Identification of Thermoacoustic Dynamics
Exhibiting Limit Cycle Behavior

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ABSTRACT

Identification of thermoacoustic dynamics that exhibit limit cycle behavior is needed to gain a better intuitive feel of the system, to design complex control strategies, and to validate modeling efforts. Limit cycle oscillations arise in thermoacoustic systems due to the coupling between a nonlinear heat release process and the acoustic dynamics of the combustor. This response arises in lean premixed gaseous power generating turbines and is a concern due to the detrimental effect of the pressure oscillations on the structural integrity of the combustor. Due to the volatile environment intrinsic in the combustor, multiple sensing apparatuses are not available. Therefore, in the current study, an identification approach is assessed considering only a single output from the thermoacoustic system.

As a means to further investigate the thermoacoustic limit cycle behavior, a scaled version of the industry-based turbine was constructed. By anchoring a flame halfway from end-to-end of a closed-open tube, a similar nonlinear response is achieved. A harmonic balance technique that linearly incorporates the nonlinearity is developed which uses frequency entrainment to offer sufficient information for the identification. Its validity is assessed on a model, which is based on known dynamics of the thermoacoustic system. The structure of the identification algorithm is based on a two-mode acoustic model with both dynamics and nonlinearity in the feedback loop. The limitations of using only a two-mode identification structure for a system with more than two modes is discussed as well as future efforts that may alleviate this problem.
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1 Introduction

1.1 Motivation

1.1.1 Systems Study

As a controls engineer, it is not only important to be aware of a system's behavior at the present moment, but it is also important to be intimate with the system perturbed from the nominal operating condition. In a linear system, small perturbations of the nominal operating condition will degrade the control system's performance minimally as small parameter perturbations of a linear system will cause only small changes in its response. On the other hand, small perturbations in the parameters of a nonlinear system may cause large changes in its response. Therefore, the control strategy as designed may not be sufficient to obtain even satisfactory results.

Efforts to identify the system and create a model help to alleviate these problems. Modeling an unknown or partially known system familiarizes the controls engineer with the physics of the process to be controlled. As such, the controlled response of a perturbed system, as well as that of a possible future design, will be more predictable.

Models derived from linearizing a nonlinear system about an operating point are beneficial but nonlinear models are essential for thorough analysis of nonlinear systems. Most often a linear model is sought for its simplicity in analysis and simulation. Fortunately, a linear model is sufficient to describe stability characteristics of both linear and nonlinear systems. This is the case for anything from a simple open-loop dynamic process to a closed-loop self-excited nonlinear system. Unfortunately, key attributes of a nonlinear system's response will be lost if a linearized model is chosen for its description.

For self-excited nonlinear systems that exhibit a stable limit cycle, its amplitude and harmonic signature will be lost in this linearization. If the linearized model predicts stable operation (no limit cycle), this is not a concern. On the other hand, if instability (leading to a limit cycle) is predicted, without a nonlinear model, the amplitude and possibly the precise frequency of the nonlinear oscillations will be undeterminable (with the exception of the quasi-linearization method of describing function analysis). Therefore, it is impossible to predict whether the self-excited response is acceptable. In the same manner, the linear model will not predict harmonic energy inherent in nonlinear
systems. Clearly a model that fails to predict these energies is not sufficient to thoroughly analyze or predict the performance of the controlled system.

Another systems-oriented motivation of this study is the further development of an occasionally used identification technique. The harmonic balance technique has been widely used in the past for analysis of systems with a periodic response. In the same manner, linear parameterization of nonlinearities is a common approach to their simplification. However, using the two in conjunction for purposes of identifying an actual self-excited system through multiple experiments is a topic that has little research associated with it. This study is therefore partially motivated by the desire to further develop a system identification tool for self-excited systems addressing the concerns associated with both complex known systems and issues that arise when it is used to identify unknown systems from data.

1.1.2 Combustion Study

As with most research endeavors, the desire to analyze and identify a dynamic system is motivated by a real world problem. Thermoacoustic instabilities occur in lean premixed gas turbines under many operating conditions. In general, these nonlinear oscillations arise when cleaner and more efficient operation of the turbine is attempted. The self-excited oscillations are exhibited by high amplitude pressure fluctuations in the combustor that are clearly detrimental to its structural integrity. Past efforts have shown that by passively altering the acoustic characteristic of the combustion chamber, these oscillations are successfully reduced. However, this control strategy will only work for the operating condition in which it is designed. As such, active control studies are being investigated to obtain control at many operating conditions. As a means to this end, a thorough understanding is needed to design stable control strategies with maximum performance for the nonlinear oscillations driven by the thermoacoustic process.

Physics of Thermoacoustic Instabilities

Though a detailed model is not currently available for the description of the thermoacoustic dynamics that provoke the limit cycle behavior, there are established physical criteria for this response. In a time-honored paper [Rayleigh 1878], Lord Rayleigh described that with sufficient heat addition in the correct location within an acoustic environment, self-excited pressure oscillations will occur. That is, whenever
heat addition is significant and in-phase with the acoustic signature, pressure fluctuations increase in an unstable manner until limited by some nonlinear characteristic. Intuitively, this can be explained by simple principles.

A flame accelerates acoustic particles (gases) in its environment. Holding a match in proper lighting and observing a smoke trace speed up as it moves toward the flame illustrates this phenomenon. Clearly the acoustic environment downstream of the flame affects this by setting up a slug of fluid that resist the flames effect on the acoustic particles. The dynamics of this slug (also known as backpressure) are similar to that of a mass-spring-damper system. As such, the acoustics of the environment not only dissipate energy, but also supply a restoring stiffness to the flame. When the flame is lit, the gases begin to accelerate until the dynamics of the acoustic environment (backpressure) overcome this source of energy. If tuned properly, this acoustically coupled thermodynamic system will begin to oscillate. If the flame were linear with an infinite supply of energy, the oscillation amplitude would increase infinitely. However, in most cases, the energy available (dictated by the fuel-to-air mixture) is finite, giving the flame a nonlinear limiting characteristic that is sufficient to support the nonlinear self-excited response often seen in lean premixed gas combustors.

This interaction between acoustics and the thermodynamics of a heat source can be considered as a closed-loop system. A block diagram of this representation is presented below (Figure 1.1).

![Figure 1.1 Block Diagram of the Thermoacoustic Coupling](image)

It should be noted that it is still a debated topic whether or not this is the dominant mechanism of instability in these systems. Sources of time delay have been identified which clearly contribute to instability in any closed-loop dynamic system. This study,
and others like it, will further the understanding in the combustion control community of the most pertinent instability mechanism for the system discussed below.

Experimental Facility

The Virginia Active Combustion Control Group (VACCG) is involved in the research of thermoacoustic instabilities including modeling, analysis, and control through both simulation and experiment. As a means to this end, a combustor was built, and designed to endorse the Rayleigh criteria introduced above. The combustor is a closed-open cylindrical tube (otherwise known as a Rijke tube) with a burner situated halfway from end-to-end. This burner, due to its ability to store heat, anchors a flame within the tube. The tube combustor was thoroughly instrumented to enable an in-depth analysis of its dynamics. Below (Figure 1.2) is a photo of the experimental rig including the tube combustor with an associated schematic of the instrumentation.

![VACCG Tube Combustor and Instrumentation Schematic](image)

Figure 1.2 VACCG Tube Combustor and Instrumentation Schematic

As described, the tube combustor was designed and tuned to support the Rayleigh criteria and therefore exhibit the self-excited thermoacoustic instability. As Rayleigh suggested, the tube supports the instability only when the heat supplied is sufficient. As such, it is possible to operate the tube in both a stable and unstable regime. Below (Figure 1.3) illustrates the spectral density of pressure for these two regimes.
**Figure 1.3 Pressure Spectrum for Stable and Unstable Tube**

In the figure above, the lower level response (blue) is that of the stable combustion process. The multiple acoustic modes are noticeable even for stable operation due to their excitation from the flow of the fuel-air mixture through the tube. As the intensity of the heat source is increased (by richening the mixture) the amount of heat that the flame provides is sufficient to motivate the instability. At this point, the amplitude of oscillation increases and the nonlinear characteristics of the flame become more prominent. This is illustrated in the fundamental limit cycle response (@ 175 Hz) and associated integer frequency harmonics imposed on the generally higher-level (green), wideband response in the above pressure spectra.

In addition to the sensing illustrated in the schematic, the current multidisciplinary research effort includes the analysis of luminous emissions from the flame. These emissions are believed to represent the time dependent heat output of the flame. With this measurement, the identification of the forward and feedback contributions of the closed-loop thermoacoustic system (Figure 1.1) is trivialized. Unfortunately, this measurement is not always available in an industrial application. In addition to this, an understanding of how this signal is precisely related to the coupling between the heat release of the flame and the acoustic environment is not completely understood. Therefore, in an effort to avoid any problems associated with using this signal, it is assumed that the only available data for the identification experiment is the pressure of the tube combustor.
Active Control Endeavors

Thus far, controlling the nonlinear oscillations in the tube combustor has been successful. Since the instability mechanism is coupled to a single acoustic mode of the tube, by introducing destructive acoustic energy at this frequency, the amplitude of the oscillations at this frequency is reduced. Since the remainder of the high-level energy seen in the unstable response of Figure 1.3 is due to the harmonic response of the nonlinearity, it is reduced as well.

The control algorithm used to reduce the oscillations in the tube revolves around theories associated with active noise control. By adding phase-locked sinusoidal energy phase-shifted by 180 degrees (at the instability frequency), the sum of the actuated energy and the tube energy at the frequency of instability is theoretically reduced to zero. To obtain the phase-shifted energy through the acoustic actuator (simple loud speaker) a pressure signal is obtained from the tube and phase shifted using a digital signal processor. Since we are primarily focused on adding energy at a single frequency, the pressure signal is band-passed to attenuate information in the pressure signal at other frequencies. With this approach, we are able to control the nonlinear response of the thermoacoustic system to that of a noise-excited acoustic response (similar to the stable response of Figure 1.3). A block diagram of this control structure is presented in Figure 1.4.

![Figure 1.4 Structure of the Phase-Shifting Controller](image)

Though the current control algorithm is primitive, establishing the ability to control the nonlinear oscillations motivates further study of control approaches for thermoacoustic instabilities. Currently, the phase shifting algorithm is often manually tuned to achieve the best performance for a specific operating condition. Clearly, for
different operating conditions, the phase and gain needed to control the oscillations will vary. In fact, for some operating conditions, the amount of energy that the current actuator provides is known to be insufficient to completely control the nonlinear response of the tube. However, achieving control of a range of operating conditions illuminates the possibility of complex control algorithms that will manually tune the control signal depending on the current operating condition. In addition to this, having the controlled response of the system increases the available data for characterization of the physical processes involved in the combustor response. Certain characteristics of the combustor are attainable from the controlled system that may not be reachable from the unstable response at the same operating condition.

1.2 Introduction to Nonlinear Systems

At this point, we emphasize how the nonlinear nature of this problem affects its analysis and ultimately the identification of the thermoacoustic dynamics. A property of nonlinear systems is that the stability or the response in general may be a function of the class of input and/or its initial conditions. As such, for one input, the response may be stable while for a slightly different (or perturbed) input the response may be unstable. This characteristic alone illuminates the complexity of the response of a nonlinear system. In general, intuition cannot be used to predict the free or forced response of a nonlinear system.

Characteristics of Nonlinear Systems

There are a few characteristics that when observed, guarantee nonlinear dynamic behavior. Here we submit four examples:

1. **Multiple equilibrium states.** The unexcited, or autonomous response of stable linear systems always returns to one location in the domain of system states (position and velocity for example). This location is called the equilibrium point of the system. Nonlinear systems may have multiple equilibrium points. The location that the autonomous nonlinear system adopts in steady-state is often dependent on its input or initial conditions.

2. **Subharmonic or superharmonic response for periodic inputs.** Steady-state oscillations in linear systems are always of the same frequency as that of the
excitation. Nonlinear systems may respond at other frequencies in steady state. Energy may arise at fractional frequencies (subharmonic) or integer multiples (superharmonic) of the excitation frequency. Due to the importance of this characteristic, for the current research study, we introduce a mathematical reasoning for this.

It is well understood that nonlinear effects on sinusoids produce energy at both integer multiple harmonics and at subharmonic frequencies at fractions of the fundamental frequency of oscillation. This can easily be described by the polynomial type gain that many nonlinearities have on the oscillation. For example, a nonlinearity that is quadratic in nature, which is $y$-axis symmetrical (or an even function), will produce an even harmonic due to the trigonometric relation: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$. Similarly, a cubic nonlinearity (origin symmetrical, or an even function) will produce odd harmonics because of the relation: $\sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$. In the same manner, a candidate nonlinearity that produces subharmonic energy may be

$$\sin\left(\frac{x}{2}\right) = \left(\frac{1 - \cos(x)}{2}\right)^{\frac{1}{2}}.$$ 

Though it is clear that the nonlinearity that occurs because of some physical process will not be purely quadratic or cubic, its polynomial nature and the above relations will contribute to a multiple harmonic frequency response.

3. **Jump resonance.** For linear resonant systems, the response to sinusoidal excitation is continuous when altering either the magnitude or frequency of the input. That is, perturbations in the amplitude and frequency of the input signal cause only perturbations in the amplitude and frequency of the output signal. Some nonlinear systems exhibit a *jump phenomenon* characterized by drastic changes in the magnitude of the output signal for small perturbations of the input signal.

4. **Limit cycle.** In linear systems, steady-state oscillations result only from oscillatory inputs and their amplitude and frequency are uniquely dependent on the excitation of the input and initial conditions. For nonlinear systems, periodic
oscillations may arise from non-oscillatory inputs. Periodic steady-state oscillations in nonlinear systems are called limit cycles. Their amplitude and frequency may be independent of initial conditions on the dynamics.

**Principle of Superposition**

The fundamental concept that separates linear and nonlinear systems is the principle of superposition. This principle states that for an input $in-A(t)$ that produces an output $out-A(t)$ and an input $in-B(t)$ with corresponding output $out-B(t)$, then for any $x, y, in-A,$ and $in-B$ the response to $x(in-A(t)) + y(in-B(t))$ will be $x(out-A(t)) + y(out-B(t))$. This principle facilitates the prediction of the systems’ response to a previously inexperienced input. For nonlinear systems, this principle does not hold. Therefore, there are no means to intuitively predict the response to an input if the response to another input is known. This in turn negates any attempt for a unified set of analytical tools sufficient to study all nonlinear systems. However, analytical tools are organized by the system they will be used on as well as the class of inputs expected.

**Types of Nonlinearities**

The most common and easily analyzed nonlinear contributions to system dynamics are static characteristics dependent explicitly on one system state. A very common type of nonlinearity in physical systems is the saturation type nonlinearity. This type of nonlinearity gives a linear output up to a certain point where the gain is no longer available and the output saturates at the largest possible gain. This often occurs in electronic equipment and is heard in the harmonically distorted output of an amplifier that is turned up too loud. Nonlinearities of this type are single-valued functions of the input variable. The output of a hysteretic type nonlinearity from a relay depends on the previous history of the input. Due to this, the characteristic will have more than one value (double-valued nonlinearity, DVNL). These two types of nonlinearities are depicted below in Figure 1.5 while the reader is referred to other sources for an outline of many more nonlinear characteristics [Atherton 75].
The effect of the double-valued characteristic is dynamic in nature. With increasing frequency, the response of the DVNL derived from a relay-type nonlinearity deteriorates. Therefore, when the response of a DVNL is important in the frequency domain, it may accurately represented as a static characteristic in series with a dynamic system [Hsu 97].

Analysis Approach

Typical behavioral analysis of nonlinear systems begins with the understanding of the free response. The response to an initial condition is then characterized as converging to, or diverging from, one of possibly many equilibrium points. Another potential response is the convergence to a periodic motion or a limit cycle.

Upon establishing the characteristics of the free or autonomous response, the system is studied with external inputs. Due to the violation of the principle of superposition in nonlinear systems, an experimental or numerical analysis of their behavior must consider many magnitudes and classes of inputs. In addition to all of these complexities, if the free response of the nonlinear system is periodic (limit cycle behavior), a sinusoidal input may quench the free response or entrain it to one related to the input signal. Chapter 2 offers tools for analysis of these characteristics.

1.3 Introduction to the System Theory

System theory is a collection of tools established for accurate analysis of dynamic systems. This analysis includes anything from modeling to characterizing their response. The approach to analyzing the system typically begins with obtaining a system model. There are many different structures that have been established for accurate representation of dynamic systems. These structures lend themselves to linear, nonlinear, time-constant and time-variant systems in both the time and frequency domains. With a model representation that sufficiently describes the dynamics of the system, the stability and
performance of the system is analyzed. Specific tools are available offering the investigator sufficient information about the stability of a system without explicitly solving the dynamic equations in time. In a specific sense, the inherent response of oscillatory systems justifies a certain class of tools for their analysis. These tools include stability assessments of free and forced, open and closed-loop systems. Chapter 2 of this manuscript offers methods of stability assessment for both linear and nonlinear systems. In addition, oscillatory theory of both linear and nonlinear, open and closed-loop systems is presented.

1.4 Introduction to Modeling

Clearly, a controls engineer requires a system model for accurate control of a nonlinear system. One or a combination of many approaches may facilitate the establishment of this model. These approaches are most often dictated by previous understanding of the physical process, experimental facilities and the complexity or accuracy needed. For a highly detailed understanding of the physical processes involved, models are derived from physical equations. For a less-detailed model, the investigator can derive a mathematical input-output representation from how the system responds to an experimental input. These two approaches are outlined and contrasted below.

1.4.1 Bottom-up Approach Using Molecular Physics

The bottom-up approach takes key physical attributes of the system into account, and with an array of assumptions, a low-order model is achieved. By low-order model, we mean a differential equation representation that is easily simulated. This often results in a well-understood model, though obtaining the necessary physical information for this approach may be more intense than what the project scope allows. Determining the prominent physical contributions of the systems response is nontrivial and requires a preexisting familiarity of the physical system. Additionally, the assumptions needed to simplify these contributions are often not clearly defined. As such, bottom up modeling techniques are beneficial to the understanding of the system but are not always a trivial, well-outlined task.

Another drawback to the bottom-up approach to modeling is the gap between the apriori equations that define the process and the use of the model. Often when the
modeling process is started, a set of somewhat inflexible (and often complex) equations is established. With the set of equations and an array of assumptions, the modeling process may develop to the point where a control input to system output representation is still not achieved. Though the detail used for this approach greatly benefits the understanding of the system, it may not be the most direct approach to control-oriented modeling.

1.4.2 Top-down Modeling Through Experiment

An alternative method to the bottom-up modeling approach is a top-down, experimentally derived model. This approach is often called system identification. Compared to the detailed bottom-up approach, the amount of apriori information is minimized and an input-output control-oriented model is immediately available. However, intimacy with the fundamental physical processes is lost, as it is not needed to obtain a useful model.

Input-output Information

To identify stable linear systems, it is necessary to have an input to excite the system so that its output is perturbed from the origin. In this manner, pertinent modes of the system are excited and information from this response characterizes the system dynamics. Intuitively, this makes sense as if the purpose of the identification is control-oriented, the response of the system to an input (feedback for example) is essential for proper controller design. The same is true for nonlinear systems in a self-excited state. Though self-excited systems have a nonstationary output, their response to an input is needed for proper controller design.

Parametric or Nonparametric Approaches

Two approaches for experimentally derived modeling include parametric and nonparametric techniques. Nonparametric techniques fit the experimentally derived response to a form of response function. On the other hand, the parametric approach uses the assumption of a low-order model structure (differential equation order, states, etc.) and with data, the coefficients to this model are found. Mathematically, this is often done to minimize error between the model’s response and the system response. By assuming a system description from the system response with minimal consideration of the physical contributions, assumptions related to the truncation of complex dynamic equations
relating to physics are avoided. In this manner very little assumptions about physical characteristics come into play and errors in this formulation do not affect the model.

Unfortunately, with the top-down approach to identifying the system from data, the investigator is at the mercy of the available data. For a nonlinear self-excited system, this causes a problem. Due to the closed-loop structure of the self-excited system, contributions of all key physical components of the system may be alluding by observing only input-output data. Unless the scientist has access to all contributions of the closed-loop system, the model may be insufficient. In other words, a specific dynamic contribution of the system may be masked by its closed-loop nature. This is not a problem for the scientist who is identifying for the purposes of controlling one operating condition. However, when operating conditions shift, the changes of a block within the closed-loop system may not be predictable.

Problems with Identifying Self-excited Systems

In addition to masking internal components of the system, a closed-loop self-excited system is difficult to identify because of its free response. As mentioned, an appropriate approach to obtaining control dependent dynamics is to force the system in question with an input similar to that of which it will be experiencing when the control strategy is activated. For a stable linear system, this is trivial as the dynamics at the output will be dependent only on the input. On the other hand, for a self-excited nonlinear system, a preexisting response is already present. Therefore, the control input may not be coherently persistent while the system is operating in its limit cycle. In other words, the energy from the input may be disguised by the limit cycle energy since modifying the free response of a self-excited system is more complex than perturbing a stable linear system from the origin.

As highlighted above, the system identification approach to modeling lends itself very well to control-oriented applications. The VACCG researchers are also involved in companion studies for modeling from fundamental physical equations. Due to this, the current study focuses on the system identification approach to modeling the thermoacoustic instability. Below we outline a few more considerations that arise in system identification studies.
1.4.3 Introduction to System Identification

Though significant information about the physics is not needed apriori, a few fundamental concerns are addressed. These concerns pertain to both the purpose of the study as well as the utilization of apriori information available about the process to simplify the identification. This section outlines topics associated with these concerns, namely the model use and accuracy needed as well as online and offline identification approaches.

Model Use

Above all, it is necessary to consider the purpose of the identification or the use of the resulting model. Two possible motives are:

1. Control-oriented. To design efficient controllers, it is necessary to be able to predict the response of a system to a certain class of inputs. Feedback control is a good example of a class of inputs. Additionally, modeling is necessary to familiarize the engineer with how response may change with operating condition. Though experimentally obtained data may not be available for many operating conditions, a good model may be able to predict controlled performance for operating conditions perturbed from that of which data is available.

2. Design-based identification. Active control may not be the deliverable in mind for the study. By obtaining an adequate model of the system, the engineer will become more intimate with the physics of the otherwise poorly understood process. With this in mind, one may identify and model the system to determine the effect of possible adjustment of design parameters.

These points are highlighted as establishing the use for the model directly impacts its complexity and therefore the intensity of the identification process.

Accuracy Needed

Identification accuracy is a measure of equivalence between the model and actual system. Certain general concerns may be addressed for this. For non-oscillatory systems accuracy can be judged by comparing the speed of response or steady-state error of a trajectory. For oscillatory systems, the measured frequency response within a bandwidth is compared to what the model produces. Specifically, for self-excited oscillatory systems, the response of the model must exhibit a stable limit cycle for a non-periodic
input with harmonic content similar to the output of the actual system. These characteristics are often described mathematically as an error function, which is minimized to obtain the best possible model.

**Online or Offline methods**

The methods described above are general approaches to system identification. Here we emphasize that these methods can be executed online or offline. The offline approach obtains data from an experiment, and processes it after the experiment is completed. The online counterpart, which is appropriate for some applications, processes the data while the experiment is being performed. Online methods use a form of recursive identification to alleviate the time needed to process large amounts of data. Due to this, computation must be accomplished within one sample interval to obtain stable identification results. Due to the advent of fast numerical facilities, this approach to system identification has gained great importance in the control community as a foundation for adaptive control. For this study, we consider only offline methods, leaving the online/adaptive identification for future studies.

**1.5 Literature Review**

An appropriate amount of information is accessed to gain an intuitive understanding of the physics of a thermoacoustic system, to review relevant system theory needed to understand its inherent limit cycle behavior and to become familiar with existing identification approaches and the complexity that arises from attempting to identify closed-loop nonlinear systems. Here we outline a sample of the literature available to benefit these technical concerns. For clarity, we subdivide this outline into its respected technical relevance. Unfortunately due to the primitive nature of this field, there does not exist a common library for the identification of thermoacoustic dynamics or of single-input, single-output limit cycling systems for that matter. As such, in this section we review the minimal work that has been done, but primarily focus on references that offer theory used to develop the algorithm described and tested in this manuscript.

**1.5.1 Thermoacoustic Physics**

The physical nature of the thermoacoustic system suggests that the acoustics of the combustor and its forcing source from the flame is schematically described as a feedback
system. Therefore, to better understand the system as a whole, these two contributions are separated. As such, we minimize this effort by studying both the physical nature of acoustics in an enclosure and the input-output characteristics of a flame within an acoustic environment.

Acoustical characterization of enclosures is a field that is associated with many research efforts. The fundamental theories used for this analysis are described in a well-respected book by Kinsler and Frey et al. [Kinsler 1982]. In addition, a reference that outlines fundamental acoustic relations with a focus on control studies is available by Nelson and Elliot [Nelson 1992]. The combination of these two works, due to their comprehensive nature, is sufficient to gain an understanding of the basic acoustics of the tube combustor.

As the focus of this study avoids detail physical modeling, an exhaustive review of combustion theory is circumvented. However, preexisting modeling efforts of thermoacoustic instabilities is available. The review of these efforts facilitates a justified construction of a simplified thermoacoustic model used to verify the identification algorithm we develop in this study. A series of references are available considering modeling the thermoacoustic behavior of a tube combustor [Annaswamy 1997, Fleifel 1997, and Rumsey 1998]. These studies offer physically based modeling efforts for a tube combustor sufficient to study its nonlinear behavior and linear control approaches. In a similar study [Culick 1987], mathematical relations governed by the physics in a combustor are outlined with their pertinence to the Rayleigh criteria. Similar to these studies, an industry derived modeling effort is available which characterizes the physics of a lean premixed high-pressure combustor [Peracchio 1998]. This effort is significant as it is intimately associated with the real problem that motivates this study.

1.5.2 System Theory

1.5.2.1 Linear System Theory

There is a tremendous amount of information available concerning linear systems. The sample utilized in this study is chosen mostly due to their convenience. A thorough outline of linear system theory is offered in Bay [Bay 1999]. This text offers system realizations and appropriate tools for their analysis. A comprehensive organization of linear system theory with respect to feedback control is available in Franklin [Franklin
1994]. These references are reviewed to understand the simple linear concepts inherent in the closed-loop thermoacoustic system.

1.5.2.2 Nonlinear System Theory

Due to the nonlinear nature intrinsic in the thermoacoustic system, it is necessary to become familiar with the basics of nonlinear systems. A mathematically sufficient collection of these theories is available in Khalil [Khalil 1996]. This text offers fundamental properties of nonlinear systems based on complex mathematics. For the study of closed-loop nonlinear systems, the reader is referred to two texts by Atherton [Atherton 1975 and 1981]. These offer tools for the stability analysis of closed-loop nonlinear systems with a focus on limit cycling systems. They present the most detailed description of describing function analysis available.

1.5.2.3 Nonlinear Oscillatory Theory

A vast collection of literature is available considering the behavior of systems that exhibit nonlinear oscillations. Though a majority of these works do not address their analysis as a means of identification, it is necessary to become familiar with the mathematical relations of these systems and tools to analyze them. To begin, we suggest a well-written text, which gathers a conceptual description of many systems that exhibit this behavior in Landa [Landa 1996]. With this intuition, a set of texts is available describing a more mathematical foundation for the analysis of nonlinear oscillations [Andronov 1949, Minorsky 1962, and Stoker 1950]. In a somewhat removed approach to the analysis of these systems, Nayfeh offers perturbation techniques for their analysis [Nayfeh 1979]. Here, many physically based mathematical systems are analyzed using this approach.

As for frequency entrainment, most literature references the work of Balth van der Pol [van der Pol 1927]. A very conceptual description of how a limit cycle oscillation falls into synchronism with another oscillation by a sequence pulses is available in Blaquiere [Blaquiere 1966]. A thorough, but easily understood outline, considering the analytical development of the resonant characteristics of self-excited systems is offered in two works by Hayashi [Hayashi 1953, and 1964]. Most of these works offer analysis of forced single degree of freedom nonlinear systems that exhibit limit cycle behavior (most often the van der Pol equation is used). The development of analytical expressions for
exterior harmonic forcing on systems with multiple degrees-of-freedom is extremely complex and is available in a collection of theory by Butenin [Butenin 1965]. A very mathematically intense presentation considering the stability characteristics of entrained nonlinear systems is available in [Summers 1996]. Again, all of the above-mentioned references, and many more like them, give accurate analysis tools for nonlinear self-excited systems with external periodic forcing. Unfortunately, none of these references apply this theory to identify the self-excited system.

1.5.3 System Identification

The theories associated with system identification begin with identification structures for open-loop linear time invariant systems. These methods are broken into two approaches: parametric and nonparametric. Nonparametric identification involves obtaining a model for a dynamic system without establishing its structure apriori. Sufficient review of these approaches is available in two references [Ljung 1987, and Wellstead 1981]. The more detailed counterpart to this approach is parametric identification. Here, a parameterized structure is formulated, and these parameters are found by minimizing an established error criterion. Two texts offer sufficient theory to understand the general approach to parametric system identification [Ljung 1987, and Ekyhoff 1974]. Since in this approach a structure is assumed, a number of concerns arise. Most notable of these concerns is attempting to identify, and subsequently separate the components of a closed-loop system.

As we will highlight by example in Section 3.2 of this document, parameter identifiability is an issue with closed-loop systems. The coupled nature of closed-loop system mathematically suggests that with some inputs, identification of the parameters in the forward and feedback paths becomes a problem. A thorough review of this issue is available in two papers [Gustavsson 1997, and Niu 1997]. These papers survey many different structures of feedback systems and classify their identifiability using different inputs. In general, it is shown that the system can be decoupled when either a measurable source is introduced in to the feedback loop, or the feedback dynamics themselves are specified. These references illustrate that close attention is needed when identifying closed-loop systems. Unfortunately, these studies do not address concerns that arise in closed-loop nonlinear systems.
The past work on closed-loop identification of non-oscillatory nonlinear systems, extends the theories discussed above, and with proper structuring, the linear and nonlinear characteristics of the system are recovered. That is, the necessity of additional information in the feedback loop is addressed while formalizing a parametric structure that lends itself to the nonlinear nature of the problem. Two sufficient examples of this approach are found in Linard and in Montasser [Linard 1997, and Montasser 1983]. Unfortunately, since they assume to have access to information in the feedback path, they do not help the current effort.

Closed-loop, single-input/single-output identification of nonlinear systems exhibiting limit cycle behavior is research area that has very little effort directed towards it. Yasuda offers a paper discussing the ability to parameterize a static nonlinearity and develop harmonic balance equations for a single degree of freedom system [Yasuda 1989]. His efforts are also extended to systems with multiple degrees of freedom [Yasuda 1993a, 1993b]. However, for both cases, the data is simulation based and for analysis of systems with high order, it is assumed that there are many valid output signals.

In a more applicable set of related studies, it is shown that the nonlinearity in an physical combustor can be recovered by use of the nonlinear harmonic balance [Casas 1997, Murray 1998, and Casas 1999]. However, in these studies, it is assumed that the forward dynamics (acoustics) are known, as they are identified by other methods. This study is very helpful as it parallels the current study, though here, we assume that the forward dynamics are unknown. As such, we conclude that more information is needed to identify the system, which in turn is alleviated by multiple experiment frequency entrainments.

1.6 Research Contributions

The overall contribution of this study is the technical investigation of the goals that motivate it. In addition to this, the topics from the disciplines that motivate the study (systems and combustion) will be outlined in a way that is easy for each other to understand. Existing gaps between systems engineering and combustion engineering perspectives will be bridged with terminology understood by researchers with both backgrounds.
The specific technical contributions are:

1. Further develop the harmonic balance method for the analysis and identification of nonlinear self-excited systems based on a single measurement, and multiple entrainment experiments.

2. Use this tool to identify the dynamics defining the thermoacoustic instability in a tube combustor, which will result in a low-order model. This model will be sufficient for analysis and high-level control design.

3. Analyze the resulting model considering known physics of the system to outline a mapping from the physical attributes to a systems representation counterpart. By this we mean the dynamic model will be described in terms of its analogy to the thermoacoustic process.

1.7 Thesis Structure

Following this introduction, the remainder of this manuscript outlines theory, describes its use, and presents the results of the identification of nonlinear systems exhibiting limit cycle behavior. To achieve this, the thesis is divided into two parts. The first part (Chapters 2 and 3) outlines system, and oscillation theory and concludes by offering an identification approach for limit cycling systems. The second part (Chapters 4 and 5) of the thesis uses this theory to identify both a simulated and physical limit cycling system. The simulated system is a model based on established physical characteristics of the tube combustor described above. The performance of the identification algorithm is illustrated with respect to exact and un-modeled dynamics in the identification structure. We conclude this article with a summary of the results and recommend an array of approaches for future efforts.
System theory is an organization of mathematically founded methods used to analyze dynamic systems. In a brief overview, system theory begins by the formalization of a model for a certain physical process considering its inherent structure (open or closed-loop for example). This model may be represented in many fashions including the time and frequency domains. Upon designating an accurate model, a library of tools is available for analyzing one of many criteria pertaining to the system response. These criteria may include the stability or some other measure of performance. Here we highlight system theory that facilitates the identification of nonlinear systems exhibiting limit cycle behavior. We outline the phase-plane method, which is an adequate means for addressing the stability of an equilibrium point for both linear and nonlinear systems. We then focus on oscillatory theory of systems. That is, we investigate methods for analyzing the free and periodically forced response of reverberant linear and nonlinear systems.

The controls engineering field includes research into many different topological system structures (open-loop, feedback, feedforward, adaptive, etc.). Analysis of oscillatory aspects of the system can be sufficiently organized by observing their effect on open-loop and feedback structures. As such, this outline is reduced to that of the input-output response (open-loop) and that of adding an oscillatory feedback signal to the reference signal and imposing this result on the input (feedback). Though we do address linear and nonlinear systems, we only consider systems whose temporal response is dependent solely on the difference between the initial time and the current time (time invariant systems).

2.1 Phase-plane Analysis

Phase-plane analysis is a tool that qualitatively depicts a systems response to initial conditions. The phase-plane facilitates the analysis of second-order systems by mapping the relation: \( x(t) = x_1 \) vs. \( \dot{x}(t) = x_2 \). However, this qualitative analysis is valid when using other state variables as well. In this analysis, the transient response, steady-state response, and the overall stability of the system is determined by observing the evolution
of phase variables with time. Because this type of analysis simply maps the time response of a system, it is valid for stability assessment in linear and nonlinear systems. Here we justify an outline of phase-plane analysis for two reasons. First, understanding the contour of a self-excited system in the phase-plane helps to motivate the understanding of its periodic response. Secondly, as we will see in Section 2.3.4, entrainment stability is established by analyzing the stability of an equilibrium point. Therefore, we merit an in-depth discussion of this theory.

There are many different ways to approximate the behavior of the system in the phase-plane. Through experimentation or numerical integration, a wide variety of initial conditions are chosen in the $x_1-x_2$ plane and the systems response is denoted graphically. In doing this, a flow-field of the systems states is established, and by visual interpolation, one can predict the response from any initial condition within the designated domain of the phase-plane. Another method graphically investigates the set of differential equations as a solution in the phase space rather than in time. These approaches include the Method of Isoclines, Lienard’s method and others [Hayashi 1964]. Aside from these two methods, a general understanding of the function of the system in the phase-plane can be determined by investigating the eigen-characteristics of the linear system. Since the response of a linear system to initial conditions is solely dependent on its eigen-realization (set of eigenvectors and associated eigenvalues), the flow of the phase-plane can be sketched knowing these characteristics.

For a linear time invariant system, this response will be composed of a linear combination of exponential functions of time. We begin with the traditional state-space realization for the system dynamics:

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}$$

where the matrices $A$, $B$, $C$, $D$ describe the entire response of the system ($x$) and are typically multidimensional. The $A$ matrix is referred to the state matrix which describes how the states ($x$) evolve in time. Through simple matrix manipulations of $A$, characteristics that generalize this response are obtained. These characteristics are represented by the eigenvalues and eigenvectors of this matrix. The input matrix ($B$) establishes how the input modifies the states. The output of the system ($y$) is composed
of the response of selected states as well as feedthrough from the input. The output matrix \((C)\) specifies which states are present in the output as not all states necessarily are included. The feedthrough matrix \((D)\) denotes contributions in the output, which are not affected by the dynamics of the system.

A general form for the solution of the linear state equations (2.1) is the convolution integral. This form describes the state evolution and output response as a function of the state transition matrix. This matrix coalesces the exponential contributions of the dynamics inherent in the state matrix \((A)\) and is represented as:

\[
\theta(t) = e^{A(t-t_0)}
\]  
[2.2]

which designates the system output as:

\[
x(t) = \theta(t)x(t_0)
\]  
[2.3]

where \(x(t_0)\) is the initial condition on the system. This illustrates that for linear systems, the response is entirely comprised of a linear combination of exponentials. This combination is represented exclusively in the state transition matrix \((\theta(t))\). Since the characteristics of the state transition matrix are uniquely described by the state matrix \((A)\), and the eigenvalues describe \(A\), the eigen-characteristics of the system thus describe the entire response of the system.

To qualitatively generalize the systems’ behavior in the phase-plane, equilibrium points are determined and their stability is established. An equilibrium point is a location in the phase-plane (or state space) where a steady-state constant solution for zero input exists. As such, for zero input, the system will remain at that point for all time. A linear system has an isolated equilibrium point at the origin of the phase-plane if the system has no zero eigenvalues. For a system with zero-valued eigenvalues, the system has a continuum of equilibrium points. These two patterns are the only option that a linear system may have considering types of equilibrium solutions. Conversely, a nonlinear system can have multiple isolated equilibrium points and the stability of each may differ [Khalil, 96].

The stability of an equilibrium point establishes how the system will respond from an initial condition near to it. For example, if the initial condition is slightly perturbed from a stable equilibrium point, the dynamics of the system will return the states of the system back to the equilibrium point.
The stability of the equilibrium point is dependent on the eigenvalues of the system. For second-order linear systems, the flow resulting from a system with a set of eigenvalues are classified as one of many types. These are summarized below in Table 2.1.

### Table 2.1 Types of Equilibrium Points

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Type of Equilibrium Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both negative real</td>
<td>Stable Node</td>
</tr>
<tr>
<td>Both positive real</td>
<td>Unstable Node</td>
</tr>
<tr>
<td>One positive real, one negative real</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>Complex conjugate (negative real)</td>
<td>Stable Focus</td>
</tr>
<tr>
<td>Complex conjugate (positive real)</td>
<td>Unstable Focus</td>
</tr>
<tr>
<td>Imaginary</td>
<td>Center</td>
</tr>
</tbody>
</table>

For a nonlinear system with multiple equilibrium points, this analysis remains valid for the most part. Unlike linear systems, the flow to and from equilibrium points is not necessarily a linear combination of exponentials. However, through linearization (which forces the response to linear), many key attributes of the stability and response of the system are attainable. Linearizing about one of the equilibrium points, by finding the Jacobian matrix for example [Bay 1999], the eigenvalues of the linearized system will then define the topology of the flow for the associated equilibrium point. In doing this, the qualitative characteristic of the flow is preserved with the exception if the linearized system suggested that the equilibrium point is a center (see Table 2.1). Below we describe why this is true.

For oscillatory systems that have a continuous exchange of energy between the states (periodic response), a closed trajectory will be evident in the phase-plane. Here we address issues and characteristics of systems that possess a closed orbit in the phase-plane. A system with an oscillating steady-state has the nontrivial periodic solution, \( x(t + T) = x(t), \forall t \geq 0 \) for \( T > 0 \). By nontrivial we mean solutions other than those corresponding to the stationary equilibrium point. This is represented in the phase-plane as a closed orbit. Purely imaginary eigenvalues of a linear system describe a frictionless system that will oscillate continuously in this manner (center in the phase-plane). When a linearized nonlinear system suggests this behavior, it maybe a misrepresentation of the actual system dynamics. This is due to the truncation of terms in the linearization.
process. In other words, depending on the characteristics of the truncated terms, the oscillations in the actual nonlinear system may either be stable or unstable \textit{[Vidyasagar 1978]}. Therefore, the type of equilibrium point resulting from the linearized approximation will be consistent with the actual nonlinear dynamics unless eigenvalues of the linearized model are purely imaginary. This is not to discount the possibility of periodic solutions to nonlinear differential equations. These solutions are called limit cycles and are discussed below.

\textbf{2.2 Oscillations in Linear Systems}

In an attempt to thoroughly analyze a system, it is essential to determine the manner in which the system will behave in response to an oscillatory input. This is necessary even if the system is expected to operate in a static or non-oscillatory manner. For example, in a well-damped DC servo system, oscillations may arise in the reference signal from noise that may excite dynamics of the system that were otherwise disregarded. Though much of the theory that assesses the oscillatory behavior of systems facilitates their identification, the focus of this section is to outline the effect that linear and nonlinear systems have on oscillatory inputs. As such, this section addresses characteristic frequency domain responses to forcing. The use of this theory to identify the dynamics of a system is postponed to later chapters (Chapter 3).

There are many simple guidelines for the analysis of oscillations in linear systems. The objective is often to study the steady-state response to periodic forcing for both open-loop and closed-loop systems. In doing this, the stability as well as other performance measures is assessed. There are many different stability measures available. For closed-loop systems, we use Bode and Nyquist analysis. These are frequency domain tools that represent the open-loop frequency response of the system and provide a prediction of characteristics of the closed-loop system based on this response.

\textbf{2.2.1 Open-loop}

For asymptotically stable linear systems, the response to an oscillatory input consists of a transient response that settles to a steady-state response at the same frequency as the input. Due to the principle of superposition, the transient response consists of free oscillations and when the system is damped in a stable manner, this response is
dominated by the onset of the steady-state forced oscillations. These steady state oscillations are uniquely described by a gain (magnification or attenuation) and a phase shift that modifies the input signal. By outlining these attributes, on a frequency-by-frequency basis, the dynamic response of the system is described. The resulting description of how the system responds to a sinusoidal input at many frequencies is known as the frequency response of the system.

To obtain the frequency response of the system, the system is forced by a sinusoid at a particular frequency and its response is noted. The output is then divided by the input to determine the magnitude and phase modification the system imposed on the input signal at this frequency. The resulting characteristics are often illustrated together on a Bode plot which is a dual representation of the magnitude and phase influences of the system on a frequency-by-frequency basis. Many deductions pertaining to system dynamics are obtained from the Bode plot to those who are familiar with them. Here to familiarize the reader, we illustrate two different typical frequency responses: flat, and modal.

1) Flat response. The system has a flat frequency response through some frequency range. This section of the frequencies is typically called the passband as the signal passes through the system with minimal modification. As such, system dynamics are considered negligible in this region. This is important as often the dynamics can be neglected if the frequencies of concern are within this region. In addition to this, it should be noted that many physical systems adopt this type of response to some degree. Natural dynamics of physical processes dictate that they cannot physically track (keep up with) an input that varies at high frequencies. Therefore a ‘roll-off’ in the frequency response at high frequencies is always expected (low-pass characteristic). Below, Figure 2.1 illustrates the flat frequency response of both a high-pass and low-pass system.
2) Modal. The frequency response of these systems varies greatly over the bandwidth of concern. This variation is due to multiple resonant characteristics of the system. Peaks and valleys in the frequency response plot illustrate this (Figure 2.2). Due to the wide variation in response over a given bandwidth, thorough frequency domain analysis is needed to become intimate with the response of the system to oscillatory inputs.

Figure 2.2 Frequency Response of a Typical Modal System

As described, the frequency response characteristics of stable linear systems are founded on straightforward complex-trigonometric relations. Here we highlight one of the most important resonant characteristics of linear systems. The steady-state response of a stable linear system when excited by a pure sinusoid (single frequency, noiseless,
etc.) will have an output that is sinusoidal and of the same frequency as the input. Though this concept may seem trivial, it is important to highlight because as we will soon see, in general, nonlinear systems violate this resonant characteristic.

### 2.2.2 Closed-loop

Closed-loop analysis typically investigates the stability and overall frequency response of an open-loop system that has information from its output imposed on its input in a feedback nature. There are many approaches to predicting closed-loop feedback performance from open-loop data. This section outlines significant aspects of these tools and their use for the analysis of the stability of a closed-loop linear system.

Frequency domain analysis of closed-loop systems usually pertains to investigating the effective stability from reintroducing the output signal into the input by adding it to the input reference signal. Often dynamics are included between the output signal and this summing junction. These dynamics may or may be due to a predetermined structure, as often they are from a controller that is added to modify the preexisting system dynamics. To assess the closed-loop stability we begin by obtaining the open-loop frequency response as described above. Now, the open-loop system includes the dynamics of the forward and feedback paths. The open-loop dynamics are obtained by ‘breaking the loop’, which typically occurs before the summer. This is illustrated in Figure 2.3.

![Figure 2.3 Typical Block Diagram for Closed-loop Stability Analysis](image)

#### Bode Analysis

A common tool for stability analysis of linear systems in the frequency domain is Bode analysis [Franklin 94]. Though often confused by terminology relating to stability margins and other aspects of the Bode plot, this tool merely focuses on the result of adding (or subtracting) two sinusoids of the same frequency. The nature of the summer in the loop must be considered for this analysis, whether it adds (positive feedback) or
subtracts (negative feedback) the output signal to the reference. Due to the resonant characteristics of linear systems, the steady-state response to an oscillatory input will be oscillatory and of the same frequency. Therefore the frequency domain analysis is reduced so that only sinusoids of similar frequencies are considered. This section describes this additive phenomenon and how to investigate the stability when this occurs.

When adding two sinusoids of the same frequency, the amplitude and phase of both sinusoids will uniquely determine the output. For two sinusoids with similar phase (both rising at the same time), their addition will be constructive and the response will be of the same frequency with amplitude of the two combined. When adding sinusoids of opposite phase (one rising while the other is falling) the resulting signal will be of the same frequency with amplitude equal to the difference of the two previous amplitudes. This is called destructive interference and is one fundamental mechanism of active noise control [Nelson 1992]. The result of adding sinusoids that are not exactly in-phase (0, +n*360 degrees) or out-of-phase (+n*180 degrees) will vary linearly depending on how close it is to destructive or constructive interference. This interaction is illustrated in Figure 2.4.

![Figure 2.4 Additive Response of Two Sinusoids](image)

Simply stated, a Bode plot illustrates the frequency response of the open-loop system in a manner that facilitates the above considerations concerning the addition of sinusoids. Below, Figure 2.5 illustrates two stable open-loop systems. The difference in the two systems is only in the gain (the phase characteristics are identical). Here, when considering positive feedback, adding a sinusoid in-phase (n*360 degrees) will
potentially destabilize the system. However, the stability of the system is challenged only when the open-loop gain is above unity and the response of the system will result in adding in-phase sinusoids at the summer. Figure 2.5 illustrates two stable open-loop systems, one of which goes unstable when a loop is closed around it because the gain is sufficient (greater than one). If the system gain is less than unity (0 dB), even though the signals will constructively add (when considering phase), the open-loop dynamics will re-attenuate the resulting signal and therefore the response will cease to grow in magnitude in an unstable manner.

![Figure 2.5 Bode Plots for Closed-Loop Stable and Unstable Systems](image)

**Figure 2.5 Bode Plots for a Closed-Loop Stable and Unstable System**

With an understanding of the Bode analysis techniques, the stability and performance of closed-loop systems is predictable. In light of this, a few additional concerns are addressed. First, the critical gain and phase locations for a type of feedback (positive or negative) should be regarded with care. In general, magnitudes exceeding –10dB should be avoided when the phase passes through a critical phase location. Similarly, the phase should be at least ± 20 degrees away from the critical phase when the magnitude is above unity.

Another situation that occurs in real systems is the phase crossing multiple critical stability points through a resonance. A simple second-order system motivates a peak in the magnitude and a phase transition of –180 degrees in a bandwidth around the natural frequency of the system. For highly dimensional systems, or systems with time delay, a transition of more than 180 degrees may occur through the resonance bandwidth.
Therefore, the phase may evolve through both stable and unstable phase crossings through the region of high gain. As such, the closed-loop response may exhibit amplification and attenuation within this frequency range. The introduction of this feedback-induced instability has been investigated for the control of thermoacoustic oscillations, and is shown to be easily predicted [Saunders 1999].

In addition to this, Bode analysis is not valid for open-loop unstable systems. There is no representation in the frequency domain for a system, whose output increases to infinity for a sinusoidal input (which is the response of an unstable system). The Bode plot is valid for only steady-state oscillations.

**Nyquist Criterion**

The Nyquist plot and its associated stability criteria illustrate the same information that is available in the Bode analysis described above. However, Nyquist analysis supports the investigation of open-loop unstable systems. Clearly this is helpful if the system is unstable and the design of a stabilizing controller is needed. Another reason for introducing this analysis tool is that the Nyquist plot is essential for the analysis of the quasilinear approximation of nonlinear systems used in describing function analysis. Here we outline the Nyquist method while the description of the describing function method is postponed until later.

The Nyquist Criteria allows the prediction of closed-loop stability with knowledge of the pole locations and frequency response of the open-loop system. The Nyquist plot is simply a representation of the open-loop frequency response in a polar plane (real vs. imaginary Fourier coefficients at each frequency). From the number of encirclements of $1.0 + 0.0i$ (for positive feedback) and the number of open-loop unstable poles, the number of unstable closed-loop poles is predicted. That is:

$$P_{CL} = N + P_{OL}$$

**Where:**

- $P_{CL}$ = # *Closed-loop Unstable Poles*
- $N$ = # *Clockwise Encirclements of $1$ (pos. feedback)*
- $P_{OL}$ = # *Open-loop Unstable Poles*

Since the Bode and Nyquist plots are essentially identical, we tabulate their analogues below.
Table 2.2 Analogues of Nyquist and Bode Plots

<table>
<thead>
<tr>
<th>Bode Plot</th>
<th>Nyquist Plot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase = 0, +(n \times 360) degrees</td>
<td>Positive Real Axis</td>
</tr>
<tr>
<td>Phase = +(n \times 180) degrees</td>
<td>Negative Real Axis</td>
</tr>
<tr>
<td>Magnitude = 0dB (1.0)</td>
<td>Unit Circle</td>
</tr>
</tbody>
</table>

With these characteristics, the stability of a closed-loop system is established from its corresponding open-loop characteristics. To clarify the similarity of the two methods of analysis, the same two systems illustrated in Figure 2.5 are denoted in the Nyquist plot of Figure 2.6. Again we see that the phase characteristics of the open-loop system are such that instability is possible, but only for sufficient gains which is when the Nyquist trajectory leaves the unit circle (exceeding \(1.0 + 0.0i\)).

![Nyquist Plots for a Stable and Unstable Closed Loop System](image)

**Figure 2.6 Nyquist Plot for a Closed-Loop Stable and Unstable System**

### 2.3 Oscillations in Nonlinear Systems

As mentioned in the introduction, nonlinear oscillatory theory is much more complex than the linear counterpart. We begin the discussion by offering tools for the analysis of linear systems with nonlinear feedback. This brings us to an in-depth outline of systems that exhibit limit cycle behavior or self-excited oscillations. We offer tools to analyze this unforced closed-loop behavior. We then discuss forced oscillations in nonlinear open-loop systems, which introduces the theory associated with forcing self-excited systems (frequency entrainment).
2.3.1 Closed-loop Analysis of Nonlinear Systems

Though there are many approaches to analyzing systems with nonlinear feedback, the most commonly used technique is the describing function method. This approach approximates the nonlinearity as an equivalent linear gain. A quasilinear approximation is established by designating an equivalent gain that is a function of both the frequency and magnitude of the expected input to the nonlinearity.

The initial assumption made for this method is that the linear dynamics are low-pass in nature compared to the harmonics that the nonlinearity produces (see Figure 2.1 for a low-pass characteristic). As described in the introduction (Section 1.2), the output of a nonlinear characteristic for an oscillatory input often contains multiple frequencies. With forward dynamics possessing low-pass dynamic nature, the higher harmonic frequency content in the output of the nonlinearity is attenuated, and can be ignored. In doing this, for odd (functional) nonlinearities, and a given input amplitude and frequency, the output of the nonlinearity is a simple gain and phase shift imposed on that system state. To further describe this procedure, we offer an example. A system containing a static, odd nonlinearity is illustrated in Figure 2.7.

![Figure 2.7 Example Closed-Loop Nonlinear System for Describing Function Analysis](image)

Using the describing function method, which considers only the fundamental harmonic output of the nonlinearity, the output of the nonlinearity \( n(t) \) is represented in the time domain as:

\[
 n(t) = A \cos(\omega t) + B \sin(\omega t) \quad [2.4]
\]
Therefore, for a purely sinusoidal input, the influence of the nonlinearity at the input frequency is a simple gain and phase shift, which through trigonometric identities can be described as:

\[ N \angle \theta = B + iA \]  \hspace{1cm} [2.5]

To obtain the quasilinear approximation of the nonlinear influence, we simply divide the complex output of the nonlinearity by the amplitude of the input \(M\) revealing:

\[ N(M, \omega) = \frac{B + iA}{M} = \frac{N \angle \theta}{M} \]  \hspace{1cm} [2.6]

Figure 2.8 illustrates this approximation.

Nonlinearity

\[ m(t) = M \sin(\omega t) \]

\[ n(t) = B \cos(\omega t) + A \sin(\omega t) \]

**Figure 2.8 Describing Function Schematic**

With this approximation to the nonlinearity, traditional Nyquist analysis (described above) predicts the onset of limit cycle behavior in the closed-loop system. For a given amplitude of the input to the nonlinearity \(M\), a frequency response curve is plotted on the Nyquist plot accompanied by the frequency response of the linear forward dynamics.

If for a given frequency, the two curves intersect \(G(\omega) = -\frac{1}{N(M, \omega)}\), a limit cycle will occur at the amplitude \(M\) that was established in defining the describing function, and for the frequency where this condition is met. With this analysis, the stability of the limit cycle may be investigated as well as the possible occurrence of multiple limit cycles. Though it may seem that significant information may be lost in the linearization process, this method has proved to be very sufficient for analyzing nonlinear systems especially when limit cycle behavior is predicted.

The above approach to nonlinear closed-loop analysis is valid only for odd nonlinearities. As mentioned in the introduction, a odd nonlinearity \(x^3\) for example) produces energy at odd harmonic frequencies, while an even nonlinearity \(x^2\) for example) produces even harmonic information. By observing the response of the thermoacoustic system (Figure 1.3), it is evident that indeed the physics that induce
nonlinearity into the thermoacoustic system producing energy at odd and even harmonics of the fundamental instability, which suggest odd and even influences. Therefore, the above-mentioned theories, as they stand, need to be used with care.

Fortunately, there exist analytical approaches concerning the describing function analysis of systems with even nonlinearities [Atherton 1975]. These theories are more complex than the above-mentioned approach as they account for a bias signal that is inherent in nonlinear systems with an even nonlinearity. In essence, the equivalent gain derived from the describing function is a function of the amplitude, frequency, and bias input to the nonlinear element. The dependence on the bias signal is due to the arousal of a bias when oscillations pass through an even nonlinearity (see section 1.2). We neglect to further discuss this approach here because as we will soon outline, the understood physics of the linear dynamics inherent in the thermoacoustic system cancel out this bias signal (through differentiation), and therefore this analysis would lead to misleading conclusions.

2.3.2 Limit Cycle Behavior

Limit cycles or self-excited oscillations occur in many physical systems. An intriguing characteristic of a system that exhibits limit cycle behavior is the steady state oscillatory output from a constant non-oscillatory input regardless of initial condition. Typically, the occurrence is due to a nonlinear relationship between friction and the relative velocity. Brake squeak, a creaking door, or the response of a violin string to a bow are examples of frictional effects that produce self-excited oscillations. Flutter, and wind-excited oscillations of transmission lines or bridges are examples as well. Intuitively, all that is needed is a destabilizing/restabilizing characteristic in the system dynamics. This occurs in a saturation type nonlinearity that is found very often in physical systems. This nonlinearity destabilizes the response until it gets to a certain amplitude in which the characteristic saturates, which in turn stabilizes the response. Though self-excited oscillations are desirable in some systems (clocks for example), if oscillatory behavior is undesirable, it is important to understand the characteristics that produce this response in order to counteract it.

A limit cycle is a response that only a nonlinear system will support. The stable limit cycle is a closed contour in the phase-plane, of which trajectories approach from within
or without from other trajectories asymptotically. This in and of itself separates the constant-amplitude oscillations of linear and nonlinear systems. The limit cycle response of a nonlinear system is represented as an isolated closed contour in the phase-plane. That is, there is a neighborhood in the phase-plane where no other periodic solutions exist. Conversely, a linear system that exhibits a closed contour response in the phase-plane (from frictionless dynamics) will have adjacent periodic solutions depending on the initial conditions of the system.

For a stable limit cycle, perturbations from the closed orbit return to the original closed orbit. Conversely, solutions of an unstable limit cycle diverge from the closed orbit. A half-stable limit cycle will possess different stability dependent on whether the perturbation is within the closed orbit or on the outside. An illustration of the response of a commonly investigated nonlinear equation exhibiting a limit cycle response (the van der Pol equation) in the phase-plane, from multiple initial conditions, is presented in Figure 2.9.

![Phase Plane for the Van der Pol Equation](image)

**Figure 2.9 Phase-plane Response of the van der Pol Equation**

2.3.3 Forced Oscillations in Nonlinear Systems

As mentioned in the introduction, nonlinear systems respond in a uniquely peculiar way to periodic forcing. The same nonlinear system with perturbed parameters, or for
different amplitude or class of input signals may respond in a way that is unrecognizable from the previous response. Here we give an example of a simple nonlinear system and describe peculiarities in its frequency response. Through example, we introduce phenomena associated with forcing an open-loop nonlinear system.

A mathematical model of a swinging pendulum captures many nonlinear characteristics and has been studied extensively [Stoker 1950, Hagedorn 1978, many others]. The restoring force from gravity is similar to a stiffness contribution, and is a sinusoidal function of the angle which the pendulum is with respect to its pivot. By approximating the nonlinear stiffness from gravity as a cubic ($\sin x \approx x - \frac{x^3}{6}$), the differential equations of motion are simplified enough to study their behavior while still supporting the nonlinear phenomenon inherent in this problem. In addition, damping is added to further capture characteristic physical traits. The resulting equation of motion is labeled the Duffing equation (with damping) from his comprehensive research, and is presented below:

$$\ddot{x} + cx^3 + \omega_0^2 x - \mu x^3 = F \cos(\omega t) \quad [2.7]$$

To obtain a qualitative understanding of the result of harmonically forcing the damped Duffing equation, we study only periodic solutions of the Equation 2.7. Specifically for this example, we study periodic solutions of $x(t)$ in which the period is equal to $2\pi/\omega$ which is the same period as that of the forcing function. We approximate the harmonic solution as a single harmonic ($x(t) = A \cos(\omega t) + B \sin(\omega t)$), differentiate this approximation and substitute it into the equation of motion. Real-valued response amplitudes ($A$ and $B$) represent stable periodic solutions for a given set of system parameters ($c$, $\omega_0$) and forcing conditions ($F$, $\omega$). Complex response amplitudes represent unstable oscillations (which cannot be compared to reality with this approximation technique) and are unique to nonlinear systems. By obtaining the magnitude of the response amplitudes ($Amp = \sqrt{A^2 + B^2}$) for different arrays of forcing amplitudes and radial frequencies ($F$, $\omega$) we establish the resonant characteristic of the nonlinear system. This response with the parameter values in Table 2.3 is illustrated in Figure 2.10.
Table 2.3 Parameter Values for the Duffing Equation

<table>
<thead>
<tr>
<th>Model Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>-0.25</td>
</tr>
<tr>
<td>$C$</td>
<td>0.06</td>
</tr>
<tr>
<td>$\omega_b$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.8 $\rightarrow$ 1.2</td>
</tr>
<tr>
<td>$F$</td>
<td>0.03, 0.045, 0.06</td>
</tr>
</tbody>
</table>

Figure 2.10 Response Curves for the Duffing Equation

This map is analogous to the magnitude response in the Bode plot, though since the nonlinear system responds differently to different forcing amplitudes, the response plot contains many traces. The qualitative resonant behavior illustrated in Figure 2.10 represents a unique physical phenomenon. Not only does the Duffing equation have unstable periodic solutions, it also has double-valued solutions for the same forcing condition. This is called a jump resonance and occurs often in physical systems that possess either a hardening or softening type stiffness characteristics. For the softening type stiffness whose response is illustrated in Figure 2.10, a bifurcation occurs in the response when forcing the system at a constant amplitude while changing the forcing frequency.
While holding the amplitude of the forcing function constant, and increasing the frequency from location \( A \), there are no stable solutions when the frequency gets to location \( B1 \). Therefore, the amplitude of the response ‘jumps’ (from \( B1 \) to \( B2 \)) to obtain the next stable solution. Upon further increase of forcing amplitude, the output continuously follows the higher amplitude response curve to the location \( D \). A similar phenomena happens when decreasing the frequency from location \( D \). While holding the amplitude constant, and decreasing the forcing frequency, when the frequency at \( C2 \) is reached, no other stable solutions (for the given forcing amplitude) are available and the response jumps to \( C1 \).

2.3.4 Forced oscillations in Self-excited Systems

At this point, we address the response of self-excited systems to external sinusoidal forcing. To analyze the frequency response of any system, we observe its output at the same frequency as the input. Since self-excited systems have a pre-established periodic output, which is irrelevant to the frequency of input, the analysis is much more complex. Due to this, for accurate input-output analysis of the system, the unforced self-excited response must be attenuated and subsequently replaced by oscillations at the frequency of the input. The phenomenon of frequency entrainment occurs when the free (self-excited) oscillation adopts the frequency of the external force. In this manner, the free oscillation falls into synchronism with that of the external force. This occurs provided that the external force has sufficient energy to impact the free oscillation and the frequencies are minimally dissimilar. If these conditions are not met, almost periodic oscillations will occur in a beating manner, which negates accurate analysis of the system.

Most often, frequency entrainment is investigated when forcing is very near to the fundamental frequency of the self-excited oscillations. However, frequency entrainment also occurs when the ratio between the natural self-excited frequency and the forcing frequency is in the neighborhood of an integer or fraction. We call such entrainments superharmonic or subharmonic respectively. The reader is directed to a thorough reference [Hayashi 1964] for their treatment, as we will only address the entrainment of the dominating frequency of self-excited oscillation or harmonic entrainment.

Again, to illustrate the phenomena associated with frequency entrainment we provide an example. A major mathematical effort into the analysis of free and forced self-excited
systems began with the study of the dynamic response of the electronic triode circuit by Balth van der Pol [van der Pol 1927]. His modeling of this circuit established the van der Pol equation, which describes an electronic circuit that exhibits limit cycle behavior. The free and forced response of this equation has been thoroughly investigated since its introduction over ¾ a century ago. Here we investigate its response to external forcing as the equation captures significant nonlinear phenomena that are worth noting.

The electronic circuit contains a nonlinear resistance, which provokes the existence of limit cycle behavior. It is assumed that the triode (nonlinear resistance) is adjusted to function symmetrically with respect to the oscillations. By solving the equations for the anode voltage, a differential equation is formed that describes many significant nonlinear phenomena for different parameters. To study the forced response, we add a periodic externally induced voltage and the differential equation becomes:

\[ \ddot{v} - \alpha \dot{v} + \gamma \frac{dv^3}{dt} + \omega_o^2 v = \omega_i^2 F \sin(\omega_i t) \]  

where \( v \) is the anode voltage, \( \omega_o \) is the radial frequency and \( \alpha \), and \( \gamma \) are positive parameters depending on the circuit elements. The forcing characteristics are specified by the forcing amplitude (\( F \)) and frequency (\( \omega_i \)).

To investigate the limit cycle response, we assume a periodic solution for the nonlinear differential equation (2.8):

\[ v(t) = A(t) \cos(\omega_i t) + B(t) \sin(\omega_i t) \]  

where \( A(t) \), and \( B(t) \) are slowly varying functions of time. That is:

\[
\begin{align*}
\frac{dA}{dt} &<< \omega_i A, & \frac{d^2 A}{dt^2} &<< \omega_i \frac{dA}{dt} \\
\frac{dB}{dt} &<< \omega_i B, & \frac{d^2 B}{dt^2} &<< \omega_i \frac{dB}{dt}
\end{align*}
\]

which makes the second time derivative of the Fourier Coefficients \( A, B \) negligible. We substitute the assumed solution into the differential equation (2.8). Here we limit the nonlinear representation to that of only the fundamental harmonic:

\[ v^3 = \frac{3}{4} \left( A^2 + B^2 \right) \left( A \cos(\omega_i t) + B \sin(\omega_i t) \right) \]  

[2.10]
By separating and equating the sine and cosine terms, we obtain a pair of linear ordinary differential equations that describe how the coefficients of the assumed solution (2.9) evolve in time. These equations are:

\[
2 \dot{A} + zB - \alpha A \left(1 - \frac{(A^2 + B^2)}{a_o^2}\right) = 0
\]

\[
2 \dot{B} - zA - \alpha B \left(1 - \frac{(A^2 + B^2)}{a_o^2}\right) = F \omega_1^2
\]

where to simplify the equations, \( z \) is chosen to represent the frequency mistuning between the frequency of the free oscillations \( \omega_o \), and the frequency of forcing \( \omega_1 \), while \( a_o \) represents the amplitude of the free oscillations. These simplifications are:

\[
z = \frac{\omega_o^2 - \omega_1^2}{\omega_1^2}, \quad a_o^2 = \frac{\alpha}{\frac{3}{4} \gamma}
\]

We assess the stability of entrainment conditions by investigating the stability characteristic of the equilibrium points of equations 2.11. As a means to this end, we set \( \dot{A} = 0, \dot{B} = 0 \) which is physically analogous to constant amplitude, phase invariant forced response of the self-excited system at the frequency of the forcing. The stability of the equilibrium point for each forcing case (amplitude \( F \) and frequency \( \omega_1 \)) describes whether the forcing is sufficient to entrain the self-excited system. We expect for large mistuning (forcing far from the limit cycle frequency) at low forcing amplitudes, to not be able to significantly affect the self-excited response. That is, the free energy of the system is much greater than that of the forcing signal, and the response of the system reflects this. Mathematically, this is represented by unstable equilibrium points of Equation 2.11. If the forcing energy is not sufficient to affect the free energy, the response of the system will be a combination of the forcing frequency and the frequency of the self-excited oscillations. Therefore, the output contradicts the assumed response for the harmonic balance of only one frequency.

Upon establishing equations for stable entrainment, we investigate different forcing conditions that are sufficient for entrainment. For the van der Pol equation, we can obtain a general condition for stability:
\[
\left( A^2 + B^2 \right) > \frac{1}{2} a^2_o
\]  

that is, stable entrainment is only reached when the amplitude of the forced response exceeds one-half the amplitude of the free response.

On a more specific level, we can solve the stability equations (2.11) using an array of forcing amplitudes and frequencies to obtain a resonant map of the self-excited system. As an example, we choose a parameter set and forcing array for the van der Pol equation. These values are tabulated below.

**Table 2.4 Parameters for the van der Pol Equation**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>1/3</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>( \omega_o )</td>
<td>1.0 rad/sec</td>
</tr>
<tr>
<td>( F )</td>
<td>0.01 → 0.3</td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>0.5 → 1.5</td>
</tr>
</tbody>
</table>

With these values for the van der Pol equation, the system exhibits a stable limit cycle response of amplitude 1.0 at 1.0 radians per second. Below are two response plots for the system with the forcing conditions of Table 2.4 using two different amplitude densities.

**Figure 2.11a,b Response Plots for the Forced van der Pol Equation**

In these figures, the stable entrainment cases are illustrated by the green data points while the unstable cases are denoted in red. As expected, as the forcing amplitude decreases, and the degree of mistuning increases, the stability of the entrained response is challenged. In addition, we validate the stability condition of 2.13 by noticing that the amplitude of the stable entrained solutions is never less than one half the amplitude of the free oscillations (1.0).
3 Parametric Identification and Nonlinear Analysis

Parametric identification is an identification approach that uses apriori assumptions of a model structure and order, and using data, the most accurate parameters for this assumed structure are found. In essence, the researcher obtains data from the system, chooses a model set, and determines the best parameters for the model set based on a criterion of accuracy. These factors are considered prior to the identification process. Upon determining the model structure and obtaining data, it is necessary to manipulate the assumed structure into a form that offers access to its parameters. That is, a set of equations functionally dependent on the parameters is established to highlight the accuracy of the identified system with respect to the parameters.

This chapter outlines apriori considerations, and methods available for parametric identification. A brief overview is offered describing issues relevant to the identification of closed-loop systems as the physics of the thermoacoustic instability suggest this structure. Additionally, the method of harmonic balance is outlined considering complications that arise due to the structure of a nonlinear self-excited system. We conclude this chapter by summarizing the benefits of frequency entrainment for identification of self-excited systems using the nonlinear harmonic balance technique.

3.1 Apriori Considerations for Identification

Input Class

The type of input affects the identification process by dictating the perturbations that the system experiences. For tracking and regulating type processes, constant step or impulsive inputs are appropriate as this is the most common input for this system. Key performance criteria such as stability, speed of response and accuracy are attainable from these inputs. Conversely, for oscillatory systems, performance is often measured in the frequency domain. Therefore, the identification process should be executed while the system is experiencing inputs that are periodic and of frequencies pertinent to the operation of the system (within its bandwidth). Regardless of the type of input, the most important consideration is to organize the input or actuation in a manner in which the system is controllable. That is, the input must excite all of the important modes of the system.
System Class

The class or model structure of the system is influenced mostly by prior knowledge of the system and its behavior. A broad overview and understanding of the dynamics should consider the topology of the system. Upon defining this topology, it is necessary to specify certain mathematical attributes defining the equations of motion.

- Topological representation

It is very important to determine topology of the system. Establishing different physical processes and their interaction with each other is very important. Specifically, it is important to establish whether the system is operating in an open or closed-loop manner. The importance revolves around the possible perturbations of parameters in the identified model. Clearly, a closed-loop system consisting of forward and feedback dynamics can be written in a manner similar to the input-output description of a simple open-loop system. As such, the forward and feedback dynamics are coupled. This coupling hides parameters essential to dynamics that are specific to the forward or feedback influences. As such, a perturbation in a parameter in either the forward or feedback paths will be misinterpreted. When either path includes nonlinear elements, specification of where they interact in the topology of the system becomes very important.

- Mathematical attributes

Fundamental mechanics suggest a differential equation system representation for most input-output dynamic systems. Though many physical processes are defined by partial differential equations, they are often simplified in some manner to obtain ordinary differential equations to ease in their analysis and simulation. Below is a typical ordinary differential equation of an input-output system:

$$a_0 \frac{d^n y}{dt^n} + \ldots + a_{n-1} \frac{dy}{dt} + a_n y = b_0 \frac{d^m u}{dt^m} + \ldots + b_{m-1} \frac{du}{dt} + b_m u + c$$  \[3.1\]

where $u$ is the system input and $y$ is the output and are functions of time. At this point, it is necessary to predict or otherwise establish the order ($m, n$) of the equation of motion. Though the ‘black box’ approach to system identification seems appealing the above representation illustrates that the neglect of available information is inefficient. In some cases, by predicting something as simple as order of the differential equation will
significantly decrease the complexity of the identification process. With this structure, the system can take one or a combination of many specific classes:

1) Linear: $a, b$ do not depend on $u, y$, or their derivatives.

2) Nonlinear: $a, b$ do depend on $u, y$ or their derivatives. In general, the principle of superposition does not apply.

3) Time invariant: $a, b$ are constant at the operating condition and duration of the identification experiment.

4) Time variant: $a, b$ vary with time.

Accuracy

An error criterion must be established before the identification process is initiated to determine when the resulting model is sufficiently accurate. Often, the criterion is set such that the output of the model ($y_m$) is as similar as possible to the output of the system ($y$) for the same input. To insure this, the criteria is often defined in a quadratic manner, namely:

$$ E(y, y_m, \theta) = \int_{t_1}^{t_f} \left( y(t) - y_m(t, \theta) \right)^2 dt $$

For a time period ($T$) and a set of identification parameters ($\theta$). To insure the best possible model, a minimum is found by finding where $\frac{\partial E}{\partial \theta_i} = 0$ and $\frac{\partial^2 E}{\partial \theta_i^2} > 0$ in the parameter space.

3.2 Issues with Close-Loop Identification

In many experimental situations, safety reasons disallow the removal of feedback processes, as they are needed for proper operation of the system. In other cases, as with the thermoacoustic system, the feedback dynamics are inherent in the entire process and therefore, access to the feedback dynamics is not available. The existence of these feedback dynamics adversely affects the accuracy of the identified system. The mathematical reasoning behind the problems with identifying closed-loop systems revolves around the redundancy of data. For a closed-loop system, due to feedback, an input at a certain time is a function of an input at a previous time [Froisy 1974]. Therefore, a persistently differing input is not imposed on the system as the input at each
time instance is correlated with previous inputs. The problems with identifying closed-loop systems are noticed in either indirect or nonparametric techniques and the more detailed parametric identification counterpart.

For nonparametric identification, the forward and feedback dynamics are essentially combined due to the feedback nature. This is easily described by example. Take the following closed-loop system, illustrated in Figure 3.1 with forward dynamics \( G \) and feedback dynamics \( H \).

![Figure 3.1 Example Closed-Loop System](image)

A nonparametric identification approach, which strives to identify the forward dynamics \( G \) from input-output data \( u \) and \( y \), will erroneously approximate the system. With only input-output data for this structure, a direct identification routine will identify the equivalent open-loop system as the inverse of the feedback dynamics or:

\[
G' = \frac{1}{H}
\]  

[3.3]

Clearly this problem becomes devastatingly significant if the presumed structure of the dynamic system does not even consider feedback (open-loop structure is assumed). Though this result is distressing, accurate forward dynamics may be estimated with the simple inclusion of an additional signal in the feedback path [Wellstead 1981].

We expect that a parametric identification technique will more accurately identify the forward and feedback dynamics. Unfortunately, this is not the case, as shown in the next example [Gustavsson 1997]. Consider the linear feedback system illustrated in Figure 3.2
The closed-loop differential equation for this system is:

$$\dot{y} + (a - k_1 k_2) y = 0$$  \hspace{1cm} [3.4]

Unfortunately, there are an infinite number of inter-related estimates for $\hat{a}$ and $\hat{k}_1$ that produce a minimum least square error for commonly used identification techniques. Mathematically, for any estimate where:

$$\hat{a} = a + \delta k_2$$

$$\hat{k}_1 = k_1 + \delta$$  \hspace{1cm} [3.5]

for any nonzero $\delta$, the actual difference equation (3.4) for data provided by the actual parameters $a$, $k_1$ and $k_2$ will be valid. In essence, what this means is that from output data, a unique solution does not exist that separates the forward from the feedback dynamics. This issue leaves the engineer with two options. Either the feedback nature can be ignored leaving the identification of the closed-loop (un-separated) dynamics, or an arbitrary feedback gain ($\delta$) can be specified to allow the decoupling of the dynamics.

As the product $\delta \hat{k}_2$ approaches $k_2$, the actual open-loop characteristics will be recovered.

As the feedback gain goes to zero, suggesting to the algorithm that no feedback is present, the closed-loop dynamics are estimated. Unfortunately, as the structure of the system stands, an error function describing the difference between the closed-loop dynamic equation from the estimated parameters and of the actual parameters will not change with this designation. That is, a least squares approach will adjust the open-loop dynamics to sufficiently account for the specified feedback contribution.

### 3.3 Available Parametric Identification Techniques

Upon assuming the model structure with a parameter vector ($\theta$ from above), the pursuit for the identification process becomes an issue of estimating a parameter vector...
that minimizes the difference in the model and the actual system. Two methods lend themselves to this optimization. In the time domain, the well-known prediction error method minimizes the prediction of the models output with the actual output of the system. The method of harmonic balance is a technique in the frequency domain that obtains an estimate for the parameter vector by equating harmonic input and output information. This section outlines these procedures.

3.3.1 Prediction Error Methods

The prediction error method is one of the most commonly used approaches to system identification. This approach begins with a parametric model \( M(t, \theta) \) where \( \theta \) is a vector of parameters to be determined. The output of the model \( y_m(t) \) is defined as the prediction of the systems response. The best set of model parameters \( (\theta) \) is then found by minimizing the difference between the output of the system and the output of the model. This minimization routine can be executed by one of many approaches. These techniques include least squares, gradient based, and recursive approaches and are thoroughly discussed in many references [see Ljung 1987 for example].

The most influential downfall of this method (with respect to the identification of nonlinear systems) is its use of discrete representation of the dynamic system in question. The vector difference equation is an appropriate approximation to linear system dynamics in the time domain and is used for this identification approach. However, obtaining this representation for a nonlinear system requires an analytical solution to the nonlinear differential equation. This solution is challenging for nonlinear systems in general and therefore, we focus on efforts on methods that utilize the data in other ways. An example of this is the harmonic balance approach and is described below.

3.3.2 Harmonic Balance

The harmonic balance method facilitates the investigation of a periodic response from any system. It provides an avenue to the identification of a parametrically defined system. This is true for any type of system whether it be linear or nonlinear, in an open or closed-loop form. The harmonic balance approach originates from fundamental differential equation theory. With an assumed parameterized differential equation representation of the system, an approximate solution or actual data, the parameters are
found in a linear manner. This procedure involves system structuring, approximating the solution, solving the linear equations and is outlined below.

**System Structuring**

The system is described in differential equation form as in Equation 3.1. For purpose of example, we define a forced linear second-order system, namely:

\[
m\ddot{x} + c\dot{x} + kx = F\sin(\omega_f t)
\]  

where \(m, c, k\) are mass, damping and stiffness. Clearly if these coefficients are linear, and the damping is nonzero, the system will have a steady non-oscillatory output for its unforced response. Therefore, a periodic forcing term is needed for the system to have a periodic output. This is addressed with the addition of the forcing amplitude \((F)\) at a radial frequency \((\omega_f)\). It should be noted that the structure of the system is irrelevant to its overall block diagram. In other words, the differential equation can be derived from any state of which you have output. Being able to choose which state the equations are founded on allows more flexibility for the user and will be discussed later.

**Solution Approximation**

Considering that the response of the system is periodic, the solution is approximated in a harmonic manner. Here we approximate the solution as a single harmonic:

\[
x(t) = Xa\cos(\omega_f t) + Xb\sin(\omega_f t)
\]  

where \(\omega_f\) is the radial frequency of the assumed solution. The coefficients \(Xa, Xb\) are commonly known as Fourier coefficients of the periodic signal. Here, for this example, these coefficients are considered time invariant \(\left(\frac{dXa}{dt} = 0, \frac{dXb}{dt} = 0\right)\). The approximation of the solution is then differentiated to allow substitution into the differential equation.

\[
\dot{x}(t) = -\omega_f Xa\sin(\omega_f t) + \omega_f Xb\cos(\omega_f t)
\]

\[
\ddot{x}(t) = -\omega_f^2 Xa\cos(\omega_f t) - \omega_f^2 Xb\sin(\omega_f t)
\]  

The next step involves substituting the approximated response into the differential equation and collecting the equivalent trigonometric terms:
The assumption that the response is at the same frequency as the input minimizes the complexity of the harmonic balance. Clearly, this may not be the case for the input-output relationship of a nonlinear system and should be considered when reducing the equations if this type of response is predicted or observed in the data. With this \((\omega_s = \omega_f = \omega)\), the equations are further reduced revealing the linear system:

\[
\begin{bmatrix}
-\omega^2 Xa & \omega Xb & Xa \\
-\omega^2 Xb & -\omega Xa & Xb
\end{bmatrix}
\begin{bmatrix}
m \\
c \\
k
\end{bmatrix}
= \begin{bmatrix}
0 \\
F
\end{bmatrix}
\]  \[
\text{[3.15]}
\]

which is in the form \([A][P] = [F]\) where \(A\) is a system matrix, \(P\) are the parameters to be identified, and \(F\) are the forcing influences.

**Equation Solving**

At this point, the assumed system structure, as well as the approximated solution, provides a representation of the system dynamics as a linear function of the unknown parameters. The linear system may be solved in many ways. A least squares approach through matrix inversions is executed by:

\[
[P] = \left( [A]^T [A] \right)^{-1} [A]^T [F]
\]  \[
\text{[3.16]}
\]

With this, the parameters are determined in a linear manner with minimal error. This error, when using exact equations and exact data, is due to numerical inaccuracies. Here we identify an error estimate, as we will find it necessary to observe this identification accuracy when attempting to identify a system whose dynamics are not in the structure of the identification algorithm. For exact (and noiseless) data, and an exact estimate of the parameter vector, the following relationship holds:

\[
[A][P] - [F] = [0]
\]  \[
\text{[3.17]}
\]

Clearly, for real data, especially when there exists dynamics outside of the presumed structure, this relationship will not hold. Additionally, for the above example, the sine and cosine equalities create two equations of information, and therefore the linear system
is underdetermined. When the parameterized system contains more than two parameters, there does not exist a unique solution for the parameter vector in the identification equations. In this case, input with more than one frequency is needed to accurately identify the system. Similarly, when using data derived from simulation and a structure that exactly matches the model that produces the data, there will still exist a non-zero error due to numerical inaccuracies. Therefore, we assign the vector norm of this nonzero value as the identification error:

\[ E = \text{norm}_2([A P] - [F]) \]  

[3.18]

As we will highlight, this error is used extensively to monitor the accuracy of the least squares solution of the harmonic balance.

The harmonic balance technique is a wonderful tool for analyzing systems with periodic outputs. However, the ease in which the dynamic equations reduce depends on the structure of the system. When nonlinear processes are present, care must be taken as to not complicate the procedure. The introduction of nonlinear characteristics in the dynamic equations in a nondestructive manner is outlined below.

3.4 Nonlinear Harmonic Balance Technique

The nonlinear harmonic balance technique offers an approach for introducing nonlinearity into the balance equations without complicating their simplification. Previous work outlines this method for simple low order models [Yasuda 1989], models with multi-degrees of freedom [Yasuda 1993], and for actual physical systems where the forward dynamics are specified [Casas 1997, Murray 1998 and Casas 1999]. The difficulty with these approaches is due to the assumed availability of data. For the multi-degree of freedom work, it is assumed that modal data is available (which requires a high degree of instrumentation for an actual system). The work by Casas, though thorough, assumes that the forward dynamics of the closed-loop system are available. Again, this requires data that has been assumed to be unavailable for the current study.

In essence, the nonlinear harmonic balance introduces the nonlinearity into the harmonic balance equations as linear functions of the unknown parameters. The nonlinear harmonic balance technique uses the same theory needed to solve for the simple second order linear system described above. In contrast, when using this
technique on nonlinear systems, special attention is taken when structuring the system. In addition, the nonlinear influences on the response of the system require additional harmonic information to satisfy the needs of the harmonic balance. These two concerns are addressed below.

3.4.1 System Structuring

3.4.1.1 Establishing the equations of motion

As with the linear case above, the differential equations of motion are established. Most notably, the order of the system and the state in which the nonlinearity acts (viscous force, stiffness, etc…) is assumed. For the purpose of this discussion, we will use a second order system with a viscous nonlinear restoring force. The equation below illustrates this structure:

\[ mx'' + cx' + kx = f(x) \]  \[ 3.19 \]

where again \( m, c, k \) are the linear mass, damping and stiffness while \( f \) is a nonlinear viscous characteristic. These parameters are chosen such that the system has a self-excited response.

3.4.1.2 Determining the input

To gather more information and to simplify the numerics associated with the harmonic balance, we force the system. The advantages of forcing the nonlinear self-excited system are twofold; to gain an input-output model of the system, and to modify the output to simplify its frequency representation. Clearly, a nonlinear system with a stable limit cycle has a nonzero (and periodic) output in its unforced state. However, if the overall purpose is to control the system, its response to an input is needed.

For nonlinear self-excited systems, the output is predetermined by the linear and nonlinear dynamics of the equation of motion. As such, the amplitude and frequency of the output oscillation is often arbitrary and non-integer in value. This complicates the representation of the signals in the frequency domain. In order to accurately capture the harmonic information by a frequency representation using Fourier coefficients, the fast Fourier transform (FFT) must have a bin for each frequency of interest. This complicates the numerics of the data processing when the output frequency content of the self-excited system is at arbitrary and of non-integer values. For example, a self-excited system may have a dominant response at 170.63 Hz (with harmonic information at 341.26 Hz, 511.89
Hz, ...). Representing this response in the frequency domain would require a very fine meshed FFT. If the superharmonic information is valid at very high frequencies, the resulting FFT would be enormous. Therefore, by forcing the system to have an output at the same frequency as the input (chosen as an integer frequency), the complexity of the data processing will be minimized. We remind the reader that the resonant characteristics of a nonlinear self-excited system are such that forcing the output to have the same frequency as the input is not a trivial task. This issue is addressed thoroughly in Section 2.3.4.

To address the above concerns, the model structure is modified to incorporate input to the system. The linear dynamics and nonlinear feedback is now considered as a combined closed-loop system with an external input. As such, the equation of motion becomes:

\[ m\ddot{x} + c\dot{x} - f(\dot{x}) + kx = F \sin(\omega_f t) \]  

where \( F \) is the amplitude of the forcing signal and \( \omega_f \) is the forcing frequency (radians/second).

3.4.1.3 Nonlinearity Parameterization

By parameterizing the nonlinearity, we are able to introduce nonlinear influences while keeping the balance equations linear. To facilitate the incorporation of the nonlinear dynamics into the system structure, the nonlinearity is parameterized as a sum of known polynomials modified by unknown coefficients. In this manner, the set of polynomials becomes a basis for the actual nonlinear characteristic. In doing this, the nonlinear element enters the equations developed by the harmonic balance in a linear manner.

Though there are an infinite number of polynomial sets that can be chosen to parameterize the nonlinearity, two lend themselves in a beneficial manner to this problem. When using simulated data, which is needed to verify the code and the theory of the identification tool, the user specifies the nonlinearity. If this nonlinearity is chosen as a sum of simple polynomials (for example: \( f(\dot{x}) = ax^3 + bx^2 + cx \)), an exact basis for this nonlinearity can be chosen. The nonlinearity would then become:
\[ f(\dot{x}) = \sum_{i=1}^{3} N_i L_i(\dot{x}) \]

where: 
\[ L_1 = \dot{x} \]
\[ L_2 = \dot{x}^2 \]
\[ L_3 = \dot{x}^3 \]

where \( N_i \) are the unknown coefficients to be determined with the harmonic balance. On the other hand, if the nonlinearity is the result of an infinite number of polynomials (saturation for example), an interpolating basis is a better option for the parameterization. Below we describe the formulation of this basis.

The preliminary step for the parameterization in an interpolating manner is the definition of a region or domain where the nonlinearity is excited. For the system described by Equation 3.20, the domain of excitation is the maximum and minimum values of the velocity (\( \dot{x} \)) signal. That is, as the nonlinearity in Equation 3.20 is a viscous force, its contribution is dependent only on the velocities that occur during the self-excited oscillations. Though the velocities that will occur may not be predicted apriori, they can be determined by observing data.

It should be noted that for the current study, the nonlinearity is assumed to be a function of a single state. The mathematical attributes associated with a nonlinearity as a function of multiple states (for example the van Der Pol equation) complicate this matter. In essence, the domain of excitation would be multidimensional. Additionally, it should be noted that the region defined as the domain of excitation of the nonlinearity need not be symmetrical about zero (dc shifts are allowed).

The domain of excitation is now discretized by a finite number of nodes (\( 0 \rightarrow k \)). An arbitrary but known polynomial is then assigned to each node. The identification process then finds a coefficient (\( N_i \)) that modifies each polynomial (\( L_i \)) so that their sum equals the actual nonlinear polynomial.

\[ f(\dot{x}) = \sum_{i=0}^{k} N_i L_i(\dot{x}) \]

The representation of the nonlinearity in this manner is similar to interpolation schemes often used in mathematics. A common such scheme is the use of Lagrangian polynomials [Burden 1997]. Each node has a polynomial \( L_i(\dot{x}) \) of order \((k-1)\) whose
value is unity at the associated node while zero at all other nodes. Due to this property, the set of polynomials are linearly independent and can be considered as basis functions for the nonlinear map. Figure 3.3 illustrates a typical basis function for a node at –3 corresponding to the nodal set [-6 -3 0 3 6].

![Graph of a typical basis function](image.png)

**Figure 3.3 Example Basis Function used to Parameterize the Nonlinearity**

With either of the above-mentioned approaches for the parameterization of the nonlinearity, the equation of motion is now:

\[
m\ddot{x} + c\dot{x} - \sum_{i=0}^{k} N_i L_i(\dot{x}) + kx = F \sin(\omega t) \tag{3.23}
\]

Certainly the map of the nonlinearity is useless to the harmonic balance in its current form. To effectively incorporate the nonlinear contributions to the system, its frequency response is needed. As a means to this end, each basis function is excited with known data (the velocity state in Equation 3.20). The frequency response of the basis functions modified by the unknown constant coefficients \(N_i\) should be equivalent to the frequency response of the actual nonlinearity. With this in mind, the parameterization of the nonlinearity facilitates its linear incorporation into the harmonic balance.

### 3.4.2 Executing the Harmonic Balance

#### 3.4.2.1 Substituting Information

The above description of system structuring assists the harmonic balance, the product being a set of linear equations sufficient to solve and identify the system. Essentially the nonlinear harmonic balance solves for unknown constants \((m,c,k,N_i)\) using only data
from system states and input \((x, \dot{x}, \ddot{x}, F, \omega)\) and the modified data from the chosen basis functions \((\mathbf{L}_i(\dot{x}))\) with simple matrix manipulations. As with the linear example above, to simplify the system matrices, the known system states are represented in the frequency domain. However, due to the nonlinear effects, a single harmonic will be insufficient to describe all pertinent information about the system's response. By addressing a tradeoff between identification accuracy, and numeric complexity, the number of harmonic coefficients of the FFT is chosen. An example of the resulting representation of the position state with the retention of \(n\) harmonics is below.

\[
x(t) = X_0 + X_{a_1} \cos(\omega t) + X_{b_1} \sin(\omega t) + \ldots + X_{a_n} \cos(n\omega t) + X_{b_n} \sin(n\omega t) \tag{3.24}
\]

As with the example above, the position state is differentiated twice to obtain the data for the other two states of the system. In a similar manner, the Fourier transform of the input is taken resulting in its representation as:

\[
F(t) = F_0 + F_{a_1} \cos(\omega_j t) + F_{b_1} \sin(\omega_j t) + \ldots + F_{a_n} \cos(n\omega_j t) + F_{b_n} \sin(n\omega_j t) \tag{3.25}
\]

In the frequency domain, the parameterization of the nonlinearity including the excited basis functions becomes:

\[
f(\dot{x}) = \sum_{i=0}^{k} N_i \left[ L_{i,0} + L_{a_i} \cos(\omega t) + L_{b_i} \sin(\omega t) + \ldots + L_{a_i} \cos(n\omega t) + L_{b_i} \sin(n\omega t) \right] \tag{3.26}
\]

Assuming that the frequency of the self-sustained oscillation is modified to be that of the forcing signal (again, care is needed in this assumption as explained in 2.3.4) the harmonic equations are reduced. By collecting the sine and cosine terms of equivalent frequency, the equations resulting from the harmonic balance become:

\[
\begin{bmatrix}
-(\omega^2) X_{a_1} & (\omega) X_{b_1} & L_{a_{i,1}} & L_{a_{i,2}} & \ldots & L_{a_{i,k}} & X_{a_i} \\
-(\omega^2) X_{b_1} & -(\omega) X_{a_1} & L_{b_{i,1}} & L_{b_{i,2}} & \ldots & L_{b_{i,k}} & X_{b_i} \\
-(2\omega^2) X_{a_1} & (2\omega) X_{b_1} & L_{a_{i,1}} & L_{a_{i,2}} & \ldots & L_{a_{i,k}} & X_{a_i} \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
-(n\omega^2) X_{a_i} & (n\omega) X_{b_i} & L_{a_{i,1}} & L_{a_{i,2}} & \ldots & L_{a_{i,k}} & X_{a_i} \\
-(n\omega^2) X_{b_i} & -(n\omega) X_{a_i} & L_{b_{i,1}} & L_{b_{i,2}} & \ldots & L_{b_{i,k}} & X_{b_i}
\end{bmatrix}
\begin{bmatrix}
X_{a_1} \\
X_{b_1} \\
\vdots \\
X_{a_i} \\
X_{b_i}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
N_0 \\
N_1 \\
\vdots \\
N_{k-1}
\end{bmatrix}
\begin{bmatrix}
F \\
0 \\
\vdots \\
0
\end{bmatrix}
\tag{3.27}
\]
Clearly the set of linear equations resulting from the harmonic balance are larger than that of the simple example considering the linear system. This is due to both the addition of harmonic information and the increased number of unknowns introduced due to the parameterization of the nonlinearity. This brings to light a problem alluded to earlier. The amount of accurate harmonic information available defines the number of equations for the harmonic balance. The desired accuracy of the identified nonlinearity establishes the number of unknowns (appended to the linear unknown parameters i.e. $m$, $c$, $k$). If the number of unknowns exceeds the number of equations, the linear set of equations (3.27) is underdetermined and the system has no unique solution. We combat this issue by extracting more information from the unidentified system. Due to the characteristics of the response as well as signal-to-noise issues, harmonic information at high frequencies is invalid and additional information is not available in a single experiment. Therefore, we force the system using slightly different inputs. In essence, these different forcing conditions excite the dynamics of the self-excited system in the same manner though different enough that the response is dissimilar. We call this approach multiple experiment harmonic balance.

3.5 Application of Multiple Experiment Frequency Entrainment to the Nonlinear Harmonic Balance

Using multiple experiments for the entrainment data promotes accuracy in the identified model. This beneficial phenomenon is highlighted considering both linear and nonlinear aspects of the identification routine. Firstly, as with any identification procedure, input-output information at many forcing conditions is needed to accurately identify a modal system. In this manner, by obtaining information within a single resonance, significant dynamics are highlighted in the data. By this we mean the rate of change in phase with respect to frequency is a function of the damping of the mode in question. This information is only available if the mesh of frequency data is sufficient to capture the slope of this change. Secondly, multiple experiment entrainment experimentally offers describing function oriented data to be used in conjunction with the harmonic balance technique. By this, we mean the nonlinearity is excited by multiple
frequencies, and due to the resonant nature of the system, it also experiences multiple amplitude inputs. Therefore, we are introducing information similar to what is used in describing function analysis. These two concerns are addressed below.

As described in the system identification overview, the design of the input-output data acquisition experiment for the identification needs to consider many factors. To obtain a physically accurate model, it is essential to excite all pertinent dynamics of the physical process. Therefore, the input for a modal system must contain many frequencies. For linear, well-conditioned systems, this input is often specified as white noise within the bandwidth of interest. For nonlinear self-excited systems, due to the preexisting oscillatory energy, this input does not lend itself as well to the identification process. Therefore, multiple frequency entrainment strives to achieve two conditions for identification. Firstly, it offers a stable output at the same frequency as the input signal. Secondly, by entraining at multiple frequencies, the modal dynamics become more relevant in the data.

The second benefit of multiple frequency entrainment deals with the describing function analysis presented in Section 2.3.1. As mentioned, the describing function analysis is a quasi-linear approximation of the nonlinearity as a function of both the input amplitude and frequency of the physical state that excites the static nonlinearity. By obtaining multiple sets of experimental data, we are in essence obtaining information needed to approximate the nonlinearity as a describing function. An explicit investigation of this response is not needed as it is absorbed in the nonlinear harmonic balance equations.

Unfortunately, as we mentioned, describing function analysis is only sufficient for investigating odd nonlinearities. Due to the presumed structure of the thermoacoustic dynamics (to be outlined in Chapter 4), the describing function analysis for even nonlinearities becomes invalid due to differentiation in the loop equations. However, we feel that using this approach, in conjunction with the nonlinear harmonic balance method with the retention of at least one harmonic past the fundamental will be sufficient to accurately identify the even nonlinearity inherent in the thermoacoustic system. To summarize this, the describing function method identifies the odd contribution of the nonlinearity. Since, as we described in the introduction, even nonlinearities breed even
harmonic information, the inclusion of the second harmonic introduces the even information. Therefore, when balancing two harmonics of information, in conjunction with multiple entrainment experiments, sufficient information is introduced into the harmonic balance to identify the nonlinearity.
4 Analytical Study – Thermoacoustic Model

At this point, there should be sufficient understanding of relevant linear and nonlinear system theory needed to understand, analyze, and ultimately identify the thermoacoustic dynamics in a tube combustor that contribute to limit cycle behavior. Therefore, we create a model with sufficient complexity to observe phenomena associated with real data, and attempt to identify the model using its simulated response. In this chapter, we outline the criteria established for the development of the model, display its free and forced response and the results of using the nonlinear multiple experiment harmonic balance technique to identify its dynamics. In addition, we conclude by illustrating the downfall in identification accuracy when neglecting to accurately designate the structure of the system that produces the data.

4.1 Creating the Model

The response of the model is to be as similar as possible to that of the thermoacoustic system to be identified. Considering this, the most important aspect of its dynamic response is that it exhibits stable limit cycle behavior. In addition to this, the harmonic signature should be as close to possible to the tube combustor. By this we mean the frequency spectrum needs to contain energy at odd and even integer harmonics of the fundamental frequency of the limit cycle (Figure 1.3).

Previous work outlines modeling efforts for thermoacoustic instabilities. A series of such work is available from the Adaptive Control Laboratory at the Massachusetts Institute of Technology [Rumsey 1998, Fleifel 1997, and Annaswamy 1997]. These efforts address obtaining accurate low-order models from known physics for the investigation of nonlinear phenomena as well as linear control strategies for thermoacoustic systems. In a similar manner, documented studies at the California Institute of Technology [Culick 1987] derive detailed physically based equations, which mathematically describe Rayleigh’s criteria. In a third effort at United Technologies Research Center [Perracchio 1998], a physically-based model is derived and verified, but for this case, the combustor operates at conditions similar to those experienced in power generating turbines. For the current study, we only need a model sufficient to include major physical influences inherent in the thermoacoustic system. As such, these efforts
are used as a basis for the thermoacoustic model while disregarding details that make the resulting model more complex than it needs to be.

4.1.1 Defining the Differential Equations

4.1.1.1 Acoustic Signature

Since the acoustic dynamics play a key role in the systems response, the basis for the model is in modal coordinates. To accurately capture the acoustically specific physics, we begin with the acoustic wave equation. This equation consists of a linear, one-dimensional (spatially) partial differential equality that defines the wave motion in the acoustic environment. By assuming solutions based on boundary conditions, its infinite dimensionality is reduced. This section explains how this is executed, resulting in a low order acoustic model described by ordinary differential equations.

We start with the one-dimensional wave equation for harmonic plane waves with dissipation [Kinsler 1982]:

\[
\left(1 + \tau \frac{\partial}{\partial t}\right) \frac{\partial^2 p(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p(x,t)}{\partial t^2}
\]

where \( p \) is the acoustic pressure at any spatial location \( x \) and time \( t \), \( c \) is the speed of sound, and \( \tau \) is an absorption coefficient. Here we remind the reader that the model we are creating will be used to test an identification routine. We are not trying to exactly model the thermoacoustic system. Therefore, for example, the dissipation term is presented to introduce damping in the acoustic model. Here, we are not concerned with whether the damping is introduced through a boundary condition, or from within the acoustic medium.

Due to both the spatial and temporal dependence of this equation, solutions are complex. By defining spatial boundary conditions to this partial differential equation, we are able to simplify it by assuming normal modes. In essence, a normal mode (or eigenfunction) is a designated response for a partial differential equation. These functions describe possible spatial solutions for the equations. As such, by designating the spatial solution, the partial differential equation is reduced to an ordinary differential equation with only temporal dependence.

For a cylindrical, closed-open tube, we choose the normal modes, or mode shapes, to be sinusoidal and of the form:
\[ \psi_i(x) = \cos(k_i x) \]  

where \( x \) is the spatial coordinate from which the one dimensional wave equation is based, and \( k_i \) is the wave number. The wave number is a spatial frequency designating the dependence from the boundary conditions on the specific \( (i^{th}) \) mode. In choosing the basis functions in this manner, the principle of orthogonality holds. That is, some properties of these basis functions are:

\[ \int \psi_m \psi_n dV = 0, \quad m \neq n \]
\[ \int \psi_m \psi_n dV = E, \quad m = n \]  

\[ \frac{d^2 \psi_i}{dx^2} = -k_i^2 \psi_i, \]

where \( V \) is the volume of the environment, \( m \) and \( n \) are different mode shapes, and \( E \) is a positive constant analogous to the energy in the mode shape.

By designating the spatial dependence of the partial differential equation, the pressure at a location in the environment at a certain time becomes:

\[ p(x,t) = \sum_{i=1}^{n} \psi_i(x) \eta_i(t) \]  

where \( \psi_i(x) \) is the mode shape and \( \eta_i(t) \) is the modal pressure. The modal pressure is analogous to the contribution of the pressure in one mode to the entire pressure in the enclosure at that location \( (x) \) and time \( (t) \). In this manner, the pressure at a certain time and location within the acoustic environment will be the sum of the influence of many modes, each of which have different amplitudes at that time.

In the same manner, we define the acoustic particle velocity \( (u) \) as:

\[ u(x,t) = \sum_{i=1}^{n} \frac{d \psi_i(x)}{dx} \frac{d \eta_i(t)}{dt} \]  

Inserting these assumptions into Equation 4.1 we get the temporal dependence (at a location \( x \)) of the wave equation as:

\[ \sum_{i=1}^{n} \psi_i(x) \dot{\eta}_i(t) - c^2 \tau_i \sum_{i=1}^{n} \frac{d^2 \psi_i(x)}{dx^2} \dot{\eta}_i(t) - c^2 \sum_{i=1}^{n} \frac{d^2 \psi_i(x)}{dx^2} \eta_i(t) = 0 \]  

Here, we can specify the degree to which we reduce the partial differential equations by specifying the number of modes that describe the spatial response. With this, we have a
unforced low-order acoustic model of the tube. To complete the model, we include the
flame dynamics, and actuation influences that force the acoustics of the combustor.

4.1.1.2 Flame Contributions

We now introduce the feedback nature that the heat release introduces into the
thermoacoustic system. Using the above mentioned references, we assume that the time
derivative of the heat release from the flame forces the modal pressure of the tube ($\eta_i$)
and is a function of the acoustic particle velocity at the location of the flame in the tube.
In addition, due to its inherent mechanical nature, we assume a first order response. That
is:

$$\dot{q}(t) + \omega_f q(t) = u(x_f, t)$$  \[4.7\]

where $q$ is the heat release, and $\omega_f$ is the characteristic frequency of the flame, and $u(x_f, t)$
is the acoustic particle velocity at the location of the flame. In addition to the linear
flame dynamics we introduce a nonlinear function that modifies the heat release. This
function is needed to obtain both a stable limit cycle and a harmonic signature similar to
what is represented in the tube combustor response. With this, we obtain a feedback
structure for the heat induced influence on the modal pressure in the tube.

$$\eta_{f,i} = k_f f(\dot{q}(t))$$  \[4.8\]

Where $\eta_{f,i}$ is the flames influence on the $i_{th}$ modal pressure and $f(.)$ is the nonlinear
function, and $k_f$ is a gain. The specifics of the nonlinearity are found through simulation
by adjusting it to achieve the desired response, and will be described later. With the
acoustic and flame dynamics, we obtain a system that responds in a similar manner to the
tube combustor. However, to obtain input-output data, we need to introduce an
appropriate actuation contribution to the closed-loop thermoacoustic system. Below we
describe how we introduce this actuation influence.

4.1.1.3 Actuation Influence

To assimilate the actual experimental facility, we choose an acoustic loud speaker as
an actuator. For simplicity, we assume that the speaker is a source of only pressure in the
tube. By forming an inertance-based model of the speaker cone, its acceleration is
directly proportional to the current applied to the speaker. That is,

$$\ddot{x}_c(t) \approx k_a i(t)$$  \[4.9\]
where \( x_c(t) \) is the position of the speaker cone, \( k_a \) is a gain, and \( i(t) \) is the actuation current. By doing this, we are neglecting any actuation dynamics. Obviously, for any control system, actuation dynamics may contribute to the overall system dynamics significantly. Here we neglect the actuator dynamics to keep the model simple. We then assume that this acceleration source directly affects the modal pressure in the tube. By this, we obtain the actuation influence as:

\[
\eta_{a,i} = k_a \sum_{i=1}^{n} \psi_i(x_a) \dot{y}(t) \quad [4.10]
\]

where \( \psi_i(x_a) \) is the mode shape evaluated at the location that the acoustic actuation is introduced.

By adding the flame and actuation influences on the acoustic environment of Equation 4.6, we have the forced closed-loop thermoacoustic system:

\[
\sum_{i=1}^{n} \psi_i(x) \eta_i(t) - c^2 \tau \sum_{j=1}^{n} \frac{d^2 \psi_j(x)}{dx^2} \eta_j(t) = k_f \psi_i(x_a) f(\dot{q}(t)) + k_a \sum_{i=1}^{n} \psi_i(x_a) \dot{y}(t) \quad [4.11]
\]

To simplify these equations, we multiply by \( \psi(x) \) and integrate over the entire volume, which decouples the acoustic modes. This in turn completes the temporal dependent, physically based thermoacoustic model:

\[
\dot{\eta}_i(t) + c^2 k_i \tau \eta_i(t) + c^2 k_i \eta_i(t) = k_f \psi_i(x_a) f(\dot{q}(t)) + k_a \psi_i(x_a) \dot{y}(t) \\
\dot{q}(t) \equiv q(t) = u(x_f, t) \\
\]

\[
u(x_f, t) = \sum_{i=1}^{n} \frac{d\psi_i(x_f)}{dx} \frac{d\eta_i(t)}{dt} \quad [4.12]
\]

With this, we have the differential equations that describe the system dynamics. To obtain the parameters that further specify the dynamics, physical attributes of the tube are used, in addition to tuning some of the remaining parameters to achieve the desired response.

### 4.2 Introducing the Model into the Simulation Environment

The next step to the model derivation is the determination of its parameters. These parameters are chosen in two ways. Some aspects of the acoustic environment are trivial to some degree. Therefore, by designating the dimensions of this environment, certain parameters are subsequently resolved. Other parameters are not as straight forward from
the preexisting knowledge of thermoacoustic system. Here, we either assume appropriate values for the parameters, or tune these parameters to obtain a self-excited response similar to what is observed from experiment. Dealing with both of these parameter designation predicaments is outlined below.

4.2.1 Physically-based Parameter Designation

The most notable and readily available influence on the model parameters is the boundary conditions of the acoustic environment. Because of the closed-open nature of the tube combustor, odd quarter wavelength approximations for the spatially dependent frequencies ($k_i$) are chosen. From observing data, we note that due to the resonant characteristics of the second mode of the tube (approximately 175 Hz), the flame couples with these dynamics in an unstable nature. Therefore, it is essential that we model this mode shape. Since the response is nonlinear, we expect harmonic energies at integer-valued frequencies of this fundamental oscillation. Clearly, as illustrated in Figure 2.2, the modal characteristics of the acoustics will modify both the amplitude and phase of the oscillations at these harmonic frequencies. Therefore the inclusion of high order modes becomes a tradeoff between the equivalence between the model and the real system, and the complexity of the model structure. To keep the model low in order, we characterize the acoustics as two modes of 3 and 5-quarter wavelengths. With this we obtain the wave number $k_i$ for the assumed mode shapes by:

$$k_i = \frac{\left(\frac{\pi}{2}\right)}{L}$$

where $L$ is the length of the tube (60.0”). For the designation of a two-mode model, the wave number for the first modeled mode is found from evaluating Equation 4.13 with $i = 3$. In a similar manner, the second modeled mode is found by designating $i = 5$. We emphasize the word modeled because the first modeled mode is actually the second acoustic mode that the boundary conditions suggest. As noted in the closed-loop Equation 4.12, the wave number multiplied by the speed of sound results in the natural frequency of the acoustic mode in question. Therefore, by specifying this part of the response, we designate the frequency of the pole locations.
Upon specifying the two possible mode shapes for the acoustic model, the modal influence factors are determined. The modal influence factors are constant gains that specify the influence of the modal pressures (or velocities) on the absolute pressure (or velocity) at the location of interest. Mathematically, these coefficients are determined by evaluating the trigonometric eigenfunctions ($\psi_i(x)$) at the location of the flame and the actuator in the tube model. These locations are respectively 30.0 and 25.0 inches from the bottom of the tube. To further clarify the assumed mode shapes as well as the locations of both the flame and actuator, we illustrate the pressure mode shapes in Figure 4.1.

![Mode shapes of the tube](image)

**Figure 4.1 Assumed Pressure Mode Shapes and the Flame and Actuator Locations**

With the wave numbers chosen, which subsequently specifies the mode shapes, a major portion of the parameters for the model are specified. Table 4.1 provides the values for the physically-based parameters.
Table 4.1 Physically-based Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>3.09</td>
</tr>
<tr>
<td>$k_2$</td>
<td>5.15</td>
</tr>
<tr>
<td>$\omega n_1$</td>
<td>168.79 Hz</td>
</tr>
<tr>
<td>$\omega n_2$</td>
<td>281.33 Hz</td>
</tr>
<tr>
<td>$\psi_1(x_f)$</td>
<td>-0.41</td>
</tr>
<tr>
<td>$\psi_2(x_f)$</td>
<td>-0.24</td>
</tr>
<tr>
<td>$\frac{d\psi_1(x_f)}{dx}$</td>
<td>-0.16</td>
</tr>
<tr>
<td>$\frac{d\psi_2(x_f)}{dx}$</td>
<td>-0.10</td>
</tr>
<tr>
<td>$\psi_1(x_a)$</td>
<td>-0.22</td>
</tr>
<tr>
<td>$\psi_2(x_a)$</td>
<td>-0.34</td>
</tr>
</tbody>
</table>

4.2.2 Response-based Parameter Designation

The remainder of the parameters for the closed-loop thermoacoustic equations are chosen from intuitive insight and by tuning them to achieve a desired self-excited response.

The flame dynamics are chosen somewhat arbitrarily. We are confident that due to the physical nature of the flame, which suggests a low-pass dynamic response, dynamics do exist in the feedback loop though their exact characteristic is unknown. Therefore, we designate the low-pass characteristic frequency ($\omega_f$ in Equation 4.7) to be at 100.0 Hz so that there is a significant contribution to the phase at the frequency of the instability. The damping characteristics of the tube are chosen considering previous acoustic characterization of the tube. These experiments suggest that the damping coefficients for the second and third modes of the tube (not the model) are approximately 0.01 and 0.05 respectively. From this information, we designate the absorption coefficient of Equation 4.12, to meet the damping requirements. The remainder of the parameters are associated with both the nonlinearity and loop-gains. The loop-gains (flame gain and linear gain in the nonlinearity) is chosen to invoke unstable system dynamics. The nonlinearity is chosen to offer an even characteristic, which will breed odd and even integer harmonics. All of these values are tweaked while in the simulation environment to achieve a
response with a power spectral density similar to Figure 1.3. The values determined by the above-mentioned approaches are tabulated below in Table 4.2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_1$</td>
<td>1.8858e-005</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>5.6572e-005</td>
</tr>
<tr>
<td>$f()$ – linear</td>
<td>22.5</td>
</tr>
<tr>
<td>$f()$ – quadratic</td>
<td>$-\frac{15000}{4000^2}$</td>
</tr>
<tr>
<td>$f()$ – cubic</td>
<td>$-\frac{200}{4000^3}$</td>
</tr>
<tr>
<td>$k_f$</td>
<td>30.0</td>
</tr>
<tr>
<td>$\omega_f$</td>
<td>100.0 Hz</td>
</tr>
</tbody>
</table>

Upon specifying all of the necessary parameters for the model, its system representation is placed in a simulation environment. Here we use Simulink, which is a MATLAB-based simulation environment. This graphical environment lends itself to the simulation of nonlinear systems in the time domain. Below (Figure 4.2) is a snapshot of the thermoacoustic model in Simulink.

![Thermoacoustic Model in its Simulation Environment](image)

**Figure 4.2 Thermoacoustic Model in its Simulation Environment**

For the purpose of simplifying discussions to follow, an equivalent block diagram for the thermoacoustic model is presented below in Figure 4.3.
With the model established, we choose forcing conditions, which are sufficient to entrain the self-excited system. The appropriate forcing conditions are found by deriving the resonant characteristic of the thermoacoustic model with the parameters designated above. Below we describe the approach for obtaining these solutions.

### 4.3 Entrainment Solutions

Upon defining the parameters of the model, we have a differential equation that represents the feedback nature of the thermoacoustic system. The coefficients of this equation are attained algebraically though notationaly complex. As such, for the remainder of this discussion, we simplify these coefficients as single variables. We write the equations at the location of the velocity input to the feedback dynamics, designating this state variable as $Y$ (see Figure 4.3). With this in mind, we have two input sources. The first input is from the feedback path termed the flame influence ($u_f$), and secondly, we have the influence from the actuation signal ($u_a$). Considering this, we can reduce the dynamic equations (Equation 4.12) into an input-output representation. In the Laplace domain, its representation is:

$$
(d_6s^4 + d_4s^2 + d_3s^2 + d_2s + d_1)Y = (n_{f,1}s^4 + n_{f,3}s^3 + n_{f,3}s^2)\left(\frac{k_f}{s + \omega_f}\right)Y + \left(n_{a,3}s^3 + n_{a,3}s^2 + n_{a,3}s\right)u_a \tag{4.14}
$$

Here, the nomenclature for the variables is chosen considering the form of the system. The denominator coefficients, which designate the pole locations, are represented as $d$’s. These coefficients designate the pole locations for both the input and forward dynamics.
In a similar manner, the numerator coefficients, which define the system zeros, for the flame and actuation inputs are designated by \( n_f \) and \( n_a \) respectively.

To assess the entrainment stability for the thermoacoustic model, we use the procedure outlined in Chapter 2. First, the harmonic solution is assumed, allowing for the possibility of time-varying Fourier coefficients.

\[
Y(t) = A(t)\cos(\omega t) + B(t)\sin(\omega t)
\]

[4.15]

We substitute this solution into the closed-loop dynamic Equation 4.14. To include the solution into the left-hand side it is differentiated four times and substituted in the equation. In the same manner, for the right hand side, we multiply the assumed solution by the nonlinear coefficients and differentiate, substituting where applicable. Here the output of the nonlinearity contains many harmonics, of which we retain only the fundamental frequency associated with the assumed frequency of the solution.

As with the example in Chapter 2, we assess the entrainment stability by observing the stability of the coefficients \( A(t) \) and \( B(t) \) of the assumed solution in the time domain. By obtaining equations for their temporal dependence, we observe the stability of their equilibrium points for an array of forcing parameters (amplitude and frequency). Due to the complexity of the system described by Equation 4.14 these equilibrium points take on multiple characteristics. Namely, they may take on the form of an unstable node, stable node, or saddle point (see Table 2.1). A mathematically intense review of these possible types of equilibrium points and their behavior with respect to forcing conditions is available for further reading [Summers 1996]. Stable entrainment solutions are associated only with equilibrium points that take on a stable node characteristic.

As mentioned, to obtain the resonant curves for the thermoacoustic model, we choose an array of forcing conditions, and observe the stability of the output at that frequency. This forcing array for the current example is presented in Table 4.3.

**Table 4.3 Forcing Conditions for the Resonance Curves of the Thermoacoustic Model**

<table>
<thead>
<tr>
<th>Frequency</th>
<th>162 : 0.1 : 176 (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude</td>
<td>1E7 : 1E9</td>
</tr>
</tbody>
</table>

Again, for these forcing conditions, we expect stable solutions to occur at frequencies close to the limit cycle frequency and at high amplitudes. Below, Figure 4.4 shows the resonant curves for these forcing conditions. The green data points are locations of stable
entrainment (both eigenvalues are negative). The forcing conditions that compel the Fourier coefficients to have a saddle point characteristic (one eigenvalue positive, one negative) are depicted by the blue and yellow data points. In the same manner, unstable solutions (having both eigenvalues positive) are shown in red.

![Resonant Response of Thermoacoustic Model](image)

**Figure 4.4 Resonant Curves for the Thermoacoustic Model**

This figure shows that for the forcing amplitudes provided, the self-excited system is entrained only at frequencies close to the limit cycle frequency (approximately 169 Hz). The resulting amplitude of the entrained system will be proportional to the respected green data point in the above figure. The region of stable solutions (green data points) will increase in size by designating larger amplitude forcing signals. Therefore, for larger forcing amplitudes, it is possible to achieve stable entrainment at frequencies more significantly removed from the limit cycle frequency.

### 4.4 Simulation Data

With a sufficient understanding of the entrainment characteristics of the thermoacoustic model, we proceed with the entrainment experiments to obtain data for the identification process. To ensure sufficient entrainment, we choose rather large
amplitude forcing conditions at four locations skirting the limit cycle frequency. These conditions are tabulated below.

Table 4.4 Forcing Conditions for the Model-Based Entrainment Data

<table>
<thead>
<tr>
<th>Forcing Amplitude</th>
<th>Forcing Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1E10</td>
<td>166 Hz</td>
</tr>
<tr>
<td>1E10</td>
<td>167 Hz</td>
</tr>
<tr>
<td>1.5E10</td>
<td>170 Hz</td>
</tr>
<tr>
<td>2E10</td>
<td>171 Hz</td>
</tr>
</tbody>
</table>

The resulting forced spectral response plotted adjacent to the free response is available in Figure 4.5 with a zoomed view of the information close to the fundamental instability frequency.

Figure 4.5a,b Spectral Density of the Forced and Free Thermoacoustic Model

Here we highlight two aspects that will help clarify characteristics in the data from multiple frequency entrainment experiments. First, we notice in Figure 4.5b that in general, the oscillation energy is localized at the forcing frequency. That is, if the self-excited system is not sufficiently entrained, energy will still exist at the limit cycle frequency. This is evident in the power spectrum of the response by twin-peaks of energy. Since these frequencies are close to each other, beating will occur which adversely affects the assumptions of the harmonic balance. This type of energy does not exist in the above response, which illustrates its validity.

Secondly, we notice that the oscillatory data is available in a bandwidth of frequencies. That is, we obtain information at four frequencies around the limit cycle frequency (166 → 171 Hz). Due to the nonlinear contributions of the system, which breeds oscillatory energy at twice the frequency of the fundamental oscillation, we obtain a second bandwidth of information in the vicinity of the second harmonic. At this
location, the information spans twice the bandwidth as those at the limit cycle frequency. That is, the second bandwidth of information is $2 \times (166 \rightarrow 171 \text{ Hz})$, which is 10.0 Hz of information (as opposed to 5.0 Hz at the fundamental frequency). This is highlighted due to the necessity to consider the linear dynamics of the system and their contribution to both the data, and the results of the identification approach. By this we mean, if the linear dynamics of the system significantly influence the data in the vicinity of the second harmonic, they will be captured in the data. At the conclusion of this chapter, we discuss why this becomes important, especially when the linear dynamics are not accounted for in the identification structure.

### 4.5 Identification of the Model

To define the structure of the resulting equations for the nonlinear harmonic balance, we adopt the procedure outlined in Section 3.4.

#### 4.5.1 System Structuring

4.5.1.1 Establishing the Equations of Motion

The majority of the effort in establishing the structure of the harmonic balance is trivial, as the system to be identified is known. Simply stated, the closed-loop differential equation of the thermoacoustic model is used for the harmonic balance. To begin, we choose the velocity input to the flame dynamics ($Y$ in Figure 4.3) as the signal to balance. This state is chosen, as it is the best tradeoff between available data in the real system and the complexity that the balancing state introduces to the identification algorithm. Unfortunately, by choosing this signal, the feedback dynamics enter the harmonic balance nonlinearly. That is, since the nonlinearity follows the feedback dynamics, their contribution is introduced in a nonlinear manner. Below we describe how this complexity is overcome.

To introduce the feedback dynamics, we adopt a novel approach to their inclusion into the identification algorithm. Because these dynamics are in the feedback loop adjacent to the nonlinearity, their parameters will enter the harmonic balance nonlinearly. Therefore, we introduce these dynamics by filtering the output signal ($Y$), resulting in a new, filtered signal $Y_f$. This signal is used in conjunction with the output of the system in the harmonic
balance. The location where the closed-loop dynamics are probed is illustrated below, Figure 4.6.

**Figure 4.6 Schematic of the Signal Locations for the Harmonic Balance**

Therefore, the parameters that define the feedback dynamics do not enter the balance equations explicitly, but their effect influences the balance through filtering. With this approach, the feedback dynamics are identified by assuming the parameters for the feedback dynamics, and minimizing an identification error with respect to these parameters. As we will show, if this filter is unknown (as is the case with the actual system) we can find its dynamic characteristics by minimizing the identification error (Equation 3.18) with respect to an assumed filter parameter.

### 4.5.1.2 Nonlinearity Parameterization

Because the nonlinearity in the model is designated as a combination of exact polynomials, we can choose the basis on which it is parameterized exactly. Therefore, the basis functions are chosen as in Section 3.4.1:

\[
L_1 = Y_f, \quad L_2 = Y_f^2, \quad L_3 = Y_f^3
\]  

[4.16]

Upon assessing the balance state, formalizing the differential equations for the linear dynamics and parameterizing the nonlinearity, we are ready to construct the harmonic balance. Here we assume that all coefficients of the closed-loop differential equations are unknown with the exception of the first coefficient \(d_0\) of the denominator characteristic equation. By doing this, we simply designate this coefficient as 1.0. With this in mind, we equate all unknown contributions to the differential equation with this one known influence. The resulting representation is:
4.5.2 Executing the Harmonic Balance

4.5.2.1 Substituting the Information

Two issues are addressed before the data obtained from the simulation is entered into the harmonic balance equations. Firstly, we address the existence of the differentiation in the forward dynamics. Recall, in Equation 4.12, that these differentiators arise from the assumption that the time derivative of heat release forces the acoustics and that the flame dynamics are a function of acoustic particle velocity (proportional to the time derivative of pressure). With these differentiating influences in the dynamics of the closed-loop system, any steady-state dc signal will be canceled. Therefore, any attempt to identify based on dc offset data will result in erroneous estimations of the thermoacoustic dynamics. With this in mind, we neglect to balance any dc information in the harmonic balance equations.

The fact that dc information (which arises from the even nonlinear characteristic) is lost within the loop dynamics also illustrates that the use of the describing function method with bias will not work. As explained in Section 2.3.1, the traditional describing method approach to nonlinear system analysis is valid for odd nonlinearities only. However, modified methods are available for even nonlinearities that consider a bias signal that arises from this type of nonlinearity. If the linear dynamics cancel this bias (or dc) signal, this alternative method becomes invalid. Therefore, we choose to disregard this method, and assume that the describing function influence on this identification algorithm will only identify the odd contributions of the nonlinearity.

Fortunately, since we are balancing harmonics, the inclusion of the harmonic energy at twice the fundamental instability frequency will add the information that arises from the even nonlinear contributions. This conclusion impacts our approach by specifying the minimum amount of harmonic information needed to accurately identify the system. Clearly for simulation-derived data, with an identification structure that matches the model structure, additional information at higher harmonics will have a positive effect on the accuracy of the identified system. This however is not the case when the
identification structure is not equivalent to the structure that produces the data (whether in simulation or through experiment) as including harmonic information at high frequencies will possibly include unexpected information about the dynamics of the system at these frequencies. By concluding that the nonlinear harmonic balance will sufficiently identify the system using two harmonics of information, we avoid the complexities resulting from this occurrence.

With this in mind, we process the multiple frequency entrainment data for its inclusion into the harmonic balance equations. As discussed, we disregard the dc component and retain oscillatory information at the fundamental limit cycle frequency and its first harmonic. In this manner, in the frequency domain, the output \((Y)\) or the balance state is:

\[
Y(t) = Y_a \cos(\omega t) + Y_b \sin(\omega t) + Y_{a2} \cos(2\omega t) + Xb_2 \sin(2\omega t)
\]  

[4.18]

In processing the input signal, we note that the self-excited system is entrained using a pure sinusoidal input. Therefore, in the frequency domain, the input signal contributes only one Fourier coefficient of data:

\[
u_a(t) = Fb_1 \sin(\omega t)
\]  

[4.19]

Recall that the harmonic information from the nonlinearity is the frequency response of the basis functions \((L_i)\) excited by the state in which the nonlinearity acts \((Y_f)\). With this in mind, the nonlinearity becomes:

\[
f(Y_f) = \sum_{i=1}^{3} N_i [L_i(Y_f)] = \sum_{i=1}^{3} N_i [L_i a_1 \cos(\omega t) + L_i b_1 \sin(\omega t) + L_i a_2 \cos(2\omega t) + L_i b_2 \sin(2\omega t)]
\]  

[4.20]

By introducing the frequency domain information into Equation 4.17, we obtain the nonlinear harmonic balance equations. To conserve space, this linear system is presented in Appendix A. Notice in this equation that the different colors represent different entrainment experiments.

At this time, we bring to light a complexity that arises due to the modal nature of the acoustic dynamics. As the absolute pressure (or velocity) at a location in the tube is a function of the sum of modal pressures (or velocities) modified by their modal influence coefficients \(\psi(x)\) (or \(\frac{d\psi(x)}{dx}\)), the resulting input-output equivalent is the combination of the modal systems in parallel. Due to this, dynamic zeros are formed. To clarify this, Figure 4.7 illustrates the input-output equivalent of a parallel system.
The analogous zero coefficients for the two-mode tube model are \( n_{f,1} \), \( n_{f,2} \), and \( n_{f,3} \). The fact that the forward dynamics contain zero dynamics challenges the identification approach as the coefficients of the nonlinearity are multiplied by these coefficients. As such, the number of unknown parameters to identify is propagated. This can be seen in the parameter vector of the linear system in Appendix A. Notice that the unknown nonlinear coefficients \( (N_i) \) appear three times. As such, with the current structure, we are identifying redundant data. That is, we know that there exists only one nonlinearity, while as the structure stands, we identify three (in the noise-free case, these will be the same).

### 4.6 Results

As mentioned in Section 3.2, identification of closed-loop systems from single-output data provides a non-unique estimation for the system dynamics. In light of this, we highlighted two options that may be used for this situation. The first option is to identify the closed-loop system. The second option specifies the feedback dynamics, which if accurate, is sufficient to mathematically decouple the forward and feedback paths. The result of using the nonlinear harmonic balance considering both of the above-mentioned approaches is outlined below.
4.6.1 Identification of Closed-Loop Dynamics

As the structure of Equation 4.17 stands, no feedback contributions of the closed-loop thermoacoustic system are specified. Therefore, we expect that the forward dynamics and the linear contribution of the feedback dynamics will be mathematically coupled. As such, we expect the identified model to have a non-unique answer. Keeping this in mind, we choose to avoid the identification of any linear feedback contributions to the closed-loop equations. Simply stated, this is done by not specifying a linear part of the nonlinearity when it is parameterized. By dictating \( L_I = 0 \) in the parameterization effort, only the nonlinear contributions of the feedback are separated from the forward dynamics. In this manner, the forward dynamics become a closed-loop system. This is illustrated in Figure 4.8 by the enclosed region.

![Block Diagram Representation for Closed-loop Identification](image)

**Figure 4.8 Block Diagram Representation for Closed-loop Identification**

Using this approach, we solve the harmonic balance equations. To determine that indeed the algorithm is accurate, we compare the identified model with both the closed-loop linear system and the nonlinear contributions to the feedback. As mentioned in Section 4.5.1, we specify the feedback (or flame) dynamics, and modify this specification until an identification error is minimized. For the current discussion, we illustrate only the identified model when the feedback dynamics are specified correctly. We postpone the discussion of how the identification error varies as a function of this designation until...
Section 4.7.1. Below, Table 4.5 and Figure 4.9 compares both the identified and actual linear closed-loop dynamics and the nonlinear contributions in the feedback loop.

For presentation purposes, it is necessary to normalize the nonlinear coefficients. That is, as the nonlinear harmonic balance equations stand, the gain associated with the forward dynamics is coupled with the gain in the nonlinearity. In this manner, the product of these gains is correct, but cannot be distinguished. Therefore, we normalize both the actual and identified nonlinear coefficients for their comparison.

Table 4.5 Estimates of the Linear Dynamics Using Closed-loop Identification

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pole-1 Frequency</td>
<td>170.39 Hz</td>
<td>170.44 Hz</td>
</tr>
<tr>
<td>Pole-1 Damping</td>
<td>-0.0063</td>
<td>-0.0061</td>
</tr>
<tr>
<td>Pole-2 Frequency</td>
<td>280.94 Hz</td>
<td>281.47 Hz</td>
</tr>
<tr>
<td>Pole-2 Damping</td>
<td>0.0543</td>
<td>0.0523</td>
</tr>
<tr>
<td>Identification Error</td>
<td>NA</td>
<td>2.2727e-006</td>
</tr>
</tbody>
</table>

Figure 4.9 Estimate of the Nonlinearity using Closed-loop Identification

Please note that the characteristics of the pole locations (frequency and damping) differ from that of Table 4.1. That is, the values tabulated above represent the closed-loop dynamics, which for any linear system differ from the open-loop dynamics. In addition, we notice indeed the nonlinear harmonic balance technique identifies the closed-loop thermoacoustic model well. That is, both the linear and nonlinear contributions of the system are recovered with minimal error.
4.6.2 Identification of Open-Loop Dynamics

In an attempt to separate the identification of the forward and feedback dynamics, we specify information in the feedback loop. This approach, as discussed in Section 3.2, suggests that once the feedback dynamics are specified correctly, the open-loop forward dynamics of the identified system will converge to the actual forward dynamics of the model. To achieve this specification, we specify both the feedback dynamics, and the coefficient on the basis function that is associated with the linear contribution of the nonlinearity in the feedback loop ($L_f$). In doing this, the closed-loop equation becomes:

\[
-\left[ n_{j,1} \frac{d^2u}{dt^2} + n_{j,2} \frac{d^2Y}{dt^2} + n_{j,3} \frac{du}{dt} \right] + \left[ d_{j,1} \frac{d^2Y}{dt^2} + d_{j,2} \frac{d^2Y}{dt^2} + d_{j,3} \frac{dY}{dt} + d_{j,4} Y \right] - \left[ n_{j,1} \frac{d^3 \sum N_i L_i(Y_f)}{dt^3} + n_{j,2} \frac{d^3 \sum N_i L_i(Y_f)}{dt^3} + n_{j,3} \frac{d^3 \sum N_i L_i(Y_f)}{dt^3} \right] = \cdots
\]

\[
\cdots = -1.0 \frac{d^4 Y}{dt^4} + \left[ n_{j,1} \frac{d^4 N_i L_i(Y_f)}{dt^4} + n_{j,2} \frac{d^4 N_i L_i(Y_f)}{dt^4} + n_{j,3} \frac{d^4 N_i L_i(Y_f)}{dt^4} \right] \quad [4.21]
\]

Using this approach, we equate the trigonometric Fourier coefficients and solve the resulting harmonic balance equation. The results are presented below (Table 4.6 and Figure 4.10).

**Table 4.6 Estimates of the Linear Dynamics Using Open-loop Identification**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pole-1 Frequency</td>
<td>168.79 Hz</td>
<td>168.8337 Hz</td>
</tr>
<tr>
<td>Pole-1 Damping</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>Pole-2 Frequency</td>
<td>281.33 Hz</td>
<td>282.6604 Hz</td>
</tr>
<tr>
<td>Pole-2 Damping</td>
<td>0.05</td>
<td>0.0467</td>
</tr>
<tr>
<td>Identification Error</td>
<td>6.5468e-006</td>
<td>na</td>
</tr>
</tbody>
</table>

![Figure 4.10 Estimate of the Nonlinearity using Open-loop Identification](image-url)
Again, we notice that the linear and nonlinear contributions of the system are identified with minimal error. Though this approach is cumbersome due to the necessity to specify information in the feedback loop, here we show that if this information is available, both the remaining characteristics of the nonlinearity, and the linear dynamics in the forward path are accurately recovered.

4.7 Identification Sensitivity

To further assess the validity of this identification approach, we investigate its sensitivity to both forward and feedback dynamics. To explore the effect of poorly modeled feedback dynamics, we illustrate the error in closed-loop identification while misrepresenting the feedback dynamics that submit filtered information to the algorithm. In a similar manner, we discuss the accuracy of the open-loop identification approach with respect to the specification of the linear contributions of the nonlinearity in the feedback loop. As a means to survey the effect of poorly modeled forward dynamics, we highlight downfalls in trying to identify a four-mode model using the identification equations derived from a two-mode model.

4.7.1 Sensitivity to Unknown Feedback Dynamics

As mentioned, we investigate the identification variance with respect to two specifications in the feedback loop. The first inquiry looks at the results of the closed-loop identification versus the specification of the feedback dynamics (or flame contributions). The second inspection considers the effect of specifying the linear contributions of the nonlinearity in the attempt to identify in an open-loop manner.

4.7.1.1 Closed-loop Identification Results vs. Feedback Dynamics

We illustrated above that with the correct feedback filter, the closed-loop dynamics of the model, as well as the nonlinear coefficients are accurately recovered. Here we illustrate that if the dynamics of this filter are unknown, they can be found by minimizing the identification error. For this, the mathematical structure is exactly as before, while obtaining solutions for many feedback dynamic pole locations. The resulting closed-loop pole locations and damping for many feedback pole locations are illustrated in Figure 4.11a,b. Additionally, the identification error as a function of the pole specification is presented in Figure 4.12 (recall that the actual pole location is 100 Hz).
As illustrated in the above figures, when the feedback dynamics are specified correctly, not only do we recover the correct closed-loop dynamics, the identification error is minimized. This suggests that this method is sufficient to determine feedback dynamics when they are not known.

4.7.2 Open-loop Identification Results vs. Nonlinear Feedback Specification

To obtain the open-loop dynamics of the model, it is necessary to specify all linear contributions in the feedback loop. Above, we showed that when we specify the feedback (linear gain from the nonlinearity and flame dynamics) accurately, indeed we get accurate estimates of the forward dynamics. Here we illustrate the effect of improperly designating this parameter. We expect that with improper specification of these parameters, the least squares approach to identification will adjust the other parameters of the model to achieve minimum error. Therefore, by changing gain in the
feedback loop we will recover the open-loop dynamics when the gain is specified correctly. Below, Figure 4.13a,b illustrates this (recall the correct feedback gain is 22.5).

![Graphs showing damping vs. feedback gain specification and pole locations vs. feedback gain specification.](image)

**Figure 4.13a,b  Open-loop Identification vs. Feedback Gain Specification**

Notice that as the gain approaches zero, the identification algorithm *thinks* that there is no feedback, and therefore, the forward dynamics are of the closed-loop system. We also highlight that the nonlinear estimates and the error remains constant with respect to this variable. That is, upon specifying different feedback pole locations, the identification algorithm simply redistributes the dynamics in the forward path. In this manner, the closed-loop characteristic equation remains the same and therefore, the error does not change.

### 4.7.2 Sensitivity to Unknown Forward Dynamics

To further assess the usefulness of this identification algorithm, we investigate its performance when attempting to identify a model that does not fit the identification structure. We recall that the structure of the nonlinear harmonic balance is associated with a two-mode model. Clearly, for a physical system, which cannot be modeled *exactly*, modes that fail to appear in the identification structure may affect the data in a way that devalidates the results of the two-mode identification routine. Here we highlight the downfall associated with attempting to identify in this manner. We postpone recommendations to overcome this problem until the end of this manuscript.

To illustrate the downfall of under representation of modal influences in the identification algorithm, we simply present the frequency response of the original two-mode thermoacoustic model adjacent to a model with four modes, and discuss the influence of the additional dynamics. The two additional modes are specified as in
Section 4.2.1, using Equation 4.13 to find the wave numbers for the 7 and 9-quarter wavelength approximations to the mode shapes. The resulting physically based parameters for the third and fourth modes are tabulated below. Note that the characteristics of the first and second mode, and all gains (including the nonlinearity) remain the same.

### Table 4.7 Physically-based Model Parameters for the Third and Fourth Modes

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>7.21</td>
</tr>
<tr>
<td>$K_2$</td>
<td>9.27</td>
</tr>
<tr>
<td>$\omega n_1$</td>
<td>393.86 Hz</td>
</tr>
<tr>
<td>$\omega n_2$</td>
<td>506.39 Hz</td>
</tr>
<tr>
<td>$\psi_1(x_f)$</td>
<td>0.17</td>
</tr>
<tr>
<td>$\psi_2(x_f)$</td>
<td>0.13</td>
</tr>
<tr>
<td>$d\psi_1(x_f)$</td>
<td>0.071</td>
</tr>
<tr>
<td>$d\psi_2(x_f)$</td>
<td>-0.055</td>
</tr>
<tr>
<td>$\psi_1(x_d)$</td>
<td>-0.032</td>
</tr>
<tr>
<td>$\psi_2(x_d)$</td>
<td>0.179</td>
</tr>
</tbody>
</table>

Clearly, the addition of the extra two modes alters the open-loop frequency response of the system. Figure 4.14a,b illustrates this difference by plotting the original two-mode frequency response adjacent to the frequency response of the four-mode model (with a zoomed view in the vicinity of twice the instability frequency).

![Frequency Response Comparison for the Two and Four Mode Models](image)

**Figure 4.14a,b  Frequency Responses of the Two-mode and Four-mode Models**

As illustrated in the above plot, the frequency response is altered mostly in the phase. This is due to the authority of the modal influence coefficients on the magnitude of the
response which in turn attenuates the contribution of the extra modes in the magnitude response. As is evident in the above plots, the open-loop dynamics differ minimally at the frequency of the limit cycle oscillation (approximately 169 Hz). Therefore, the amplitude and frequency of the limit cycle oscillation remains similar. However, due to the impact of the extra dynamics at high frequencies, the harmonic signature is altered. This is evident in the power spectrum of the limit cycle oscillation (Figure 4.15a,b).

![Power Spectral Density of Velocity for Two and Four Modes](image1)

![Power Spectral Density of Velocity for Two and Four Modes](image2)

**Figure 4.15a,b** Power Spectrum of the Two-mode and Four-mode Models

Note that the spectrum at the fundamental oscillation frequency is very similar. On the other hand, oscillations at two and three times this frequency are altered significantly (this information is available in the zoomed view).

At this point, we identify why the addition of this high frequency dynamic information adversely affects the result of the nonlinear harmonic balance. As illustrated in the frequency response of Figure 4.14a,b, the phase is significantly modified at the twice the frequency of the of the limit cycle oscillation. Recall that this information is needed to obtain the even characteristic of the nonlinearity. By dictating the harmonic balance to have only two modes, its least squares solution becomes a tradeoff between identifying a nonlinearity that will produce the amplitude, and linear dynamics that offer phase in the region of the second harmonic (the multiple experiments offer information in a bandwidth). Therefore, the degree in which the resulting identification is accurate is dependent on the difference in phase that the two-mode model can produce and the phase in the actual system. Clearly when identifying a physical system, this impact may be very significant.
To overcome this obstacle, we propose to identify the first mode accurately while adding additional dynamics in the forward path as a residual system that will modify the linear dynamics in a manner that corrects the phase. By this, we can find a residual dynamic, which has essentially no physical accuracy but will modify the phase and amplitude such that the phase at the locations where there is entrainment information is correct. As to not alter the nonlinear harmonic balance equations as they stand, we introduce this dynamic through filtering, similar to the above explanation considering the feedback dynamics. Below, Figure 4.16 illustrates the location within the thermoacoustic block diagram where the residual dynamic is introduced.

![Figure 4.16 Block Diagram for the Inclusion of Residual Dynamics](image)

**Figure 4.16 Block Diagram for the Inclusion of Residual Dynamics**

Though this approach seems feasible, the introduction of additional parameters to minimize the identification with respect to complicates the identification process. Clearly, if the residual dynamics needed to correct the linear dynamics is dependent on a single parameter (unknown pole location for example), and the feedback dynamics are unknown, we can specify a two-dimensional parameter space and choose their locations based on the identification error in this space. However, if the residual dynamics needed to correct for the unmodeled modes contain more than one parameter and the feedback dynamics are unknown, the parameter space, and thus computation time grows substantially based on the minimization technique used. With this in mind, we have illustrated the downfall of the two-mode identification algorithm as it stands.

When using data created by the four-mode thermoacoustic model, we find that indeed the accuracy of the identification is very dependent on the accuracy of the residual dynamics. Due to the modal nature of the model, the addition of two acoustic modes breed two complex poles and two complex zeros. The sensitivity of the resulting
identification is very dependent on these dynamics. This is due to the interlaced frequencies that the modal system exhibits. That is, the additional dynamics are not only close in frequency to each other, their frequencies are in the bandwidth of the harmonic information used for the harmonic balance. Since the needed residual dynamics contain eight parameters (frequency and damping for four sets of complex roots), it is intuitively obvious that identifying these dynamics through a minimization approach is not a good idea. This in turn, clearly de-justifies the effectiveness of the identification algorithm developed in this manuscript. However, alternative approaches to deal with the current problem are presented in the closing chapter.
5 Case Study – Thermoacoustic Experiment

With sufficient understanding of the behavior of the identification algorithm, we attempt to use it to identify the physical tube combustor. This chapter explains how we obtain and process the data, assesses predicted concerns with respect to unmodeled dynamics and presents the best results achieved for this study.

5.1 Experimental Data

As explained in the introduction, the Virginia Active Combustion Control Group has a tube combustor whose thermoacoustic dynamics exhibit limit cycle behavior. Fortunately, since this experiment was considered while creating the identification algorithm, its general structure is retained while using actual data. Below we describe the tube conditions, and the approach used to obtain the acoustic particle velocity at the flame.

5.1.1 Tube Conditions

Clearly, the operating condition of the tube must be in a regime where limit cycle behavior occurs. That is, we avoid conditions of stable combustion. In addition, we choose an operating condition where the instability is not so severe. Since we are using a loudspeaker actuator, we only have a finite amount of actuation pressure available. Therefore, at some operating conditions that demonstrate high amplitude pressure oscillations, we expect attempts at stable entrainment to fail. Therefore, we choose an operating condition with a clearly evident, but somewhat mild instability amplitude. Table 5.1 presents the two variables that dictate the operating condition of the tube combustor, where phi is the actual fuel-air ratio divided by that for stoichiometric equivalence, and the total flow rate includes the air and fuel mass flow rates.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total flow rate</td>
<td>125 cc/sec</td>
</tr>
<tr>
<td>Phi</td>
<td>0.55</td>
</tr>
</tbody>
</table>

5.1.2 Obtaining Particle Velocity at the Flame

As we mentioned, the acoustic particle velocity at the location of the flame is the most worthwhile system state to balance in the nonlinear harmonic balance technique.
Unfortunately, as the experiment stands, this signal is not readily available. Although there are sensors that measure fluid particle velocity, they will not withstand the combustion environment. Therefore, we derive the velocity measurement from two pressure measurements. This section describes this approach, and summarizes the hardware involved in the data acquisition effort.

5.1.2.1 Establishing a Velocity Signal from Two Pressure Signals

The two-microphone method of determining the net rate of flow of acoustic energy per unit area (velocity) is based on a fundamental acoustic equality. This approach to measuring velocity, and its validity is outlined in a reference from Purdue University [Waser 1984]. By rearranging Euler’s equation [Kinsler 1982]:

\[ \rho \frac{\partial u}{\partial t} = -\nabla p \]  \[5.1\]

we obtain a relation for the time-dependent acoustic particle velocity as a function of two, spatially differing pressure measurements. This becomes:

\[ u(t) = -\frac{1}{\rho} \int \frac{p(x_2,t) - p(x_1,t)}{x_2 - x_1} \]  \[5.2\]

where \( u(t) \) is the velocity, \( p(x,t) \) is a pressure measurement and \( \rho \) is the density of the medium. Clearly, to achieve the most accurate results, we need to take these two measurements as close as possible to the location of the flame. This is achieved by introducing pressure probes at 2.0 and 4.0 inches upstream of the flame location. These \( \frac{1}{4}'' \) microphones (B&K model 4136) are of very high quality and the output of their conditioning amp (B&K model 2690) is assumed to be calibrated to one another (no phase introduced between the signals). Upon taking these two measurements, Equation 5.2 is evaluated off-line to achieve the approximate velocity measurement three inches upstream from the flame location.

5.1.2.2 Implementing Acoustic Actuation

Obviously, to entrain the limit cycle response of the tube combustor, we need to introduce an actuation source. This is achieved by attaching a loudspeaker at a location just below (upstream of) the flame. To direct the acoustic energy, a plenum is constructed which funnels the acoustic energy from the loudspeaker through a port in the tube. The loudspeaker is a 3”, 15-Watts RMS midrange tweeter (Radio Shack #40-
1289A). It is attached to a cylindrical plenum, 2.75” in diameter, 2.75” long connected to the tube with a 5/16” diameter, 3/4” long pipe fitting. Figure 5.1 further captures its representation.

![Figure 5.1 Acoustic Actuator Mounted on the Tube Combustor](image)

To excite the actuator, a signal generator supplying a pure sinusoid is amplified by a Peavey (model CS800S) amplifier. This high quality amplifier amplifies the actuation signal while adding negligible phase. The output of this amplifier is acquired as the entrainment (or forcing) signal.

5.1.2.3 Data Acquisition Equipment

The remainder of the experimental arrangement deals with the collection of the signals in an accurate manner. Each of the above-mentioned signals necessary for the entrainment-based identification (forcing signal, and two pressure signals) are filtered to prevent aliasing when sampled. Low-pass 8-pole elliptic filters are used for this purpose (Frequency Devices #9002). The cut-off frequency is set for 1.5kHz for each signal as this leaves an appropriate bandwidth of valid information. The most important consideration when acquiring the signals is that they are simultaneously sampled. If not, a phase difference, although small, will be introduced to the signals, which will influence any analysis of the data in a destructive manner. To overcome this issue, a National Instruments Simultaneous-Sampling Differential Amplifier (model SC-2040) is used to gather signals at exactly the same time instance. A personal computer based multi-channel data acquisition card then acquires the output of this conditioner. To achieve this measurement with a high signal-to-noise ratio and flexible sampling options, a National Instruments 16-bit, 100KS/s board is used (model PCI M10-16E-10). With this, we have
all components necessary for the experiment. Below, Figure 5.2 illustrates these components in a graphical manner.

![Figure 5.2 Experimental Schematic for the Entrainment Experiments](image)

### 5.2 Entrainment Conditions and Data

The forcing conditions sufficient to entrain the thermoacoustic system are found by experimental observation. That is, since we lack a model of the system, we cannot predict which amplitude and frequency combinations for the forcing signal will entrain the limit cycle oscillation in a stable manner. Therefore, we choose an array of frequencies near the limit cycle frequency and increase its amplitude until the limit cycle oscillation is entrained. We justify stable entrainment when the amplitude of the fundamental oscillation is decreased by approximately 20 dB at that frequency (which is replaced by energy at the entrainment frequency). For four entrainment experiments, we find the amplitude of the signal generator voltage that is necessary to achieve this. These conditions are tabulated below.

**Table 5.2 Entrainment Conditions for the Tube Combustor**

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Amplitude (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>172</td>
<td>1.50</td>
</tr>
<tr>
<td>173</td>
<td>1.25</td>
</tr>
<tr>
<td>179</td>
<td>0.50</td>
</tr>
<tr>
<td>180</td>
<td>0.63</td>
</tr>
</tbody>
</table>
Using these forcing conditions, the data is taken as described above. Upon capturing a window of information (1.0 seconds of data, sampled at 10.0 kHz), the pressure signals are processed to obtain an approximate for the velocity at the flame. The resulting spectral density (with a zoomed view) is available in Figure 5.3a, b.

![Figure 5.3a, b Spectral Response of the Forced Tube Combustor](image)

Again, as in the forced response of the tube model, we notice that the entrained energy close to the limit cycle frequency is pure. That is, there is only one spike in the power spectrum, which is associated with the frequency of the forcing signal. In addition, a bandwidth of information is available due to the multiple experiments. Specifically, 8.0 Hz of information is available at the fundamental frequency, while 16.0 Hz is valid near the frequency of the second harmonic.

### 5.3 Establishment of Possible Unmodeled Dynamics

As we mentioned in Section 4.7, dynamics that are not included in the identification structure significantly affect the results. Therefore, it is essential that we consider possible dynamics that may not be included when deriving the nonlinear harmonic balance equations from the thermoacoustic model. In summary, we feel that additional dynamics may arise from the actuation system, the path between the microphones and the flame, and unmodeled acoustics. In this section, we describe how these traits of the physical system may arise.

#### 5.3.1 Additional Unmodeled Actuation Dynamics

As it stands, the actuation input relates the current to the speaker directly to pressure perturbations at the location of the actuator in the tube. However, it is intuitively obvious
that just from considering the physical configuration of the speaker and plenum combination, additional dynamics may arise. It is known that a well-modeled acoustic loudspeaker has dynamics due to the inductance in its windings and from the mechanical properties of the speaker cone. This can be modeled as a single low frequency pole from the electrical influence and a pair of complex poles from the mechanical resonance. In addition to this, the plenum modifies the signature due to its geometry. With this in mind, we expect at least one more pair of complex poles contributing to the actuation system as a whole. In order to become more familiar with the actuator, its bench-top experimentally obtained frequency response (between pressure at the plenum exit and applied voltage) is presented in Figure 5.4.

![Bench-top Frequency Response of the Actuator](image)

**Figure 5.4 Bench-top Frequency Response of the Acoustic Actuator**

The purpose of the current discussion is not to identify or completely characterize the above frequency response, as we only want to present information that will help to direct our attention to reasons why the identification algorithm, in its current state, may fail. It should be noted that the above frequency response (between actuation voltage and pressure output) is from a bench-top test. When implementing the actuator on the tube combustor, the dynamics will be loaded and the above frequency response will change. In addition to this, the model structure specifies an actuation pressure proportional to actuation current. Due to this, if the bench-top experimental analysis is to be useful, it needs to consider the phase difference arising from converting a voltage signal to a current signal. Therefore, the actual dynamic influences that enter the signals used in the identification routine will differ.
5.3.2 Additional Dynamics Between the Sensor Location and the Flame

As mentioned, to obtain the acoustic particle velocity at the flame, two microphones are used and the velocity is subsequently derived offline. Here we suggest, that since these sensor locations are removed from the actual location of the flame, additional phase is introduced. That is, we expect that through this approach, the velocity derived will be most accurate at a location halfway between the two microphones. This, due to limitations in the experimental apparatus is 3.0” inches upstream from the flame location. We feel that this distance may be accurately modeled by additional dynamics. Based on the eigenfunctions shown previously, it seems likely that the primary influence of the unmodeled phasing will arise from the effect of the unmodeled dynamic zeros rather than unmodeled poles. These dynamics append themselves to the feedback dynamics in the structure of the thermoacoustic model.

5.3.3 Additional Dynamics from the Acoustics

As previously mentioned, the physical equations that govern the acoustic nature of any system are infinite in dimension. By assuming modes in the system, these equations are simplified in manner that facilitates their analysis. By this, we weigh the tradeoff between accuracy and numerical complexity. Currently we have not derived a criteria for the degree of which we are allowed to simplify the acoustic properties of the system. We do not know the minimum number of modes that are needed to accurately describe the acoustics of a physical system when considering the nonlinear harmonic balance approach to identification. Here we simply highlight that indeed their influence will affect the results of the identification process.

The above-mentioned considerations are not brought forth to devalue the current identification approach. There discussion simply brings to light topics, that if ignored, may contribute to erroneous results. Here we suggest an alternative that takes into account the additional unmodeled dynamics.

As we previously discussed, the monitoring of the identification error is a powerful tool to analyze the results of the nonlinear harmonic balance. We suggest that while keeping the two-mode identification structure, we can arbitrarily introduce additional dynamics, that upon minimizing the identification error, will be sufficient to accurately capture the above-mentioned influences. These dynamics are introduced as before
through filtering. Possible locations for the inclusion of these residual dynamics are presented in Figure 5.5, highlighted by shading.

![Figure 5.5 Locations for the Inclusion of Arbitrary Dynamics](image)

#### 5.4 Identification Results

At this point, we have sufficiently described why we feel that there may be some error in the results when using the actual data. The possible contribution of any of the aforementioned unmodeled dynamics is sufficient to confuse the two-mode structure of the identification algorithm. To overcome this issue, we outlined an approach to introduce additional dynamics without altering the structure of the identification algorithm. Here, we discuss the results we expect to obtain, and effect of introducing the residual dynamics.

The experiment, as it stands, offers no information about the linear or nonlinear contributions in the feedback loop. As we discussed in Section 4.6.2, accurate information in the feedback loop is necessary to accurately identify the forward dynamics. Here, since we do not have this information, we identify the coupled forward and feedback contributions as a closed-loop system. With this in mind, we expect to identify an unstable pair of dynamic poles close to the frequency of the instability. This physically signifies the unstable second mode of the tube due to the positive feedback from the flame. Since we have no prior knowledge of the nonlinear characteristics of the flame, concluding that the identification algorithm is producing accurate results is not as straightforward. However, we expect that there exists only one nonlinearity. Therefore,
the three nonlinear characteristics should be equivalent. Using these criteria, we introduce different combinations and forms of residual dynamics in the candidate locations described above.

The most significant issue related to the modification of the identification routine by introducing the residual dynamics is due to the limited time available for this study. That is, there are many different dynamic systems that can be placed into one of the candidate locations for the residual dynamics. Considering this, and the fact that we have identified three such locations, there exist many possible combinations. In addition to this, for each set of residual dynamics that we choose, we limit ourselves to two minimization parameters. That is, for example, we can specify low-pass residual dynamics for the actuation input and the feedback path. Of these two specifications, we can vary the pole locations and observe the identification error (minimizing on two parameters). Once this error is minimized, we have the best possible solution for this structure.

In light of this, we try many combinations of residual structures, and while varying two parameters, we analyze the best solution for this structure. When the best solution approaches the above-mentioned results that we expect, we conclude that this structure is sufficient to modify the identification algorithm in a manner that addresses unmodeled dynamics.

Upon investigating many possible residual structures, we find that the dynamic zeros of the residual forward dynamics most significantly affect the results. That is, the solutions become most accurate (considering what we expect) when residual dynamics are introduced here. Due to this, we choose to introduce a pair of complex zeros, of which, their damping and natural frequency is parametrically modified to minimize the identification error. To make this dynamic system proper, we specify a pair of complex poles in the denominator. These dynamics remain constant with the minimization routine. Below, Table 5.3 is the characteristics of the residual dynamics, while Figure 5.6 presents the error surface for this residual structure.

<table>
<thead>
<tr>
<th>Dynamic</th>
<th>Frequency (Hz)</th>
<th>Damping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complex zero</td>
<td>175.0</td>
<td>-0.003</td>
</tr>
<tr>
<td>Complex pole</td>
<td>340.0</td>
<td>0.005</td>
</tr>
</tbody>
</table>
Figure 5.6 Identification Error Surface for the Inclusion of Residual Dynamics

Note that the characteristics of the complex zero are found by minimization. Using the residual dynamics tabulated above, the harmonic balance equations are inverted to obtain the remaining characteristics of the system. The linear characteristics are tabulated in Table 5.4, while the nonlinearity is presented in Figure 5.7.

Table 5.4 Estimates of the Closed-loop Linear Dynamics of the Tube Combustor

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pole-1 Frequency</td>
<td>176.39 Hz</td>
</tr>
<tr>
<td>Pole-1 Damping</td>
<td>-0.0043</td>
</tr>
<tr>
<td>Pole-2 Frequency</td>
<td>94.38 Hz</td>
</tr>
<tr>
<td>Pole-2 Damping</td>
<td>0.8953</td>
</tr>
<tr>
<td>Identification Error</td>
<td>0.0117</td>
</tr>
</tbody>
</table>
Figure 5.7 Identified Nonlinear Characteristic of the Tube Combustor
As illustrated above, the results, while using these residual dynamics agree with what we expect. That is, we obtain an unstable complex pole (or mode) at the frequency of the instability. In addition, the nonlinear characteristics are virtually identical. This suggests that the combination of the identified modes in the harmonic balance, with the addition of the residual dynamics, accurately represents the dynamics of the tube combustor at the frequencies where it is identified (entrainment frequencies and their first harmonic). Unfortunately, both the residual dynamics, and the second identified mode have little physical significance. This issue is clearly important for an in-depth, control-oriented identification study. As such, we discuss approaches to overcome this problem in the conclusion of this manuscript.
6 Conclusions and Future Effort

The nonlinear balance approach with multiple experiment frequency entrainment identifies a thermoacoustic system exhibiting limit cycle behavior when the assumed structure is close to the actual structure that produces the data. By parameterizing the nonlinearity in the feedback loop, its inclusion into the harmonic balance equation becomes linear. With the additional information provided by multiple experiment frequency entrainment, inverting the harmonic balance equations provides a correct estimate of both the dynamics and nonlinear characteristic that produces the data. Unfortunately, when the dynamic structure assumed for the harmonic balance equation differs from the actual structure of the system, accurate results of the identification algorithm are challenged. We introduced methods that may account for this variance while keeping the original structure of the model. Here, we offer different, more intensive approaches to modify the algorithm in a manner that captures the additional dynamics inherent in the system.

We previously suggested possible efforts to passively modify the identification algorithm to account for additional, unmodeled dynamics in the data. At this point we submit two approaches, which change the effort significantly, as they may be necessary for the correct identification of the actual system. In essence, these suggestions include more information in the harmonic balance equations. The first approach specifies modal information inherent in the acoustics of the system. In a similar, but removed manner, we suggest that by changing the identification structure to include more modes, the least squares solution from the harmonic balance will not be restrained to the placement of only two modes which as we found, results in an inaccurate identification. Below we outline the methods to address these two approaches.

6.1 Specification of Modal Information

By specifying correct modal information, the forward dynamics of the thermoacoustic system are designated, which deters ambiguity in the identification approach. This modal information, if obtained properly, will offer the acoustic pole locations and damping of the modes. In addition, the modal influence coefficients will be specified, which in turn designates the zero locations of the acoustic model. With this information, if accurate,
the nonlinear harmonic balance technique is simplified significantly. Two methods are available with respect to designating the modal influences of the acoustic environment. The first method includes specifying the mode shapes of the acoustic enclosure. The second method uses intense instrumentation to experimentally derive the same information. Both of these approaches were considered prior to the study, but were ignored due to the acoustic complexity of combustors in industry, and because numerous sensors are not always available. However, as we have highlighted with this study, this information may be necessary for accurate identification of the system at hand.

6.2 Increasing the Dimensions of the Nonlinear Harmonic Balance

With the inclusion of additional modes in the identification structure, the algorithm is not restrained and therefore, it is possible to accurately identify the thermoacoustic system. Unfortunately, due to the modal nature of the system, the number of unknown parameters to identify increases tremendously when including additional modes. Due to the parallel nature that these dynamics are associated, the inclusion of a complex pair of poles (one-mode) creates another complex pair of zeros. As we discussed, the coefficients of the nonlinearity breed with the coefficients of the zero dynamics of the forward path in the system. Therefore, considering the structure of the two-mode thermoacoustic model presented in Chapter 4, the addition of a single acoustic mode increases the unknown coefficients by 5 (two from the poles and three from the zero-nonlinear products). Clearly, there is only one nonlinearity in the system and therefore, we will be identifying redundant information. We feel that through mathematical analysis, or novel recursive algorithms, this complexity can be overcome allowing for the creation of an identification structure with more modal content with out significantly increasing the complexity of the algorithm.
## Appendix A

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REFERENCES


Vita

Bryan Eisenhower was born in Charleston, South Carolina February 14, 1975. Many of his early days were spent in the water, enjoying the wonders of wind-driven water sports, and achieving the second fastest time in the state of South Carolina in swimming (butterfly stroke). Upon moving to Clifton, Virginia (suburb of Washington DC) in 1985, Bryan became actively involved in the outdoor adventure and achieved his Eagle Scout rank at the age of 13.

The repair of his motorcycle engine and the mechanics of his sisters Volkswagen motivated a hobby that would eventually take much of his time. Prior to receiving his driver’s license, he was given a Porsche, which had experienced an engine fire. Throughout high school, he focused much of his time on the repair of these vehicles. While a senior in high school, Bryan competed in guitar competitions, was introduced to the art of ceramics, and landed a job at a Porsche and Mercedes service shop, which heightened his knowledge of auto mechanics. With all of these outlets, he was unsure what his future held so he enrolled in a Community College general education program while maintaining the auto-related employment and musical hobbies. During the short years of Community College, Bryan purchased the equipment to construct his own pottery studio, was promoted by his employer to a technician status, and purchased many music-related equipment. Upon receiving recognition for achievements in physics and in engineering fundamentals classes, he decided that he should focus his efforts on formal education and enrolled as a Mechanical Engineer at Virginia Tech in the spring of 1996.

In the years following his transfer to Virginia Tech, Bryan focused all of his efforts on schoolwork, leaving music, cars, and pottery behind. Upon graduating at the top of his class and with honors in 1998, he decided to continue his work with a professor he had been working with. In the two years of graduate school, Bryan revisited his affection for automobiles by purchasing and maintaining antique Type-3 Volkswagens, and also enrolled in a local pottery class. Here he was turned onto an ancient Japanese glazing technique (Raku) that has re-inspired his desire to make pottery. Upon receiving his Masters Degree, Bryan plans to obtain a job in engineering while maintaining his hobbies of ceramics and auto repair.