CHAPTER 3
BACKGROUND AND RELATED WORK

This chapter will provide the background knowledge required for the rest of this thesis. It will start with the definition of the input signal function and terminology related to the triangulation mesh. This will be followed by a discussion of the triangular representation, the image reconstruction method and its error metric. The definitions of Delaunay and data dependent triangulation will also be presented in this chapter. Finally, this chapter will end with a discussion of graph theory for triangular mesh applications.

3.1 Input and Output Data

Usually a natural gray-scale image contains a lot of redundant information. This redundancy in data can be used for file compression so that less storage or transfer capacity will be required. The similarity in intensity and rate of change at different scales are some of the duplications in data that can be used in compression. Since it is possible to segment some regions into triangles without substantially degrading the quality of the image, a triangular mesh is a good approach for data compression for storage reduction and data processing.

3.1.1 Gray-Scale Image

A gray-scale image is usually presented in the two-dimensional domain. It can be viewed as a set of finite intensity (or magnitude) values being sampled at finite intervals with time (or space) domain. This can be mathematically expressed as:

\[ V = \{ v_{ij} \mid v_{ij} = f(x_i, y_j) \in \Omega, i = 1, \ldots, N_1, j = 1, \ldots, N_2 \} \] (3.1)

where \( \Omega \) represents the parameter domain and \( N_1 \) and \( N_2 \) represent the number of samples along each dimension of the image. Meanwhile, the intensity of the image, which can be found at each of the sample points, can be defined as:
where $z_y$ represents the quantized value of the intensity at sample location $(x_i, y_j)$. Normally this value is in an integer that ranges from 0 to 255.

3.1.2 Triangulation

A triangulation is a partitioning of the 2D image plane into a set of triangles. It may be represented using a set of appropriate vertices, which are located on the original grid of sample points, and a set of edges that connects those vertices. A vertex can be represented by three parameter variables as

$$V_i = (x_{i}, y_{i}, I_{i}) \quad 1 \leq i_1 \neq i_2 \neq i_3 \leq N_v$$

where $N_v$ is the number of vertices found in the triangulation. Note that $(x_{i}, y_{i})$ is the sampled location of the vertex $V_i$ and $I_i$ is the quantized intensity at that location.

In addition to vertex information, a triangulation requires vertex connection information. This relationship is usually defined in the form of edges. An edge is a line segment that connects two vertices and it can be expressed by

$$E_i = (V_{i_1}, V_{i_2}), \quad 1 \leq i_1 \neq i_2 \leq N_e$$

A triangle is a planar patch, which is determined by three non-collinear points that are connected by three line segments. This patch can be defined by

$$T_i = (V_{i_1}, V_{i_2}, V_{i_3}), \quad 1 \leq i_1 \neq i_2 \neq i_3 \leq N_T$$

where the three vertices $V_i, V_j, V_k$ are located at three different sample points in the two-dimensional $x-y$ plane and $N_T$ is the number of triangles. The triangulation, $\Gamma_v$, in a finite parametric domain, $\Omega \subset R^2$, must satisfy four conditions:

(i) Let $V$ be the set of all vertices found in $\Gamma_v$ such that $V_i \cap V_j = \phi$.

(ii) $E_i \cap E_j$ is either $\phi$ or a vertex if $i \neq j$.

(iii) $\Omega = \bigcup_{i=1}^{N_T} T_i$, where $N_T$ is the number of triangles in a given triangulation.

(iv) $T_i \cap T_j$ is either $\phi$ or a common edge $E_k$, or a common vertex $V_k$ if $i \neq j$.
Therefore the triangulation $\Gamma_v$ can be defined as

$$
\Gamma_v = \{ T_i | i = 1, ..., N_T \}
$$

(3.6)

Since it is common to define an operator for the number of elements in a set as $|\cdot|$, we define $N_T = |\Gamma_v|$ as the number of triangles in the triangulation $\Gamma_v$, which covers the parameter domain $\Omega \subset \mathbb{R}^2$ and $N_v = |V|$ as the number of vertices in the triangulation $\Gamma_v$.

Figure 3.1 and 3.2 illustrate some of the regular sampled grid triangulations and how the sampling can affect the quality of the image. The original image of a crater lake is shown in Figure 3.1 (a) while (b) shows an example of the locations of points on the 16x16 regular sample grid. Figure 3.2 (a-c) show the triangulations that are sampled at different regular finite intervals of 4, 8 and 16 respectively. Since the image size is 256x256, the grid sizes are 64x64, 32x32 and 16x16 respectively while the reconstructed images are shown in Figure 3.2 (d-f). These images are reconstructed using the Gouraud shading technique, which will be explained in Section 3.3. Although denser triangulation $\Gamma_v$ gives a better approximation of the original image, it requires more vertices and triangles for image representation.

### 3.2 Delaunay and Data-Dependent Triangulations

There has been much research concerning the minimization of error using refinement methods. Some of these methods are greedy insertion [De Floriani et al. 85], feature-based [Scarlatos and Pavlidis 92], and hierarchical subdivision [De Floriani et al. 84]. The greedy insertion algorithm scans for a vertex location where total error reduction is achieved. This method produces a good approximation of the original dataset. However it is time-consuming to search for an optimal point. This drawback motivates the search for a faster algorithm. A feature-based method is proposed using the features to determine the vertices or edges of the triangulation [Chen and Schmitt 93]. Edge detectors and Laplacian filters have been proposed to detect some of these features for vertex insertion in mesh refinement [Southard 91].
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Figure 3.1 – Regular sample grid. (a) Original image of crater lake of size 256x256. (b) Locations of sample points taken by the regular sampling grid of size 16x16.

Figure 3.2 - Approximation of an image using a regularly sampled triangular mesh. Triangular meshes with regular sampling grid size of (a) 64x64, (b) 32x32 and (c) 16x16. Reconstructed images using Gouraud shading on the regular sampling grid of (d) 64x64, (e) 32x32 and (f) 16x16 respectively.

Another approach is hierarchical subdivision [Scarlatos and Pavlidis 92], which recursively subdivides a triangle into smaller sub-triangles to represent a better approximation of the original dataset. The advantage of this method is its fast computation and use of multiresolution modeling. Its reconstructed image quality, however, is no better than that of the greedy insertion method.

Besides these refinement processes, one of the equally important decisions for triangle representation is to determine the choice of triangles without considering error metrics. One of the proposed methods for choosing triangles is Delaunay triangulation [Kropatsch and Bischof 01]. Generally, this type of triangulation is known as the dual of the Voronoi
diagram, which segments the parameter domain into regions in such a way that points lying in the same region are located nearest to the vertex of its region. A Voronoi polygon is defined by

\[ \kappa(p) \cap H(p_i, p_j) \]  

(3.7)

where \( p_i \) is a discrete point (or vertex) of interest and \( H(p_i, p_j) \) is a half-plane that contains the set of points that lie in \( R^2 \) and are located nearer to \( p_i \) than to \( p_j \). Therefore, if there are \( N \) vertices in the domain, each Voronoi polygon can result from the intersection of at most \( N - 1 \) half-planes. These polygons are convex with no more than \( N - 1 \) sides.

Delaunay triangulation can be achieved from a Voronoi diagram by connecting vertices, whose regions in the Voronoi diagram intersect. The result of using the Delaunay scheme is the triangulation that maximizes the minimum angles of all triangles. In other words, in a Delaunay triangulation, the circle that circumscribes three vertices of any triangle contains no other vertices. Figure 3.3 below shows an example of a Voronoi diagram with ten vertices of interest and its corresponding Delaunay triangulations, while Figure 3.4 shows an example of Delaunay triangulation and non-Delaunay triangulation.

Figure 3.3 - Example of a Voronoi diagram and its Delaunay triangulation. (a) Voronoi diagram and its points (or vertices) of interest. (b) Corresponding Delaunay triangulation.
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Figure 3.4 – Concept of Delaunay triangulation. (a) In a Delaunay triangulation, no other vertices lie in the circumscribing circle of each triangle. Circle $C_1$ does not include vertex $V_4$ and circle $C_2$ does not contain vertex $V_1$. (b) Non-Delaunay triangulation. Circle $C_1$ contains more than three vertices. It includes vertices $V_1$, $V_2$, $V_3$ and $V_4$.

Although the Delaunay triangulation method is fast in regularizing the triangulation, it does not guarantee an optimal solution to the approximation. In many applications, the choice of Delaunay triangulation can significantly increase the approximation error. It is found that non-Delaunay triangles can produce a better result in most cases because long, sliver triangles can give a better approximation, especially along the edges, where large changes in intensity occur.

Figure 3.5 compares the results of reconstructed images of Lena using both Delaunay triangulation and non-Delaunay triangulation. An edge swap operation, which will be discussed in Section 5.4.1, is performed on the 16x16 regular grid triangulation to improve the reconstruction image quality. It can be observed that the Delaunay triangulation tends to produce a lot of inappropriate triangles along the edge in the image. This is because it does not allow the thin, sliver triangles in the set. However, the data-dependent triangulation, which permits sliver triangles, yields a better rough approximation to the image data. The reason is that a change in intensity can be represented more accurately by thin, sliver triangles by short, ‘fat’ triangles. However it is possible to combine these two important parameters to achieve a better result. The choice of combining Delaunay and data-dependent triangles will be discussed in Section 5.4.1.
Figure 3.5 – Comparison of Delaunay triangulation and non-Delaunay triangulation. (a) Original image of Lena. (b-c) A Delaunay triangulation and its reconstructed image. (d-e) A non-Delaunay triangulation and its reconstructed image.
3.3 Gouraud Shading

There are many techniques to perform shading of a triangle. The choice of rendering not only helps the triangulation-based image look more realistic, but also smooths the edges between adjacent triangles. The shading technique used in this thesis is based on the Gouraud shading technique [Watt 93]. Given a sequence of three vertices \((V_1, V_2, V_3)\) and their intensity values \((I_1, I_2, I_3)\), it is possible to perform Gouraud shading on a triangle by bilinear interpolation. Figure 3.6 shows the concept of how Gouraud shading can be performed.

To render a triangle, first, its maximum and minimum values \((y_{\text{max}} \text{ and} y_{\text{min}})\) in the \(y\) (vertical) direction are determined. These two values are used for the vertical scan limitation. Next, for each row \(y_s\) starting from \(y_{\text{max}}\) to \(y_{\text{min}}\), its horizontal boundaries \((x_a \text{ and } x_b)\) and their intensities \((I_a \text{ and } I_b)\) are calculated by interpolating a necessary pair of vertices, whose edges give the limitation of the horizontal boundaries.

\[
x_a = (y_s - y_1) \frac{x_2 - x_1}{y_2 - y_1} + x_1 \tag{3.8}
\]

\[
x_b = (y_s - y_1) \frac{x_3 - x_1}{y_3 - y_1} + x_1 \tag{3.9}
\]

\[
I_a = (y_s - y_1) \frac{I_2 - I_1}{y_2 - y_1} + I_1 \tag{3.10}
\]

\[
I_b = (y_s - y_1) \frac{I_3 - I_1}{y_3 - y_1} + I_1 \tag{3.11}
\]

These interpolated values not only smooth the intensity between two vertex points but also smooth the connection between two connected triangles. This is because the two connected triangles share the two vertices of same location and intensity. Finally, in the same way, the intensity inside the triangle can be calculated by the horizontal scan of \(x_s\). With each value of \(x_s\) from \(x_a\) to \(x_b\), the intensity inside the triangle, \(I_s\), can be calculated as

\[
I_s = (x_s - x_a) \frac{I_b - I_a}{x_b - x_a} + I_a \tag{3.12}
\]
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For computational efficiency, incremental calculation is preferable:

\[ \Delta I_s = \frac{\Delta x}{(x_b - x_a)} (I_b - I_a) \]  
(3.13)

\[ I_{s,n} = I_{s,n-1} + \Delta I_s \]  
(3.14)

where \( \Delta I_s \) is the incremented change in intensity and \( n \) is the value from \( x_a \) to \( x_b \). This method significantly improves the calculation time by replacing multiplication operator with addition. However one of its drawbacks is its imprecision. Since there is round off in \( \Delta I_s \), the error tends to get larger as \( n \) increases.

One of the disadvantages of employing Gouraud shading is a visual side effect known as Mach banding. The change in intensity triggers the human visual system to perceive a brighter or darker band of intensity. This is due to the high sensitivity in the human visual system to the first derivative of intensity, which is used to detect and enhance edges [Watt 93]. Figure 3.7 shows the Mach band phenomenon. Notice the overshoot that occurring in the perceived signal relative to the actual ramp signal.
3.4 Error Metric

Since a reconstructed image is an approximation of the original, there will be an error between the original and the reconstruction. Therefore it is necessary to define an error metric to measure the quality of the reconstructed image. Let \( f_{T_i} \) be a planar patch that approximates a triangular region, \( T_i \), in an image. For \( i = 1,2,...,N_T \), \( f_{T_i} \) must satisfy a plane equation defined by \( P_i(x,y) = a_i x + b_i y + c_i \), which can be determined by the three vertices forming the triangle \( T_i \). In the parameter domain, this approximated patch can be defined as:

\[
\begin{align*}
  f_{T_i}(x,y) &= \begin{cases} P_i(x,y) & (x,y) \in \Omega(T_i) \\ 0 & \text{otherwise} \end{cases} \\
  P_i(x,y), &= g(z_i) \quad i = i_1, i_2, i_3
\end{align*}
\]

where \( \Omega(T_i) \) defined the boundary of the parameter domain of the triangulations and \( i_1, i_2 \) and \( i_3 \) are the vertices of triangle \( T_i \). The function \( g(z) \) is used for a better approximation when noise or a high frequency component is present in the data. Since the wavelet transform yields both approximation and detail of the original data, the function \( g(z) \) can take
advantage of this computation. With the definition of the approximation of triangles, the approximation of the triangulation can be defined as

\[ f_{\Gamma_r} = \sum_{i=1}^{N_T} f_{\Gamma_i} \]  \hspace{1cm} (3.17)

which accounts for the approximation of \( N_T \) non-overlapping triangles. The error metric, which is shown in Figure 3.8, can be denoted by \( \|f - f_{\Gamma_r}\| \). It is normally measured by the average of the sum of squared errors, which is normally called mean-squared error (MSE). MSE can be defined as

\[ \bar{\Delta}^2 = \frac{1}{\Delta x \Delta y} \int \int_{\Delta x \Delta y} \left| f(x,y) - f_{\Gamma_r}(x,y) \right|^2 dx dy \]  \hspace{1cm} (3.18)

This is computationally equivalent to

\[ MSE = \frac{1}{MN} \sum_{x=1}^{M} \sum_{y=1}^{N} \left[ f(x,y) - f_{\Gamma_r}(x,y) \right]^2 \]  \hspace{1cm} (3.19)

where \( M \) and \( N \) denote the height and width of the image respectively. Another popular error metric is the peak-signal-to-noise ratio (PSNR). It is defined by

\[ PSNR = 20 \log_{10} \frac{I_{\text{max}}}{\sqrt{MSE}} \]  \hspace{1cm} (3.20)

where \( I_{\text{max}} \) is the possible maximum intensity value in the image. Usually, this value is 255 for a gray-scale image. In reconstruction, a PSNR, that is larger than 30 dB, will often appear to have little or no visible degradation. However, this still depends on the image itself.

3.5 Graph Theory

Since the approximation algorithm in this thesis is based on triangulation, the two-dimensional image has to be segmented into many triangular regions, which are represented by three vertices. One of the solutions to present this type of data structure is to use an undirected graph. A graph \( G = (V, E) \) consists of a set of vertices, \( V \), and a set of edges, \( E \), which defines a relationship between the vertices.
To construct a valid triangulation, there are many cautions that should be made. First, instead of a general graph, the data structure should preserve the simple planar graph. Otherwise, it violates the fourth condition of the definition of a triangulation, which states that the intersection of two triangles should be equal to zero or an edge. Figure 3.9 shows example of a general graph and a simple planar graph. Notice a self-loop, \( z \), an edge that connects a vertex to itself, is found in general graph. This self-loop also must not exist in a triangulation.

Figure 3.8 – Error calculation for the Gouraud shading reconstructed image.

Figure 3.9 – General and simple planar graphs. (a) General graph. (b) Simple planar graph.
Since the graph is mathematically defined by its vertices and their binary relation, the positions of the vertices and the curvature of the edge are not factors used to consider the equivalence of the two graphs. Figure 3.10 shows two equivalent graphs, which have different representations.

![Figure 3.10 – Equivalent graphs. (a) Original graph. (b) Equivalent graph with different representation.](image)

It is possible to show that these two graphs are equivalent by the bijection mapping of the vertex and edge names. The following shows the equivalences to show that they are the same graphs.

<table>
<thead>
<tr>
<th>Vertex Equivalent</th>
<th>Edge Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A \rightarrow M )</td>
<td>( v \rightarrow q )</td>
</tr>
<tr>
<td>( B \rightarrow N )</td>
<td>( w \rightarrow r )</td>
</tr>
<tr>
<td>( C \rightarrow O )</td>
<td>( x \rightarrow s )</td>
</tr>
<tr>
<td>( D \rightarrow P )</td>
<td>( y \rightarrow t )</td>
</tr>
<tr>
<td></td>
<td>( z \rightarrow u )</td>
</tr>
</tbody>
</table>

Two graphs \( G = (V, E) \) and \( \tilde{G} = (\tilde{V}, \tilde{E}) \) are isomorphic if there exists at least one bijection \( f : G \rightarrow \tilde{G} \) such that \( (U, V) \in E \) if and only if \( (f(U), f(V)) \in \tilde{E} \). This vertex and edge bijection can be defined by:

\[
 f_v : V_G \rightarrow \tilde{V}_{\tilde{G}} \quad \text{and} \quad f_e : E_G \rightarrow \tilde{E}_{\tilde{G}} \quad (3.21) 
\]

Furthermore, the composite function \( f \) also inherits the isomorphism property from the two isomorphic functions. The composite functions of two isomorphic functions can be denoted by

\[
 f = f_1 \circ f_2 = f_1(f_2) \quad (3.22)
\]
In the special case of a simple graph, if there exists a vertex bijection $f: V_G \rightarrow \tilde{V}_G$ such that $f(x)$ is adjacent to $f(y)$ if and only if $x$ is adjacent to $y$ for all $(x, y) \in V_G$, the two graphs are isomorphic. Therefore only vertex bijection is sufficient to show isomorphism of the two simple graphs. This property is essential to designing templates for initial triangulation, which will be discussed in Section 4.2.3.