Application of Lyapunov Exponents to Strange Attractors and Intact & Damaged Ship Stability

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(ABSTRACT)

The threat of capsize in unpredictable seas has been a risk to vessels, sailors, and cargo since the beginning of a seafaring culture. The event is a nonlinear, chaotic phenomenon that is highly sensitive to initial conditions and difficult to repeatedly predict. In extreme sea states most ships depend on an operating envelope, relying on the operator’s detailed knowledge of headings and maneuvers to reduce the risk of capsize. While in some cases this mitigates this risk, the nonlinear nature of the event precludes any certainty of dynamic vessel stability.

This research presents the use of Lyapunov exponents, a quantity that measures the rate of trajectory separation in phase space, to predict capsize events for both intact and damaged stability cases. The algorithm searches backwards in ship motion time histories to gather neighboring points for each instant in time, and then calculates the exponent to measure the stretching of nearby orbits. By measuring the periods between exponent maxima, the lead-time between period spike and extreme motion event can be calculated. The neighbor-searching algorithm is also used to predict these events, and in many cases proves to be the superior method for prediction.

In addition to the ship stability research, the Lyapunov exponents are used in conjunction with bifurcation analysis to determine regions of stable behavior in strange attractors when the system parameters are varied. The boundaries of stability are important for algorithm validation, where these transitions between stable and unstable behavior must be accounted for.
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1. Introduction

1.1 Lyapunov Exponents

The motion of a single point on an attractor can be defined as chaotic if exhibits sensitivity to infinitesimally small changes in initial conditions (Ott et al 1994). A simple but telling example of this condition would be to place a ball on a hill and give it a small push in one direction. No matter how precise the push may be, the ball will always follow a different orbit down the hill because of the miniscule differences in the force being applied and the terrain it follows. The elaborate orbit structure that comes as a result of vast number of possible orbits, as well as the “stretching” of minute displacements of the orbit (initial condition sensitivity), can be modeled with Lyapunov exponents (Ott et al 1994).

The starting point for defining a Lyapunov exponent is a flow field:
\[ \dot{x} = v(x) \] (1)

In addition to this flow field, a trajectory \( x(t) \) is defined, as well as small deviations from that trajectory, \( \delta x \). After forming a matrix of derivatives, \( L_{ij} = \frac{\partial v_i}{\partial x_j} \), an equation for the changing nature of the flow can be defined as:
\[ \delta \dot{x} = L(x(t))\delta x \] (2)

Therefore, for all initial trajectories and initial displacements a maximal Lyapunov exponent for the system can be defined as follows (Eckhard and Yao, 1993):
\[ \lambda_\infty = \lim_{T \to \infty} \frac{1}{T} \log \frac{\|\delta x(t)\|}{\|\delta x(0)\|} \] (3)

This exponent is normally assumed to exist on an attractor, and it should be noted that for some cases an attractor may not exist for the system. As was mentioned previously, the exponent measures the stretching of nearby orbits in phase space; this stretching can come in the form an expanding or contracting nature, and may best be visualized by a ball of initial condition points. Because of the local deformations, or stretching, of the flow, the ball of initial condition points in \( k \) dimensions will become a \( k \)-dimensional ellipsoid whose axes are deforming exponentially as defined by these Lyapunov exponents (Wolf et al 1985).

The number of exponents for the system is determined by the number of state variables governing the system. Generally for any continuous time-dependent dynamic system without a fixed point, there will be a zero exponent reflecting the slowly changing principal axis, a positive exponent reflecting an expanding axis, and a negative exponent reflecting a contracting axis (Wolf et al 1985).
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A finite-time Lyapunov exponent (FTLE) is simply the Lyapunov exponent defined over a short time interval, rather than over an infinite continuous time series. It can be defined as (Eckhardt and Yao, 1993):

$$\lambda(x(t),\delta x(0)) = \frac{1}{T} \log \frac{\|\delta x(t+T)\|}{\|\delta x(t)\|}$$  \hspace{1cm} (4)

The FTLE allows for a more meaningful measure of real-time changes of the expanding and contracting nature of the elliptical axes, and is the form of the Lyapunov exponent that will be used in the analysis of the damaged stability cases. The Lyapunov exponent as defined over an infinite continuous time series, whereas capsize is a finite time event. In the FTLE calculation the Jacobian is being calculated locally at each instant in the time series, and thus the FTLE is reacting to the changes as they occur.

1.2 Wolf Algorithm

One approach used to calculate the Lyapunov exponent spectrum for this work is the algorithm developed by Wolf et al (1985), which determines the exponents directly from the equations of motion. Wolf’s method follows the long-term changes along a principle axis, or “fiducial trajectory”, in order to calculate the largest positive exponent values, and maintains space orientation using a Gram-Schmidt reorthonormalization procedure (Wolf et al 1985).
1.3 Sano and Sawada Algorithm

The algorithm developed for Lyapunov exponents by Sano and Sawada (1985) is used to calculate both the Lyapunov spectrum and FTLE values. The Sano and Sawada approach begins with the same steps as presented in equations 1-3, but also defines a linear operator, $A'$:

$$\delta x(t) = A'\delta x(0)$$  \hspace{1cm} (5)

Given the time series measured at the discrete time interval $\Delta t$, $x_j = x(t_0 + (j-1)\Delta t)$, the k-dimensional ellipsoid as described above can be defined by a displacement vector $y^i$, and a displacement vector over a time interval $\tau = m\Delta t$, given by $z^i$. The details of the derivation of these vectors are thoroughly outlined in Sano and Sawada (1985). With these vectors defined, the evolution of the ellipsoid can be represented by:

$$z^i = A_j y^j$$  \hspace{1cm} (6)

Where the matrix $A_j$ is an approximation of the flow map $A'$, from equation 5. Using a least-error-algorithm, which minimizes the average of the squared error norm between $z^i$ and $A_j y^j$ with respect to all components of the matrix $A_j$ (Sano and Sawada, 1985), the Lyapunov exponents can be found as follows:

$$\lambda_i = \lim_{n \to \infty} \frac{1}{n\tau} \sum_{j=1}^{n} \ln \|A_j e^j_i\|$$  \hspace{1cm} (7)

In this equation $n$ is the number of data points, and $e$ is a set of orthonormal basis vectors that are renormalized using the Gram-Schmidt procedure (Sano and Sawada, 1985).
1.4 Verification and Validation

“[I]n a meaningful though overly scrupulous sense, a ‘Code’ cannot be Validated, but only a Calculation (or range or calculations with a code) can be Validated. However, it is clear that physical problems and their solutions present more-or-less continuum responses in their parameter spaces. Although parameter ‘transition’ boundaries do occur, at which solution properties can change discontinuously or rapidly, these parameter transition boundaries are at least countable, and are usually few. The determination of the parameter transition boundaries is the task of the entire professional community (experimental, theoretical, computational) working in the subject area” (Roache, 1998, pp. 280-281).

Chaotic attractors are often used as an algorithm verification benchmark for the calculation of the Lyapunov exponent spectrum. Verification is the process by which the researcher demonstrates that the numerical model implementation and solution matches the developed theoretical model; this process works hand in hand with validation, which confirms that the model is an accurate representation of the physical reality of the system (McCue, 2008).

Previous work done by Rosenstein et al (1992) investigated Lyapunov sensitivity to various changes made in the Lorenz system, as a method for verifying a new algorithm for the calculation of the largest Lyapunov exponent. The work investigated the effect of embedding dimension, time series length, reconstruction delay, and additive noise on the spectrum of exponents, but did not investigate the effect of attractor parameter variation on the spectrum.

There has been little research done in regard to Lyapunov sensitivity to small-scale changes in the parameters of three and four-dimensional strange attractors from the verification and validation perspective. By investigating the effects of these parameter variations on the computed attractors, the transition boundaries between stable and unstable regions of the attractor can be determined; these transitions have been explored before, but never with respect to the Lyapunov spectrum as a V&V tool. These boundaries are an important piece of algorithm validation, where the sensitivity to experimental error can lead to incorrect results if these transitions are not taken into account.

The use of both Wolf and Sano/Sawada algorithms is a key part of the verification and validation of simulated vs. experimental results. In fields of research such as ship motions, data sets are created both from experimental tank testing as well as directly from the equations of motion. Lyapunov exponents are a robust tool for validating both types of models; there should be excellent agreement between exponent values if the underlying physics of the experimental model agrees with the experimental data.
1.5 Lyapunov Application to Ship Capsize

Ship capsize is often a chaotic phenomenon with capsize/non-capsize conditions demonstrating high sensitivity to initial conditions. Many mathematical, statistical, and numerical methods have been employed to determine the likelihood of ship capsize in specified sea conditions. However, because of the chaotic nature of the problem it is exceedingly difficult to consistently and robustly predict ship capsize in a series of random waves. Additionally there is the issue of rarity, where disparate time intervals between roll period and loss of stability lead to difficulties in numerical simulations for capsize cases. The average time before stability failure is very large compared to natural roll period, and therefore sets of reconstructed wave data must be very long to capture all possible dynamics, presenting a numerical challenge when working with the comparatively small time scale of roll period (Belenky 2007). Recent innovation in hull design has been heightening the awareness of these issues, and has pushed further investigation into the nature of ship capsize. This very real problem is where the mathematical study of chaotic processes may one-day allow for the real-time prediction of whether a ship is facing imminent capsize.

Capsize research is currently being performed both experimentally and numerically at government and academic research institutions worldwide, with powerful numerical tools being developed for the purpose of analyzing nonlinear ship motions. The LAMP program was used to complete numerical time-domain work in the 1990's by Lin and Yue (1990,1993), and began in 1988 as a DARPA project for the simulation of nonlinear ship motions (Belenky 2002). The use of this tool has been continued by Belenky, Weems, and Liut et al. (2002) to simulate criteria such as water-on-deck, impact and whipping, and wave-loads. The nonlinear strip-theory code FREDYN has been significantly used by De Kat et al. (2000,2001)to make motion predictions for both intact and damaged stability, as well as progressive flooding and sloshing.

Lyapunov exponents have seen limited use in the field of naval architecture and ship dynamics. Some of the earliest work was done by Papoulias, investigating the behavior of a mooring system for tankers in a three degree-of-freedom model. The Lyapunov spectrum was used in this case to confirm the onset of chaotic behavior, and thus instability in the model (Papoulias, 1987) Early work by Falzarano calculated the Lyapunov spectrum for the capsize case of the fishing vessel Patti-B. It was concluded that the exponent can serve as both a qualitative and quantitative measure of chaos, with a positive value both confirming chaotic behavior and associating a number with that expansion (Falzarano 1990). Spyrou’s work investigated Lyapunov exponents in connection with large-amplitude ship motion in quartering waves; rudder angle was used as a control parameter, with a focus on oscillatory behavior and transitions to chaotic regions. The exponents were used in conjunction with bifurcation analysis to detect the transition boundary between stable and chaotic behavior in a controls-fixed ship, focusing on positive values of the first exponent (Spyrou 1996). Work by Murashige and collaborators examined the role of chaotic behavior in a flooding box-barge model in waves; the model was coupled two-dimensionally with roll and flooding. Their results determined that the roll response of a flooded vessel can exhibit chaotic behavior in regular waves, supported by the existence of a positive Lyapunov exponent (Murashige 1998a; 1998b; 2000). Arnold et al. correlated measurements of the positive
Lyapunov exponent in the spectrum to capsize results in a one degree-of-freedom roll model. Their results concluded that for their model the attractor disappears in a capsize case, usually while the first exponent is negative; this signifies a stable periodic attractor rather than a chaotic region. The research produced some capsize cases that were the result of a positive exponent, demonstrating that the numerical model can exist as a chaotic attractor before capsize (Arnold 2003).

Recent work by McCue et. al. has used Lyapunov exponents for deterministic research on ship motions and capsize. The work used a finite-time form of the exponent (FTLE) to examine its ability to predict capsize cases in a single degree-of-freedom roll model. The research concluded that the Lyapunov approach can accurately predict impending capsize in regular and random seas (McCue 2005). Further work by McCue, Bassler, and Belknap continued the use of FTLE’s to indicate capsize in experimental time series data from wave tank tests conducted at the Naval Surface Warfare Center in Carderock, MD. This research is a continuation of that work, examining the feasibility of FTLE’s for both intact and damaged stability, and the application of different algorithms to improve the response time for capsize prediction.
2. Identification of Parameter Transition Boundaries with Lyapunov Exponents

2.1 Bifurcation analysis

The sensitivity to parameter changes for strange attractors can be achieved through a number of methods, the most traditional of which is bifurcation analysis. The definition of this analysis as outlined by Crawford (1989):

“Bifurcation theory studies these qualitative changes in the phase portrait, e.g., the appearance or disappearance of equilibria, periodic orbits, or more complicated features such as strange attractors. The methods and results of bifurcation theory are fundamental to an understanding of nonlinear dynamical systems...”

Bifurcation analysis was used to further validate the Lyapunov transition boundaries for large-scale parameter changes. The AUTO program, originally developed by Doedel (2008), is a freeware software package with built-in algorithms for calculating Hopf bifurcations for the Lorenz system. These bifurcations occur during the transitional phases of the Lorenz oscillator, from stable to unstable behavior, and are useful in verifying the transitions captured by the Lyapunov exponent spectrum.

2.2 Time-Series Length

Both time-series length and time-step are critical in order to allow for the long-term convergence of the Lyapunov exponent. Abarbanel et al investigated the effect of local Lyapunov exponents and their governance of small perturbations along an orbit based on a finite number of steps. Their work concluded that as L, the number of steps along the orbit, grows to infinity, variations about the mean of the Lyapunov exponents approaches zero (Abarbanel et al 1991). Further work studying the effects of time series length with regards to exponent deviation was completed by Rosenstein et al; they also explored variations in embedding dimension, reconstructive delay, and additive noise using the same Lorenz system investigated by Wolf et al. The findings concluded that the best results for Lyapunov calculation were achieved using a long time-series and closely spaced samples; they also saw similar results using long observation time and widely-spaced samples (Rosenstein et al, 1992). All data sets in this study are 2000 seconds in length with a step size of 0.1 seconds, providing a sample size of 20,000 points. Generally for any continuous time-dependent dynamic system without a fixed point, there will be a zero exponent reflecting the slowly changing principle axis, a positive exponent reflecting an expanding axis, and a negative exponent reflecting a contracting axis (Wolf et al 1985). Convergence for the positive and zero exponents occurs quickly along the time series, while the third exponent, which is generally considered to be the most unstable of the three, takes a number of iterations to converge, as seen in Figure 2. By the conclusion of the time series, all three exponents oscillate around their final value on the order of 0.1-0.3%. 
2.3 Lorenz System

Two different variations on the Lorenz system were investigated, each having been previously solved for the Lyapunov spectra using different approaches. The algorithm used for this work was verified against the published values. The first system is based on a familiar model that was the result of a study of convection in the lower atmosphere (Abarbanel et al 1991). These parameter values for the Lorenz system were used by Wolf et al in their direct method algorithm, and have been used more recently in other approaches to determine Lyapunov spectra for strange attractors. The Wolf system is defined by the following parameters: $\sigma = 16$, $R = 45.92$, $b = 4.0$ The parameters of the second system are as defined by Lorenz himself in his original work on the attractor as follows (Lorenz 1963): $\sigma = 10$, $R = 28.0$, $b = 8/3$. This system is widely considered to be the classic example of a
Lorenz oscillator, as seen in Figure 3.

Figure 3: Lorenz oscillator system for $\sigma = 10, R = 28.0, \beta = 8/3$

2.4 Rossler System

The three dimensional Rossler attractor, as proposed by Rossler in 1976, is a three-dimensional system that has a positive, zero, and negative exponent spectrum when stable; two variations in parameter values were compared, those by Wolf et al. ($a = 0.15, b = 0.20, c = 10.0$), as seen in Figure 4, and those by Sano and Sawada ($a = 0.20, b = 0.20, c = 5.7$). The Rossler hyperchaos system, a 4-D hyperchaotic flow proposed in 1979 and seen in Figure 52 in Appendix A, was also investigated; it contains a second positive exponent reflecting the expanding axis of the extra unseen dimension. It is extremely sensitive to parameter inputs, and therefore there is generally the following are the only parameters used for numerical study ($a=0.25, b=3.0, c=0.05, d=0.5$).
2.5 Small-scale Parameter Variations

The definition of “small-scale” for this study were parameter perturbations of 10% of the parameter value in each direction.

2.5.1 Lorenz System

2.5.1.1 $\sigma$ (small-scale)

Parameter changes of this scale show small changes in the exponent values, but at no point are they significant enough to force the system into a difference phase or orbit. Figure 5 shows the Lyapunov exponent spectrum changes for both Wolf and Classical variations, noting that the difference in parameters between the two accounts for the apparent gap in measurement value.
The third exponent shows appreciable change for both the Wolf and Classical variations, but its linear nature shows belies no series phase change or transitional period. The standard deviations for the $\sigma$ case are seen in Table 1:

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.007445</td>
<td>0.0006189</td>
<td>0.9992</td>
</tr>
<tr>
<td>Sano/Sawada</td>
<td>0.005233</td>
<td>0.0009968</td>
<td>0.5216</td>
</tr>
</tbody>
</table>

Table 1: Standard deviations for Lyapunov spectrum for small-scale changes in $\sigma$

2.5.1.2 R (small-scale)

The R-parameter has little to no effect on the system for changes of this degree. As can be seen in Figure 50 and Table 14 of Appendix A, the Wolf variation shows a little more change in the third exponent than the Classic variation, though both deviations are small; though the third exponent has been the most erratic of the three, for R variations it remains moderately constant. The second exponent variation is again on the scale of computational error, oscillating around zero, and the first exponent increases in a linearly positive direction for the duration of the calculations.
2.5.1.3 $b$ (small-scale)

All three exponents trend similarly to changes in sigma, with the third exponent decreasing linearly, the first exponent linearly increasing to a small degree, and the second oscillating around zero. There is no appreciable transition or phase change.

2.5.2 Rossler System

Given the similarity between parameters of the Wolf and Sano/Sawada systems, and little numerical change in either system for small-scale parameter change, only the Wolf variation was used.

2.5.2.1 $a$ (small-scale)

The Rossler system showed no obvious phase changes or transition boundaries for $a$:

![Graph showing Lyapunov spectrum changes for small-scale variations in $a$ for Wolf system](image)

Figure 6: Lyapunov spectrum changes for small-scale variations in $a$ for Wolf system

<table>
<thead>
<tr>
<th>System</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.01860</td>
<td>0.001806</td>
<td>0.03411</td>
</tr>
</tbody>
</table>

Table 2: Standard deviations for Lyapunov spectrum for small-scale changes in $a$
All three exponent deviations are relatively uniform, with no increasing or decreasing trends among them. All three exponents fluctuate along the length of the parameter change, and the system is stable throughout, with no transitions.

2.5.2.2 $b$ (small-scale)

Changes in $b$ have even less of an affect on the system, as can be seen in Figure 51 and Table 16 of Appendix A.

2.5.2.3 $c$ (small-scale)

Changes in parameter $c$ reflect those seen in the Lorenz system:

![Graph showing Lyapunov spectrum changes for small-scale variations in $c$ for Wolf system](image)

**Figure 7: Rossler Lyapunov spectrum changes for small-scale variations in $c$ for Wolf system**

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.005389</td>
<td>0.0002207</td>
<td>0.3316</td>
</tr>
</tbody>
</table>

**Table 3: Standard deviations for Rossler Lyapunov spectrum for small-scale changes in $c$**

The first two exponents show very little deviation along the length of changes in $c$. The third exponent shows a slightly negative linear change, but nothing significant enough to be considered transition behavior.
2.5.3 Rossler Hyperchaos System

2.5.3.1 $a$ (small-scale)

The initial system parameter given by Wolf was $a=0.25$, and changes were attempted for 10% of this value in each direction. However, for early values $a=0.225-0.23$, the system would not converge to a stable fourth exponent. The convergence began at $a=0.2325$ and continued through $a=0.2575$. After this point the ODE solver failed to compute for all further time iterations. Figure 8 shows the variations along $a$ that would converge to a stable fourth exponent:

![Graph showing variations in $a$ for small-scale changes in $a$ for Wolf variation](image)

**Figure 8:** Rossler Hyperchaos Lyapunov spectrum changes for small-scale changes in $a$ for Wolf variation

When comparing all three attractor systems, the Rossler Hyperchaos system is the most reactive of all three when making small-scale variations, at least in regards to the fourth exponent. This exponent is extremely sensitive to parameter change, and is the driving force behind the convergence failure for the attractor. The standard deviations belie the erratic nature of the negative exponent:
<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
<th>Exponent 4 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.006431</td>
<td>0.003710</td>
<td>0.0006582</td>
<td>13.41</td>
</tr>
</tbody>
</table>

Table 4: Standard deviations for Lyapunov spectrum for small-scale changes in $a$

The deviations in the first three exponents are on the same scale as similar measurements for the other two attractor systems. Given this result, it does not appear that the system is undergoing any major phase change or transition period. The negative exponent is classically the most unstable of the three, and thus all three exponents must be taken into account when considering whether the system is crossing a transition boundary.

2.5.3.2 $b$ (small-scale)

Though the negative exponent shows slightly oscillatory behavior, the other three exponents remain constant, similar to the case for changes in $a$. Therefore, there appear to be no apparent transition boundaries; the data for $b$ can be found in Appendix A, Figure 52 and Table 17.

2.5.3.3 $c$ (small-scale)

The changes in parameter $c$ mirror those in $a$; instead of a positively trending erratic negative exponent, it trends rapidly negative. As with the other parameters, there is little to no change in the first three exponents, and therefore no reason to suspect a phase change in the system. See Figure 53 and Table 18 in Appendix A for the visuals and deviations.

2.5.3.4 $d$ (small-scale)

Unlike the other two attractors, the Rossler Hyperchaos system has a fourth dimension, and therefore a fourth parameter to vary. The changes in parameter $d$ cause oscillations in the fourth exponent similar to changes in the $b$ parameter, as well as an increasing trend similar to changes in the $a$ parameter; the exponent failed to converge for any value of $d$ higher than 0.545. The first three exponents are again primarily uniform throughout their length, so no transition boundaries were observed. See Appendix A for further data.
2.6 Large-Scale Parameter Variations

The parameter perturbations on the large scale were on the order of $10^1$ of the original parameter value, with approximately 20 different parameter values for each system. For example, the Rossler attractor’s original Wolf parameters were $a = 0.15$, $b = 0.2$, $c=10.0$. The $a$ parameter was varied from 0.5-0.35 in a step size of 0.5, with extra points near the 0.35 mark; this was due to the instability of the attractor beyond a value of $a = 0.38$, at which point the MATLAB ODE solver fails. The other two parameters are held constant while the $a$ parameter is being varied. Similarly, $a$ and $c$ are held constant while $b$ is varied from 0.05-1.0, and $a$ and $b$ are held constant while $c$ is varied from 1-20. This wide range for each parameter value allowed for the qualification of any significant changes in Lyapunov exponent solution due to increasing chaos in the attractor, thus determining the parameter transition boundaries.

2.6.1 Lorenz System

2.6.1.1 $\sigma$ (large-scale)

The $\sigma$ value was the most significant driver for changes in the Lyapunov spectra of the Lorenz oscillator, particularly in the third exponent; as mentioned earlier, this exponent is the least stable of the three, and therefore the one most prone to changes in the system. Figure 9 shows the spectrum progression in all three exponents for changes in $\sigma$ for the Wolf variation:
Figure 9: Lorenz Lyapunov spectrum for changes in $\sigma$ for the Wolf Variation
\[ (\sigma = 16, R = 45.92, b = 4.0) \]

The classical system behaves in a very similar manner, with large-scale changes in sigma prompting a rapid decrease in the value of the third exponent. The standard deviations for the three exponents show that the positive exponent (1) is affected more than the zero exponent (2) for all three systems, with the negative (3) exponent varying by a much larger factor, as seen in Table 5:

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.5444</td>
<td>0.1002</td>
<td>6.612</td>
</tr>
<tr>
<td>Classic</td>
<td>0.7171</td>
<td>0.3538</td>
<td>6.577</td>
</tr>
</tbody>
</table>

Table 5: Standard deviation for Lorenz Lyapunov spectrum for large-scale changes in $\sigma$

The first and second exponents show qualitatively similar parameter transition boundaries in both Lyapunov spectrum and bifurcation analysis, as seen in Figures 10 and 11:

Figure 10: Lorenz first Lyapunov exponent change and Hopf bifurcation for large-scale change in $\sigma$
Figure 11: Lorenz second Lyapunov Exponent change for large-scale changes in $\sigma$

Both exponents show a rapid increase for early sigma changes, transitioning to a relative plateau of stability with little to no significant change, and finally a decrease at increasing values of sigma; the initial values for both variations rest in the stable area of the parameter range. The results indicate that a parameter transition boundary does exist at both low and high ends of the sigma spectrum, with high instability in Lyapunov values for small values of sigma. The Hopf bifurcations mark the boundaries in most cases, though for the first exponent it occurs earlier than the Lyapunov transition indicates.
The third exponent shows a linear decrease as $\sigma$ increases:

![Graph](image)

**Figure 12**: Lorenz third Lyapunov exponent changes for small-scale variation in $\Sigma$

There is only a slight transitionary period for the third exponent, with both variations stabilizing for the middle range of sigma values; the Classic variation appears to transition again at the end of the sigma range. Again the bifurcations indicate the transition periods, though more subtle than those indicated by the first and second exponents.
Figure 13: Lorenz first Lyapunov exponent change and phase-space for large –scale change in $\sigma$ for the Classic variation

The Lorenz phase space transitions through three distinct forms as the $\sigma$ parameter is changed in the Classical variation. The first phase, seen in Figure 14, is a unit cycle; in this region of $\sigma$ the equations of motion have a steady state solution.
Figure 14: Lorenz Classic variation steady-state phase-space for $\sigma=2.0$

After the transition to chaotic behavior, the oscillator phase space looks like Figure 2, with two distinct lobes and classic attractor behavior. After the second transition the phase space returns to a stable fixed-point attractor, and the orbits all converge to one point, as seen in Figure 15:
Figure 15: Lorenz Classic variation unstructured phase-space for $\sigma=19.0$
2.6.1.2 R Variation

Changes in the R parameter affect the system far less significantly than those of sigma, both in overall Lyapunov variation and in recognition of clear parameter boundary transitions with bifurcation analysis. The change in R for both variations can be seen in Figures 16 and 17:

Figure 16: Lorenz Lyapunov spectrum changes for R in the Wolf Variation
\( \sigma = 16, R = 45.92, b = 4.0 \)
As seen in the previous figures and Table 6, there is little change amongst the exponents compared to those changes made with Sigma variations.

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.1319</td>
<td>0.0005465</td>
<td>0.1322</td>
</tr>
<tr>
<td>Classic</td>
<td>0.1227</td>
<td>0.0002564</td>
<td>0.1231</td>
</tr>
</tbody>
</table>

Table 6: Standard deviations for Lorenz Lyapunov Spectrum for small-scale changes in $R$

Though there is a slight linear increase for the positive exponent (1) in both cases, and conversely a more pronounced linear decrease for the negative exponent (3), there are no clear transitions in phase space.
2.6.1.3  $b$ (large-scale)

For variations in the $b$ parameter, the Classic variation showed significant differences in both parameter transition and overall exponent behavior. The three exponent variation figures and standard deviation table are detailed below:

Figure 18: Lorenz first Lyapunov exponent changes for large-scale variations in $b$
Figure 19: Lorenz second Lyapunov exponent changes for large-scale variations in $b$
Figure 20: Lorenz third Lyapunov exponent changes for large-scale variations in $b$

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.2104</td>
<td>0.0003961</td>
<td>1.572</td>
</tr>
<tr>
<td>Classic</td>
<td>0.5659</td>
<td>0.1569</td>
<td>0.6625</td>
</tr>
</tbody>
</table>

Table 7: Standard deviations for Lorenz Lyapunov spectrum for small-scale changes in $b$

The Classic variation shows significant deviations in all three exponents through the range of beta values, quickly trending negative as the beta value increases. Though the higher beta values for this variation are on the same scale as the Wolf variation, the combination of high beta values with the other two parameters causes the system to become unstable, as can be seen in Figure 18. For the first two exponents there is a notable transition boundary at $b=3$, where the Lyapunov exponent values rapidly trend below zero. The Wolf variation shows none of these boundaries, and remains stable throughout the range of parameter values. The Classic variation bifurcation sits in the Lyapunov transition zone for all three exponents, while the Wolf bifurcation slightly precedes the transition.

For validation interest the small range of permissible $b$ values for the Classic variation should be noted. As the transition occurs, the Lorenz system quickly unravels from classic chaotic behavior to the stable fixed-point convergence structure seen in Figure 15. The following figure shows the system as transitions between the two phases:
Figure 21: Lorenz Classical variation transition between chaotic and stable behavior for changes in $b$
2.6.2 Rossler System

2.6.2.1 \( a \) (large-scale)

Like the Lorenz system, the Rossler attractor is a three-dimensional system that can break down from its typical structure when parameter changes become too pronounced. It should be noted that AUTO had difficulty calculating the bifurcation points for this system. This may have been due to the need for a negative \( a \) parameter to find the steady state solution, from which the chaotic bifurcations may be calculated. Changes in Lyapunov spectra from variation in \( a \) can be seen in Figures 22-24:

![Figure 22: Rossler first Lyapunov exponent changes for large scale changes in \( a \)](image-url)
Figure 23: Rossler first Lyapunov exponent changes for large scale changes in $a$
Similarly to the Lorenz system, changes in the first parameter are most evident in the third exponent. It shows the highest level of variation among the three, as seen in the standard deviation values:

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.09301</td>
<td>0.03894</td>
<td>0.7853</td>
</tr>
<tr>
<td>Sano/Sawada</td>
<td>0.0798</td>
<td>0.09224</td>
<td>0.4550</td>
</tr>
</tbody>
</table>

Table 8: Standard deviations for Rossler Lyapunov spectrum for small-scale changes in $a$

One significant change from the Lorenz oscillator is the movement of the third exponent: rather than trending linearly negative, it has a positive trend as $a$ increases for both systems. Also, for small values of $a$, the system begins to converge to a fixed point, at which point all exponents are negative. As $a$ increases to 0.1, the Lyapunov distribution is $(0,-,-)$, signifying a unit cycle of period 1; after 0.1 the distribution returns to a familiar $(+,0,-)$, a standard strange attractor. Figure 25 shows the attractor at an $a$ value of 0.05, where it converging from a fixed point to a unit cycle.

The second and third exponents show clear parameter transition boundaries, though the range of exponent stability is mirrored. The second exponent stabilizes as $a$ increases, leveling off to a stable value around $a=0.2$, while the third is moderately stable at low values,
only to increase after the 0.2 mark. The first exponent is the least defined of the three, with no apparent transition areas at any point in the range of parameter values.

Figure 25: Rossler oscillator for $a = 0.05, b = 0.2, c = 10$

2.6.2.2 $b$ (large-scale)

Variations in $b$ for the Rossler system were at or below the order of accuracy for the Lyapunov algorithm. As seen in Figure 26, the values for all three exponents were relatively stable across the entire range of changes in $b$: 
The oscillations seen in the three exponents were on a level small enough to be on the order of accuracy, and therefore the $b$ parameter shows reasonable stability has no apparent transition boundaries at any point in the measured range of values:

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.02961</td>
<td>0.009072</td>
<td>0.03233</td>
</tr>
<tr>
<td>Sano/Sawada</td>
<td>0.02859</td>
<td>0.02681</td>
<td>0.06090</td>
</tr>
</tbody>
</table>

Table 9: Standard deviations for Rossler Lyapunov spectrum for small-scale changes in $b$
2.6.2.3 \( c \) (large-scale)

For this parameter, the system evolved similar to that of the Lorenz oscillator. The third exponent underwent a familiar linear decrease as \( c \) increased, the first exponent stayed positive for almost the entire spectrum, and the second exponent oscillated around 0 for most of the duration, seen in Figure 27:

![Figure 27: First Lyapunov exponent changes for small-scale variations in \( c \)](image-url)
Figure 28: Second Lyapunov exponent changes for small-scale variations in $c$
Figure 29: Third Lyapunov exponent changes for small-scale variations in $c$

The first exponent shows oscillatory behavior, with no apparent transition boundaries; it appears that the exponent slightly stabilizes between $c=10$ and $c=15$, but then begins to oscillate again. The second exponent shows more delineated behavior, with parameter transition boundaries at both low and high ends of the spectrum of $c$ values. There are no apparent transitions for the third exponent, which shows linear behavior throughout its length.

Notably, both variations undergo brief periods of periodicity throughout the extent of parameter variation, returning to a chaotic state an instant later. Figure 30 shows the phase space behavior for differing exponent values:
Figure 30: Rossler first Lyapunov exponent change and phase-space for large -scale change in c

For small values of c, the system exhibits periodicity. Figure 31 was run for 500 seconds, with an 0.001 step size for the time interval:
Figure 31: Rossler oscillator periodicity for $a = 0.2, b = 0.2, c = 4$

Compare the previous figure to Figure 32, which run in the same manner for a $c$ value of 20:
Figure 32: Rossler oscillator periodicity for \( a = 0.2, b = 0.2, c = 20 \)

For this case the attractor exhibits chaotic behavior, and does so for most values of \( c \). It is apparent that the results of Figure 32 are not a result of the number of orbits, but rather their path through phase space.

The standard deviations for changes in this parameter appear very similar to the Lorenz case, with small changes in the first two exponents following by significant changes in the third:

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 ( \sigma )</th>
<th>Exponent 2 ( \sigma )</th>
<th>Exponent 3 ( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.0387</td>
<td>0.1457</td>
<td>5.948</td>
</tr>
<tr>
<td>Sano/Sawada</td>
<td>0.04787</td>
<td>0.04840</td>
<td>5.863</td>
</tr>
</tbody>
</table>

Table 10: Standard deviations for Rossler Lyapunov spectrum for small-scale changes in \( c \)
2.6.3 Rossler Hyperchaos System

The hyperchaos system is extremely sensitive to parameter inputs, and as such is impossible to change on a large scale like the other two systems. As seen in section 2, there is a very small range for which the system can be solved via an ODE solver in MATLAB. It appears that the system has no transition boundaries, and only exists for a very small range of parameter values.
3. Application of Lyapunov Exponents to Intact & Damaged Ship Stability Cases

3.1 Application of FTLEs to Dynamic Ship Motion

The Lyapunov Exponent approach was used for two different scenarios: damaged stability data for a commercial passenger Ro-Ro ship model, and the intact stability of notional destroyer DTMB hull model 5514. Both analyses use roll and pitch data that has been normalized with respect to the mean and standard deviation; the roll-velocity and pitch velocity was then calculated based on these normalized values.

3.2 Damaged Stability of a Commercial Passenger Ro-Ro Ship

The data for the damaged stability analysis was provided by Dr. Andrzej Jasionowski of the Ship Stability Research Center of the Universities of Strathclyde and Glasgow (Jasionowski, 2001). The model tests were performed at the Denny Tank at the University of Strathclyde on a 1:40 scale model of a passenger Ro-Ro vehicle (Jansonowsky, 2001).

For this data set, FTLE time histories were calculated along with the period between neighboring FTLE maxima, and both plotted vs. time as shown in Figures 1 and 2. The period calculations were employed in an attempt to provide instantaneous qualitative and quantitative methods for determining the time for advance warning of extreme ship motions. The period measurements are made using a reverse difference method, in order to simulate real-time data collection. This method searches backwards in the time series to find neighboring points to populate the displacement vectors \( y' \) and \( z' \); the backwards approximation was used to more closely approximate a realistic on-board scenario where the only available data would be logged time-histories for previous ship motions.
3.2.1 Period Measurement

Figure 33 shows the full range of roll, period, and FTLE data that was calculated for Damaged Stability Run 101:

![Graphs of Roll, Period, and FTLE vs. Time](image)

**Figure 33: Damaged Stability Run 101. From top to bottom: Roll vs. Time, Period vs. Time, FTLE vs. Time**

The four Lyapunov exponents measured for this system were 
\( \lambda_1 = 0.4882, \lambda_2 = -0.08540, \lambda_3 = -0.6922, \lambda_4 = -2.854 \), qualitatively identifying the system with one expanding axis and two contracting axes in the ball of initial condition points; the second exponent is the slowly changing principal axis, and would likely trend to a zero value in an infinite time series. In general a positive exponent reflects a chaotical system, a zero exponent identifies a stable orbit, and a negative exponent characterizes a periodic orbit; however, the existence of any positive exponent identifies it as a chaotic, rather than stable or periodic.

In this analysis, the measured period value is the \( \Delta t \) between neighboring FTLE maxima, calculated with a backwards approximation; this delta value is indicated by the red arrow in Figure 34:
Figure 34: Damaged Stability Run 101. Closeup of FTLE values and period measurement

Though it appears from the scale of Figure 33 that there is a period measurement at every time step, period measurements only occur at each maxima; the greater the spacing between maxima point, the greater the period. The period measurement of the FTLE’s is closely linked to the drop-out points in the FTLE measurements; these drop-outs occur where the code can not find enough neighboring points to fill the $y'$ and $z'$ vectors, and the algorithm automatically applies an arbitrarily high value to the FTLE, as seen in Figure 35:
Figure 35 makes it apparent that the largest period measurements are directly tied to lack of neighbors, rather than any “stretching” of the FTLE values; though directly measuring the number of neighbors proved to be a better solution, the period measurements provide simple visual cues for extreme motion, and were adequate advance indicators for large roll amplitudes, as illustrated in Table 11. The maximum roll amplitudes for each time series were recorded, along with the time at that point and the time of the preceding period spike.
<table>
<thead>
<tr>
<th>Run ID</th>
<th>Max. Roll Amplitude (Degrees)</th>
<th>Time of Max. Roll (Seconds)</th>
<th>Time of Period Max. (Seconds)</th>
<th>Lead-Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 101</td>
<td>-10.368</td>
<td>575.44</td>
<td>542.97</td>
<td>32.47</td>
</tr>
<tr>
<td>Run 101</td>
<td>-15.983</td>
<td>1083.19</td>
<td>1022.67</td>
<td>60.52</td>
</tr>
<tr>
<td>Run 102</td>
<td>-13.98</td>
<td>1144.13</td>
<td>1112.50</td>
<td>31.63</td>
</tr>
<tr>
<td>Run 366</td>
<td>-11.613</td>
<td>1050.93</td>
<td>1055.14</td>
<td>-4.21</td>
</tr>
<tr>
<td>Run 398</td>
<td>-18.296</td>
<td>715.87</td>
<td>685.51</td>
<td>30.36</td>
</tr>
<tr>
<td>Run 399</td>
<td>-16.49</td>
<td>1122.41</td>
<td>1108.92</td>
<td>13.49</td>
</tr>
<tr>
<td>Run 400*</td>
<td>-29.345</td>
<td>1993.9</td>
<td>1959.95</td>
<td>33.95</td>
</tr>
<tr>
<td>Run 400*</td>
<td>-29.345</td>
<td>1993.9</td>
<td>2001.70</td>
<td>-7.80</td>
</tr>
<tr>
<td>Run 401</td>
<td>-12.52</td>
<td>502.4</td>
<td>436.48</td>
<td>65.92</td>
</tr>
<tr>
<td>Run 402</td>
<td>-17.46</td>
<td>500.5</td>
<td>436.91</td>
<td>63.59</td>
</tr>
</tbody>
</table>

Table 11: Lead time for period correlation of maximum roll amplitudes

The predictive results show a great deal of variation; the average lead-time is 31.99 seconds, the standard deviation 26.26 seconds, and the variance 689.6, with the average being slightly skewed towards the larger values. However, there are some cases where the period spikes are not predictive at all; they are merely reacting to the large motions after they occur, as represented by the negative values in the table. Some of this inconclusively is due the variations in neighbor vectors for different roll series; the neighbor vectors can vary greatly based on the preceding ship motion. For example, if a series of data undergoes large amplitude motions twice during its duration, then the second motion will find more neighboring points to populate the vectors and only a very extreme roll or pitch motion will cause a loss in the number of neighboring points. In a shipboard application, the code could potentially have a vast number of data points to sort through to find neighboring points. With a large database of data at hand, only significant events would contain roll values where very few neighbors could be found, e.g. irregular large-amplitude ship motions. Another cause of these negative values is the time-delay in period calculation, where a loss of neighbors takes a number of time-steps before it is reflected in the period spike, as can be seen in the major period spikes of Figure 35.

Accurately determining which period spike is a flag for the large amplitude roll is the most significant challenge of the period measurement techniques. On a typical run, each large-amplitude roll motion can create multiple large spikes in FTLE period. Run 402 is a good example of the difficulty inherent in using the period-measurement method as a predictor for the most extreme motions.
Figure 36: Roll vs. Time and FTLE period for Capsize Run 402

The figure above is qualitatively similar to the period-measurement results for most of the analyzed data sets; period spikes were seen early in all runs, as a result of the initial lack of data from which to pull neighboring point. Examination of the roll time series makes it apparent that many of the period spikes are either reacting to or slightly predicting large local variations in roll. While these local variations are important, this study is most concerned with predicting the extreme variations, and therefore the data in Table 11 was compiled with the largest roll value in mind; for the case of Run 402, the largest amplitude occurs 500 seconds into the run, and the first major preceding period spike at 436 seconds, as seen below:
Figure 37: Marked period indicator for largest amplitude motion, Run 402

It is apparent that some of the period spikes are reacting to the ship motions, but difficult to ascertain their predictive nature. The spike at 243 seconds could be an indicator for the large-amplitude motions to come, or it could be reacting to the quick roll oscillation at that point in the time series. The period markers proved to be inconclusive predictors compared to similar predictions made by calculating the number of neighbors, given the time-delay inherent in a reverse-approximation method. Ultimately, neighborhood measurements prove to be a superior predictive method.
3.2.2 Neighbor Measurement

While the period measurements do an adequate job of predicting the extreme roll motions, Figure 35 shows that the large period spikes are reacting to the loss of FTLE neighbors, which is in turn reacting to the upcoming large motion amplitudes. This observation led to a modification of the algorithm that solves only for neighboring data points, rather than the actual FTLE values themselves. The threshold for the number of neighbors was set at 50; if more than 50 neighboring points are found to fill the $y'$ and $z'$ vectors, then the code continues to iterate. Below this value the neighboring points are counted and graphed in relation to the roll motions. The following figure presents a typical run with the counting of neighbor values:

![Figure 38: Damaged Stability Run 101 Roll vs. Number of Neighbors.](image-url)
Figure 39: Damaged Stability Run 101 zoom of neighbor counting

Figure 39 gives a better illustration of what is occurring as the neighbors are being counted. The blue data line is the roll amplitude, the green data line the number of neighbors. The drop in neighbor count contributes directly to the spikes in measured period values; in the previous algorithm a complete loss of neighbors (number of neighbors decreases to 0) causes the FTLE values drop out to a set value of -500, and the period amplitude increases due to these large gaps between FTLE maxima. Figures 38 and 39 show why the neighbor count is ultimately more useful than the period measurements. There does not need to be a complete loss of neighbors for warning flags to go up regarding lack of neighboring points. In the case above, any value that is falling below a neighbor count of 50 can be seen as a warning flag with respect to large amplitude motions. Whereas the time of max period for this run was flagged at 1022.67 seconds, the drop of neighborhood count below 50 neighbors occurs at 1016.56 seconds. While this is not a huge increase in lead-time, 9 seconds can be a significant amount of time in regards to split-second decision-making by a captain or crew, and any increase in warning time will be to their advantage.

As seen in Figure 38, loss of neighbors occurs erratically across the entire time series. Determining which neighborhood loss to mark as the indicator for a particular maximum amplitude is somewhat of a qualitative decision; the algorithm must take periods of stable behavior where there are no drop-outs into account. In an attempt to quantify this
neighborhood loss and find a computational solution that wouldn’t require a visual inspection of the data, a summation was used to flag a “danger” marker. For every step in time where the number of neighbors fell below 50, the variable “flag” was increased by 1; therefore, the more steps in time that were progressing with a lack of neighbors, the steeper the slope of the flag variable, as seen in Figure 40:

![Figure 40: Damaged Case Run 101 Roll vs. Flag](image)

The flag variable, increasing in value across the entire time series, experiences drastic increases in slope where there is a lack of neighboring points, as a response to increasing ship motion amplitudes. When the slope reaches a certain steepness, as seen in Figure 38 just past the 1000 second mark, a “danger” marker is flagged as a sign of increased amplitude motion. The danger markers provide a more concrete loss-of-neighbor indicator, and can be seen with regards to roll motion in Figure 41:
Figure 41: Damaged Case Run 101 Roll vs. Danger indicator

Table 12 replicates the results of Table 11, using the danger indicator rather than the FTLE period as the metric for lead-time:
<table>
<thead>
<tr>
<th>Run ID</th>
<th>Max. Roll Amplitude (Normalized)</th>
<th>Time of Max. Roll (Seconds)</th>
<th>Time of Danger Flag(Seconds)</th>
<th>Lead-Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 101</td>
<td>-1.7587</td>
<td>575.44</td>
<td>539.80</td>
<td>35.64</td>
</tr>
<tr>
<td>Run 101</td>
<td>-15.983</td>
<td>1083.19</td>
<td>1016.56</td>
<td>66.63</td>
</tr>
<tr>
<td>Run 102</td>
<td>-13.98</td>
<td>1144.13</td>
<td>1110.39</td>
<td>33.74</td>
</tr>
<tr>
<td>Run 366</td>
<td>-11.613</td>
<td>1050.93</td>
<td>1039.54</td>
<td>11.39</td>
</tr>
<tr>
<td>Run 398</td>
<td>-18.296</td>
<td>715.87</td>
<td>671.82</td>
<td>44.05</td>
</tr>
<tr>
<td>Run 399</td>
<td>-11.085</td>
<td>330.40</td>
<td>321.78</td>
<td>8.62</td>
</tr>
<tr>
<td>Run 399</td>
<td>-16.49</td>
<td>1122.41</td>
<td>1103.50</td>
<td>18.91</td>
</tr>
<tr>
<td>Run 400*</td>
<td>-29.345</td>
<td>1993.90</td>
<td>1899.01</td>
<td>94.89</td>
</tr>
<tr>
<td>Run 401</td>
<td>-12.52</td>
<td>502.40</td>
<td>340.12</td>
<td>101.99</td>
</tr>
<tr>
<td>Run 402</td>
<td>-16.46</td>
<td>453.60</td>
<td>431.42</td>
<td>36.52</td>
</tr>
</tbody>
</table>

*Capsize Case

Table 12: Lead time for neighbor correlation of maximum roll amplitudes

The average lead-time for the neighborhood-loss “danger-flag” method is 45.24 seconds, a 13.25 second improvement over the period measurement method. The standard deviation and variance both increase, to 32.70 seconds and 1069.0 respectively. However, more importantly, the “danger” spikes are a more concrete quantitative indicator than the period measurement method. Multiple losses of neighbors is still a hurdle; like the period spikes, in some runs it can be difficult to determine which “danger” spike is reacting to which large amplitude motion, though the clusters of spikes tend to signify a larger amplitude roll event. The flag summation approach removes much of the ambiguity and subjectivity of the period and simple neighbor counting methods, but there are still cases where multiple “danger” spikes occur before a large amplitude event, and which one to designate as the true warning spike requires a decision on part of researcher. For future shipboard applications, the algorithm would need to determine when to signal a warning without any human input.
3.3 Application to Notional Hullform 5514 Capsize Cases

The period and neighborhood methods were applied to the 5514 hullform data in a similar manner to the damaged stability case. Unlike the damaged stability data, all of the 5514 runs that were analyzed were capsize runs. In order to provide a full-set of data for the neighbor-finding process, all 37 different runs were analyzed for neighbor points, rather than attempting to draw neighbors from the limited set of data contained in one capsize run. This technique provided an excellent example of how this process could be used in real-world applications, where there would be many hours of ship rolling data to use for the neighbor searching process.

Previous approaches by McCue et al. explored the FTLE and Lyapunov exponent values exclusively, and used roll/roll-velocity and pitch/pitch-velocity as their state-space variables (McCue et al. 2006). This research furthers their work, with changes in neighbor counting methods and application of new warning algorithms.

As with the damaged stability cases, neighborhood measurements were better predictors for capsize than period indicators. For the capsize runs the neighborhood size dropped precipitously near the beginning of the run; for this study a threshold of fifty neighboring points was used. Figure 42 illustrates a typical loss of neighbors for a Hullform 5514 run, where the neighborhood size falls below fifty as the instabilities approach:
Figure 42: Hullform 5514 Run 216 Roll vs. Number of Neighbors
The contrast between these two figures reinforces the strength of the neighborhood counting code versus the period indicators. For the case of Run 216, the drop in number of neighbors precedes the lead period spike by 5.5 seconds, a lead-time advantage that carried through all of the 5514 capsize runs.

For run 216, the capsize event occurs at 11.88 seconds, the first period spike at 9.75 seconds, and the first drop of neighborhood size at 2.17 seconds. The period spike results in a lead-time of 2.13 seconds, and the neighborhood loss a lead of 9.71 seconds. While at first glance a 2 to 9 second warning appears to be a trivial amount of time, it is worth noting that both the first period spike and loss of neighbors occurs in the realm of stability for the ship, as seen in Figures 44 and 45:
Figure 44: Hullform 5514 Run 216 Roll vs. Roll Velocity basin of stability for period indicators
The period measurement method for run 216 has three markers, each of which represents a significant period spike in the time series; note that the last two markers in Figure 43 are not good indicators, given that the ship has already capsized based on the roll time-history. Each basin of stability for the neighbor counting method has two highlighted markers: the most optimistic indicator, and a conservative alternative. In the case of the neighborhood counting method, both markers sit well inside the basin. For the period indicators, the first period spike sits within the basin, but the second two occur after capsize, and well outside the basin of stability. Given these results, the proceeding discussion will only involve the neighborhood counting method. While the period indicators are an interesting study, neighborhood counting consistently provides a longer lead-time indicator for capsize cases. Figures 46 and 47 detail the loss in number of neighbors for run 327, another 5514 capsize case:
Figure 46: Hullform 5514 Run 327 Neighborhood loss
Again both the optimistic and conservative marker points lie well within the basin of stability for the capsize run. The lead time for these points were 4.91 and 4.12 seconds, respectively, and are a number of iterations from the point where the ship deviates from stable behavior. Figure 46 is an excellent example of the conservative and optimistic marker points; the first loss of neighbors recovers quickly, but the second consistently falls below the twenty neighbor threshold. Figure 48 shows the capsize case roll vs. roll velocity data for the other three capsize runs:
Figure 48: Hullform 5514 Runs 220,331,333 Roll vs. Roll Velocity Neighborhood loss markers
Each of the other cases show neighbor losses occurring well within the realm of stability for the capsize case, often many cycles before the ship falls outside of that stability basin. For the case of Run 333 it appears that the marker sits outside of the basin of stability, but much closer examination reveals it to be an anomaly that returns to the basin for numerous cycles. For these cases the actual lead time given by the loss of neighbors is less than what was seen for the damaged stability, but is still of consequence:

<table>
<thead>
<tr>
<th>Run ID</th>
<th>Time of Capsize (s)</th>
<th>Time of Neighbor Loss (s)</th>
<th>Lead Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 216 (Marker 1)</td>
<td>11.88</td>
<td>2.17</td>
<td>9.71</td>
</tr>
<tr>
<td>Run 216 (Marker 2)</td>
<td>11.88</td>
<td>4.17</td>
<td>7.71</td>
</tr>
<tr>
<td>Run 220 (Marker 1)</td>
<td>35.63</td>
<td>2.04</td>
<td>33.59</td>
</tr>
<tr>
<td>Run 220 (Marker 2)</td>
<td>35.63</td>
<td>5.79</td>
<td>29.84</td>
</tr>
<tr>
<td>Run 327 (Marker 1)</td>
<td>9.29</td>
<td>4.38</td>
<td>4.91</td>
</tr>
<tr>
<td>Run 327 (Marker 2)</td>
<td>9.29</td>
<td>5.17</td>
<td>4.12</td>
</tr>
<tr>
<td>Run 329 (Marker 1)</td>
<td>57.21</td>
<td>10.58</td>
<td>46.63</td>
</tr>
<tr>
<td>Run 329 (Marker 2)</td>
<td>57.21</td>
<td>30.08</td>
<td>27.13</td>
</tr>
<tr>
<td>Run 331 (Marker 1)</td>
<td>32.71</td>
<td>5.88</td>
<td>26.83</td>
</tr>
<tr>
<td>Run 331 (Marker 2)</td>
<td>32.71</td>
<td>23.08</td>
<td>9.63</td>
</tr>
<tr>
<td>Run 333 (Marker 1)</td>
<td>53.46</td>
<td>7.42</td>
<td>46.04</td>
</tr>
<tr>
<td>Run 333 (Marker 2)</td>
<td>53.46</td>
<td>35.13</td>
<td>18.33</td>
</tr>
</tbody>
</table>

Table 13: Lead time for neighborhood loss correlation of Hullform 5514 capsize cases

Using the optimistic “marker 1” cases, the average lead-time to capsize is 27.95 seconds; the more conservative “marker 2” scenario shows a 12-second drop, with 16.13 seconds lead-time. Though not as good as the damaged stability results, it should be noted that the Hullform 5514 time series were much shorter, with a smaller pool of time history to draw neighbors from. While these times may not seem like a large enough time for any ship captain to react, 18-24 seconds enough time to make one maneuver, or a course correction that might mean the difference between a large amplitude event and a capsize event.

The “flag” summation method was also tested for the DDG51 data; in this case each run was analyzed for the maximum flag value obtained, and normalized with the time length for the run. The average result for this normalized value on a capsize run was 12.12 flags/second, whereas the average normalized value for a non-capsize run was 5.68 flags/second. This makes it apparent that the capsize runs are finding significantly less neighbors than the non-capsize runs, which translates to more danger flags going up in the algorithm. The
normalized capsize values ranged from 5.22 flag/s to 21.27 flag/s; the higher the value, the more often the number of neighbors is falling below the 50-neighbor threshold. The non-capsize cases had a number of anomalous runs, with large normalized values- many of these runs came extremely close to capsize, but regained stability at the last instant. The values for the non-capsize runs typically ranged from 0.42 flag/s to 5.04 flag/s, with most of the values lying in the 0.5-1.0 range. The anomalous runs ranged in value from 6.23 flag/s to 23.40 flag/s, which was large enough to mark it as a capsize run. When analyzing the non-capsize runs, 7 of the 31 cases were anomalous, and flagged mistakenly as a capsize run.
4. Application of Neighbor Searching Method to Real-time Ship Motions

4.1 Motivation

The next step in application of the predictive neighbor method was to apply the algorithm in a real-time setting, with a data acquisition system in place to measure roll and pitch, with the neighbors being counted at each instant in time. The objective was to record the roll and pitch values for the damaged ship data in real-time, and replicate the neighbor counting method as seen in Figures 38 and 39. Motivation for this is to eventually implement a similar system on naval or fishing vessels, with neighbors being counted in an attempt to predict large-amplitude motions at sea.

4.2 Experimental Setup

4.2.1 Data-collection

The data from the damaged ship case was replicated on the motion platform (MOOG 6DOF2000E) located in the Virginia Tech CAVE (Automatic Virtual Environment). The MOOG is a 6 D.O.F. hydraulic motion platform, with freedom of 20 degrees in both roll and pitch. The platform can be seen in Figure 49:

![Figure 49: MOOG motion platform](image)

The MOOG is controlled by an SGI/IRIX system, with a position vector written to a DTK shared memory segment. The pitch and roll values for the damaged ship were fed to the
platform, and were then read to a Dell Latitude D610 laptop via a Crossbow CXTILT02EC tilt sensor, as seen in Figure 50:

![Crossbow tilt sensor mounted on motion platform](image.jpg)

**Figure 50: Crossbow tilt sensor mounted on motion platform**

The tilt sensor is accurate to within 0.2 degrees, with digital output via a RS-232 serial interface. The data from the serial port was fed directly into the neighbor counting algorithm in Matlab, which recorded the roll and pitch values in addition to the number of neighbors.

4.2.2 Algorithm/Data modification

The roll and pitch velocities were calculated using a backwards approximation from the roll and pitch values, similarly to the neighbor counting methods used in section 3. Unlike the measurements of that section, there was no normalizing of roll and pitch values; the normalizations for that section were performed using mean and standard deviations, and the objective for this section were to obtain neighbor measurements in real-time without any sort of standardizing. Future work could include some sort of method to normalize values in real-time based on previously collected data, but the results of this study were satisfactory without it. Additionally, further work could be done in the following areas:

- Normalize values in real-time. A mean and standard deviation would have to be calculated for each ship, based on large sets of previously collected data; these would be used to perform a standard normalization of the data.
• Optimize the algorithm for long data sets, knowing that it will need to perform calculations for days, weeks, or even months at sea. Not only will data buffers need to be routinely cleared, but the data needs to be databased for the neighbor searches.

• Databases would need to be searched for neighbors efficiently enough to react in real-time, a significant task when dealing with tens of thousands of data points.

• A hardware system with simple warnings, a “black-box” so to speak, would need to be developed for ship operators. Captains could not be expected to read complicated outputs in a critical situation- the warning system would need to be simple but effective.

The algorithm used in section 3 could not be implemented directly for real-time neighbor calculations. With the data being collected on the order of one value per 0.03 seconds, a number of efficiency and storage problems arise. With the old algorithm the computer had to search through the entire history of data values at each instant in time to find neighbors; while there is no issue with this when operating on a prescribed set of data, a laptop like one that would eventually be used on a shipboard application is not computationally quick enough to perform a search of all previous points on the time scale described above.

Instead, the algorithm was modified to only search the time-history for neighbors when a new area of phase space was entered. The new algorithm only counts neighbors for a roll value that has not been encountered before; if the roll value has been logged in the history, it defaults to the neighbor value previously recorded. This reduced an 80,000+ step time series to only 800-1000 actual neighbor searches, greatly increasing the efficiency of the algorithm.

However, even with the algorithm running more efficiently than before, memory limitations became an issue. With multiple matrices over 100,000 points in size, the algorithm began to fail after about 80,000 steps into the time series. Therefore, the results seen in the following section will be missing the very end of each time series. Luckily, all of the large-amplitude motions in every run occur earlier than this cutoff point, so the data can be compared directly.
4.3 Real-time Neighbor Counting Results

A side-effect of the efficient algorithm was that only significant neighbor losses were recorded. For example, compare Figure 38 in section 3 to the following figure:

![Figure 51: Damaged Stability Run 101, data recorded from MOOG platform](image)

As seen in Figure 51, the modified algorithm is much more efficient at recording losses of neighbors than before, while still capturing the major neighbor losses that occur at significant motion events in the time-series. These neighbor dropouts function the same way as section 3.2.2, just in real-time; though at first glance it appears that the dropouts are purely reactive, further examination of time series makes it apparent that they are identifying significant changes in ship behavior, not just large amplitude motions. The two beginning neighbor losses signify a new regions of unstable behavior for the ship, and the numbers reflect these future motions.
Knowing that the data qualitatively satisfactory to the results of section 3, the next step was to compare lead-times to that of Tables 11 and 12. It should be noted that the warning-flag method was attempted, but abandoned when numerous runs produced garbage data. This is a result of the computer attempting to iterate nested for-loops every 0.03 seconds, at which point it failed to even record the correct roll and pitch values. Therefore, the lead-time represents every major loss of neighbors, where a warning flag would have certainly occurred in the previous algorithm; a conservative estimate was used in every case, and even at that the real-time method produced some startling results, as seen in Table 14:

<table>
<thead>
<tr>
<th>Run ID</th>
<th>Max. Roll Amplitude (Degrees)</th>
<th>Time of Max. Roll (Seconds)</th>
<th>Time of Neighbor Dropout (Seconds)</th>
<th>Lead-Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 101</td>
<td>-10.368</td>
<td>575.44</td>
<td>514.11</td>
<td>61.33</td>
</tr>
<tr>
<td>Run 101</td>
<td>-15.983</td>
<td>1083.19</td>
<td>1009.79</td>
<td>73.40</td>
</tr>
<tr>
<td>Run 102</td>
<td>-13.98</td>
<td>1144.13</td>
<td>1102.91</td>
<td>41.22</td>
</tr>
<tr>
<td>Run 366</td>
<td>-11.613</td>
<td>1050.93</td>
<td>975.48</td>
<td>75.45</td>
</tr>
<tr>
<td>Run 398</td>
<td>-18.296</td>
<td>715.87</td>
<td>642.36</td>
<td>73.51</td>
</tr>
<tr>
<td>Run 399</td>
<td>-16.49</td>
<td>1122.41</td>
<td>1008.83</td>
<td>113.58</td>
</tr>
<tr>
<td>Run 400*</td>
<td>-29.345</td>
<td>1993.90</td>
<td>1924.80</td>
<td>69.10</td>
</tr>
<tr>
<td>Run 401</td>
<td>-12.52</td>
<td>502.40</td>
<td>412.11</td>
<td>90.29</td>
</tr>
<tr>
<td>Run 402</td>
<td>-17.46</td>
<td>500.50</td>
<td>435.20</td>
<td>65.30</td>
</tr>
</tbody>
</table>

Table 14: Lead time for real-time neighborhood loss correlation of Damaged Stability cases

For the real-time cases the average lead-time is 73.69 seconds, with a standard deviation of 19.90 seconds and a variance of 396.1; the latter two values are lower for this case than in either of the other previous methods outlined for Tables 11 and 12. Though this result was initially surprising, the changes made to the algorithm and method in which the data is collected point to the significant improvements in lead-time values. The data is being collected at a rate much higher than the original time-series, and thus the algorithm is collecting neighbors at a much higher rate. This, combined with the changes outlined earlier,
allow the algorithm to react more quickly to the ship entering a previously unseen area of phase space. It is a promising set of results for future applications.
5. Conclusions

5.1 Verification and Validation

Chaotic attractors can be extremely sensitive to inputs by nature. The Lorenz and Rossler systems, both three-dimensional chaotic attractors, can undergo very large changes in parameters without losing their standing as a strange attractor. For certain combinations of parameter values both systems have the potential to shift from attractors to fixed points, or to show varying levels of periodicity.

These factors become important in validating code for both numerical and experimental research. While bifurcation analysis is a useful tool for determining regions of chaotic behavior from a numerical approach, it is limited in its application to experimental time series. The Lyapunov exponent proves to be a very robust tool in this regard; multiple methods of calculating the exponent exist, both for numerical and experimental data. This research has shown that the Lyapunov approach accurately captures changes in phase space for chaotic behavior, and can do so with similar accuracy to other methods like bifurcation analysis. By comparing exponent spectrums, the researcher can effectively validate the underlying physics of the developed theoretical model, making the Lyapunov approach a very important piece of the verification and validation framework.

5.2 Application to Ship Capsize

The data presented shows that the Lyapunov/neighbor counting method proves to be a valid way to predict capsize and large amplitude motions for a given time series of experimental data. The damaged ship data shows that the algorithms prove useful for large-motion analysis, but it appears that the method is much more useful for capsize cases like the ones presented by the 5514 data. While the lead-times given by the methods may not be on a scale of minutes, but rather seconds, it may often be the case that if a ship captain knows a capsize event is about to occur, a single drastic course correction or maneuver could be undertaken.

5.3 Future Work

While the application of Lyapunov Exponents to ship capsize in a numerical and controlled experimental environment is a good start, there is still much work to be done in order to provide a useful and reliable tool in the field to assist ship captains in extreme sea states. Realistically the Lyapunov method is but one approach being taken in regards to predicting capsize or large-amplitude motions, and can be viewed as another tool in the nonlinear dynamic analysis toolbox. The next step for this research is to implement a system on a test vessel at sea, and being to acquire sets of numerical data from which to draw neighbors. This presents a few numerical problems, including data storage and algorithm efficiency. The task of predicting nonlinear ship dynamics is a complicated one, but hopefully work such as this in academic and commercial institutions around the world will eventually lead to safer environments for both passengers and crew at sea.
Appendix A

1. Figures

Figure 52: Rossler hyperchaotic attractor for $a = 0.25, b = 3.0, c = 0.05, d = 0.5$
Figure 53: Lorenz Lyapunov spectrum for small-scale changes in R

Figure 54: Lorenz Lyapunov spectrum for small-scale changes in b
Figure 55: Rossler Hyperchaos Lyapunov spectrum changes for small-scale changes in $b$ for Wolf system
Figure 56: Rossler Hyperchaos Lyapunov spectrum changes for small-scale changes in $c$ for Wolf system
Figure 57: Rossler Hyperchaos Lyapunov spectrum changes for small-scale changes in $d$ for Wolf system
2. Tables

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.03228</td>
<td>0.000394</td>
<td>0.2804</td>
</tr>
<tr>
<td>Classic</td>
<td>0.04187</td>
<td>0.001866</td>
<td>0.04219</td>
</tr>
</tbody>
</table>

Table 14: Standard deviations for Lorenz Lyapunov spectrum for small-scale changes in $\sigma$

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.01722</td>
<td>0.0009249</td>
<td>0.1817</td>
</tr>
<tr>
<td>Classic</td>
<td>0.03228</td>
<td>0.0093941</td>
<td>0.2804</td>
</tr>
</tbody>
</table>

Table 15: Standard deviations for Lorenz Lyapunov spectrum for small-scale changes in $b$

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.003200</td>
<td>0.0005432</td>
<td>0.005284</td>
</tr>
</tbody>
</table>

Table 16: Standard deviations for Rossler Lyapunov spectrum for small-scale changes in $b$

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
<th>Exponent 4 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.004902</td>
<td>0.002564</td>
<td>0.000474</td>
<td>1.380</td>
</tr>
</tbody>
</table>

Table 17: Standard deviations for Rossler Hyperchaos Lyapunov spectrum for small-scale changes in $b$

<table>
<thead>
<tr>
<th>Variation</th>
<th>Exponent 1 $\sigma$</th>
<th>Exponent 2 $\sigma$</th>
<th>Exponent 3 $\sigma$</th>
<th>Exponent 4 $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>0.005657</td>
<td>0.003972</td>
<td>0.0007657</td>
<td>3.77</td>
</tr>
</tbody>
</table>

Table 18: Standard deviation for Rossler Hyperchaos Lyapunov spectrum for small-scale changes in $d$
Appendix B: Choosing D.O.F. Parameters for best Neighbors/FTLE results

The author conducted a parameter search into the influence of chosen DOF for calculating the FTLE values. The most pronounced ship movement was in roll, and subsequently it proved to be the most robust degree-of-freedom for generating FTLE values; further runs determined that including roll-velocity vs. roll provided even better predictive results than measuring the values based on roll alone. The two figures below show the FTLE measurements for the same data run, the first only generating for the roll degree-of-freedom by itself, and the second for roll vs. roll velocity.

Figure 58: Damaged Stability Run 101 Roll vs Time and non-dimensionalized FTLE period measurement
Figure 59: Damaged Stability Run 101 Roll/Roll Velocity vs. Time and non-dimensionalized FTLE period

The graphs above both show the same roll time series, but with different embedded parameters. The top set of data in blue is the roll amplitude, while the bottom set of data in green is the period measurement of the FTLE points that were computed for the time series; each series is represented by its own y-axis. Figure 12 shows the period measurements for Roll vs. Time, without roll velocity being embedded in the solution for the FTLE’s, while Figure 13 shows the same period measurement when roll velocity is embedded. Figure 13 shows a much better correlation between large roll motions and marked increases in period measurement in the FTLE values as compared to the erraticism of the data in the one state-space variable case seen in Figure 12.

FTLE and period values were also calculated for pitch and pitch velocity. Figure 14 shows the period calculations for a pure pitch case:
The figure shows the erratic nature of the pitch measurements for the data; the pitch motions show none of the extreme motions of the roll data, and thus is not as robust for predicting large amplitude motions. The calculation of period values for pitch-pitch velocity was very similar in regards to period measurement:
The pitch/pitch velocity case seen in Figure 15 is just an inconclusive an indicator as the single variable pitch case. Other combinations of these degrees of freedom were explored, including integrating the four DOF case of roll, roll-velocity, pitch, and pitch-velocity, as seen in Figure 16:
Figure 62: Damaged Stability Run 101 Pitch/Pitch Velocity & Roll/Roll Velocity vs. Time and non-dimensionalized FTLE period

Other degrees of freedom were considered, but after scrutiny it appeared that the 2 D.O.F. case of pitch and roll, extended to four state-space variables with the roll/pitch velocity calculations, was more than sufficient to capture the major changes in phase-space.
References

1. Chapter 1


2. Chapter 2


