)$, the desired signal covariance matrix is

$$
R_s = R_s^{\theta\theta} + R_s^{\phi\phi} + R_s^{\theta\phi} + R_s^{\phi\theta},
$$

(2.12)

where

$$
R_s^{\theta\theta} = \begin{bmatrix}
A_{\theta\theta} \langle |E_\theta[k]|^2 \rangle & \cdots & A_{\theta\theta} \langle E_\theta[k]E_\theta[k-(M-1)] \rangle \\
\vdots & \ddots & \vdots \\
A_{\theta\theta} \langle E_\theta[k-(M-1)]E_\theta[k] \rangle & \cdots & A_{\theta\theta} \langle |E_\theta[k-(M-1)]|^2 \rangle
\end{bmatrix},
$$

(2.13a)

$$
R_s^{\phi\phi} = \begin{bmatrix}
A_{\phi\phi} \langle |E_\phi[k]|^2 \rangle & \cdots & A_{\phi\phi} \langle E_\phi[k]E_\phi[k-(M-1)] \rangle \\
\vdots & \ddots & \vdots \\
A_{\phi\phi} \langle E_\phi[k-(M-1)]E_\phi[k] \rangle & \cdots & A_{\phi\phi} \langle |E_\phi[k-(M-1)]|^2 \rangle
\end{bmatrix},
$$

(2.13b)

$$
R_s^{\theta\phi} = \begin{bmatrix}
A_{\theta\phi} \langle E_\theta[k]E_\phi[k] \rangle & \cdots & A_{\theta\phi} \langle E_\theta[k]E_\phi[k-(M-1)] \rangle \\
\vdots & \ddots & \vdots \\
A_{\theta\phi} \langle E_\theta[k-(M-1)]E_\phi[k] \rangle & \cdots & A_{\theta\phi} \langle E_\theta[k-(M-1)]E_\phi[k-(M-1)] \rangle
\end{bmatrix},
$$

(2.13c)

$$
R_s^{\phi\theta} = \begin{bmatrix}
A_{\phi\theta} \langle E_\phi[k]E_\theta[k] \rangle & \cdots & A_{\phi\theta} \langle E_\phi[k]E_\theta[k-(M-1)] \rangle \\
\vdots & \ddots & \vdots \\
A_{\phi\theta} \langle E_\phi[k-(M-1)]E_\theta[k] \rangle & \cdots & A_{\phi\theta} \langle E_\phi[k-(M-1)]E_\theta[k-(M-1)] \rangle
\end{bmatrix},
$$

(2.13d)

$$
A_{\theta\theta} = a_\theta(\psi_0)a_\theta^T(\psi_0), \quad A_{\theta\phi} = a_\phi(\psi_0)a_\phi^T(\psi_0),
$$

(2.13e)

$$
A_{\phi\phi} = a_\phi(\psi_0)a_\phi^T(\psi_0), \quad A_{\phi\theta} = a_\theta(\psi_0)a_\phi^T(\psi_0),
$$

(2.13f)

$$
a_\theta(\psi_0) = [a_{1\theta}^\theta(\psi_0) \ a_{2\theta}^\theta(\psi_0) \ \cdots \ a_{N\theta}^\theta(\psi_0)]^T, \quad \text{and}
$$

(2.13g)

$$
a_\phi(\psi_0) = [a_{1\phi}^\phi(\psi_0) \ a_{2\phi}^\phi(\psi_0) \ \cdots \ a_{N\phi}^\phi(\psi_0)]^T.
$$

(2.13h)
For the $j^{th}$ interference signal incident from $\psi_j$, the associated covariance matrix is

$$R_j = R_j^{\theta\theta} + R_j^{\phi\phi} + R_j^{\theta\phi} + R_j^{\phi\theta}, \quad (2.14)$$

where $R_j^{\theta\theta}$, $R_j^{\phi\phi}$, $R_j^{\theta\phi}$, and $R_j^{\phi\theta}$ have expressions analogous to those for the signal of interest, and the incident direction is $\psi_j$.

**Covariance matrix for external noise**

The $MN \times MN$ covariance matrix $R_z$ can be partitioned into $M^2 \times N$ submatrices as

$$R_z = \begin{bmatrix}
R_{11} & \cdots & R_{1M} \\
\vdots & \ddots & \vdots \\
R_{M1} & \cdots & R_{MM}
\end{bmatrix}, \quad (2.15)$$

where the $(n, n')^{th}$ $(n, n' = 1, \cdots, N)$ element of the submatrix $R_z^{pq}$ $(p, q = 1, \cdots, M)$ represents the correlation between external noise at the $p^{th}$ tap of sensor $n$ and external noise at the $q^{th}$ tap of sensor $n'$. Due to the possible correlation of external noise (e.g., Galactic noise) among sensors, the off-diagonal elements of submatrix $R_z^{pq}$ are not safely assumed to be zeros. Under the assumption that external noise is wide-sense stationary, we have

$$R_z^{pq} = P_z R_z[q - p], \quad p, q = 1, \cdots, M, \quad (2.16)$$

where the $(n, n')^{th}$ element of $P_z$ is the correlation of external noise between sensors $n$ and $n'$, and $R_z[\cdot]$ is the normalized auto-correlation function of external noise. Assuming all the sensors are identical, a model proposed by Ellingson in [7] can be used to compute the correlation of external
noise between sensors \( n \) and \( n' \):

\[
P_z^{[n,n']} = \frac{k\eta}{\lambda^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[ a_\theta^n(\psi)a_\theta^{n'}(\psi) + a_\phi^n(\psi)a_\phi^{n'}(\psi) \right] T_e(\psi) \sin \theta d\theta d\phi ,
\]

(2.17)

where \( k \) is Boltzmann’s constant \( (1.38 \times 10^{-23} \text{ J/K}) \), \( \eta \) is the impedance of free space, \( \lambda \) is the wavelength, and \( T_e(\psi) \) is the external noise brightness temperature in the direction \( \psi \).

Covariance matrix for internal noise

Similarly with the external noise covariance matrix, the internal noise covariance matrix can also be partitioned into \( M^2 \times N^2 \) submatrices as

\[
R_u = \begin{bmatrix}
R_{11} & \cdots & R_{1M} \\
\vdots & \ddots & \vdots \\
R_{M1} & \cdots & R_{MM}
\end{bmatrix}
\]

(2.18)

where the \((n, n')^{th}\) \((n, n' = 1, \cdots, N)\) element of the submatrix \( R_{pq} \) \((p, q = 1, \cdots, M)\) represents the correlation between internal noise at the \( p^{th} \) tap of sensor \( n \) and internal noise at the \( q^{th} \) tap of sensor \( n' \). Because (1) we do not expect the internal noise associated with any sensor to be significantly correlated with that of any other sensor, and (2) internal noise is assumed not to be scattered into the system, each submatrix \( R_{pq} \) is a diagonal matrix in this situation. Assuming internal noise is wide-sense stationary, the submatrices are

\[
R_{pq} = P_u R_u[q-p] , \quad p, q = 1, \cdots, M,
\]

(2.19)
where \( R_u(\cdot) \) is the normalized auto-correlation function of internal noise, and \( P_u \) is an \( N \times N \) diagonal matrix whose non-zero elements are:

\[
P_u^{[n,n]} = k T_{p,n} R_L . \tag{2.20}
\]

Here, \( T_{p,n} \) is the input-referred internal noise temperature associated with the \( n^{th} \) sensor. If the difference in the noise of sensor electronics is negligible, it is reasonable to assume that all the electronics are identical such that \( T_{p,n} = T_p \).

## 2.2 Beamforming Techniques

The beamforming weights may be selected using different approaches depending on the desired objectives. In this section, three beamforming techniques are discussed and the associated beamforming weights are given.

### 2.2.1 Delay-and-Sum Beamforming

A delay-and-sum beamformer generates the desired beam by delaying the signal from each sensor by an appropriate amount and then summing them together. Typically, the delay associated with the individual sensor is determined by the array geometry and the desired pointing direction. In this case, the filter associated with each sensor in Figure 2.1 is a delay FIR filter.

For a given sample period \( T_s \), the time delay \( D \) can be interpreted as

\[
D = d T_s + \tau , \tag{2.21}
\]
where $dT_s$ is the integer sample period delay (coarse delay) with $d$ being an integer, and $\tau$ is the fractional sample period delay (fine delay) satisfying $0 \leq \tau/T_s < 1$. An implementation of this scheme is shown in Figure 2.3, where FIFOs are used to implement the coarse delay and FIR filters are used to implement the fine delay. In the following, we will derive the prototype digital filter implementing the fractional sample period delay.

Consider a continuous-time signal $x(t)$ sampled every $T_s$ seconds in a manner satisfying the Nyquist criterion, resulting in a time series $x[k]$ where $k$ is an integer indexing time samples. The original continuous-time signal can be recovered using the reconstruction formula as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} x[k] \text{sinc} \left( \frac{t-kT_s}{T_s} \right), \quad (2.22)$$

where $\text{sinc}(t)$ is defined as $\sin(\pi t) / (\pi t)$. Delaying $x(t)$ by a time equal to $\tau$ yields output as follows:

$$x_d(t) = x(t-\tau) = \sum_{k=-\infty}^{\infty} x[k] \text{sinc} \left( \frac{t-\tau-kT_s}{T_s} \right). \quad (2.23)$$
Sampling the delayed continuous-time signal, at the time instants \( t = nT_s \), we obtain

\[
x_d[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k],
\]

(2.24)

where

\[
h[k] = \text{sinc} \left( k - \frac{\tau}{T_s} \right), \quad k \in (-\infty, \infty).
\]

(2.25)

This is the prototype for the fine delay filter, with \( h[k] \) being the ideal impulse response of the fine delay filter.

When the delay determined by the delay FIR filter is associated with the directional delay as described in Equation (2.1), the weights for delay-and-sum beamforming consist of the delay FIR filter coefficients.

2.2.2 Wideband Optimum-SNR Beamforming

The objective of wideband optimum-SNR beamforming is to achieve maximum possible SNR, even in the case where noise is correlated between sensors. This is equivalent to delay-and-sum beamforming only if the noise is uncorrelated between sensors. At the output of beamformer, the SNR is

\[
\text{SNR} = \frac{w^T R_s w}{w^T R_n w},
\]

(2.26)

where \( R_n = R_a + R_z \). The maximum possible SNR that can be achieved is equal to the maximum eigenvalue of \( R_n^{-1} R_s \), and is achieved by selecting \( w \) to be the corresponding eigenvector [33].
2.2.3 Optimum-SNR Beamforming with Null Constraints

Neither delay-and-sum beamforming nor optimum-SNR beamforming provide explicit protection against interference. One possible way to obtain beamforming weights that achieve interference nulling simultaneously with SNR optimization is to extend the optimum-SNR beamforming solution to accommodate null constraints. This can be done by computing optimum-SNR beamforming weights in the subspace orthogonal to the interference subspace. The method is as follows: Assume one has knowledge of the incident directions associated with the desired signal and interference signals, respectively. The procedure for nulling interferers is to project the array outputs onto the subspace orthogonal to the interference subspace $[34, 35]$; i.e.,

$$x_P = P_V^\perp x,$$

(2.27)

where the projection matrix $P_V^\perp$ is the $NM \times NM$ matrix given by

$$P_V^\perp = I - V \left( V^T V \right)^{-1} V^T,$$

(2.28)

with $V$ being the $NM \times J$ matrix formed by concatenating the $J \ [NM \times 1]$ interference signal vectors. The problem of wideband optimum-SNR beamforming after this projection becomes to choose $w$ according to

$$\max_w \frac{w^T \tilde{R}_s w}{w^T \tilde{R}_n w},$$

(2.29)

where $\tilde{R}_s = P_V^\perp R_s \left( P_V^\perp \right)^T$, and $\tilde{R}_n = P_V^\perp R_n \left( P_V^\perp \right)^T$. The weight vector thus is the eigenvector of $\tilde{R}_n^{-1} \tilde{R}_s$ corresponding to the maximum eigenvalue.
There are three criteria for this method to be feasible: 1) The number of interference signals should be much less than the number of sensors in the array, 2) The interference is far away from the main lobe of the desired beam, and 3) Each interference signal should be localized (i.e., not distributed) in direction of arrival.

2.2.4 Comparison Between Beamforming Techniques

In this section we compare the above three beamforming techniques in the context of LWA1. In the experiments, we consider a simplified version of LWA1 consisting of $N = 256$ isotropic sensors whose properties are identical over $10 - 88$ MHz. The arrangement of these sensors is shown in Figure 2.4. For this study, mutual coupling is neglected. Instrumental effects including dispersion in cables and dispersion by antennas are ignored initially. The desired pointing direction is $22^\circ$ away from the zenith toward the east (i.e., $\theta = 22^\circ$ and $\phi = 0^\circ$). The internal noise covariance matrix is given in Equations (2.18) – (2.20), where $T_{p,n} = 250$ K and $R_L = 100$ $\Omega$ for LWA1. The external noise covariance matrix is as described in Equations (2.15) – (2.17). Since the bandwidth of interest for LWA1 ($10 - 88$ MHz) is about 81% of the Nyquist bandwidth with respect to the sampling frequency $f_s = 196$ MHz, here, we assume that both signal of interest and external noise have the bandwidth 81% of the Nyquist bandwidth. The bandlimiting filter is a bandpass filter with bandwidth equal to 78 MHz and transition bands from 8 MHz to 10 MHz and 88 MHz to 90 MHz, when the sampling frequency is 196 MHz. The desired stopband attenuation is 60 dB and the passband ripple is no greater 0.1 dB. Although $T_e(\psi)$ varies considerably both as a function of $\psi$ and a function of time of day due to the rotation of the Earth, we assume $T_e(\psi)$ is uniform over the sky ($\theta \leq \pi/2$) and zero for $\theta > \pi/2$. This assumption provides a reasonable standard condition

26
for comparing Galactic noise-dominated antenna systems, as explained in [36] and demonstrated in [37] and [38]. Using this model, $T_e(\psi)$ toward sky is found as a function of frequency; that is,

$$T_e(\psi) = \frac{1}{2k} I_v \frac{c^2}{f^2} ,$$  

(2.30)

where $k$ is Boltzmann’s constant, $c$ is the speed of light in free space, $f$ is frequency in Hz, and $I_v$ is intensity having units of $W \cdot m^{-2} \cdot Hz^{-1} \cdot sr^{-1}$. From observation of the Galactic polar region by Cane (1979) [39], we have

$$I_v = I_g f_{\text{MHz}}^{-0.52} + I_{eg} f_{\text{MHz}}^{-0.80} ,$$  

(2.31)

where $I_g = 2.48 \times 10^{-20}$, $I_{eg} = 1.06 \times 10^{-20}$, and $f_{\text{MHz}}$ is frequency in MHz. Under these assumptions, the noise covariance matrix is determined.
First we study two beamforming techniques without the considerations of interference or nulling: (1) Delay-and-sum beamforming, in which weights are determined by geometrical delays; and (2) Wideband optimum-SNR beamforming, in which weights are selected to maximize the SNR of the array over the bandwidth of interest. The beamforming techniques are implemented using the structure shown in Figure 2.1, and are compared in terms of the improvement in SNR over that of a single sensor. To find the relationship between the performance of the beamforming technique and the filter length $M$, the correlation of the external (Galactic) noise between sensors is initially ignored. The two beamforming techniques are expected to enhance the SNR by a factor of $N = 256$ (or $10 \log_{10} N$ on a decibel scale) if the delays are perfect (true only if the filter length is infinite). The degradation of SNR improvement due to finite filter length is shown in Figure 2.5. The results show that the SNR improvement increases with the increase of the filter length $M$. Under the condition of finite filter length, wideband optimum-SNR beamforming is always superior to delay-and-sum beamforming. For the same increase in filter taps, the improvement in SNR by delay-and-sum beamforming over the SNR of a single sensor is larger than that by wideband optimum-SNR beamforming; i.e., additional taps are more important for delay-and-sum beamforming than for wideband optimum-SNR beamforming. The additional filter taps do not have a significant effect on the performance of wideband optimum-SNR beamforming when the filter length is larger than some value; e.g., it is observed as expected that the improvement in SNR by wideband optimum-SNR beamforming approaches to the desired result as shown in Figure 2.5(d). However, the delay-and-sum beamforming result has a 0.9 dB degradation for $M = 28$. Note that the correlation of external noise between sensors is considered in the calculation of the optimum-SNR beamforming weights, which leads to the better performance in optimum-SNR beamforming.
Figure 2.5: Impact of the filter length $M$ on the SNR improvement over that of a single sensor. The correlation of external noise between sensors is ignored in these results.
Now we consider the correlation of external noise to see how the external noise correlation and finite filter length affect the SNR improvement by beamforming techniques. Figure 2.6 illustrates the SNR improvement by delay-and-sum beamforming and wideband optimum-SNR beamforming in the presence of correlated external noise. In comparison with Figure 2.5, the correlation of external noise between sensors significantly degrades the improvement in SNR by 0.5 – 7.5 dB due to the correlation of external noise between sensors. It is observed as expected that the increase in number of filter taps improves the SNR performance at the output of the beamformer. The increase in SNR is less at high frequencies. This is because the “electrical” distance between sensors decreases with increasing frequency, which leads to smaller correlation of external noise between sensors.

Next we investigate wideband optimum-SNR beamforming with null constraints in the presence of interferers incident from directions different from the beam pointing direction. The per-sensor FIR filters for the beamforming are selected to have $M = 28$ taps consistent with reasonable performance shown in Figures 2.5 and 2.6. Also, LWA1 uses 28-tap delay FIR filters for delay-and-sum beamforming [26]. To find what part of degradation in SNR is due to the interference signals, we consider four different cases as shown in Table 2.1. Except Case 5, all interference signals are far away from the desired incident direction of $\theta_0 = 22^\circ$ and $\phi_0 = 0^\circ$. Figure 2.7(a) shows the effect of the number of nulls on the SNR. The results show that the improvement in SNR over that of a single sensor by beamforming degrades with the increasing number of nulls. This is expected, because each new interference projection (null) reduces the available number of “degrees of freedom” available for beamforming by one. Figure 2.7(b) illustrates how the direction of interference signals affect the SNR-improvement at the output of beamformer. There is a large degradation in SNR for Case 5 since the interference signal comes from a direction near the desired
Figure 2.6: Impact of external noise correlation and filter length on the SNR improvement.
pointing direction. This is expected as the interference projection affects the desired signal when the null is too close. For Case 1, there is no interference signal and no null constraints, so it is equivalent to wideband optimum-SNR beamforming as shown in Figure 2.6(d).

### 2.3 Implementation of Fine Delay FIR Filters

In Section 2.2, we assumed the simple reconstruction filter (see Equation (2.25) with $M$ (as opposed to $\infty$) taps). We now consider in more detail the effect of filter length $M$, and we consider alternative “prototype” delay filters for use in beamforming.
Figure 2.7: Impact of interference on SNR of the array systems. (See Table 2.1 for details.)

2.3.1 Selection of Truncation Length in Reconstruction-based Delay Filter

Reducing the limits of $k$ in Equation (2.25) from $\pm\infty$ to $\lceil\pm(M-1)/2\rceil$, in order to obtain an implementable filter results in a FIR filter having $M$ taps; i.e.,

$$h[k] \cong \text{sinc}\left(\frac{k - \tau}{T_s}\right), \quad \left\lceil -\frac{M-1}{2} \right\rceil \leq k \leq \left\lceil \frac{M-1}{2} \right\rceil.$$ 

(2.32)

In order to demonstrate the relationship between $M$ and the performance (in terms of producing exactly the desired delay) of the fine delay FIR filter, a simple example using the LWA1 operating frequency band (10 to 88 MHz) and sample rate (196 MSPS) is employed. The fine delay FIR filter is implemented using the simple truncation technique implied by Equation (2.32). (Alternative design methods will be discussed in Section 2.3.2.) The sample period $T_s \approx 5.1$ ns. Let us consider the problem of implementing a delay equal to $\tau = 1.6$ ns, which is approximately $0.31T_s$. If fine
delay were not implemented, the corresponding phase errors would be up to 50.7° and 5.8° at 88 MHz and 10 MHz, respectively. Such errors can be expected to significantly desensitize the array and increase sidelobe levels. Following [40], we require that the maximum acceptable phase error at any given frequency is 10°, and that phase errors no larger than 1° are preferred.

Figures 2.8(a) – 2.8(f) show the frequency response of the fine delay FIR filter, for filter length $M$ between 4 and 84. In each figure, the dash-dot rectangular box in the bottom panel indicates the 1° phase error specification over 10 – 88 MHz. In all cases except $M = 4$, the maximum phase error is less than 10° and the magnitude response appears to be quite reasonable. However, only the $M = 84$ FIR filter achieves phase accuracy of 1.0° over 10 – 88 MHz. These results lead to two conclusions. First, the maximum phase error clearly decreases as $M$ increases. Second, increasing $M$ increases the rate of ripple in the phase response. A very fast ripple is undesirable even if the phase deviation requirement is met, because for spectroscopy (and possibly other applications) any ripple needs to be calibrated out. Since larger filter length also increases system complexity and power consumption, there is a trade-off consideration in the selection of $M$.

### 2.3.2 Alternative Designs for the Fractional Sample Period Delay FIR Filter

Various known methods for fractional sample period delay FIR filters are presented in Table 2.2 (see also [1]). The well-known Windowing technique can reduce Gibbs phenomenon (i.e., the ripple observed in Figure 2.8), a big problem in the prototype truncation method. The Windowing technique is simple, but lacks precise control. The main lobe width and sidelobe level depend on the window function and the associated parameters. A comprehensive review of window functions was presented by Harris [41]. The Kaiser window and the Chebyshev window are two candidates
Figure 2.8: Frequency response deviation of the fine delay FIR filter from the ideal for various $M$. 
(a) $M = 4$. 
(b) $M = 24$. 
(c) $M = 44$. 
(d) $M = 64$. 
(e) $M = 74$. 
(f) $M = 84$. 

35
Table 2.2: Design techniques for fractional sample period delay FIR filters.

<table>
<thead>
<tr>
<th>Design Technique</th>
<th>Filter Design</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prototype truncation</td>
<td>$h[k] = \text{sinc}\left(k - \frac{\tau}{T_s}\right)$</td>
<td>Uncontrolled Gibbs phenomenon.</td>
</tr>
<tr>
<td>Windowing technique$^a$</td>
<td>$h[k] = W\left(k - \frac{\tau}{T_s}\right) \text{sinc}\left(k - \frac{\tau}{T_s}\right)$</td>
<td>Fast and simple design; Different window functions perform best for different requirements [41].</td>
</tr>
<tr>
<td>Lagrange interpolation</td>
<td>$h[k] = \prod_{n=0, n \neq k}^{M} \frac{\tau/T_s - n}{k - n}$</td>
<td>Easy explicit formula [42–44]; Optimally flat frequency response [45, 46]; Longer filter length required for the certain specifications over the frequency band of interest [47].</td>
</tr>
<tr>
<td>MMSE optimization</td>
<td>Constrained optimization problem</td>
<td>The same symmetry properties as the ideal filter [48]; Performance dependent on the choice for the bandwidth [49]; Performance criterion: minimum mean-square error.</td>
</tr>
<tr>
<td>Minimax optimization</td>
<td>Peak error minimization beginning with a prototype filter</td>
<td>Equal-ripple error pattern [50, 51]; Performance dependent on the cutoff frequency [52]; Performance criterion: minimize the worst-case estimation error.</td>
</tr>
</tbody>
</table>

$^a$ $W[k]$ is the window sequence with length $M$. 
for fine delay FIR filter designs, where the Kaiser window allows the control of the peak ripple using one parameter, and the Chebyshev window can minimize the sidelobe level for a given main-lobe width [53].

Another design method is the Lagrange interpolation method. This approach yields maximally flat frequency response with low effort. However, this approach results in longer filter length than necessary for the required filter specifications such as phase errors over the given bandwidth [47]. Since an objective of this dissertation is to design the filter with minimum filter length while achieving the required specifications, the Lagrange interpolation method is not suited to the application of interest in this research.

Optimization approaches can also be considered. One is to minimize the mean square error of the frequency response with respect to the same ideal response over the bandwidth of interest (i.e., MMSE optimization), and another is to minimize the maximum error of the frequency response over the bandwidth of interest (i.e., minimax optimization). D’Addario (2008) studied these two design criteria in the context of the LWA1 problem and found that the performance of the minimax filter is superior to the MMSE filter of the same length [54]. Minimax optimization is formulated as a Chebyshev approximation problem [52], so the resulting filter has equal ripple level in both the passband and the stopband.

In this dissertation, we will focus on the use of the windowing and minimax optimization methods for the delay FIR filter design.
2.4 Summary

This chapter presented a brief review of three beamforming techniques and compared their performance in terms of the SNR improvement. Mathematical expressions for the filter-and-sum beamforming weights in bandpass representation were derived for radio astronomy applications. The design approaches for the fractional sample period delay FIR filter were discussed and demonstrated in the context of LWA1.
Chapter 3

Cable Distortion and Correction

This chapter describes the problem of cable dispersion and proposes a wideband solution. In Section 3.1 ("Transmission Line Theory"), we introduce the general theory to characterize a transmission line. Section 3.2 ("Characteristics of Coaxial Cables") analyzes the problem of loss and dispersion in coaxial cables specifically, and proposes a method to correct the distortion. In Section 3.3 ("Application to KSR200DB Cable"), we apply the theory to a special case of Kingsignal\(^2\) Part Number KSR200DB coaxial cable, which is used in LWA1. Finally, Section 3.4 ("Summary") summarizes this chapter.

3.1 Transmission Line Theory

An electrical transmission line can be modeled as a two-port network consisting of four elementary components, as shown in Figure 3.1 [55]. In this representation, any infinitesimal length of trans-

\(^2\)http://www.kingsignal.com/en
mission line is modeled with a resistance ($R$ in $\Omega/m$) and inductance ($L$ in $H/m$) in series, and a capacitance ($C$ in $F/m$) and conductance ($G$ in $S/m$) in parallel. If properly terminated at both ends of the transmission line, the voltage at distance $z$ ($+z$ represents the direction of the wave propagation) can be written in phasor (time harmonic) form as:

$$V(z) = V(0)e^{-\gamma z}, \quad (3.1)$$

where $V(0)$ is a complex-valued voltage (phasor) at the beginning of the transmission line, and $\gamma$ is the complex-valued “propagation constant” given by

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} . \quad (3.2)$$

Separating $\gamma$ into real and imaginary parts $\gamma = \alpha + j\beta$, we define the voltage transfer function from input voltage $V(0)$ to output voltage $V(z)$, as

$$H(z) \triangleq \frac{V(z)}{V(0)} = e^{-\alpha z}e^{-j\beta z} , \quad (3.3)$$

where the first and second factors describe attenuation and phase along the transmission line.
at distance \( z \), respectively. This transfer function also characterizes the transmission line in the frequency domain as

\[
|H(\omega)| = e^{-\alpha z},
\]

(3.4a)

and

\[
\angle H(\omega) = -\beta z \text{ (radians)}.\]

(3.4b)

### 3.2 Characteristics of Coaxial Cables

Coaxial cable is a commonly-used type of transmission line, and its typical composition is as shown in Figure 3.2. One advantage of coaxial cable over other types of transmission line (e.g., balanced twin wire lines) is that in an ideal coaxial cable, the electromagnetic field carrying the signal exists only in the space between the inner and outer conductors and is thereby shielded from the outside.

#### 3.2.1 Cable Loss and Cable Dispersion

From Equation (3.3), the frequency response of a coaxial cable of length \( l \) is

\[
H_c(\omega) = e^{-\alpha l} e^{-j\beta l}.
\]

(3.5)

The attenuation of voltage along the coax of length \( l \) is given by

\[
A = e^{-\alpha l}.
\]

(3.6)
Figure 3.2: Typical composition of a coaxial cable.

and the associated delay is

$$\tau_c = -\frac{d\angle H_c(\omega)}{d\omega} = \frac{d\beta}{d\omega}l . \quad (3.7)$$

It is noted that the delay along the cable depends on length (as expected), but also possibly on frequency.

For an ideal coaxial cable, the shunt conductance and the series resistance are much smaller than the capacitive reactance and the inductive reactance, respectively; that is, $G \ll \omega C$, and $R \ll \omega L$. Under these conditions, the propagation constant is approximately imaginary-valued:

$$\gamma \cong \sqrt{(j\omega L)(j\omega C)} = j\omega \sqrt{LC} . \quad (3.8)$$

The frequency response for the ideal coaxial cable is then

$$H_{ic}(\omega) = e^{-j\omega(\sqrt{LC})l} , \quad (3.9)$$

where the magnitude response is unity and the delay along the cable is proportional to $(\sqrt{LC})l$; that is, independent of frequency. Hence the ideal coaxial cable has no loss and is dispersionless.
In practice, \( \alpha \) is non-zero and frequency-dependent, giving the propagation loss as shown in Equation (3.6); and \( \beta \) is a non-linear function of \( \omega \), giving the dispersion due to the frequency-dependent delay as shown in Equation (3.7). The source of dispersion in coaxial cables is that the center conductor and shield are not perfectly conductive, so that part of the current travels in the metal where propagation speed is frequency-dependent [55]. Unequal cable lengths in an array thus result in dispersion which might vary significantly between sensors.

### 3.2.2 Equalization of Cable Loss and Cable Dispersion

For the problem of loss and dispersion in the coaxial cable, we consider a filter to correct the distortion:

\[
H_{cd}(\omega) \cdot H_c(\omega) = H_{ic}(\omega)
\]

where \( H_{cd}(\omega) \) represents the frequency response of the correction filter. The correction filter in the frequency domain is thus

\[
H_{cd}(\omega) = e^{\alpha l} e^{j(\beta - \omega \sqrt{LC})l}
\]

where the first and second factors correct the loss and dispersion in the coaxial cable, respectively.

We can obtain the impulse response \( h_{cd}(t) \) of the correction filter by taking the inverse Fourier transform of \( H_{cd}(\omega) \).

If an appropriate set of samples of \( H_c(\omega) \) are known, the impulse response of the correction filter can be obtained by taking the inverse Fourier transform of \( H_c^{-1}(\omega)H_{ic}(\omega) \) numerically. However, this approach provides only numerical results (i.e., time samples), rather than a mathematical expression for \( h_{cd}(t) \). For the purpose of correction filter design and implementation, however, the
latter is often more useful. In order to find an explicit expression for \( h_{cd}(t) \), we return to the circuit
model given in Figure 3.1 and attempt to find appropriate expressions for coaxial cables. In the
calculation for the values of the circuit elements, the following symbols are used: \( a \) and \( b \) are the
radii of the inner conductor and the facing surface of the outer conductor, respectively; \( \sigma_a \) and \( \sigma_b \)
are the conductivities of the inner conductor and outer conductor, respectively; \( \epsilon \) is the permittivity
of the medium between the inner and outer conductor; and \( \mu \) is the permeability of the medium
between the inner and outer conductor. Note that the following values of the circuit elements are
specified per unit length. The shunt capacitance per unit length is given in [55] as

\[
C = \frac{2\pi\epsilon}{\ln(b/a)} .
\] (3.12)

Generally, \( \epsilon = \epsilon_0 \epsilon_r \) where \( \epsilon_0 \approx 8.85 \times 10^{-12} \text{ F/m} \) is the vacuum permittivity and \( \epsilon_r \) is the relative
permittivity, which is a property of the material, and which may be assumed frequency-independent
over the range of frequencies of interest. Thus, \( C \) is frequency-independent. The shunt conductance
per unit length described in [55] is

\[
G = \frac{2\pi\sigma}{\ln(b/a)} ,
\] (3.13)

where \( \sigma \) is the conductivity of the dielectric between the conductors at the operating frequency.
However, the shunt conductance is negligible for well-designed transmission line. For example, for
KSR200DB cables, \( C \) is about 80.4 pF/m and \( G \) is about 200 \( \mu \)S/m, so at 10 MHz the capacitive
reactance is \( \omega C \approx 5000 \mu \text{S/m} \); i.e., already more than 25 times greater than \( G \) and increasing with
frequency. Thus Equation (3.2) can be written as

\[
\gamma \approx \sqrt{(R + j\omega L)j\omega C} .
\] (3.14)
The series inductance per unit length, $L$, accounts for two sources of inductance. One is the ideal inductance that is associated with the magnetic component of the field between the conductors [55], given by

$$L_0 = \frac{\mu}{2\pi} \ln\frac{b}{a} ,$$

(3.15)

where $\mu = \mu_0 \mu_r$ with $\mu_0 = 4\pi \times 10^{-7}$ H/m being the permeability of free space and the relative permeability $\mu_r$ being essentially 1 for the dielectric media typically used as spacing material. Another is associated with the magnetic component of the field interior to the inner and outer conductors, which is possible because they are not perfectly conductive. For the case that the skin depth is much thinner than the material thickness, the second source of series inductance is for the inner and outer conductors respectively [55]:

$$L_a = \frac{\mu \delta_a}{4\pi a} , \quad \text{and}$$

(3.16a)

$$L_b = \frac{\mu \delta_b}{4\pi b} , \quad \text{and}$$

(3.16b)

where $\delta_a$ and $\delta_b$ are the skin depths for the inner and outer conductors respectively. It is known that

$$\delta_a = (\pi \mu \sigma_a f)^{-1/2} , \quad \text{and}$$

(3.17a)

$$\delta_b = (\pi \mu \sigma_b f)^{-1/2} .$$

(3.17b)

The total series inductance per unit length $L$ is the sum of $L_0$, $L_a$ and $L_b$. With substitutions and some algebra, we have

$$L = L_0 + L_s \delta f^{-1/2} ,$$

(3.18)
where
\[ L_{s0} = \frac{\mu^{1/2}}{4\pi^{3/2}} \left( \frac{\sigma_a^{-1/2}}{a} + \frac{\sigma_b^{-1/2}}{b} \right). \]  (3.19)

The series resistance per unit length arises from the exact same current associated with \( L_a \) and \( L_b \). For good conductors it is known that real and imaginary parts of wave impedance are equal [55]; thus
\[ R = \omega(L_a + L_b) = 2\pi L_{s0} f^{1/2}. \]  (3.20)

Applying substitutions in Equation (3.14), we find
\[ \gamma = j\beta_0 \sqrt{1 + (1 - j) \frac{L_{s0}}{L_0} f^{-1/2}}, \]  (3.21)

where \( \beta_0 = \omega\sqrt{L_0C} \) is the wavenumber for an ideal coaxial cable. It is noted that any frequency dependence is due to the current interior to the conductors, which manifests as non-zero \( R \) and frequency-dependent \( L \). The second term under the radical in Equation (3.21) is small compared to 1; see the example demonstrated in [32]. Applying the “small \( x \)” approximation \( \sqrt{1 + x} \approx 1 + \frac{1}{2}x \) to Equation (3.21), we obtain
\[ \gamma = \beta_0 \frac{1}{2} \frac{L_{s0}}{L_0} f^{-1/2} + j\beta_0 \left( 1 + \frac{1}{2} \frac{L_{s0}}{L_0} f^{-1/2} \right). \]  (3.22)

The real part of \( \gamma \) is then
\[ \alpha = \text{Re}\{\gamma\} = \beta_0 \frac{1}{2} \frac{L_{s0}}{L_0} f^{-1/2}, \]  (3.23)

The imaginary part of \( \gamma \) is found to be
\[ \beta = \text{Im}\{\gamma\} = \beta_0 \left( 1 + \frac{1}{2} \frac{L_{s0}}{L_0} f^{-1/2} \right). \]  (3.24)
After substituting Equations (3.23) and (3.24) into Equation (3.11) and applying some algebra, the frequency response of the cable correction filter is then

\[ H_{cd}(\omega) = e^{\alpha l} e^{j(\beta_0 - \beta_0)l} = \exp \left( (1 + j) \frac{\beta_0 l}{2} \frac{L_{s0}}{L_0} f^{-1/2} \right). \]  

(3.25)

This is the physical description of the cable correction filter, which is dependent upon the geometry and materials of the coaxial cable. Written in terms of physical parameters of the cable, we have

\[ H_{cd}(\omega) = \exp \left[ (1 + j) \sqrt{\frac{\epsilon}{8}} \left( \frac{\delta a^{-1/2}}{a} + \frac{\delta b^{-1/2}}{b} \right) \left( \ln \frac{b}{a} \right)^{-1} l \sqrt{\omega} \right]. \]  

(3.26)

We can also compute \( \alpha \) and \( \beta \) by using Equations (3.6) and (3.7) directly from datasheet values and measurements: From Equation (3.23), the attenuation in a coaxial cable of length \( l \) at frequency \( f \) can be modeled as

\[ A = e^{-\alpha_0 l \sqrt{f/f_0}}, \]  

(3.27)

where \( \alpha_0 \) is the real part of the propagation constant specified at frequency \( f_0 \). Recalling that the attenuation at any other frequency is \( A = e^{-\hat{\alpha} l} \) as given in Equation (3.6), we obtain

\[ \hat{\alpha} = \frac{\alpha_0}{\sqrt{2\pi f_0}} \sqrt{\omega}. \]  

(3.28)

Now we consider the delay associated with a cable of length \( l \). From Equation (3.24), the delay in the cable at any frequency \( f \) is

\[ \tau_c = t_0 + t_1 \frac{l}{l_1} \left( \frac{f}{f_1} \right)^{-1/2}, \]  

(3.29)

where the first term \( t_0 \) is the propagation delay in a dispersion-free cable, and the second term is
the excess delay due to dispersion. Here, \( t_1 \) is the excess delay measured at frequency \( f_1 \) for length \( l_1 \). \( t_0 \) is independent of frequency and is given by

\[
t_0 = \frac{l}{cV_p},
\]

(3.30)

where \( c \) is the speed of light in free space, and \( V_p \) is the velocity factor for the given cable. Recalling the delay as described in Equation (3.7), we have

\[
\dot{\beta} = \frac{1}{l} \int \tau_c d\omega = \frac{1}{cV_p} \omega + \frac{t_1 \sqrt{8\pi f_1}}{l_1} \sqrt{\omega}.
\]

(3.31)

Since \( \beta_0 cV_p = \omega \), we obtain

\[
\dot{\beta} = \beta_0 + \frac{t_1 \sqrt{8\pi f_1}}{l_1} \sqrt{\omega}.
\]

(3.32)

The associated frequency response of the cable correction filter is then

\[
\hat{H}_{cd}(\omega) = e^{\alpha l} e^{j(\beta - \beta_0)l} = \exp \left( \frac{\alpha_0}{\sqrt{2\pi f_0}} l \sqrt{\omega} \right) \exp \left( j \frac{t_1 \sqrt{8\pi f_1}}{l_1} l \sqrt{\omega} \right).
\]

(3.33)

Both Equations (3.26) and (3.33) show that \( H_{cd}(\omega) \) is a function of cable length and frequency.

The generalized expression for the frequency response of the cable correction filter is thus

\[
H_{cd}(\omega) = e^{\xi l \sqrt{\omega}} e^{j\kappa l \sqrt{\omega}} = e^{(\xi+j\kappa)l \sqrt{\omega}},
\]

(3.34)

where \( \xi \) and \( \kappa \) are constants, in m\(^{-1}\)·Hz\(^{-1/2}\), dependent upon the physical parameters of the cable.

For the case that \( H_{cd}(\omega) \) is determined from Equation (3.26), we have
\[ \zeta = \kappa = \sqrt{\frac{\epsilon}{8}} \left( \frac{\delta_a^{-1/2}}{a} + \frac{\delta_b^{-1/2}}{b} \right) \left( \ln \frac{b}{a} \right)^{-1}. \] (3.35)

For the case that \( H_{cd}(\omega) \) is determined from Equation (3.33), we have

\[ \zeta = \frac{\alpha_0}{\sqrt{2\pi f_0}}, \quad \text{and} \] (3.36a)

\[ \kappa = \frac{t_1 \sqrt{8\pi f_1}}{l_1}. \] (3.36b)

### 3.2.3 Time Domain Filter for Correction of Cable Distortion

The impulse response of the time domain correction filter, \( h_{cd}(t) \), is the inverse Fourier transform of \( H_{cd}(\omega) \). One approach is to obtain the numerical values for \( H_{cd}(\omega) \) at many \( \omega \), and then apply the inverse discrete Fourier transform. However, this approach provides only numerical values (e.g., time samples), rather than a mathematical expression for the impulse response. In the following, we try to derive a mathematical formulation of the impulse response.

Since there is no explicit closed form of the inverse Fourier transform of Equation (3.34) in the “common” literature, we choose to expand \( e^{g\sqrt{\omega}} \) (where \( g = (\zeta + j\kappa)l \) is a constant independent of frequency) as a polynomial to obtain an explicit closed-form expression for the impulse response of the cable dedispersion filter. A candidate method is to use a Taylor series expansion to obtain the desired polynomial. As Equation (3.34) illustrates that the frequency response of the correction filter is non-linear, neither the one-term nor the two-term Taylor series expansion is appropriate in this case. With increasing number of terms for Taylor series expansion, the inverse Fourier transform becomes more and more complicated, which is counter the intention. I therefore assume
the three-term Taylor series expansion to be appropriate. More details about the Taylor series are available in Appendix A. Using a three-term Taylor series to expand \( e^{g\sqrt{\omega}} \) around point \( \omega = \omega_c \), we obtain

\[
e^{g\sqrt{\omega}} \approx e^{g\sqrt{\omega_c}} \left( \left( \frac{g^2}{8\omega_c} - \frac{g}{8\sqrt{\omega_c^3}} \right) \omega^2 + \left( \frac{3g}{4\sqrt{\omega_c^3}} - \frac{g^2}{4} \right) \omega + \left( \frac{g^2\omega_c}{8} - 5g\sqrt{\omega_c} + 1 \right) \right) .
\] (3.37)

The approximation of \( H_{cd}(\omega) \) given in Equation (3.25) is then

\[
\tilde{H}_{cd}(\omega) = c_0\omega^2 + c_1\omega + c_2 ,
\] (3.38)

where

\[
c_0 = \left[ \frac{(\zeta + j\kappa)l^2}{8\omega_c} - \frac{(\zeta + j\kappa)l}{8\sqrt{\omega_c^3}} \right] e^{(\zeta + j\kappa)l\sqrt{\omega_c}} ,
\] (3.39a)

\[
c_1 = \left[ \frac{3(\zeta + j\kappa)l}{4\sqrt{\omega_c^3}} - \frac{(\zeta + j\kappa)l^2}{4} \right] e^{(\zeta + j\kappa)l\sqrt{\omega_c}} ,
\] (3.39b)

\[
c_2 = \left[ \frac{5(\zeta + j\kappa)l\sqrt{\omega_c}}{8} + 1 \right] e^{(\zeta + j\kappa)l\sqrt{\omega_c}} .
\] (3.39c)

The time domain filter \( \tilde{h}_{cd}(t) \) for the correction of cable distortion can be obtained by taking inverse Fourier transform of Equation (3.38). If sampled at the rate of \( f_s \) in the time domain, the complex-valued filter is then obtained. Using the method as described in Appendix B, the real-valued prototype filter \( \tilde{h}_{cd} \) is, in terms of its taps \( k \in (-\infty, \infty) \):

\[
\tilde{h}_{cd}[k] = -j(2c_0\pi^2 f_s^2 + c_1\pi f_s + c_2)k^2 + (2c_0\pi f_s^2 + c_1 f_s)k + 2jc_0 f_s^2 e^{j\pi k} + \frac{j(2c_0\pi^2 f_s^2 + c_1\pi f_s + c_2)k^2 + (2c_0^*\pi f_s^2 + c_1^* f_s)k - 2jc_0^* f_s^2}{2\pi k^3} e^{-j\pi k} + \frac{j(c_2 - c_0^*)k^2 - (c_1^* c_s f_s k - 2j(c_0 - c_0^*) f_s^2)}{2\pi k^3} ,
\] (3.40)
where $k$ indexes samples, and the operator “$*$” denotes conjugation. Reducing the limits in Equation (3.40) from $\pm \infty$ to $\left\lceil \pm (M - 1)/2 \right\rceil$ results in a $M$-tap FIR filter using the prototype truncation method. Other design approaches for the cable correction filter include windowing and minimax optimization, as described in Section 2.3.2.

### 3.3 Application to KSR200DB Cable

The cables used in LWA1 are Kingsignal Part Number KSR200DB. In this section, we will apply the transmission line theory described in Section 3.1 to characterize KSR200DB cables, and apply the procedure described in Section 3.2 to design the associated correction filters.

#### 3.3.1 Characteristics of KSR200DB Cable

It was shown that the attenuation in a KSR200DB cable of length $l$ can be modeled as Equation (3.5). Although independent measurements of KSR200DB cable are not available, the result found in [56] gives an excellent fit (within 0.1 dB at 150 MHz) to the 150 MHz, 450 MHz, and 900 MHz values provided in the KSR200DB data sheet. In terms of our model, $\zeta = 5.4 \times 10^{-7}$ m$^{-1}$ Hz$^{-1/2}$.

The velocity factor of KSR200DB cable is advertised to be 0.83 times the speed of light in vacuum$^3$, which has been confirmed by direct measurement of cables using time-domain reflectometry, and also independently confirmed by measurement data [57]. The additional dispersive delay is

$$\tau_d = (2.4 \text{ ns}) \left( \frac{l}{100 \text{ m}} \right) \left( \frac{f}{100 \text{ MHz}} \right)^{-1/2}. \quad (3.41)$$

In terms of our model, we have $\kappa = 3.8 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2}$.

Figure 3.3 summarizes the distortion of KSR200DB cables having lengths of 50 m, 100 m, and 150 m, respectively, using our model with the above values of $\zeta$ and $\kappa$. Figure 3.3(a) shows the attenuation in KSR200DB cables of different lengths. The results show that cable loss is dependent on cable length and frequency. Figure 3.3(b) illustrates the dispersive delay due to dispersion in KSR200DB cables. The results show that the problem of such excess delay due to cable dispersion becomes acute for large radio-frequency arrays operating below a few hundred megahertz with large fractional bandwidth. Thus, unequal cable lengths result in distortion (cable loss and dispersive delay) which may vary significantly between sensors.

Recall that the frequency response of any cable according to our model is

$$H_c(\omega) = e^{-\zeta \sqrt{\omega}} e^{-j \left( \frac{\pi c}{V_p} \omega + \kappa l \sqrt{\omega} \right)}, \quad (3.42)$$

whereas the frequency response of an ideal cable (e.g., $\zeta = \kappa = 0$) is

$$H_{ic}(\omega) = e^{-j \frac{\pi c}{V_p} \omega}. \quad (3.43)$$

To demonstrate the impact of distortion in coaxial cables on array systems, we again consider the simplified version of LWA1 described in Section 2.2.4, but now include cables that connect sensors to the beamformer, and further we assume they are of unequal lengths as indicated in Appendix C. Using the delay-and-sum beamforming scheme, the SNR improvement over the SNR of a single isotropic sensor is analyzed in two cases: (1) cables are assumed to be ideal ($\zeta = 0$ and $\kappa = 0$); and (2) cables are KSR200DB ($\zeta = 5.4 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2}$ and $\kappa = 3.8 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2}$).
Figure 3.3: Response of KSR200DB cables having different lengths, using Equation (3.5) with \( \zeta = 5.4 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2} \) and \( \kappa = 3.8 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2} \).

Using 28-tap filters for delay-and-sum beamforming, Figure 3.4 shows that unequal cable losses and dispersive delays degrade the SNR performance by about 0.2 – 0.9 dB.

### 3.3.2 Correction Filter for KSR200DB Cables

KSR200DB cable having length \( l \) and the associated correction filter can be characterized in the frequency domain. Taking \( l = 150 \text{ m} \) as an example, the results are shown in Figures 3.5 and 3.6. Figure 3.5(a) describes the KSR200DB cable frequency response as described in Equation (3.42) with \( \zeta = 5.4 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2} \) and \( \kappa = 3.8 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2} \). Figure 3.5(b) presents the cable loss and cable dispersion, where the ideal frequency response as described in Equation (3.43) is removed from the frequency response shown in Figure 3.5(a). It is seen that both the magnitude and phase response are frequency-dependent, due to the cable loss and cable dispersion. An approach to correct such distortion in cables is using a filter whose frequency response is shown in Figure 3.6, which is simply the inverse of the Figure 3.5(b) result.
Figure 3.4: Impact of cable dispersion on the array system. Delay-and-sum beamforming scheme is used.

One way to compute the impulse response is taking inverse discrete Fourier transform of sampled values of the frequency response. Using this method, the correction filter $h_{cd}$ is obtained in the form of numerical values. In general, these coefficients are complex-valued. The procedure for calculating the real-valued filter coefficients from the complex-valued filter is described in Appendix B. In the context of LWA1 where the sample rate is 196 MSPS, the corresponding sample period for $h_{cd}$ is 5.1 ns. We choose a frequency span of about 1.495 kHz for the sampled values of $H_{cd}(\omega)$, yielding $2^{17} = 131072$ bins. Figure 3.7 shows a $2^{17}$-tap correction filter (obtained via the $2^{17}$-point inverse Fourier transform) for the KSR200DB cable of length 150 m. Similarly, the impulse response of the cable, $h_c$, can also be obtained in the form of numerical values. Figure 3.8 gives the frequency response of the convolution of $h_c$ and $h_{cd}$, which now exhibits only the dispersionless delay as expected. The results confirm that a filter can correct the loss and dispersion in coaxial cables.
(a) $H_c(\omega)$. 

(b) Cable distortion $H_c(\omega)H_i^{-1}(\omega)$.

Figure 3.5: Frequency response of a KSR200DB cable having length of 150 m. $H_c(\omega)$ is computed using the model given in Equation (3.42), and $H_i(\omega)$ is computed using the model given in Equation (3.43).

Figure 3.6: Frequency response of the correction filter for a KSR200DB cable of length 150 m.
Figure 3.7: Close-up view of the $2^{17}$-tap impulse response of the correction filter obtained using the numerical inverse Fourier transform method. The sample period is 5.1 ns.

Figure 3.8: Frequency response of the convolution of $h_c$ and $h_{cf}$. 
We now employ the closed-form formulation described in Section 3.2.3. As Equation (3.38) shows, a three-termed Taylor series is used to obtain $\tilde{H}_{cd}(\omega)$, the approximation of $H_{cd}(\omega)$. In the context of LWA1, the center frequency for the Taylor series expansion should be chosen to yield minimum frequency response error over the bandwidth between 10 MHz and 88 MHz. By trying various frequencies over $10 - 88$ MHz, it is found that $f_c = 39.42$ MHz was best. Figure 3.9 illustrates the frequency response errors over the frequencies $10 - 88$ MHz using this choice. The results illustrate that the three-termed Taylor series expansion around the frequency 39.42 MHz is suitable between 10 MHz and 88 MHz. From this, the impulse response of the FIR correction filter, $\tilde{h}_{cd}$, is obtained using Equation (3.40) and shown in Figure 3.10.

In order to further verify the correctness of $\tilde{h}_{cd}$, we compare the differences between $h_{cd}$ and $\tilde{h}_{cd}$ in the frequency domain. The results are shown in Figure 3.11, where the maximum magnitude error and maximum phase error are found to be 0.3 dB and 0.43° over $10 - 88$ MHz, respectively. The sources of errors include the approximation of $H_{cd}(\omega)$ and the finite length of the correction filter.

We now consider the relationship between the filter length $M_{cd}$ and the performance of this cable correction filter. Figures 3.12(a) through 3.12(d) show the results for $M_{cd}$ ranging from 20 to 140, respectively. In each case, the correction filter is designed for the KSR200DB cable having length of 150 m. Each figure consists of three panels: The top panel gives the impulse response, whereas the middle and bottom panels show the deviation of the frequency response $\tilde{H}_{cd}(\omega)$ from the ideal frequency response. Note that the unavoidable pipeline delay has been removed in the phase response of the correction filter. Increasing the number of taps decreases the maximum magnitude response error and phase response error, as expected. It is found that a 140-tap FIR is required to achieve a phase accuracy of 1.0° over $10 - 88$ MHz, although $M_{cd} \gtrsim 60$ comes pretty close. Since
Figure 3.9: Frequency response errors in the cable correction filter using the closed-form formulation described in Equation (3.38).

Figure 3.10: Close-up view of the impulse response of the correction filter from Equation (3.40). Compare to Figure 3.7.
3.3.3 Alternative Design Methods for the KSR200DB Cable Correction Filter

In this section, we consider the performance of cable correction FIR filters obtained by different design methods, in terms of the frequency deviation from the actual frequency response $H_{cd}(\omega)$. In the following, we discuss three methods (see Section 2.3.2) for the design of the cable correction filter: Prototype truncation, windowing, and minimax optimization. For windowing, we consider the Kaiser and Chebyshev windows. The associated parameters are selected to design an FIR filter with sidelobe height $-60$ dB; that is, $\beta = 5.65$ for Kaiser window, and $r = 60$ dB for Chebyshev window. In each approach, $M_{cd}$ is selected as the minimum filter length which can achieve phase accuracy of $1.0^\circ$ over $10 - 88$ MHz. A KSR200DB cable of length 150 m is once again used as an example for the design of the cable correction filter. Figure 3.13 presents the results: Figure 3.13(a)
Figure 3.12: Performance of KSR200DB cable correction filters of different filter lengths. The filters are used to correct the distortion in a KSR200DB cable of length 150 m. $M_{cd}$ represents the number of taps of the cable correction filter.
Figure 3.13: Performance of the cable correction FIR filter using different design methods. The filters are designed to correct the distortion in a KSR200DB cable of length 150 m.
shows the results for a 140-tap cable correction FIR filter using the prototype truncation method repeated from Figure 3.12(d), Figure 3.13(b) shows the results for a 27-tap cable correction FIR filter using the Kaiser window with $\beta = 5.65$, Figure 3.13(c) shows the results for a 29-tap cable correction FIR filter using the Chebyshev window with $r = 60$ dB, and Figure 3.13(d) shows the results for a 19-tap cable correction FIR filter using minimax optimization. In each figure, the top panel gives the impulse response of the cable correction filter, the middle panel shows the deviation of magnitude response from the ideal, and the bottom panel shows the deviation of phase response from the ideal. The dash rectangular box in the bottom panel of each figure indicates the $1.0^\circ$ phase error specification over the bandwidth between 10 MHz and 88 MHz. A summary of these results is given in Table 3.1. The magnitude response appears to be quite reasonable in all cases, although the magnitude ripple for prototype truncation could be troublesome for spectroscopy. The results illustrate that windowing techniques significantly improve ripple in the phase response. In comparison with windowing techniques, for a given specification (i.e., $1.0^\circ$ phase error over $10 - 88$ MHz), minimax optimization yields the filter with smallest filter length, but increases the rate of ripple in the frequency response.

3.3.4 Demonstration

The longest cable of LWA1 has the length of 149 m. If no correction is implemented, the phase errors due to dispersion are up to $76.28^\circ$ and $25.71^\circ$ at 88 MHz and 10 MHz, respectively. In this research, the maximum acceptable phase error due to dispersion in any cable of LWA1 is taken to be $1.0^\circ$ over $10 - 88$ MHz. In this section, we consider implementing cable correction FIR filters to improve the SNR of LWA1 beamformer in the presence of cable dispersion. The previous section
Table 3.1: Summary of the performance of cable correction FIR filters using different design approaches.

<table>
<thead>
<tr>
<th>Design methods</th>
<th>Filter length $(M_{cd})$</th>
<th>Maximum magnitude error over 10 – 88 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prototype truncation</td>
<td>140</td>
<td>0.3 dB</td>
</tr>
<tr>
<td>Kaiser Window ($\beta = 5.65$)</td>
<td>27</td>
<td>0.6 dB</td>
</tr>
<tr>
<td>Chebyshev window ($r = 60$ dB)</td>
<td>29</td>
<td>0.6 dB</td>
</tr>
<tr>
<td>Minimax optimization</td>
<td>19</td>
<td>0.4 dB</td>
</tr>
</tbody>
</table>

shows that windowing can yield reasonable cable correction filters. Cable correction filters using windowing can also be described by closed-form mathematical expressions. For these reasons, we now consider correction filters for LWA1 using the Kaiser window. The associated filter length $M_{cd}$ for each sensor is selected to be 27 as before. After implementing the 27-tap per-sensor cable correction FIR filters to LWA1, the corresponding improvement in SNR using the delay-and-sum beamforming technique is shown in Figure 3.14. The FIR filter for the delay-and-sum beamforming is separate and has the length of 28. The results show that the proposed design can provide a benefit of about $0.1 - 0.5$ dB, and overcomes most of the degradation resulting from dispersion, especially above 30 MHz or so.

3.4 Summary

This chapter presented the problem of frequency-dependent distortion in cables using a rigorous description and proposed a wideband solution to the correction of cable distortion. The next chapter will discuss the problem of dispersion by antennas, which can be corrected using a similar approach.
Figure 3.14: SNR improvement at the output of the delay-and-sum beamformer accounting for the effect of unequal cable losses and dispersive delays in different cases.
Chapter 4

Antenna Dispersion and Dedispersion

This chapter discusses the problem of antenna dispersion and offers a solution. In Section 4.1 (“Relevant Antenna Theory”), we introduce the relevant theory of antennas and present a general model for an antenna. Taking a simple antenna as an example, we show that unequal antenna dispersion degrades the SNR of an array system. Section 4.2 (“Design of a Per-Sensor Antenna Dedispersion FIR Filter for a Thin Straight Dipole”) presents a correction approach for dispersion by a thin straight dipole. In Section 4.3 (“Description of a Single LWA1 Stand”), we provide an electromagnetic model for a single stand of LWA1, compute the antenna parameters including self-impedance and effective length, and characterize the dispersion of an antenna in a single, isolated (standalone) stand in terms of the variable group delay. In Section 4.4 (“Design of a Per-Sensor Antenna Dedispersion FIR Filter for a Single LWA1 Stand”), we apply designs of the antenna dedispersion filter to an antenna in a standalone stand (i.e., ignoring the rest of stands in the array). In Section 4.5 (“Design of a Per-Sensor Antenna Dedispersion FIR Filter for a LWA1 Antenna Stand in the Presence of a Second LWA1 Antenna Stand”), we present a simple array consisting of two stands and correct
the dispersion of an antenna in this case using a per-sensor antenna dedispersion filter. Finally, Section 4.6 (“Summary”) summarizes this chapter.

4.1 Relevant Antenna Theory

In this section we describe how to model an antenna in receive mode. This section also presents a procedure for the calculation of antenna parameters (e.g., self-impedance and vector effective length). The sources of dispersion in antennas are considered. At the end of this section, we apply the theory to a dipole antenna to show the impact of antenna dispersion on array systems.

4.1.1 Equivalent Circuit Model

An antenna can be modeled as a Thevenin equivalent circuit consisting of a voltage source in series with the self-impedance of the antenna [58]. Figure 4.1 shows this model for an antenna, where $Z_L$ represents the input impedance of the receiver, $Z_A$ is the impedance of the antenna, and $V_A$ is the open-circuit voltage at the terminals of the open-circuit antenna. The current flowing through this series circuit is then

$$I_L = \frac{V_A}{Z_A + Z_L},$$

(4.1)

Here, $V_A$ is in response to the incident electric field $\mathbf{E}_i$ as

$$V_A = \mathbf{E}_i \cdot \mathbf{l}_e,$$

(4.2)

where $\mathbf{l}_e$ is the vector effective length (VEL) of the antenna and is generally dependent on frequency and the direction from which the electric field arrives. If the angle between $\mathbf{E}_i$ and $\mathbf{l}_e$ is denoted as
\[ V_A = E^i l_e \cos \varphi, \quad (4.3) \]

where \( E^i \) is the (possibly complex-valued) magnitude of the electric field incident on the antenna, and \( l_e \) is the scalar effective length of the antenna.

A transfer function describing the relationship between the output current \( I_L \) and the scalar incident electric field \( E^i \) is then

\[ H_a(\omega) = \frac{I_L}{E^i} = \frac{l_e \cos \varphi}{Z_A + Z_L} \text{ (m/Ω) }. \quad (4.4) \]

This function characterizes the antenna in the frequency domain, since both \( l_e \) and \( Z_A \) are frequency-dependent. The associated group delay is given by

\[ \tau_g = -\frac{d\angle H_a(\omega)}{d\omega}, \quad (4.5) \]

where \( \angle H_a(\omega) \) represents the phase of \( H_a(\omega) \).

A source of antenna dispersion is the frequency-varying impedance mismatch between the antenna
and receiver, since this leads to frequency-dependent group delay [29]. When antennas are arranged in a non-uniform geometry, the way that mutual coupling changes each antenna’s impedance $Z_A$ is different, resulting in variable dispersion between antennas. This can be seen in simulations by Ellingson (2008) [31]. Dispersion by antennas may also result from variation in the current distribution as a function of frequency [30].

One approach to correct the dispersion by antennas is to use a time domain filter. A possible frequency response of the antenna dedispersion filter is obtained from Equation (4.4) as

$$H_{ad}(\omega) = H^{-1}_a(\omega) = \frac{Z_A + Z_L}{l_e \cos \varphi} \text{ (\Omega/m)}.$$  (4.6)

It is not important whether the signal to which this filter is applied represents current or voltage, since these are related by the constant input impedance $Z_L$.

It is a non-trivial problem to determine the vector effective length for arbitrarily-shaped antennas. A general procedure for an arbitrarily-shaped antenna can be derived using the Reciprocity Theorem of Electromagnetics [58] as described in [59]: Let the shape of the wire antenna depicted in Figure 4.2 be defined by $\hat{p}_a(l_a)$, where $l_a$ is arclength along the wire. If a test current $I^t_a$ is applied to the antenna terminals, then the resulting current distribution $I^t_a(l_a)$ when transmitting from the terminals can be found using any suitable approach; e.g., the Method of Moments (MoM). For an incident field $\mathbf{E}^i$, let $\hat{r}$ be the unit vector pointing from the antenna terminals toward the direction of incidence. The VEL of the antenna for the direction from which $\mathbf{E}^i$ arrives is then

$$l_e(\hat{r}) = \frac{1}{I^t_a} \int_{\text{wire}} I^t_a(l_a) \left[ \hat{p}_a(l_a) \times \hat{r} \times \hat{r} \right] e^{-j\frac{2\pi}{\lambda} w(l_a) \cdot \hat{r}} dl_a,$$  (4.7)

where $w(l_a)$ is the coordinates of the wire at position $l_a$, and $\lambda$ is the wavelength of the incident wave. Also note Equation (4.7) can be applied to any antenna which can be described as a wire
Figure 4.2: Calculation for the VEL of an arbitrarily-shaped wire antenna.

4.1.2 Example: A Thin Straight Dipole

In this section, a thin straight dipole antenna in free space is used as a simple example to show the dispersion by antennas. It is known from Equation (4.7) that VEL of an antenna is generally dependent on frequency and the direction from which the electric field arrives. For thin straight dipoles, VEL is a real-valued quantity which is proportional to the length of the antenna and oriented in parallel with the antenna. \( Z_A(\omega) \) can be obtained from an equivalent circuit of the impedance of an antenna. Several circuit models for dipoles include \([60-62]\). Here, we use the method of Tang, Tieng, and Gunn (1993) \([61]\) to compute the impedance for a thin straight dipole, because this approach can accurately model the impedance of a dipole antenna over a broad bandwidth using a simple circuit. In the following, this method will be referred to as “TTG”. As Figure 4.3 shows, in the TTG approach, the antenna is represented as an equivalent circuit of four components. The
values of the circuit elements are

\[
C_{31} = \frac{12.0674h}{\log(2h/a) - 0.7245} \text{ (pF)},
\]

\[
C_{32} = 2h \left\{ \frac{0.89075}{[\log(2h/a)]^{0.806} - 0.861} - 0.02541 \right\} \text{ (pF)},
\]

\[
L_{31} = 0.2h \left\{ [1.4813 \log(2h/a)]^{1.012} - 0.6188 \right\} \text{ (µH)},
\]

\[
R_{31} = 0.41288[\log(2h/a)]^2 + 7.40754(2h/a)^{-0.02389} - 7.27408 \text{ (kΩ)},
\]

(4.8)

where half-length \( h \) and radius \( a \) are both in meters. The impedance of the dipole antenna is then given by

\[
Z_A(\omega) = \frac{1}{j\omega C_{31}} + \frac{1}{j\omega L_{31}} + \frac{1}{R_{31} + j\omega C_{32}}
\]

\[
= \frac{1}{j\omega C_{31}} + \frac{j\omega R_{31}L_{31}}{R_{31}(1 - \omega^2L_{31}C_{32}) + j\omega L_{31}}.
\]

(4.9)

Now we consider a straight dipole antenna having the total length of \( 2h = 3.7 \text{ m} \) and radius of \( a = 5 \text{ mm} \) over the frequency bandwidth between 10 MHz and 88 MHz. Figure 4.4 presents the...
Figure 4.4: Calculated impedance of a straight dipole of length 3.7 m and radius 5 mm. The bottom panel is a close-up from the top panel to show the frequency of resonance.

It is shown that the dipole antenna resonates at about 35 MHz.

Assuming the incident electric field is perfectly co-polarized with the antenna, the open-circuit voltage at the antenna terminals is then

\[ V_A = E_i l_e . \]  \hspace{2cm} (4.10)

Equation (4.4) becomes in this case

\[ H_a(\omega) = \frac{l_e}{Z_A + Z_L} \text{ (m/Ω)} . \]  \hspace{2cm} (4.11)

The group delay will now be determined by using Equation (4.5), taking \( Z_L = 100 \Omega \). Figure 4.5
Figure 4.5: Group delay due to antenna dispersion.

presents the associated group delay. Note that the dispersion is worst around the resonance of the antenna. There is about 33 ns variation in the group delay over the frequency range between 10 MHz and 88 MHz; this is significant compared to the free-space propagation time across the diameter of the LWA1 array (100 m), which is about 333 ns.

4.2 Design of a Per-Sensor Antenna Dedispersion FIR Filter for a Thin Straight Dipole

Now we consider using a filter to correct the per-sensor dispersion by antennas. The frequency response of the ideal dedispersion filter is described in Equation (4.6). Closed-form expressions typically do not exist for $H_{ad}(\omega)$ for most practical antennas, however, the associated impulse response can still be obtained by taking the inverse discrete Fourier transform of samples representing
frequency response $H_{ad}(\omega)$.

We intend to use a real-valued filter to correct the dispersion by antennas. A general procedure for calculating suitable real-valued coefficients from a complex-valued dedispersion filter is as described in Appendix B and demonstrated in Section 3.3.2. The alternative design methods described in Section 2.3.2 (e.g., windowing and minimax optimization) are also applicable, and will be considered later in this chapter.

In the following example, a real-valued antenna dedispersion FIR filter is designed for the dipole antenna as described at the end of Section 4.1.2. To make this relevant to LWA1, the sampling rate is 196 MSPS. This antenna is of course quite different from the LWA1 antenna (see Figures 4.20 and 4.21), but it has a similar resonant frequency (approximately 38 MHz). As in LWA1, the antenna is in series with a load $Z_L = 100 \Omega$. If the incident electric field and the VEL are co-polarized (i.e., $\varphi = 0$), the associated frequency response of the dipole antenna is given by Equation (4.11). The results depicted in Figure 4.6 show that the magnitude response has a peak at the resonant frequency, as expected, and the slope of phase is variable, indicating dispersive delay. The frequency response of the associated correction filter is

$$H_{ad}(\omega) = H_{a}^{-1}(\omega) = \frac{Z_A + Z_L}{l_e} \ (\Omega/m) . \quad (4.12)$$

Figure 4.7 shows the antenna dedispersion filter in the frequency domain. Applying the procedure as described in Appendix B to samples of $H_{ad}(\omega)$, the real-valued impulse response of the antenna dedispersion filter $h_{ad}$ is obtained in terms of data samples. In the context of LWA1, the time interval is about 5.1 ns since the sample rate is 196 MSPS. Here we choose a frequency span of about 1.495 kHz for the sampled values of $H_{ad}(\omega)$, therefore the number of samples is $2^{17} = 131072$. Figure 4.8 presents the results. Similarly, $h_a$, the truncated impulse response of $H_a(\omega)$, can also
Figure 4.6: Frequency response $H_a(\omega)$ of the dipole antenna, as described in Section 4.1.2.

Figure 4.7: Frequency response $H_{ad}(\omega)$ of the antenna dedispersion filter.
be obtained by performing the procedure as described in Appendix B on $H_a(\omega)$ followed by a truncation. The convolution between $h_a$ and $h_{ad}$ is expected to be approximately equal to a Kronecker delta function. Figure 4.9 shows $h_a \ast h_{ad}$ with different $M_{ad}$, confirming the calculations are correct and that accuracy improves with increasing number of taps. Figures 4.10(a) – 4.10(d) show the frequency response errors of the antenna dedispersion filter for $M_{ad}$ between 200 and 426. In each figure, the dash-dot rectangular box in the bottom panel indicates the 1° phase error specification over the span of 10 – 88 MHz. The results are also summarized in Table 4.1. In all cases, the maximum magnitude error over 10 – 88 MHz are reasonable; i.e., no larger than 0.5 dB. However, only the $M_{ad} = 426$ FIR filter achieves the phase accuracy of 1.0° over the bandwidth of interest.

The above results use prototype truncation. We now consider the performance of dedispersion FIR.
Figure 4.9: Verification of the dipole antenna dedispersion FIR filter in the time domain for different filter lengths $M_{ad}$. The bottom panel is a close-up from the top panel.
Figure 4.10: Frequency response errors of the dipole antenna dedispersion FIR filters from the ideal for different filter lengths.

(a) $M_{ad} = 200$.

(b) $M_{ad} = 300$.

(c) $M_{ad} = 400$.

(d) $M_{ad} = 426$. 
Table 4.1: Summary of performance of the dipole antenna dedispersion FIR filters for different filter lengths.

<table>
<thead>
<tr>
<th>Filter length ((M_{ad}))</th>
<th>Frequency response derivation from the ideal over 10 – 88 MHz</th>
<th>Peak magnitude error</th>
<th>Peak phase error</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td></td>
<td>0.5 dB</td>
<td>2.6°</td>
</tr>
<tr>
<td>300</td>
<td></td>
<td>0.3 dB</td>
<td>1.8°</td>
</tr>
<tr>
<td>400</td>
<td></td>
<td>0.2 dB</td>
<td>1.4°</td>
</tr>
<tr>
<td>426</td>
<td></td>
<td>0.2 dB</td>
<td>1.0°</td>
</tr>
</tbody>
</table>

filters obtained using different methods: Windowing, and minimax optimization. For windowing, the Kaiser and Chebyshev windows are selected. The associated parameters are selected to design an FIR filter with sidelobe height –60 dB; that is, \(\beta = 5.65\) for Kaiser window, and \(r = 60\) dB for Chebyshev window. In each approach, \(M_{ad}\) is selected to be the minimum filter length which achieves phase error less than 1.0° over 10–88 MHz. Figure 4.11 shows error with respect to the ideal frequency response: Figure 4.11(a) shows the result for the 426-tap antenna dedispersion filter using the prototype truncation method repeated from Figure 4.10(d), Figure 4.11(b) shows the result for the 69-tap antenna dedispersion filter using the Kaiser window with \(\beta = 5.65\), Figure 4.11(c) shows the result for the 79-tap antenna dedispersion filter using a Chebyshev window with \(r = 60\) dB, and Figure 4.11(d) shows the result for the 34-tap antenna dedispersion filter using minimax optimization. The dash rectangular box in the bottom panel of each figure indicates 1.0° phase error limits over the frequency span of 10–88 MHz. A summary of results is presented in Table 4.2. Although all three methods show considerable improvement relative to prototype truncation, the results illustrate that a dipole antenna dedispersion filter designed by minimax optimization results in the shortest filter length. This design exhibits much better ripple response than prototype truncation, although not quite as good as Windowing. Again, a tradeoff between windowing and minimax methods is evident.
Figure 4.11: Performance of the dipole antenna dedispersion FIR filters using different design approaches.

Table 4.2: Minimum number of taps required for less than 1° phase error over 10 – 88 MHz for dipole antenna dedispersion FIR filters designed using different methods.

<table>
<thead>
<tr>
<th>Design methods</th>
<th>Filter length ((M_{ad}))</th>
<th>Maximum magnitude error over 10 – 88 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prototype truncation ( \beta = 5.65 )</td>
<td>426</td>
<td>0.24 dB</td>
</tr>
<tr>
<td>Kaiser Window ( \beta = 5.65 )</td>
<td>69</td>
<td>0.12 dB</td>
</tr>
<tr>
<td>Chebyshev window ( r = 60 dB )</td>
<td>79</td>
<td>0.11 dB</td>
</tr>
<tr>
<td>Minimax optimization</td>
<td>34</td>
<td>0.21 dB</td>
</tr>
</tbody>
</table>
4.3 Description of a Single LWA1 Stand

In this section, the characteristics of LWA1 antennas are studied. LWA1 consists of 512 antennas arranged into 256 “stands”, with each stand consisting of a pair of orthogonally-aligned dipoles over a wire mesh ground screen, as shown in Figure 1.4. The conducting ground screen is used to reduce loss through absorption into the ground and also to stabilize the system temperature by isolating the antenna from variable ground (e.g., dry versus wet) conditions [63]. The arms of each dipole are angled downward at approximately 45° to form an inverted-vee shape. The dipoles are linearly polarized and resonant around 38 MHz. Each of the two antennas in a stand is connected to an “active balun” which presents a balanced input impedance of $Z_L = 100 \, \Omega$.

4.3.1 Electromagnetic Model

As the antennas and ground screen comprising each stand consist of interconnected metal wire segments, it is well-suited to wire grid modeling using MoM. In this research, we employ the 4NEC2 implementation of MoM to analyze the performance of these antennas.

The wire grid model used to simulate a LWA1 stand is shown in Figure 4.12. It is the same model used in [7]. The dimensions and parameters used to model the antenna are provided in Figure 4.13. The dipole arms are bent 45° downward from the junction with the center wire. The wire is modeled as a perfectly conducting wire having radius of 1.2 cm. The wire grids representing each of the two antennas in a stand are vertically separated by three times the wire radius to prevent the feeds from intersecting. The mean height of the highest points on each antenna is

\[ \text{http://home.ict.nl/~arivoors} \]
Figure 4.12: Wire grid model of a LWA1 stand. The Cartesian coordinate system is selected as described in Section 2.1.

Figure 4.13: Geometry of wire grid used to model the dipole. Dimensions are in meters. Number of segments used in the MoM model are indicated next to each wire.
1.5 m above ground. Simulations and experiments have demonstrated that neither the center mast nor the non-metallic structure supporting the dipole arms have a significant effect on the relevant properties of the dipoles, and therefore no attempt is made to model them. The two dipoles in a stand are orthogonally-aligned; i.e., aligned East-West and North-South, respectively. Using the same Cartesian coordinate system as given in Section 2.1, the dipole aligned along the $x$-axis (i.e., East-West) is referred to as the $x$-aligned dipole, whereas the dipole aligned along the $y$-axis (i.e., North-South) is referred to as the $y$-aligned dipole. The ground screen is modeled using a $3 \times 3$ m wire grid with spacing $10 \times 10$ cm and wire radius of 1 mm, which is very close to the actual dimensions. The modeled ground screen is located 1 cm above ground to account for the significant but irregular gap that exists because of ground roughness. The ground itself is modeled as an infinite homogeneous half-space with relative permittivity of 3 and conductivity of $100 \mu$S, which is appropriate for “very dry ground” which predominates in New Mexico, where LWA1 is located.

### 4.3.2 Self-Impedance

We first consider a single antenna stand by itself; i.e., a “standalone” stand. Various numerical methods, including the MoM and the finite difference time domain (FDTD) method, can be used to calculate antenna impedance for complex or realistic antennas for which simple expressions are not available. In this section, we use MoM (NEC) model defined above to obtain the impedance of the antennas in a standalone stand.

A general method to compute the impedance of a standalone antenna is modeling the antenna in transmit mode. The impedance is calculated by applying a test voltage to the antenna terminals, calculating the resulting current, and then $Z_A$ is the ratio. Figure 4.14 shows that the LWA antennas
Figure 4.14: Calculated impedance of the antenna in a standalone stand in transmit mode.

as modelled are resonant at about 35 MHz.

For the LWA1 array, however, antennas operate in receive mode. Whereas this should not make a difference for a single antenna, this is a consideration for antennas in an array, due to mutual coupling. To demonstrate a receive-mode procedure, we compute the impedance of the antenna in a standalone stand. In the NEC model, both dipoles are always present in all cases, and the dipole not of interest is loaded with 100 Ω at the feedpoint. First, the antenna of interest is loaded with 0 Ω at the feedpoint. The NEC model is run with the antenna of interest illuminated by a 1 V/m plane wave. This generates the short-circuit current $I_{sc}$ at the antenna terminals. Next, the antenna of interest is loaded with $10^7$ Ω at the feedpoint, approximately an open circuit but yielding a measurable current $I_{oc}$. The NEC model is run with the antenna of interest exposed to a 1 V/m plane wave. This gives $I_{oc}$ at the antenna terminals. Finally, the impedance of the antenna
of interest in receive mode is obtained by Ohm’s Law as:

\[ Z_A = \frac{I_{sc}}{I_{sc}} \cdot 10^7 \ (\Omega) . \tag{4.13} \]

This result does not depend on the direction from which \( E^i \) arrives or the polarization of \( E^i \). To demonstrate this, \( Z_A \) is computed using the above method for different incident signals. The results are shown in Figure 4.15.

As further confirmation of the validity of this approach, we also consider the impedance of the antenna in a standalone stand using the transmit-mode method and receive-mode method. In this comparison, the incident signal used in the receive-mode calculation is selected as 1 V/m \( \theta \)-polarized plane wave incident from the zenith. The comparison result is presented in Figure 4.16. We observe that the impedance of the antenna in a standalone stand is essentially identical in both calculations.

### 4.3.3 Effective Length

In the NEC model for the calculation of effective length \( l_e \), the two antennas in a standalone stand are always present and are both loaded with \( Z_L = 100 \ \Omega \) at the feedpoint, which is the actual situation for the LWA1. The NEC model is run using a 1 V/m plane wave excitation. This yields the current \( I_L \) at the antenna terminals, from which \( l_e \) can be calculated as

\[ l_e = \frac{I_L Z_L}{E^i} . \tag{4.14} \]

Unlike \( Z_A \), \( l_e \) is dependent upon the polarization and incident direction of \( E^i \).
Figure 4.15: Calculated impedance of an antenna in a standalone stand in receive mode for various incident signals. $E_\theta^i$ and $E_\phi^i$ represent the $\theta$- and $\phi$-polarized components of the incident plane wave $E^i$, respectively.

(a) Zenith incidence.  
(b) $\theta = 74^\circ, \phi = 45^\circ$.

Figure 4.16: Comparison of the impedance of the antenna in a standalone stand calculated using the transmit-mode and receive-mode methods.

(a) $x$-aligned antenna.  
(b) $y$-aligned antenna.
Figures 4.17–4.19 show calculated $l_e$ of the antenna in a standalone stand for signals incident from different directions with different polarization components. The results are expressed in terms of the magnitude and phase of the effective length. Magnitude is expressed in dB relative to 1 m. It is noted that the effective length introduces dispersion, since there is a variable phase slope. The physical source of this dispersion is the variation in time required for currents excited on different parts of the antenna to arrive at the feedpoint. Despite this, the magnitude response to $\theta$- and $\phi$-polarization are identical for $x$- and $y$-aligned antennas if $\phi = 45^\circ$ (see Figures 4.17 and 4.18), which is the expected result. When the signal is incident along the $x$-axis as shown in Figure 4.19, we see the magnitude response of the $x$-aligned antenna is suppressed compared to that of the $y$-aligned antenna for the $\phi$-polarization, and the magnitude response of the $y$-aligned antenna is suppressed compared to that of the $x$-aligned antenna for the $\theta$-polarization, again as expected.

4.3.4 Group delay

The group delay $\tau_g$ of $H_a(\omega)$, defined in Equation (4.5), depends on $Z_A$ and $l_e$. We have observed that the impedance of the antenna in a standalone stand is independent of the polarization and direction of the incident signal, but that $l_e$ depends on the polarization and direction of the incident signal. Figures 4.17 and 4.18 also illustrate that $l_e$ for $x$- and $y$-aligned antennas are identical if the signal is incident from the “diagonal” plane between the $x$- and $y$-axis ($\phi = 45^\circ$). In the following, we consider the group delay of the antenna in a standalone stand due to dispersion with respect to a signal incident from $\phi = 45^\circ$. Figures 4.20 and 4.21 show that the group delay associated with $x$- and $y$-aligned antennas are identical in this case, consistent with the results in Figures 4.17 and 4.18. We also note that the group delay increases for low-elevation incidence (i.e., $\theta$ is large),
Figure 4.17: Calculated effective length of the antenna in a standalone stand with the incident direction of $\theta = 0^\circ$ and $\phi = 45^\circ$.

Figure 4.18: Calculated effective length of the antenna in a standalone stand with the incident direction of $\theta = 74^\circ$ and $\phi = 45^\circ$. 
Figure 4.19: Calculated effective length of the antenna in a standalone stand with the incident direction of \( \theta = 74^\circ \) and \( \phi = 0^\circ \).

Figure 4.20: Group delay of an antenna in a standalone stand due to a signal incident from \( \theta = 0^\circ \) and \( \phi = 45^\circ \).
although the variation in group delay is not much affected. As expected, the group delay always has peak near the resonance frequency, independent of polarization or the incident direction.

4.4 Design of a Per-Sensor Antenna Dedispersion FIR Filter for a Single LWA1 Stand

Applying the theory as described in Sections 4.1 and 4.2, the correction filter for a single LWA1 antenna by itself (i.e., with no mutual coupling) as described in Section 4.3 is obtained.

The frequency response of the antenna is given in Equation (4.4), where $Z_A$ and $l_e$ are obtained as in Section 4.3. Calculating $H_a(\omega)$ on a dense frequency grid is problematic due to the large computational effort associated with MoM. Instead, the procedure used here is to calculate $H_a(\omega)$ at selected frequencies using MoM, and then to obtain $\widetilde{H}_a(\omega)$ at all other frequencies using piecewise cubic Hermite interpolation [64–66]. This method is selected because it has no overshoot and yields
small oscillation even if the data are not smooth [65]. In the following experiments, we take the $x$-aligned antenna in a standalone stand as an example. The incident signal is $\theta$-polarized and from the direction of $\theta = 74^\circ$ and $\phi = 45^\circ$. $\tilde{H}_a(\omega)$ is obtained by interpolating between the calculated samples of the magnitude and phase of $H_a(\omega)$. Figure 4.22 shows $H_a(\omega)$ and $\tilde{H}_a(\omega)$. The real-valued impulse response $h_a$ can be obtained by performing the method in Appendix B on samples of $\tilde{H}_a(\omega)$. In the context of LWA1 where the sample rate is 196 MSPS, the corresponding sample period for $h_a$ is thus 5.1 ns. We choose a frequency span of about 1.495 kHz for the sampled values of $H_{ad}(\omega)$, therefore the number of taps is $2^{17} = 131072$. Figure 4.23 shows a close-up view of the resulting real-valued filter.

Using Equation (4.6), a possible frequency response of the antenna dedispersion filter, $H_{ad}(\omega)$ is obtained and shown in Figure 4.24. Using the procedure as described in Appendix B to samples of $H_{ad}(\omega)$, the real-valued impulse response $h_{ad}$ is obtained in terms of data samples. Figure 4.25 shows a close-up view of $h_{ad}$. If the number of taps for the impulse response is large enough, the convolution between $h_a$ and $h_{ad}$ should be a Kronecker delta function. Figure 4.26 shows $h_a \ast h_{ad}$ with different $M_{ad}$, confirming the validation of the solution.

Now we consider the relationship between the filter length $M_{ad}$ and the performance of the antenna dedispersion filter $h_{ad}$. Note that, as Figure 4.27 shows, on the order of a few hundred taps are needed to accurately model the antenna dedispersion filter using the prototype truncation method. Figure 4.27 shows the frequency response error with respect to the ideal, for filter lengths $M_{ad}$ between 97 and 177. In each figure, the dash-dot rectangular box in the bottom panel indicates the $1.0^\circ$ phase error specification over the span of $10 - 88$ MHz. The results are also summarized in Table 4.3. Only the $M_{ad} = 177$ filter achieves phase error no larger than $1.0^\circ$ over the bandwidth
Figure 4.22: Frequency response of the antenna in a standalone stand.

Figure 4.23: Close-up view of the impulse response $h_a$ of $\tilde{H}_a(\omega)$. 
Figure 4.24: Frequency response $H_{ad}(\omega)$ of the antenna dedispersion filter for the antenna in a standalone stand.

Figure 4.25: Close-up view of the impulse response $h_{ad}$ of the antenna dedispersion filter for the antenna in a standalone stand.
Figure 4.26: Verification of the antenna dedispersion FIR filter for the antenna in a standalone stand in the time domain for different filter lengths $M_{ad}$. The bottom panel is a close-up view from the top panel.
Figure 4.27: Frequency response error of the antenna dedispersion FIR filters for the antenna in a standalone stand using the prototype truncation method from the ideal for different $M_{ad}$.

(a) $M_{ad} = 97$.
(b) $M_{ad} = 137$.
(c) $M_{ad} = 167$.
(d) $M_{ad} = 177$.

The direction of incidence is $\theta = 74^\circ$ and $\phi = 45^\circ$. 

94
Table 4.3: Summary of performance of the antenna dedispersion FIR filters for the antenna in a standalone stand using prototype truncation for different $M_{ad}$.

<table>
<thead>
<tr>
<th>Filter length ($M_{ad}$)</th>
<th>Frequency response derivation from the ideal over 10 – 88 MHz</th>
<th>Peak magnitude error</th>
<th>Peak phase error</th>
</tr>
</thead>
<tbody>
<tr>
<td>97</td>
<td></td>
<td>0.7 dB</td>
<td>7.2°</td>
</tr>
<tr>
<td>137</td>
<td></td>
<td>0.3 dB</td>
<td>1.9°</td>
</tr>
<tr>
<td>167</td>
<td></td>
<td>0.3 dB</td>
<td>1.2°</td>
</tr>
<tr>
<td>177</td>
<td></td>
<td>0.2 dB</td>
<td>1.0°</td>
</tr>
</tbody>
</table>

of interest. In all cases, the maximum error over the bandwidth of interest is near first resonance, where large errors are possible because the self-impedance is changing rapidly with frequency in this region. The maximum magnitude error over 10 – 88 MHz appears to be reasonable in all cases except $M_{ad} = 97$. These results confirm that the frequency response error decreases as the filter length $M_{ad}$ increases. Increasing the number of taps, however, increases the rate of ripple in the phase response. Therefore there appears to be no obvious optimal choice of the filter length $M_{ad}$, although $M_{ad} \geq 177$ meets the phase error specification.

We now consider the alternative design methods as described in Section 2.3.2 to obtain a FIR filter with smaller filter length: Windowing and minimax optimization. For windowing, the Kaiser and Chebyshev windows are selected. The associated parameters are selected to design a FIR filter with sidelobe height $-60$ dB; that is, $\beta = 5.65$ for the Kaiser window, and $r = 60$ dB for the Chebyshev window. In each approach, $M_{ad}$ is selected to be the minimum filter length achieving phase error no larger than 1.0° over 10 – 88 MHz. Figure 4.28 shows the results and Table 4.4 presents a summary of results. Note that 149 taps are required for Kaiser window, 161 taps are required for Chebyshev window, and 100 taps are required for minimax optimization. Windowing has low rate of ripple in the phase response over the bandwidth of interest. However, minimax optimization yields the smallest filter length. Note again that the largest frequency response errors are near resonance.
Figure 4.28: Performance of the antenna dedispersion FIR filters for the antenna in a standalone stand using windowing and minimax optimization. The incident direction is $\theta = 74^\circ$ and $\phi = 45^\circ$. 

(a) Kaiser window ($M_{ad} = 149$).

(b) Chebyshev window ($M_{ad} = 161$).

(c) Minimax optimization ($M_{ad} = 100$).
Table 4.4: Summary of performance of the antenna dedispersion FIR filters for the antenna in a standalone stand.

<table>
<thead>
<tr>
<th>Design methods</th>
<th>Filter length ($M_{ad}$)</th>
<th>Maximum magnitude error over 10 – 88 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kaiser Window ($\beta = 5.65$)</td>
<td>149</td>
<td>0.26 dB</td>
</tr>
<tr>
<td>Chebyshev window ($r = 60$ dB)</td>
<td>161</td>
<td>0.26 dB</td>
</tr>
<tr>
<td>Minimax optimization</td>
<td>100</td>
<td>0.31 dB</td>
</tr>
</tbody>
</table>

4.5 Design of a Per-Sensor Antenna Dedispersion FIR Filter for a LWA1 Antenna Stand in the Presence of a Second LWA1 Antenna Stand

In the previous section, we considered the performance of an LWA1 stand in the absence of any other stand. In this section, we consider the effect of a second identical stand 5.5 m away from the stand of interest. This corresponds to the minimum distance between two stands in the LWA1 array. The wire grid model for the array consisting of two identical stands is shown in Figure 4.29. Antennas and the per-stand ground screens are modeled using the same dimensions and parameters as described in Section 4.3. The self-impedance and effective length for an antenna in a two-stand pair are determined using the same method as described in Section 4.3.

Figure 4.30 compares the antenna self-impedance for one antenna in a two-stand pair to that of an antenna in a standalone stand. The results show that the difference of the self-impedance is less than 0.06 dB in magnitude and 0.05° in phase over the frequency range of interest. Thus we may reuse the calculated antenna impedance from the standalone stand result in this analysis.

Figure 4.31 compares the effective length $l_e$ due to a signal incident from the zenith in the diagonal plane (i.e., half-way between the E- and H-planes) in the two-stand pair and standalone stand cases.
Figure 4.29: Wire grid model of an array consisting of two identical stands. The Cartesian coordinate system is selected as described in Section 2.1.

Figure 4.30: Difference between the self-impedance of antenna in a two-stand pair and the same antenna in a standalone stand.
Figure 4.31: Difference between the effective length of an antenna in a two-stand pair and the same antenna in a standalone stand for the incident direction $\theta = 0^\circ$ and $\phi = 45^\circ$.

The difference is larger than that seen in $Z_A$, but nevertheless it is small. Figure 4.32 repeats this experiment for a signal incident from $\theta = 74^\circ$ in the diagonal plane. Note that difference between the two-stand pair and standalone stand cases is significantly larger in this case. Thus, antenna dispersion correction filters should account for the effect of mutual coupling, polarization, and direction of arrival on the effective length $l_e$.

Figure 4.33 shows the performance of various antenna dedispersion filters using different design approaches for a signal incident from $\theta = 74^\circ$ and $\phi = 45^\circ$. The results are summarized in Table 4.5. Once again, the number of taps is selected to achieve phase error no larger than $1.0^\circ$ over $10 - 88$ MHz. Comparing the results to Table 4.4 (standalone stand), we see only a few more taps are required to account for the mutual coupling present in the two-stand pair case.
Figure 4.32: Same as Figure 4.31, except for $\theta = 74^\circ$.

4.6 Summary

This chapter analyzed the problem of dispersion by antennas and addressed the solution for LWA1 antennas singly and in pairs. A complete solution including all antennas over all frequencies and directions of incidence can be obtained in the same manner, but requires very burdensome moment method calculations. This matter is addressed further in Chapter 6.
Figure 4.33: Performance of antenna dedispersion FIR filters for an antenna in the two-stand pair using different design approaches. The incident direction is $\theta = 74^\circ$ and $\phi = 45^\circ$.

Table 4.5: Summary of performance of antenna dedispersion FIR filters for an antenna in the two-stand pair using different design approaches.

<table>
<thead>
<tr>
<th>Design methods</th>
<th>Filter length $(M_{ad})$</th>
<th>Maximum magnitude error over $10 - 88$ MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prototype truncation</td>
<td>178</td>
<td>0.13 dB</td>
</tr>
<tr>
<td>Kaiser Window ($\beta = 5.65$)</td>
<td>151</td>
<td>0.24 dB</td>
</tr>
<tr>
<td>Chebyshev window ($r = 60$ dB)</td>
<td>163</td>
<td>0.25 dB</td>
</tr>
<tr>
<td>Minimax optimization</td>
<td>105</td>
<td>0.29 dB</td>
</tr>
</tbody>
</table>
Chapter 5

Combination Approaches

Chapters 2, 3, and 4 presented per-sensor filters for beamforming, cable correction, and antenna dedispersion, respectively. Using the shortest filters as determined to be suitable in each chapter, a total of 147 (i.e., $28 + 19 + 100$) taps is required for the “concatenation” scheme described in Figure 1.2(a). We now aim to reduce the total number of taps, thereby reducing hardware resources and power consumption. This chapter presents two approaches to obtain a single per-sensor filter which performs all three of these functions, as shown in Figure 1.2(b). Section 5.1 (“Direct Method”) describes the method based on convolution and truncation, and Section 5.2 (“Optimization Method”) describes an alternative approach using minimax optimization. The results are summarized in Section 5.3 (“Comparison”). In Section 5.4 (“Application to a Simplified Version of LWA1”), we apply the combined filter using the proposed designs to a simplified version of LWA1 which has been employed in previous chapters, and compare filters designed using the two methods in terms of SNR improvement with respect to the filter length. Finally, Section 5.5 (“Summary”) summarizes this chapter.
5.1 Direct Method

The straightforward approach to combining filters is using convolution followed by truncation. In this section, we consider the performance of this method in terms of frequency response errors over the bandwidth of interest with respect to the filter length.

Let $h_b$, $h_{cd}$, and $h_{ad}$ be the per-sensor FIR filters for filter-and-sum beamforming, cable correction, and antenna dedispersion, respectively. An equivalent single filter is obtained by convolution as

$$ h_1 = h_b \ast h_{cd} \ast h_{ad}, \quad (5.1) $$

where the operator “$\ast$” represents convolution. The length of $h_1$ computed in this manner is $M_1 = M_b + M_{cd} + M_{ad} - 2$, where $M_b$, $M_{cd}$ and $M_{ad}$ are the lengths of FIR filters $h_b$, $h_{cd}$, and $h_{ad}$, respectively. The final filter length $\tilde{M}_1$ can then be determined by truncating the combined filter $h_1$ to the minimum length that satisfies requirements such as frequency response errors over the given bandwidth.

5.1.1 Combined Filter for Sensor Dedispersion

We begin with implementing a single FIR filter to simultaneously compensate cable distortion and antenna dispersion, for the purpose of the correction of per-sensor dispersion. The desired frequency response of the ideal combined filter, $H_{sd}(\omega) = H_{cd}(\omega)H_{ad}(\omega)$, is shown in Figure 5.1, where the frequency response $H_{cd}(\omega)$ of a cable correction filter for a KSR200DB cable of length 150 m is given in Figure 3.6, and the frequency response $H_{ad}(\omega)$ of an antenna dedispersion filter for an antenna
Figure 5.1: Desired frequency response $H_{sd}(\omega)$ of the ideal combined filter. Note that integer multiple of $360^\circ$ (e.g., $360^\circ$) has been removed from the phase response.

In a standalone stand is presented in Figure 4.24. By firstly taking the inverse Fourier transform of $H_{sd}(\omega)$ numerically with a following truncation, a close-up view of the impulse response is shown in Figure 5.2. With different filter lengths, the associated frequency response errors from $H_{sd}(\omega)$ are summarized in Figure 5.3, where the dash rectangular box in the bottom panel of each figure specifies the $1.0^\circ$ phase error over $10 - 88$ MHz. It is found that only the 352-tap FIR filter can meet our specification. The 352-tap FIR filter yields the maximum magnitude error of $0.22$ dB between $10$ MHz and $88$ MHz. The magnitude error also decreases with the increasing filter length; however, the ripple of rate in frequency response becomes more acute for longer filter.

Since that the impulse response $h_{cd}$ for the cable correction filter is obtained in Section 3.3.3 and the impulse response $h_{ad}$ for the antenna dedispersion filter is obtained in Section 4.4, the combined filter using the convolution is then

$$h_{sd} = h_{cd} * h_{ad} .$$

(5.2)
Here we firstly choose minimax optimization to design $h_{cd}$ and $h_{ad}$, as minimax optimization can yield the minimum filter length given an error specification. From Tables 3.1 and 4.4, the filter lengths for $h_{cd}$ and $h_{ad}$ are $M_{cd} = 19$ and $M_{ad} = 100$, respectively. The length of the combined filter $h_{sd}$ using Equation (5.2) before truncation is thus $M_{sd} = 118$. Figure 5.4(a) provides a close-up view of the impulse response $h_{sd}$. Figure 5.4(b) shows the derivation of the associated frequency response from the ideal as shown in Figure 5.1. It is found that the maximum magnitude error over the bandwidth between 10 and 88 MHz is about 1.45 dB, and the maximum phase error over 10 – 88 MHz is $2.27^\circ$, larger than the acceptable phase-error tolerance of $1.0^\circ$. Note that the convolution of individual minimax-derived filters cannot meet requirement, even before truncation. Recall that the 118-tap $h_{sd}$ described in Figure 5.3(a) has the maximum magnitude error of 0.66 dB and the maximum phase error of $3.24^\circ$ over 10 – 88 MHz. For the same filter length, $h_{sd}$ yield smaller maximum phase error than that of $h_{sd0}$, although $h_{sd}$ has larger maximum magnitude error.
Figure 5.3: Frequency response errors of $h_{sd0}$ from $H_{sd}(\omega)$ with different filter lengths.
Figure 5.4: The 118-tap combined filter $h_{sd}$ using convolution. The individual filters $h_{cd}$ and $h_{ad}$ are designed using minimax optimization.

Next, we consider windowing for the individual filters. The results in Tables 3.1 and 4.4 show that Kaiser window is superior to Chebyshev window in this case. Using the Kaiser window with $\beta = 5.65$ for the individual filter designs, the filter lengths for $h_{cd}$ and $h_{ad}$ are $M_{cd} = 27$ (see Table 3.1) and $M_{ad} = 149$ (see Table 4.4), respectively. The length of the combined filter $h_{sd}$ obtained by Equation (5.2) before truncation is thus $M_{sd} = 175$. Figure 5.5(a) provides a close-up view of the impulse response of the combined filter using convolution. The error of the frequency response from the ideal is presented in Figure 5.5(b). The maximum magnitude error over frequencies $10 – 88$ MHz is about $0.56$ dB and the maximum phase error over frequencies $10 – 88$ MHz is about $0.86^\circ$. Thus the convolution of individual filters designed using windowing can meet our error requirements before truncation. For the specification of phase errors less than $1.0^\circ$ over $10 – 88$ MHz, the minimum filter length is found to be $\tilde{M}_{sd} = 104$ after truncation. As shown in Figure 5.6, the final filter yields the maximum magnitude error $0.57$ dB over the frequency band between $10$ MHz
Figure 5.5: The 175-tap combined filter $h_{sd}$ using convolution. The individual filters $h_{cd}$ and $h_{ad}$ are designed using Kaiser window with $\beta = 5.65$.

Figure 5.6: Frequency response errors of $h_{sd}$ described in Figure 5.5 after truncation. The number of taps after truncation is $\hat{M}_{sd} = 104$. 

108
and 88 MHz. The results show that the reduction in filter length has greater effect on the phase response than that on the magnitude response. Since $\widetilde{M}_{sd} \approx 0.6 M_{sd}$, a significant reduction in number of taps has been achieved. In this case, the minimum filter length of $h_{sd}$ is much shorter than that of $h_{sd0}$ for the specified phase errors over the bandwidth of interest.

For a given specification of phase accuracy over the bandwidth of interest, the combined FIR filter by the direct method is always shorter than that obtained by directly taking the inverse Fourier transform of the ideal frequency response. The above results also illustrate that the performance of the combined filter obtained by convolution may not satisfy our specifications even if the individual filters achieve the requirements.

5.1.2 Combined Filter Including Beamforming and Sensor Dedispersion

In this section, we consider the combined filter including all the functions of delay-and-sum beamforming, per-sensor cable correction, and per-sensor antenna dedispersion. The desired frequency of the ideal combined filter, $H(\omega) = H_b(\omega)H_{cd}(\omega)H_{ad}(\omega)$, is shown in Figure 5.7, where the filter $H_b(\omega)$ for the beamforming is used to implement a beamforming delay equal to $\tau = 2.5$ ns for the sample rate of 196 MSPS (i.e., about half the sample period), the frequency response $H_{cd}(\omega)$ of a cable correction filter for a KSR200DB cable of length 150 m is given in Figure 3.6, and the frequency response $H_{ad}(\omega)$ of an antenna dedispersion filter for an antenna in a standalone stand is presented in Figure 4.24. Comparing with the combined filter $H_{sd}(\omega)$ described in Figure 5.1, the magnitude response remains the same as $H_b(\omega)$ has the unit magnitude (see Section 2.3). By firstly taking the inverse Fourier transform of $H(\omega)$ numerically and then truncating to a minimum length that meets our specification, the performance of a 109-tap combined filter $h_0$ is shown in
Figure 5.7: Desired frequency response $H(\omega)$ of the ideal combined filter that implements 2.5 ns delay at the sample rate of 196 MSPS and sensor dedispersion. Note that integer multiples of 360° (e.g., 720°) have been removed from the phase response.

Figure 5.8. The maximum magnitude error over 10 – 88 MHz is about 0.11 dB.

Considering Kaiser windowing for the individual filters, the filter lengths for $h_{cd}$ and $h_{ad}$ are $M_{cd} = 27$ (see Table 3.1) and $M_{ad} = 149$ (see Table 4.4), respectively. It is found that 27 taps are required for the beamforming filter $h_b$ to achieve phase error no larger than 1.0° over 10 – 88 MHz, as shown in Figure 5.9. Now according to Equation (5.1), the length of $h_1$ before truncation is $M_1 = 201$ (i.e., $27 + 27 + 149 – 2$). Figure 5.10(a) is a close-up view of the impulse response of $h_1$, and Figure 5.10(b) shows the errors of frequency response from the ideal. It is found that the maximum magnitude error over 10 – 88 MHz is about 1.15 dB and the maximum phase error over 10 – 88 MHz is about 1.15°. The results show that the combined filter using convolution of windowing-derived individual filters cannot meet our error specifications, even before truncation.

To find whether the combined filter using minimax optimization is dependent on the beamforming
Figure 5.8: The 109-tap combined filter $h_0$ obtained using the inverse Fourier transform of $H(\omega)$ described in Figure 5.7.

Figure 5.9: The 27-tap delay FIR filter $h_b$ using Kaiser windowing.
delay, we consider a combined filter implementing a 1.6 ns delay at the sample rate of 196 MSPS (i.e., about one third the sample period), cable correction and sensor dedispersion. The desired frequency response is then shown in Figure 5.11. The results are similar to the Figure 5.7 result as the frequency response $H_b(\omega)$ has unit magnitude and linear phase. By taking the inverse Fourier transform of $H(\omega)$ numerically and truncating to a minimum length that meets our specification, the performance of a 206-tap combined filter $h_0$ is shown in Figure 5.12. The maximum magnitude error over $10 - 88$ MHz is about 0.20 dB.

We still consider using Kaiser windowing to design the individual filters. It is found that 31 taps are in need for the beamforming filter $h_b$ to achieve phase error no larger than $1.0^\circ$ over $10 - 88$ MHz, as shown in Figure 5.13. From Equation (5.1), the length of $h_1$ before truncation is $M_1 = 205$ (i.e., $31 + 27 + 149 - 2$). Figure 5.14(a) is a close-up view of the impulse response of $h_1$, and Figure 5.14(b) shows the frequency response errors from the ideal. It is found that the maximum phase error over $10 - 88$ MHz is about $1.12^\circ$, and the maximum magnitude error over $10 - 88$ MHz is about
Figure 5.11: Desired frequency response $H(\omega)$ of the ideal combined filter that implements 1.6 ns delay at the sample rate of 196 MSPS and sensor dedispersion. Note that integer multiple of $360^\circ$ (e.g., $360^\circ$) has been removed from the phase response.

Figure 5.12: The 206-tap combined filter $h_0$ obtained using the inverse Fourier transform of $H(\omega)$ described in Figure 5.11.
Figure 5.13: The 31-tap delay FIR filter $h_b$ using Kaiser windowing.

Figure 5.14: The 205-tap combined filter $h_1$ using convolution. The individual filters $h_b$, $h_{cd}$ and $h_{ad}$ are all designed using Kaiser window with $\beta = 5.65$. 
0.72 dB. For the beamforming delay of about one third the sample period, the combined filter using convolution of windowing-derived individual filters cannot meet the specification of phase errors no larger than 1.0° between 10 MHz and 88 MHz, even before truncation.

For the direct method, the beamforming delay does not play a significant effect on the performance of the combined filter. The above results confirm one conclusion described in Section 5.1.1 that the performance of the combined filter obtained by convolution may not satisfy our specifications even if the individual filters achieve the requirements. However, the results show that the combined filter obtained using the direct method does not always perform better than that obtained using the inverse Fourier transform of the ideal frequency response, which does not correspond to the results concerning $h_{sd}$ and $h_{sdb}$. This is simply because there is nothing in the direct combining of filters that accounts for the goal of constraining magnitude and phase errors.

### 5.2 Optimization Method

The direct method described in Section 5.1 is simple but does not explicitly minimize magnitude and phase errors. Optimization with respect to our error specifications might yield a better solution to this nonlinear-phase FIR filter design problem. In this section, we seek a better solution using an optimization algorithm applied to the ideal combined frequency response. For the same reasons identified earlier, the minimax optimization method is used here.

The desired frequency response of the ideal combined filter is the product of the frequency response of individual filters. The prototype for the combined filter can be obtained by taking the inverse Fourier transform of the desired frequency response and then using sampling and truncation. Now
we can perform minimax optimization with respect to the desired frequency response starting from the prototype. The pseudocode is shown in Algorithm 1.

**Algorithm 1**

1: Given \( H_b(\omega), H_{cd}(\omega), H_{ad}(\omega), \Delta M, M_0, \epsilon, f_s, \) and \( \Omega \)
2: \( H(\omega) \leftarrow H_b(\omega)H_{cd}(\omega)H_{ad}(\omega) \)
3: \( h_0 \leftarrow F^{-1}\{H(\omega)\} \) followed by sampling
4: Truncate \( h_0 \) to \( M_0 \) taps
5: Obtain \( M_0 \)-tap \( h_2 \) using
6: \( H_2(\omega) \leftarrow F\{h_2\} \)
7: if \( \max_{\omega \in \Omega} |\angle H_2(\omega) - \angle H(\omega)| > 1.0^\circ \) then
8: \( M_0 \leftarrow M_0 + \Delta M, \) and go back to step 5
9: else if \( \max_{\omega \in \Omega} |\angle H_2(\omega) - \angle H(\omega)| \leq (1.0^\circ - \epsilon) \) then
10: \( M_0 \leftarrow M_0 - \lfloor \Delta M/2 \rfloor, \) and go to step 5
11: else if \( (1.0^\circ - \epsilon) < \max_{\omega \in \Omega} |\angle H_2(\omega) - \angle H(\omega)| \leq 1.0^\circ \) then
12: \( M_2 \leftarrow M_0, \) \( h_2 \) is found as \( M_2 \) taps
13: return
14: end if

In this algorithm, \( M_0 \) is the initial filter length (typically obtained using prototype truncation), \( \Delta M \) is the step size for each iteration (typically equal to 5), \( \epsilon \) is the tolerance of the phase error (typically equal to 0.01\(^\circ\)), \( f_s \) is the sampling frequency, and \( \Omega \) is the frequency range of interest.

### 5.2.1 Combined Filter for Sensor Dedispersion

We firstly use Algorithm 1 for the design of the combined filter that can simultaneously correct cable distortion and antenna dispersion. The desired frequency response \( H_{sd}(\omega) \) is shown in Figure 5.1,
and the prototype for the combined filter is presented in Figures 5.2 and 5.3(c). Using \( M_0 = 352, \Delta M = 5 \) and the tolerance of \( \epsilon = 0.01^\circ \), Algorithm 1 yields a 102-tap FIR filter. The result is shown in Figure 5.15. This filter yields maximum magnitude error of 0.26 dB over 10 – 88 MHz. Note that the maximum magnitude error still happens around the first resonance of the antenna. The minimax method is superior to the direct method, which yields a filter with 104 taps in this case (see Figure 5.6 in Section 5.1.1).

5.2.2 Combined Filter Including Beamforming and Sensor Dedispersion

In this section, we consider the combined filter including all the functions of delay-and-sum beamforming, per-sensor cable correction, and per-sensor antenna dedispersion. For a beamforming delay equal to 2.5 ns at the sample rate of 196 MSPS (i.e., about half period), the desired frequency of the ideal combined filter is shown in Figure 5.7, and the prototype for the combined filter is presented in Figure 5.8. Using \( M_0 = 109, \Delta M = 5, \) and \( \epsilon = 0.01^\circ \), Algorithm 1 yields a 99-tap FIR filter. Figure 5.16 shows that this filter yields maximum magnitude error of 0.35 dB over 10 – 88 MHz.

To find whether the combined filter using minimax optimization is dependent on the beamforming delay, we consider a combined filter implementing the beamforming delay of 1.6 ns (i.e., about one third the sample period), cable correction and sensor dedispersion. The desired frequency response is shown in Figure 5.11, and the prototype for the combined filter as shown in Figure 5.12 needs at least 206 taps to meet our specification. We start with \( M_0 = 206, \Delta M = 5, \) and \( \epsilon = 0.01 \). Algorithm 1 yields a 101-tap combined filter, as shown in Figure 5.17. The maximum magnitude error between 10 MHz and 88 MHz is about 0.45 dB.

The above results show that the combined filter using the optimization method can achieve our
Figure 5.15: The 102-tap combined filter using the optimization method. The combined filter is used to correct the cable distortion and antenna dispersion.

Figure 5.16: The 99-tap combined filter using the optimization method. The combined filter is used to simultaneously implement a delay equal to 2.5 ns at the sample rate of 196 MSPS and sensor dedispersion.
Figure 5.17: The 101-tap combined filter using the optimization method. The combined filter is used to simultaneously implement a delay equal to 1.6 ns at the sample rate of 196 MSPS and sensor dedispersion.

specifications with much smaller filter lengths, as opposed to the direct method. The results also reveal that beamforming delay does not significantly affect the performance of the combined filter using the minimax optimization.

5.3 Comparison

In this section, we summarize the results described in Sections 5.1 and 5.2, and compare the different design methods in terms of the filter length with respect to the maximum phase error over 10 – 88 MHz. For filters that combine the cable correction and antenna dedispersion, the performance with respect to the design method is presented in Table 5.1. The results show that the optimization method is superior to the direct method and the prototype truncation approach. Using different design methods for the individual filters, the performance of the combined filter using the direct method varies significantly.
Table 5.1: Summary of the performance of combined filters for sensor dedispersion versus different design methods.

<table>
<thead>
<tr>
<th>Design methods</th>
<th>Filter length</th>
<th>Maximum frequency error over 10 – 88 MHz</th>
<th>Peak magnitude error</th>
<th>Peak phase error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prototype truncation⁹</td>
<td>118</td>
<td>0.7 dB</td>
<td>3.2⁰</td>
<td></td>
</tr>
<tr>
<td></td>
<td>352</td>
<td>0.2 dB</td>
<td>1.0⁰</td>
<td></td>
</tr>
<tr>
<td>Direct method</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>minimax-derived⁶</td>
<td>118</td>
<td>1.5 dB</td>
<td>2.3⁰</td>
<td></td>
</tr>
<tr>
<td>windowing-derived⁶</td>
<td>104</td>
<td>0.6 dB</td>
<td>1.0⁰</td>
<td></td>
</tr>
<tr>
<td>Optimization method</td>
<td>102</td>
<td>0.3 dB</td>
<td></td>
<td>1.0⁰</td>
</tr>
</tbody>
</table>

⁹ It means taking inverse Fourier transform of the desired frequency response followed by sampling and truncation.
⁶ It means that the individual filters are designed using minimax optimization.
⁶ It means that the individual filters are designed using Kaiser windowing with $\beta = 5.65$.

Next, we summarize the performance of combined filters, for beamforming delay and sensor dedispersion, with respect to the design methods in Table 5.2. For phase error no larger than 1.0⁰ over 10 – 88 MHz, the optimization method yields the combined filter with the smallest filter length, as opposed to the direct method and the prototype truncation approach. The beamforming delay affects the combined using prototype truncation approach significantly, but not the combined filter using either direct method or optimization method. For the same specification, each design method yields a combined filter with smaller filter length when the fractional sample period delay is around one half, as opposed to one third.

5.4 Application to a Simplified Version of LWA1

In this section, we apply the direct and optimization methods to a simplified version of LWA1, and compare the resulting improvement in SNR due to beamforming for each design. The objective is once again to figure out which method yields a smaller combined filter with comparable performance than that of others, and to understand the tradeoffs. In the following analysis, the combined filter
Table 5.2: Summary of the performance of combined filters for beamforming delay and sensor
dedispersion versus different design methods.

<table>
<thead>
<tr>
<th>Design methods</th>
<th>Filter length</th>
<th>Maximum frequency error over 10 – 88 MHz</th>
<th>Magnitude</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prototype truncation</td>
<td>109</td>
<td>0.1 dB</td>
<td>1.0°</td>
<td></td>
</tr>
<tr>
<td>Direct method&lt;sup&gt;a&lt;/sup&gt;</td>
<td>201</td>
<td>1.2 dB</td>
<td>1.2°</td>
<td></td>
</tr>
<tr>
<td>Optimization method</td>
<td>99</td>
<td>0.4 dB</td>
<td>1.0°</td>
<td></td>
</tr>
<tr>
<td>Prototype truncation</td>
<td>206</td>
<td>0.2 dB</td>
<td>1.0°</td>
<td></td>
</tr>
<tr>
<td>Direct method&lt;sup&gt;a&lt;/sup&gt;</td>
<td>205</td>
<td>0.7 dB</td>
<td>1.1°</td>
<td></td>
</tr>
<tr>
<td>Optimization method</td>
<td>101</td>
<td>0.5 dB</td>
<td>1.0°</td>
<td></td>
</tr>
</tbody>
</table>

<sup>a</sup> For the direct method, the individual filters are all designed using Kaiser windowing with $\beta = 5.65$.

Using the convolution method described in Section 5.1 is represented as $h_1$, and the combined filter using the optimization method described in Section 5.2 is represented as $h_2$.

Now we use the direct and optimization methods to implement the combined FIR filter for each sensor of LWA1 to see the improvement in SNR at the output of the “combined” beamformer. The per-sensor combined filter length is now selected to be 28, which is what LWA1 actually has. For this simplified version of LWA1, the array geometry and cable lengths are illustrated in Appendix C. Pattern multiplication is assumed to be applicable; i.e., mutual coupling is ignored.

The direction of incidence is 22° away from the zenith in the $\phi = 0°$ plane, same as in Section 2.2.4.

Figure 5.18 shows the SNR improvement over that of a single sensor by delay-and-sum beamforming in four different cases: The asterisk-line curve gives the SNR of the array with respect to that of a single sensor in the ideal case where the effects of unequal cable distortion and antenna dispersion are either not significant or always perfectly compensated; the cross-line curve shows the SNR improvement in the case which accounts for unequal cable and antenna dispersion but with no correction scheme implemented; the circle-line curve presents the result using the direct method to correct unequal cable distortion and antenna dispersion; and the dot-line curve describes the
result using the optimization method for the correction of unequal cable distortion and antenna dispersion. Figure 5.19 describes the results of different combination methods with respect to the “ideal” result in Figure 5.18 (i.e., the asterisk-line). Using either the direct method or the optimization method to compensate the unequal cable distortion and antenna dispersion, there are still about 0.12–0.59 dB degradation in SNR performance. This is mainly due to the finite length of the combined filter. The optimization method is superior to the direct method over the bandwidth of interest. Figure 5.20 presents the results of different combination methods with respect to the “w/o correction” result in Figure 5.18 (i.e., the cross-line). The results show that the optimization method can provide a benefit of about 0.05–0.28 dB over that by the direct method, although this improvement is limited to frequencies above 30 MHz. There is a very slight performance penalty below 30 MHz, due to the insufficient length of the combined filter. These minor improvements in SNR are significant for large arrays and weak signal applications.
Figure 5.19: Same as Figure 5.18, except now relative to the “ideal” result.

Figure 5.20: Same as Figure 5.18, except now relative to the “w/o correction” result.
Now we repeat the experiment for the optimum-SNR beamforming under the same conditions described previously. Figure 5.21 shows the SNR improvement over that of a single sensor by optimum-SNR beamforming in four different cases: The asterisk-line curve gives the SNR of the array with respect to that of a single sensor in the ideal case where the effects of unequal cable distortion and antenna dispersion are either not significant or always perfectly compensated; the cross-line curve shows the SNR improvement in the case which accounts for unequal cable and antenna dispersion but with no correction scheme implemented; the circle-line curve presents the result using the direct method to correct unequal cable distortion and antenna dispersion; and the dot-line curve describes the result using the optimization method for the correction of unequal cable distortion and antenna dispersion. The performance penalty due to the unequal cable distortion and antenna dispersion is larger than the Figure 5.18 result. This is because that unequal cable loss and antenna attenuation are more important in optimum-SNR beamforming than in delay-and-sum beamforming. Figure 5.22 describes the results of different combination methods with respect to the “ideal” result in Figure 5.21 (i.e., the asterisk-line). Due to the finite length of the combined filter, there are about $0.12 - 0.74$ dB degradation in SNR performance by using the combined FIR filters. Figure 5.23 presents the results of different combination methods with respect to the “w/o correction” result in Figure 5.21 (i.e., the cross-line). The results show that the optimization method can provide a benefit of about $0.05 - 0.36$ dB over that by the direct method, a little better than the Figure 5.20 result. There is performance penalty at low frequencies, due to the insufficient filter length of the combined filter.

It is interesting to note that SNR improvement as a function of frequency is not monotonic. This is because the correlation of external noise between sensors shows a Bessel function-like behavior.
Figure 5.21: Optimum-SNR beamforming performance for different methods in the condition described in the text of Section 5.4.

Figure 5.22: Same as Figure 5.21, except now relative to the “ideal” result.
Figure 5.23: Same as Figure 5.21, except now relative to the “w/o correction” result.

with respect to wavelength [7].

The above results illustrate that the combined filter using the optimization method is superior (i.e., provide more improvement in SNR over the bandwidth of interest with the same filter length) to that using the direct method. Additionally, the optimization method can also generate constrained solution, which can not be ensured by the direct method. Therefore we will apply the optimization method to the design of the combined filter for the application of LWA1.

We now consider the computational burden associated with obtaining the coefficients of combined filters. The simplest case is implementing delay-and-sum beamforming for the array, which presumably must be done. For the LWA1 array, the computational effort for the calculation of 256 28-tap delay FIR filters (designed by Kaiser.windowing) is about 0.23 seconds under MATLAB 7.1 (R14). For the 256 28-tap combined (beamforming delay + cable correction) filters, the computational effort is about 0.67 seconds for the direct method of which the individual filters are designed using
Kaiser windowing, and the computational effort is about 11108.3 seconds (sim3.09 hours) for the optimization method. The results show that the computational burden associated with the calculations of combined filters coefficients using the optimization method is quite heavy; however, the filter coefficients can be computed off-line.

5.5 Summary

This chapter presented two methods for the design of combined filters, and analyzed their characteristics. The best design approach to the combined filter is selected in terms of the filter performance (e.g., frequency response errors or/and SNR performance over the bandwidth of interest) with respect to the filter length. The results show that the optimization method is better both generally and the application to LWA1.
Chapter 6

Application to LWA1

Previous chapters considered aspects of the overall problem: Chapter 2 described several beamforming techniques, Chapter 3 described the correction scheme for cable dispersion, and Chapter 4 presented the design of per-sensor antenna dedispersion filters. Now we wish to consider an application where all of these are considered together, as an application for the “combined” beamforming scheme as advocated in Figure 1.2(b) and Chapter 5. We will again use LWA1 as an example in this chapter. LWA1 was introduced in Section 1.1. Appendix C describes the LWA1 array geometry and cable lengths. In Sections 2.2.4 and 3.3.4, we did some preliminary studies on a simplified version of LWA1 assuming isotropic antennas. In Sections 4.3 – 4.5, we showed the actual LWA1 antenna designs, and demonstrated per-sensor antenna dedispersion filters required for dispersion of signals from single stands and pairs of stands.

In this chapter, we apply delay-and-sum beamforming (see Section 2.2.1) and optimum-SNR beamforming (see Section 2.2.2) techniques to LWA1. We consider only one frequency at a time. This is because large computation time is required for calculation of the LWA1 array manifold, as ex-
plained in [7]. Thus, delay-and-sum beamforming becomes “phase-and-sum” beamforming, and optimum-SNR beamforming becomes “complex multiply-and-sum” beamforming. We will analyze the improvement in SNR by beamforming at only three frequencies (i.e., 20 MHz, 38 MHz, and 74 MHz). Our implementation of phase-and-sum beamforming neglects the effects of mutual coupling and spatial noise correlation, thus the weights are determined using only the array geometry and direction of interest. By contrast, our implementation of optimum-SNR beamforming accounts for the effects of mutual coupling and spatial noise correlation, and the beamforming weight magnitudes vary from sensor to sensor. These were considered previously by Ellingson (2011) [7], but we will extend the work by considering the effects of unequal cable losses and dispersive delays, and the degree to which it can be corrected. ([7] already accounts for the effect of unequal antenna dispersion.) As an extension, we will consider explicitly a third beamforming possibility: Optimum-SNR beamforming which ignores the correlation of spatial noise, but accounts for antenna mutual coupling. Since the beamforming weights (filter coefficients) in this case depend only on the azimuth and elevation of pointing, and not time of day, the number of filter taps that must be predefined can be dramatically reduced. For these reasons, this option is of practical interest even though it may not yield optimum performance.

The organization of this chapter is as follows: First we estimate the system equivalent flux density (SEFD) for a LWA1 standalone stand in Section 6.1 (“SEFD for a LWA1 Standalone Stand”). These results serve as a baseline for beamforming performance, which can be expressed in terms of SNR improvement over that of a single stand. Section 6.2 (“LWA1 Array Manifold”) describes the computation of the LWA1 array manifold; i.e., the electromagnetic response of all antennas of the array simultaneously. In Section 6.3 (“LWA1 Beamforming”), we study the three beamforming
techniques described above. Finally, Section 6.4 ("Summary") summarizes this chapter.

### 6.1 SEFD for a LWA1 Standalone Stand

SEFD is a commonly-used metric of the sensitivity of radio telescopes. SEFD is defined as the power flux spectral density, having units of $\text{W} \text{m}^{-2} \text{Hz}^{-1}$, which yields SNR equal to unity at the beamformer output. The SNR at the output of the beamformer is given in Equation (2.26). In this section, we calculate the SEFD of a LWA1 standalone stand, which will serve as a metric for beamforming performance.

First we consider the relationship between the flux density and the incident electric field. Let $S(\psi)$ be the flux density associated with the signal of interest, having units of $\text{W} \text{m}^{-2} \text{Hz}^{-1}$. For the electric field $E(\psi, t)$ of the signal of interest, having units of $\text{V} \text{m}^{-1} \text{Hz}^{-1/2}$, we have

$$S(\psi) = \left\langle |E(\psi, t)|^2 \right\rangle \eta ,$$

(6.1)

where $\eta$ is the impedance of free space, and

$$E(\psi, t) = E_\theta(\psi, t) + E_\phi(\psi, t)$$

(6.2)

with $E_\theta(\psi, t)$ and $E_\phi(\psi, t)$ being the $\theta$- and $\phi$-polarized components of $E(\psi, t)$, respectively. It is reasonable to assume the signal of interest is unpolarized to characterize the sensitivity of LWA1, since most astronomical sources are unpolarized or, at most, weakly polarized. In this case, we have

$$\left\langle E_\theta(\psi, t)E_\phi(\psi, t) \right\rangle = 0 , \text{ and}$$

(6.3a)
\[ \langle |E_{\theta}(\psi, t)|^2 \rangle = \langle |E_{\phi}(\psi, t)|^2 \rangle = \frac{\eta}{2} S(\psi) . \]  

(6.3b)

Recalling Equations (2.12) – (2.13d) with \( M = 1 \) (i.e., the implementation of delay-and-sum beamforming in terms of “phase-and-sum” beamforming, and the implementation of optimum-SNR beamforming in terms of “complex multiply-and-sum” beamforming), the covariance matrix of the signal of interest is an \( N \times N \) matrix as given by

\[ \mathbf{R}_s = S(\psi) \frac{\eta}{2} \left( \mathbf{A}_{\theta\theta} + \mathbf{A}_{\phi\phi} \right) , \]  

(6.4)

where the \( N \times N \) matrices \( \mathbf{A}_{\theta\theta} \) and \( \mathbf{A}_{\phi\phi} \) are described in Equations (2.13e), (2.13g) and (2.13h). In the case of \( M = 1 \), the internal noise covariance matrix \( \mathbf{R}_u \) becomes an \( N \times N \) diagonal matrix whose non-zero elements are described in Equation (2.20). Similarly, the external noise covariance matrix \( \mathbf{R}_z \) also becomes an \( N \times N \) matrix of which the \((n, n')\)th element \( \mathbf{R}_z^{[n, n']} \) is the correlation of external noise between sensors \( n \) and \( n' \) as described in Equation (2.17). The sensitivity of LWA1 in terms of SEFD is the value of \( S(\psi) \) in Equation (6.4) required to double the power at the beamformer output (i.e., \( \text{SNR} = 1 \)); thus

\[ \text{SEFD} = \frac{2}{\eta} \frac{\mathbf{w}^H \left( \mathbf{R}_z + \mathbf{R}_u \right) \mathbf{w}}{\mathbf{w}^H \left( \mathbf{A}_{\theta\theta} + \mathbf{A}_{\phi\phi} \right) \mathbf{w}} , \]  

(6.5)

where \( ^H \) denotes the conjugate transpose operator.

The LWA1 antenna and stand designs are shown in Section 4.3. The same assumption about ground conditions and the same sky model are used in the following analysis. We start by considering the SEFD for a single dipole of a LWA1 standalone stand; in this case, are \( x \)-aligned dipole. To analyze an antenna using MoM, we need to find the current at the dipole terminals with respect to the
incident plane wave. We firstly illuminate the antenna with a $\theta$-polarized 1 V/m plane wave to find the resulting current, and then repeat the process with a $\phi$-polarized 1 V/m plane wave. For each case, the incident direction varies from $0^\circ$ to $90^\circ$ in $\theta$ with an increment of $0.5^\circ$, and from $0^\circ$ to $315^\circ$ in $\phi$ with an increment of $45^\circ$. Since the antenna is connected to an “active balun” whose impedance is $Z_L = 100 \ \Omega$ [7], the array manifold of the antenna is calculated by multiplying the current with the load impedance $Z_L$. Using the calculated array manifold and the noise covariance matrices as described in Section 4.4, the resulting SEFD, having units of Jy where $1 \ Jy = 10^{-26} \ W \ m^{-2} \ Hz^{-1}$ for the $x$-aligned dipole can be computed from Equation (6.5), where the beamforming weight is $w = [1]$ in this case. Figure 6.1(a) shows the result in the $\phi = 0^\circ$ plane, and Figure 6.1(b) shows the result in the $\phi = 90^\circ$ plane. Note that lower SEFD is better (more sensitive). As expected, the response becomes very small at the horizon. Since the two dipoles in a stand are perpendicular, the array manifold of $x$-aligned dipole in the $\phi = 0^\circ$ plane corresponds to that of $y$-aligned dipole in the $\phi = 90^\circ$, and vice versa.

Next we consider the SEFD for a complete LWA1 stand; i.e., both $x$- and $y$-aligned dipoles are included and combined using the beamforming weights $w = [1 \ 1]^T$. The procedure to analyze the array manifold of a single stand is similar to that for a single antenna; however, the difference is that we analyze both dipoles in a stand simultaneously. Each element of the $N = 2$ array response vector is the product of the current and the load impedance $Z_L$ for the associated dipole. Figures 6.2(a) and 6.2(b) show the resulting SEFD in the $\phi = 0^\circ$ and $\phi = 90^\circ$ planes, respectively. The Figure 6.2(a) result corresponds to and agrees with Figure 3 of [7]. The Figure 6.2 result shows that the performance for a standalone stand is identical in different planes, as expected, since the dipoles are nominally orthogonally polarized. Also, the SEFD is not significantly different from
Figure 6.1: Calculated SEFD for the $x$-aligned dipole of a LWA1 standalone stand in different planes.
Figure 6.2: Calculated SEFD for a LWA1 standalone stand in different planes.
SEFD for the single dipole. This is because both the signal and the noise are uncorrelated between polarizations. It is found that the SEFD performance at 38 MHz and 74 MHz is very similar, even though there is a large difference in frequency. The reason is that both the Galactic noise temperature and the effective aperture of the antennas decrease with frequency at approximately the same rate [24]. It is also seen that the calculated SEFD performance at 20 MHz is dramatically worse. This is because the loss associated with the Earth ground becomes large at 20 MHz and the ground screen becomes electrically-tiny at 20 MHz, as explained in [7].

The calculated SEFD for a LWA1 standalone stand will serve as a reference for beamforming performance: The SNR improvement over that of a single stand by beamforming is expressed in terms of the ratio of the SEFD for a single stand to the SEFD for the beamformer.

### 6.2 LWA1 Array Manifold

To obtain the frequency response of LWA1 antennas, we need data for all 512 antennas in the array (i.e., the array manifold) over the bandwidth of interest. We could use the method described in [7] to calculate the LWA1 array manifold at given frequencies, and then interpolate to obtain the LWA1 array manifold at all other frequencies. However, the method described in [7] requires approximately one month continuous computation using a cluster of four computers to obtain one frequency, so we would need about 40 months to compute every 2 MHz between 10 MHz and 88 MHz. This is not feasible. Instead, we will use data already generated for the three frequencies (i.e., 20 MHz, 38 MHz, and 74 MHz) reported in [7]. These data describe the response of the antenna embedded in the LWA1 array. From Equation (4.6), the response of the antenna dedispersion filter in this case is then determined as the reciprocal of the response of the antenna.
6.3 LWA1 Beamforming

Ellingson (2011) [7] discussed delay-and-sum beamforming and optimum-SNR beamforming techniques for LWA1 in the presence of spatially-correlated Galactic noise and unequal antenna dispersion, but did not consider the degradation in SNR due to unequal cable losses and dispersion characteristics. In this section, we repeat Ellingson’s analysis but express the result in terms of SNR improvement relative to a single stand. Then we extend [7] by considering the effect of unequal cable losses and dispersive delays, and the degree to which it can be corrected. For implementation purposes, we also consider a third beamforming possibility: optimal-SNR beamforming neglecting the correlation of spatial noise between sensors.

6.3.1 Confirmation of Results from Previous Work

Using the array manifold obtained as described in Section 6.2, the SEFD can be calculated using Equation (6.5). As a first step, we repeat the analysis as described in [7] to confirm we get the same results. Figure 6.3 shows the SEFD results in the $\phi = 0^\circ$ plane. The associated SNR improvement by beamforming is shown in Figure 6.4. If the principle of pattern multiplication applies, the result from Figure 6.3 is expected to be identical to the result from Figure 6.2 divided by the number of stands (i.e., $N = 256$). However, the Figure 6.4 result shows that pattern multiplication does not exactly apply to LWA1. This is due to non-uniform antenna mutual coupling, and external noise correlation. The SNR improvement shown in Figure 6.4 is smaller than the result predicted by pattern multiplication by about $1 - 6$ dB for $20^\circ \leq \theta \leq 80^\circ$, and is different (not consistently better or worse) for $\theta < 20^\circ$ and $\theta > 80^\circ$. To determine which (i.e., mutual coupling or noise
Figure 6.3: SEFD for various beamforming techniques in the $\phi = 0^\circ$ plane. For each frequency, the upper (dotted) curve is the result for phase-and-sum beamforming, and the lower (solid) curve is the result for optimum-SNR beamforming.

Figure 6.4: SNR improvement over that of a single stand by various beamforming techniques in the $\phi = 0^\circ$ plane. This result is essentially Figure 6.3 divided by Figure 6.2. The dashed line is the result predicted by pattern multiplication.
correlation) is responsible, we can remove the effect of the correlation of external noise by setting 
\( R_{z}^{[n,n']} \) to zero for any \( n \neq n' \). We show a recalculation of the Figure 6.4 result with the external noise correlation zeroed in Figure 6.5. The associated SNR improvement is now relatively close to the result predicted by pattern multiplication, which indicates that the correlation of external noise between antennas, as opposed to antenna mutual coupling, is primarily responsible for the reduced SNR improvement.

6.3.2 Effects of Unequal Cable Losses and Dispersive Delays

The above analysis and the analysis of [7] account for the degradation due to external noise correlation and unequal antenna dispersion, but ignores the effect of unequal cable losses and dispersive delays. We now consider these effects on the array SNR performance. Recall that Appendix C provides cable lengths of the LWA1 array. Lengths vary between 43 m and 149 m in the LWA1 cable system. For KSR200DB cables which are now used for LWA1, the parameters are \( \zeta = 5.4 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2} \) and \( \kappa = 3.8 \times 10^{-7} \text{ m}^{-1} \text{ Hz}^{-1/2} \), as explained in Section 3.3.1. Since the velocity factor of the KSR200DB cable is 0.83, the travel time (i.e., \( t_0 \) described in Section 3.2.1) along LWA1 cables is distributed in the range between 172.7 ns and 598.4 ns. Figure 6.6 shows the distribution among 50 bins with centers evenly distributed between the minimum and maximum travel time. Although the travel time scatters in a large range, it is easily corrected since it is independent of frequency. Figures 6.7 – 6.9 show the distribution of cable losses (i.e., \( A \) described in Section 3.3.1) and dispersive delays (i.e., \( \tau \) described in Section 3.3.1) in the LWA1 cable system at 20 MHz, 38 MHz, and 74 MHz, respectively. All the histograms represent the distribution among 50 bins with centers evenly distributed between the minimum and maximum values. At 20 MHz,
Figure 6.5: Same as Figure 6.4, except the external noise correlation zeroed. Note that the vertical scale is much smaller relative to that in Figure 6.4.

Figure 6.6: Distribution of constant delays in LWA1 cable system.
Figure 6.7: Distribution of cable losses and dispersive delays at 20 MHz.

Figure 6.8: Distribution of cable losses and dispersive delays at 38 MHz.
cable losses vary between $-7.8$ dB and $-2.3$ dB, and dispersive delays vary between 0.73 ns and 2.53 ns; at 38 MHz, cable losses vary between $-10.8$ dB and $-3.1$ dB, and dispersive delays vary between 0.53 ns and 1.83 ns; at 74 MHz, cable losses vary between $-15.1$ dB and $-4.3$ dB, and dispersive delays vary between 0.38 ns and 1.32 ns. The results show that cable losses are scattered over a larger range at higher frequencies whereas dispersive delays are scattered over a larger range at lower frequencies. In the following discussion, we will analyze the effects of unequal cable losses and dispersive delays.

We start with considering the effect of unequal cable losses and dispersive delays with no correction scheme implemented. We do this by modifying the array manifold elements (i.e., $a_n^\theta(\psi_0)$ and $a_n^\phi(\psi_0)$ associated with sensor $n$) magnitude by cable loss $A$ and phase by dispersive phase $\phi_d$ in the calculation of SEFD. Figure 6.10 gives the results using phase-and-sum beamforming and optimum-SNR beamforming in this case, which show that optimum-SNR beamforming is still superior to phase-and-sum beamforming when accounting for the effect of unequal cable losses and
dispersive delays without implementing any correction scheme. Figure 6.11 shows the associated SNR performance penalty due to the effect of unequal cable losses and dispersive delays. Note that there is about 0.1 – 0.9 dB degradation in SNR due to uncompensated unequal cable losses and dispersive delays. The degradation in SNR increases with increasing frequency, due to larger cable losses and dispersive phases at higher frequencies.

To consider our third beamforming scheme, optimum-SNR beamforming neglecting external noise correlation, we zero the off-diagonal elements of the external noise covariance matrix $\mathbf{R}_z$ in the solution to beamforming weights, but retain $\mathbf{R}_z$ with non-zero off-diagonal entries in the calculation of SEFD. The array manifold elements are modified as above to account for unequal cable losses and dispersive delays. Figure 6.12 shows the SNR improvement in this case, and the differences from Figure 6.10 are shown in Figures 6.13 and 6.14. The Figure 6.13 result shows that the degradation
Figure 6.11: Difference between Figures 6.10 and 6.4.

Figure 6.12: SNR improvement over that of a single stand by optimum-SNR beamforming neglecting external noise correlation, in the presence of unequal cable losses and dispersive delays, but without correction.
Figure 6.13: Difference between the Figure 6.12 result and the optimum-SNR beamforming result in Figure 6.10.

Figure 6.14: Difference between the Figure 6.12 result and the phase-and-sum beamforming result in Figure 6.10.
in SNR as opposite to the optimum-SNR beamforming result in Figure 6.10 is due to the correlation of external noise that is not corrected by beamforming. Note that the degradation is potentially large for low-elevation pointing. The Figure 6.14 result shows that optimum-SNR beamforming neglecting external noise correlation is significantly better than phase-and-sum beamforming for $20^\circ \leq \theta \leq 80^\circ$. This is because our third beamforming scheme can compensate the effects of mutual coupling, but phase-and-sum beamforming can not.

### 6.3.3 Correction of Unequal Cable Losses

Cable losses in LWA1 are easily equalized by varying the gains of the analog receivers. Thus it is of interest to repeat the analysis assuming only dispersive delays exist and are uncompensated. In this section, we repeat the analysis of the previous section, but now assume cable losses only are perfectly corrected, whereas dispersive delays are not. The results using phase-and-sum and optimum-SNR beamforming in this case are shown in Figure 6.15, and the associated SNR performance penalty due to unequal cable dispersion is shown in Figure 6.16. Note that there is about less than 0.5 dB degradation in SNR in this case. The Figure 6.17 result shows that the degradation in SNR is smaller than that in the Section 6.3.2 study in which no correction scheme was implemented. This confirms that the correction of unequal cable losses is beneficial for array performance, even if the effect of unequal cable dispersive delays is present and uncorrected.

We again consider optimum-SNR beamforming neglecting external noise correlation. The SNR improvement in this case is shown in Figure 6.18, and the associated differences from Figure 6.15 are presented in Figures 6.19 and 6.20. Figure 6.21 compares Figures 6.19 and 6.13. With the perfect correction of unequal cable losses, the SNR degradation due to external noise correlation (i.e.,
Figure 6.15: Same as Figure 6.10, but now perfectly correcting the effects of unequal cable losses. The effects of unequal dispersive delays are not compensated.

Figure 6.16: Difference between Figures 6.15 and 6.4. Compare also to Figure 6.11.
Figure 6.17: Difference between Figures 6.16 and 6.11.

Figure 6.18: Same as Figure 6.12, but now perfectly correcting the effects of unequal cable losses. The effects of unequal dispersive delays are not compensated.
Figure 6.19: Difference between the Figure 6.18 result and the optimum-SNR beamforming result in Figure 6.15.

Figure 6.20: Difference between the Figure 6.18 result and the phase-and-sum beamforming result in Figure 6.15.
neglecting correlation of external noise in calculation of the solution to beamforming weights) becomes larger. This is apparently because that the effects of external noise correlation become more significant when unequal cable losses are perfectly corrected. Figure 6.22 shows the differences between Figures 6.20 and 6.14. The perfect correction of unequal cable losses does not significantly affect the SNR improvement of optimum-SNR beamforming neglecting external noise correlation over phase-and-sum beamforming. This is because that unequal cable losses may not be significant for the effects of mutual coupling.

### 6.3.4 Correction of Unequal Cable Losses and Dispersive Delays

In this section, we consider implementing the correction scheme for both unequal cable losses and dispersive delays. If the correction scheme is perfect, then we simply obtain the same results in Section 6.3.1 (Figure 6.4). For optimum-SNR beamforming neglecting external noise correlation,
the results are shown in Figure 6.23, and the performance penalty as opposed to the optimum-SNR beamforming result in Figure 6.4 is presented in Figure 6.24. This performance penalty results from the neglect of external noise correlation in the calculation of beamforming weights.

However, the correction scheme is actually implemented using per-sensor FIR filters. Thus the effects of unequal cable losses and dispersive delays cannot be perfectly corrected, resulting in a performance penalty due to the finite number of filter taps. The situation in this case can be interpreted as perfect correction, but with a new error corresponding to multiplication by the error with respect to the response of the ideal correct filter. It was shown in Chapter 5 that this error could reasonably be constrained to about 0.5 dB in magnitude and 1.0° in phase. To ascertain the impact on performance, we can repeat the analysis of Section 6.3.1 (Figure 6.4) and Figure 6.23 many times, each time introducing random per-sensor magnitude and phase errors, and then consider the variation in the results. In the following experiment, the random per-sensor
Figure 6.23: Same as Figure 6.4, but now for optimum-SNR beamforming neglecting external noise correlation.

Figure 6.24: Difference between the Figure 6.23 result and the optimum-SNR beamforming result in Figure 6.4.
magnitude error is uniformly distributed between $-0.5$ dB and 0.5 dB, and the random per-sensor phase error is uniformly distributed between $-1.0^\circ$ and $1.0^\circ$. By repeating the previous analysis 100 times, the results are shown in Figures 6.25 – 6.27, respectively. As expected, the results are close to the corresponding results in Figures 6.4 and 6.23. Next, we consider the variation in these results by dividing Figures 6.25 – 6.27 by the perfect correction results. Figures 6.28 – 6.30 shows the corresponding variations due to the finite number of filter taps (random response errors). It is found that the variations are less than 0.05 dB. When the correction filter can constrain the response errors to about 0.5 dB in magnitude and $1.0^\circ$ in phase, the finite number of filter taps will not play a significant impact on the array performance, especially for optimum-SNR beamforming as shown in Figure 6.29.

6.3.5 Discussion

In this section, we summarize the results in Sections 6.3.1 – 6.3.4 to compare the performance of three beamforming schemes. Table 6.1 shows the range of SNR improvement by beamforming in different situations at three frequencies (i.e., 20 MHz, 38 MHz, and 74 MHz) in the $\phi = 0^\circ$ half plane. The results show that optimum beamforming neglecting external noise correlation is superior to phase-and-sum beamforming, but is inferior to optimum-SNR beamforming, as expected. An advantage of optimum-SNR beamforming neglecting external noise correlation is that this method can provide acceptable performance without the need to recompute the coefficients for all possible sidereal times, as explained in the beginning of Chapter 6.

Table 6.2 summarizes the results in Figures 6.28 – 6.30. For optimum-SNR beamforming, the impact of the finite number of correction filter taps on array performance is very small, if the correction
Figure 6.25: SNR improvement by phase-and-sum beamforming repeated 100 times with random magnitude and phase errors.

Figure 6.26: SNR improvement by phase-and-sum beamforming repeated 100 times with random magnitude and phase errors.
Figure 6.27: SNR improvement by optimum-SNR beamforming neglecting external noise correlation repeated 100 times with random magnitude and phase errors.

The results of phase-and-sum and optimum-SNR beamforming neglecting external noise correlation are almost the same, and are inferior to optimum-SNR beamforming results. This is probably because of the significant effect of external noise correction on LWA1 array.

6.4 Summary

This chapter applied three beamforming schemes (i.e., phase-and-sum beamforming, optimum-SNR beamforming, optimum-SNR beamforming neglecting external noise correlation) to the realistic LWA1 array. Unequal cable losses and dispersive delays were found to significantly degrade the SNR of LWA1. The solution to the problem of unequal cable distortion using per-sensor correction FIR filters was proved to be beneficial for the LWA1 array.
Figure 6.28: Differences between Figure 6.25 and the phase-and-sum beamforming results in Figure 6.4.
Figure 6.29: Differences between Figure 6.26 and the optimum-SNR beamforming results in Figure 6.4.
Figure 6.30: Differences between Figures 6.27 and 6.23.
Table 6.1: Summary of SNR improvement by different beamforming schemes in the $\phi = 0^\circ$ half plane.

<table>
<thead>
<tr>
<th>Different cases</th>
<th>Beamforming schemes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Phase-and-sum</td>
</tr>
<tr>
<td>Ideal cables / Perfect correction</td>
<td>17.6 – 24.4 dB</td>
</tr>
<tr>
<td>Unequal cable losses and dispersive delays without correction</td>
<td>17.4 – 24.2 dB</td>
</tr>
<tr>
<td>Unequal dispersive delays only$^b$</td>
<td>17.5 – 24.3 dB</td>
</tr>
<tr>
<td>Correction with FIR filters</td>
<td>17.6 – 24.3 dB</td>
</tr>
</tbody>
</table>

$^a$ i.e., our third beamforming scheme, optimum-SNR beamforming neglecting external noise correlation.

$^b$ i.e., unequal cable losses are perfectly corrected and unequal dispersive delays are not compensated.

Table 6.2: Variations due to the finite number of correction filter taps with respect to different beamforming schemes at three frequencies.

<table>
<thead>
<tr>
<th>Beamforming schemes</th>
<th>Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20 MHz</td>
</tr>
<tr>
<td>Phase-and-sum</td>
<td>&lt; 0.03 dB</td>
</tr>
<tr>
<td>Optimum-SNR</td>
<td>&lt; 0.01 dB</td>
</tr>
<tr>
<td>Revised optimum-SNR$^a$</td>
<td>&lt; 0.03 dB</td>
</tr>
</tbody>
</table>

$^a$ i.e., our third beamforming scheme, optimum-SNR beamforming neglecting external noise correlation.
Chapter 7

Conclusions

Array systems operating below a few hundred megahertz may suffer from spatially-correlated noise, interference, and unequal dispersion by antennas and cables. Some aspects of these problems have been individually investigated, however strategies for combining them do not seem to have been previously considered. In this dissertation, we considered the use of a single per-sensor filter to perform all functions (as shown in Figure 1.2(b)), instead of a cascade of per-sensor filters respectively for beamforming, cable correction and antenna dedispersion (as shown in Figure 1.2(a)).

In the following, the principal findings and future research topics are discussed.

7.1 Findings

The principal findings in this dissertation are as follows:

1. A wideband solution to the correction of frequency-dependent cable losses and dispersive delays was rigorously derived. The prototype for the cable correction filter was obtained
using physical parameters of cables. In Section 3.3.4, the per-sensor cable correction FIR filter was used to compensate the SNR degradation resulting from unequal cable distortion for a simplified version of LWA1 (described in Section 2.2.4). As shown in Figure 3.14, this work provided a benefit of $0.1 - 0.5$ dB above 30 MHz. This method is applicable to systems which suffer from unequal cable losses and dispersive delays.

2. The combined effect of correlation of noise between sensors and unequal sensor dispersion was analyzed, which has not been considered previously. Taking LWA1 as an application example, Figure 6.10 shows that the combined effect results in $0.9 - 6.5$ dB degradation at all frequencies for $10^\circ < \theta < 75^\circ$ for phase-and-sum beamforming, and $0.5 - 5.5$ dB degradation at all frequencies for $10^\circ < \theta < 75^\circ$ for optimum-SNR beamforming. This shows that the combined effect is significant for the system of interest and the strategy for the correction of the combined effect is useful.

3. A strategy for the simultaneous implementation of noise correlation compensation and sensor dedispersion using a single FIR filter per-sensor (Figure 1.2(b)) was developed, which has not been investigated before. The results in Figures 5.20 and 5.23 show that the combination scheme is beneficial for the system of interest as it can provides a benefit of $0.05 - 0.72$ dB improvement in SNR in the presence of noise correlation and unequal sensor dispersion. We also compared two methods (direct and optimization methods) for the combination scheme in Section 5.4 in order to find which one could yield a smaller combined filter with comparable performance (i.e., $1.0^\circ$ peak phase error over $10 - 88$ MHz) than that of others. The optimization method was found to be better.

4. A general description of filter-and-sum beamforming (Figure 2.1) in \textit{bandpass representation},
using \textit{a priori} information such as the desired beam pointing direction, was developed in Section 2.1.

5. It was shown in Table 6.1 that optimum-SNR beamforming neglecting external noise correlation was a suitable beamforming strategy for LWA1. As opposed to delay-and-sum beamforming, it is better in array performance; whereas as opposed to optimum-SNR beamforming, it is better in the sense that the beamforming weights are independent of sidereal time and thus the number of sets of coefficients required is dramatically reduced.

Although the findings in this research are derived and confirmed in the context of LWA1, they are also applicable to array systems used in other radio science applications, including HF/VHF direction finding arrays, radar arrays for measuring the atmosphere or ionosphere, and riometers.

7.2 Future Work

The recommended future investigations are as follows:

1. The external noise characterization in Section 2.2 assumes a uniform brightness temperature distributed over the sky, however the actual situation is somewhat different. A realistic sky model at 38 MHz is illustrated in Figure 3 of [67] using the low-frequency sky model as described in [68]. The results demonstrate that brightness temperature is not uniformly distributed over the sky. Therefore the external noise covariance matrix should account for non-uniform brightness temperature if possible. This could significantly affect the results in Chapter 6.
2. In Section 6.3, the SNR performance of LWA1 was analyzed only at three frequencies, due to the large computation time acquired for calculation of the array manifold. While the results are consistent with our assertion that the effects of dispersion by antennas and cables are significant and can be largely corrected, further work should emphasize broadband confirmation of these findings, which will require array manifold calculations at many more frequencies.

3. In this dissertation, we did not discuss practical implementation issues including proper number of bits for filter coefficients, relationship between power consumption and the number of taps (similarly, area and cost with respect to the number of taps), and so on. These issues should be considered prior to final decision on filter designs (especially the number of taps) for systems under consideration.
Appendix A

Taylor Series Expansion

A one-dimensional Taylor series as an expansion of a function $f(x)$ that is infinitely differentiable in a neighborhood of a point $x_0$ is the power series

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

where $n!$ represents the factorial of $n$, and $f^{(n)}(x_0)$ denotes the $n^{th}$ derivative of function $f(x)$ evaluated at the point $x_0$.

Now we take the function $f(x) = e^{g\sqrt{x}}$, where $g$ is an arbitrary constant, as an example to observe how many terms of the Taylor series are needed to accurately approximate the function $f(x)$ around the point $x_0$. Here we consider four different cases: The one-term Taylor series, the two-term Taylor series, the three-term Taylor series, and the four-term Taylor series. The expressions for the Taylor polynomials are given as follows: The one-term Taylor series around $x_0$ is
\[ f_1(x) = e^{g\sqrt{x_0}}, \] 
(A.2)

the two-term Taylor series around \( x_0 \) is

\[ f_2(x) = e^{g\sqrt{x_0}} + \frac{g e^{g\sqrt{x_0}}}{2\sqrt{x_0}} (x - x_0), \] 
(A.3)

the three-term Taylor series around \( x_0 \) is

\[ f_3(x) = e^{g\sqrt{x_0}} + \frac{g e^{g\sqrt{x_0}}}{2\sqrt{x_0}} (x - x_0) + \frac{1}{2} \left( \frac{g^2 e^{g\sqrt{x_0}}}{4x_0} - \frac{g e^{g\sqrt{x_0}}}{4\sqrt{x_0}} \right) (x - x_0)^2, \] 
(A.4)

and the four-term Taylor series around \( x_0 \) is

\[ f_4(x) = e^{g\sqrt{x_0}} + \frac{g e^{g\sqrt{x_0}}}{2\sqrt{x_0}} (x - x_0) + \frac{1}{2} \left( \frac{g^2 e^{g\sqrt{x_0}}}{4x_0} - \frac{g e^{g\sqrt{x_0}}}{4\sqrt{x_0}} \right) (x - x_0)^2 + \]
\[ \frac{1}{6} \left( \frac{g^3 e^{g\sqrt{x_0}}}{8\sqrt{x_0}} - \frac{3g^2 e^{g\sqrt{x_0}}}{8x_0^2} + \frac{3ge^{g\sqrt{x_0}}}{8\sqrt{x_0}^3} \right) (x - x_0)^3. \] 
(A.5)

In the context of LWA1 (i.e., bandwidth of interest is between 10 and 88 MHz), we consider \( f(x) \) in the range of \( 10 \text{ MHz} \leq x/(2\pi) \leq 88 \text{ MHz} \) and select \( x_0 \) to be the middle point \( 98\pi \times 10^6 \text{ rad/s} \).

Let the constant \( g \) be

\[ g = \frac{1 + j}{\sqrt{2}} \times 10^{-5}, \] 
(A.6)

which is roughly the same order as and proportional to what we expect for \((\zeta + j\kappa)l\) in Equation (3.34) for the KSR200DB cable as shown in Figure 3.5. The Taylor series for \( f(x) = e^{g\sqrt{x}} \) with different terms are then determined by Equations (A.2) - (A.5), and shown in Figure A.1.

The approximation errors due to the finite terms of Taylor series are presented in Figure A.2.
Figure A.1: The Taylor series for $f(x) = e^{g \sqrt{x}}$ with different terms, where $g$ is given in Equation (A.6). All the Taylor series are expanded around $x_0 = 98\pi \times 10^6 \text{ rad/s}$.

Figure A.2: Approximation error of the Taylor series for $f(x) = e^{g \sqrt{x}}$ where $g$ is given in Equation (A.6). All the Taylor series are expanded around $x_0 = 98\pi \times 10^6 \text{ rad/s}$. 
expected, the Taylor series having higher degrees yield better approximation. The three-term and four-term Taylor series can provide a good approximation to the function $f(x) = e^{\sqrt{x}}$ in the vicinity of $x_0 = 98\pi \times 10^6$ rad/s. In comparison with the four-term Taylor series, the three-term Taylor series is a simpler expression with lower order resulting in simpler inverse Fourier transform, at the cost of larger approximation errors. Over the bandwidth of interest, the three-term Taylor series yield maximum magnitude error less than 0.08 dB and maximum phase error less than 0.5°, which are good enough for LWA1. Therefore, we select the three-term Taylor series as the approximation to the frequency response of the cable correction filter as described in Section 3.2.3.
Appendix B

Real-Valued Filter

The desired impulse response of a correction filter is generally found to be complex-valued. There is no difficulty with this if the signal being corrected is also complex-valued, and a complex-valued result is acceptable. However, in some systems, such as LWA1, the input and output signals are real-valued, and it is desired to do the correction entirely with real-valued quantities. In those cases, it is desired to find a real-valued impulse response which is in some sense equivalent to complex-valued impulse response.

Let $p(t)$ be a complex-valued impulse response, and let $P(\omega)$ be its Fourier transform; i.e.,

$$P(\omega) = \int_{-\infty}^{\infty} p(t)e^{-j\omega t} \, dt \text{.}$$  \hspace{1cm} (B.1)

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\omega)e^{j\omega t} \, d\omega \text{.}$$  \hspace{1cm} (B.2)

We seek a real-valued impulse response $q(t)$ such that the Fourier transform of $q(t)$, $Q(\omega)$, is equal
to \( P(\omega) \) for \( \omega \geq 0 \) at least.

Since \( q(t) \) is real valued, \( Q(\omega) = Q^*(-\omega) \); thus

\[
Q(\omega) = \begin{cases} 
  P(\omega) & \text{for } \omega \geq 0 \\
  P^*(-\omega) & \text{for } \omega < 0
\end{cases}
\]

which can be written as

\[
Q(\omega) = \frac{1}{2} P(\omega) + \frac{1}{2} \text{sgn}(\omega) P(\omega) + \frac{1}{2} [1 - \text{sgn}(\omega)] P^*(-\omega) ,
\]

where

\[
\text{sgn}(\omega) \triangleq \begin{cases} 
  +1 & , \omega > 0 \\
  0 & , \omega = 0 \\
  -1 & , \omega < 0
\end{cases}
\]

The fact that this expression gives

\[
Q(0) = \text{Re}\{P(0)\}
\]

as opposed to \( Q(0) = P(0) \) for \( \omega = 0 \) is of no consequence as long as we are not interested in the “DC” value of the frequency response.

Taking the inverse Fourier transform of Equation (B.4), we have

\[
q(t) = \frac{1}{2} p(t) + \frac{1}{2} \frac{j}{\pi t} * p(t) + \frac{1}{2} \left[ 1 - \frac{j}{\pi t} \right] * p^*(t) \\
= \frac{1}{2} [p(t) + p^*(t)] + \frac{j}{2\pi t} * [p(t) - p^*(t)]
\]

\[
= \text{Re}\{p(t)\} + \frac{j}{\pi t} * j\text{Im}\{p(t)\} ,
\]

168
where “∗” represents the convolution operation. Let $p_r(t) = \text{Re}\{p(t)\}$ and $p_i(t) = \text{Im}\{p(t)\}$. Then

$$q(t) = p_r(t) + \frac{j}{\pi t} * j p_i(t)$$

$$= p_r(t) - \frac{1}{\pi t} * p_i(t).$$

(B.8)

Note that the second term in the above equation is the Hilbert transform of $p_i(t)$, $\hat{p}_i(t)$. The equivalent real-valued impulse response $q(t)$ is thus

$$q(t) = p_r(t) - \hat{p}_i(t).$$

(B.9)

The discrete finite impulse response $q[k]$ can be developed using the following procedure: (1) Sample $P(\omega)$ and take the $M$-point discrete Fourier transform (DFT) to obtain $p_r[k]$ and $p_i[k]$; (2) Obtain $\hat{p}_i[k]$ of length $M$ using

$$\hat{p}_i[k] = \text{Im}\left\{\mathcal{F}^{-1}\left\{h[k] \# \mathcal{F}\{p_i[k]\}\right\}\right\},$$

(B.10)

where the operator “#” represents elementwise multiplication, and

$$h[k] = \begin{cases} 1, & k = 1, \frac{M}{2} + 1 \\ 2, & k = 2, 3, \ldots, \frac{M}{2} \\ 0, & k = \frac{M}{2} + 2, \ldots, M \end{cases}$$

(B.11)

Therefore, we have the real-valued FIR filter as follows:

$$q[k] = p_r[k] - \hat{p}_i[k].$$

(B.12)
Now we take the cable correction filter shown in Figure 3.6 as an example to confirm the above conclusion. We begin with obtaining the 131072-point complex-valued filter by taking the 131072-point inverse Fourier transform of the frequency response. Then we use Equation (B.12) to obtain the corresponding 131072-tap real-valued filter \( h(t) \), as shown in Figure B.1. The frequency response deviation of the real-valued filter \( h(t) \) from the ideal frequency response described in Figure 3.6 is presented in Figure B.2. The results confirm that the real-valued filter described in Equation (B.12) is sufficient to represent the desired frequency response at positive frequencies; i.e., the real-valued filter is suitable to substitute for the complex-valued filter obtained by simply taking the inverse Fourier transform of the frequency response.
Figure B.1: Impulse response of the filter as shown in Figure 3.6. The right figure is a close-up view of the left one.

Figure B.2: Differences between the frequency response of the filter described in Figure B.1(a) and the Figure 3.6 result.
## Appendix C

### LWA1 Stand Positions and Cable Lengths

Table C.1: LWA1 stand positions and cable lengths.

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