MIXED-FIELD FINITE-ELEMENT COMPUTATIONS

by

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Abstract

A new method called the Direct Method is developed to solve for the propagating modes in waveguides via the finite-element method. The variational form of the Direct method is derived to ensure that an extremum is reached. The Direct method uses Maxwell’s equations directly, both zero and first-order, scalar and vector bases that are used in the finite-element formulation. The direct solution method solves for both the magnetic and electric fields simultaneously. Comparisons are made with the traditionally used vector-Helmholtz equation set. The advantages and disadvantages of the newly developed method is described as well as several results displayed using the WR-90 waveguide and a circular waveguide as test waveguides. Results include a partially filled dielectric loaded rectangular waveguide. The effects of including the divergence of the fields in the functional as penalty terms on the quality of results obtained by the Direct method and the vector-Helmholtz method is explored. The quality of results is gauged on the accuracy of the computed modes as well as the elimination or a significant reduction in the number of ’spurious modes’ that are often encountered in solutions to waveguide problems. It is shown that computational time for the solution and computer storage requirements exceed the typically used Helmholtz equation method but the results obtained can be more accurate. Future work may include developing a sparse eigenvalue solution method that could reduce the solution time and storage requirements significantly.

The Direct method of solution in dynamics resulted after an initial search in magnetostatics for methods to solve for the magnetic field without using the magnetic-vector potential using finite-element methods. A variational derivation that includes the boundary conditions is developed for the magnetic-vector potential method. Several techniques that were used to attempt accurate solutions for the magnetostatic fields with multiple materials and without the use of the magnetic-vector potential are described. It was found that some of the newly developed general techniques for magnetostatics are only accurate when homogeneous media are present. A method using two curl equations is developed which is a Direct method in magnetostatics and reveals the interaction between the bases used. The transition from magnetostatics to dynamics is made and similar Direct methods are applied to the waveguide problem using different bases.
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Dedication

This Dissertation and the entire duration of my work at Virginia Tech is dedicated to my wife Mieko. Without her I would not have been able to accomplish several other things as well besides this degree. It was she who encouraged me to continue to pursue my education at Virginia Tech when I almost quit the program a few years back.
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There are several people and organizations that I am grateful to for helping me reach this milestone. Almost the entire time I was a student at Virginia Tech, I was also employed at Kollmorgen Corp., Radford which is now a part of Danaher Corp. (I would have liked to be still employed there but I was laid off). I thank Kollmorgen for the assistance provided with tuition as well as the time that I was given even during working hours for the drive from Radford to Blacksburg and back including the time for attending classes and hunting for parking space at Tech.

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during the time I spent at Tech and then introduced me to my present Advisor as an opportunity for a solution, if not for Dr. Brown I would not be writing these lines either.

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Bangalore, India
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Chapter 1

Introduction
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1.1 Introduction and Outline of the Dissertation

The focus of this research is to investigate alternatives, to the commonly used vector Helmholtz equation in finite elements to obtain solutions to the propagating modes in waveguides. Some alternatives improve the quality of the results and offer a variational view of finite-element methods. A new technique, called the Direct method, is developed that solves for both the electric and magnetic fields simultaneously.

Finite-element methods (FEM) are widely used in electromagnetics in almost every discipline. Applications include solutions for the magnetic field and flux density in magnetostatics for electric motor, lifting electromagnet, and actuator design. Both statics and dynamics are used for eddy current computations in both high and low frequency electromagnetics. In dynamics, finite-element analysis (FEA) is often used to obtain solutions to the propagating modes in waveguides and their field patterns, or to find the resonant frequencies of cavities. Finite-element methods are also used in scattering by arbitrarily shaped objects, radiation, and antenna design.

This research is a result of an initial search to obtain solutions to magnetostatic problems without directly using the magnetic-vector potential in the finite-element method. It is also an extension of the research by Bunting and Davis [1999] in dynamics for obtaining solutions via new and unique methods to the propagating modes in waveguides. Almost all finite-element solution methods in magnetostatics available at the present time use the magnetic-vector or scalar potential as a means to obtain the solution. However, in dynamics applied to waveguides, the presently used techniques do not rely on the vector or scalar potential but solve Maxwell’s equations directly or use the vector or scalar Helmholtz equation. The latter being the most commonly used technique for waveguide analysis. Satisfactory reasons, other than the results being accurate, for the specific use of the magnetic-vector potential for solutions in magnetostatics via the finite-element method could not be found in the literature that was surveyed.

For general magnetostatic solution methods, Maxwell’s equation themselves were used to develop functional forms based on the weighted-residual and least-square approximations. This development and the results obtained are described in detail. It was found that the general equations developed fail to provide accurate results when multiple media are introduced. A solution that only works well in a homogeneous medium is unsuitable for practical use. Further investigation into the
magnetostatic solution using the magnetic-vector potential revealed a specific relationship in the structure of the basis functions that are used for the expansions. It is this relationship between the basis functions used in the finite-element formulation that is applied to dynamics to obtain solutions to the propagating modes of waveguides.

Waveguide modes are typically computed as an eigenvalue problem of the form $Ax = \lambda Bx$ where $\lambda$ represents the propagating modes or the cutoff wavenumber, $k_c$. Here, $x$ represents the magnetic and/or electric fields expansion coefficients. Several solutions to waveguide modes use the vector-Helmholtz equation in one form or another. Spurious modes, which are false modes that corrupt the results, are common when certain bases are used.

The Direct method for Maxwell’s equations attempts to provide accurate computed eigenvalues as robust solutions that represent the modes of propagating fields in waveguides. The Direct method also attempts to reduce or eliminate the spurious modes that occur. It is shown that direct use of Maxwell’s equations provide a variational form for solution. This variational form is required to ensure that an extremum has been reached and that energy minimization has occurred. The resulting equations, both in magnetostatics and dynamics, have the same form as a general weighted-residual formulation. The functional is developed in general form so that several different basis functions can be applied to each component of the magnetic and electric fields. The equations that define the Direct method can also be applied to several other disciplines in electromagnetics that seek solutions via the finite-element method such as in scattering and radiation.

Only zero-order and first-order basis functions that are either scalar or vector in nature are used throughout this document. Higher order basis functions are not considered as the analysis of the first and zero-order bases can be carried over to the higher order bases in a straightforward manner. The focus is on fundamental methods using different techniques to improve existing solutions. The rectangular WR-90 waveguide and a circular waveguide of 1 m radius are used as test waveguides. Different basis function sets are used and the process of applying the basis functions, the boundary conditions, and the results obtained are described.

This dissertation can be divided into four parts. The first contains a small chapter with a brief introduction into Delaunay mesh generation and descriptions of the basis functions used. The mesh generation forms the backbone of any finite-element method. The choice of basis functions used dictates the performance and quality of the computed results. The second part covers the research
accomplished in magnetostatics including a variational derivation of the functional that includes the boundary conditions in the functional itself. Also included are the general solution techniques that use Maxwell’s equations directly applied to magnetostatics and the results obtained. The third part covers the research in dynamics, including a detailed overview of the vector-Helmholtz equation method for solutions to the propagating modes in waveguides, the development of the Direct method as an alternative and unique technique, comparison of results from the Direct methods to the Helmholtz method, and results obtained by using scalar, vector, and a combination of bases in the Direct method. The final part covers the analysis of results in dynamics by the Direct method, analysis of a waveguide partially filled with dielectric, the conclusion, and recommendations for future work.

1.2 Literature Survey

Literature in magnetostatics into the justification of the use of the magnetic-vector potential is not easily found. Most text books, papers, and publication focus on the use of the magnetic-vector potential along with finite elements to solve a particular problem. A detailed implementation of the magnetic-vector potential applied to the use of finite elements was summarized by Albanese and Rubinacci [1990]. A detailed account of the use of the magnetic-vector potential in three-dimensional eddy current computations can be found in Bíró and Preis [1989]. Another implementation with the magnetic-vector potential can be found in Manges and Cendes [1995]. Work by Bíró et al. [2000] contains a transient skin effect solution using the magnetic-vector potential. Both scalar nodal bases and vector edge bases have been used to solve for magnetostatic fields.

Typically nodal basis functions are used for two-dimensional problems with the source present only in the longitudinal or $\hat{z}$ direction. Excellent details of using the magnetic potential in linear and nonlinear magnetostatic problems in lifting electromagnets and electric motor analysis can be found in Silvester and Ferrari [1996] for first-order finite elements. These principles can be extended to higher-order basis functions as well.

Several solutions to the problem of computing the propagation constants of modes in waveguides are available, most using the vector-Helmholtz equation. A comprehensive overview with useful references can be found in the text by Peterson et al. [1998], as well as details on basis functions and
the application of finite-elements to several other disciplines in electromagnetics. Fernandez and Lu [1991] provides a variational formulation to obtain solutions to the modes in dielectric waveguides based on the magnetic-field form of the vector-Helmholtz equation. Hayata et al. [1988] provides a method to solve for the modes in lossy waveguides using the magnetic-field form of the vector-Helmholtz equation. Later work by Lu and Fernandez [1993] provides an extension to their earlier research by including lossy waveguides and use of the magnetic form of the vector-Helmholtz equation. The vector-Helmholtz equation is used by Hano [1984] in what he calls a variational form to solve for the modes of dielectric loaded waveguides.

Rahman and Davies [1984b] make use of a penalty term to include the divergence of the magnetic field to identify the spurious modes, again using the magnetic-field form of the vector-Helmholtz equation. A variant of this penalty term is also applied to the developed Direct method in dynamics. Koshiba et al. [1985] provide results that eliminate spurious modes that are greater than the free space wavenumber, \( k_0 \), and a mechanism to easily identify those that remain.

Collin [1991] contains references to certain other methods that involve waveguides that contain anisotropic media. With reflection symmetry, the methods have been used in specific problems. These formulations involve only the use of the transverse component of the fields and have a variational form. Both the vector-Helmholtz method and the Direct method to be introduced can be written only in terms of the transverse fields by expressing the longitudinal fields in term of the transverse fields. These methods reduce the size of the final eigenvalue matrices but involve pre-processing and post-processing in the finite-element formulation.

The method described by Angkaew et al. [1987] differs from the methods used in previous citations in dynamics. They incorporate Maxwell’s equations instead of the vector-Helmholtz equation. By expressing the longitudinal or \( \hat{z} \) directed electric and magnetic fields in terms of the transverse fields, the eigenvalue problem is reduced to just the transverse electric and magnetic fields being the unknowns. However, it has been found during the course of this research that solutions based on Maxwell’s equations alone are also sensitive to the basis sets used and do not completely eliminate spurious modes without additional modifications. In contrast, the Direct method to be introduced also uses Maxwell’s equations, but are not reduced to just the transverse fields. The Direct method also allows for the application of appropriate bases to any component of the magnetic or electric fields.
Research by Lee et al. [1991] use the electric-field form of the Helmholtz equation to solve for the solutions in dielectric waveguides. Boyse et al. [1992] use Nodal finite elements, Maxwell’s equations, and vector potentials to derive equations of the Helmholtz type along with the use of a Lorentz like gauge that helps to eliminate spurious modes. Bunting and Davis [1999] introduced a new functional that partially eliminates spurious modes. Additional details and references on obtaining solutions based on the electric and magnetic form of the Helmholtz equation and via the vector-magnetic potential can be found in textbooks by Peterson et al. [1998], Silvester and Ferrari [1996], Jin [1993], and Volakis et al. [1998].

It is to be noted that an abundance of publications among those available for solutions to waveguide problems use the vector-Helmholtz equation as the starting point. Maxwell’s equations have been used by Angkaew et al. [1987], but specifically with first-order edge type elements, for the convenience of applying boundary conditions. This does reduce the burden of computer memory and storage requirements, but increases preprocessing as well as other computational overhead. Also, it does not remove spurious modes when scalar bases are used as will be shown in subsequent chapters.

The fundamental electromagnetic concepts used in this research are available in Balanis [1991], Harrington [2001] (for analytical solutions to the propagating modes in waveguides), and Van Bladel and Van Bladel [1988]. Integral and vector calculus identities are available in Van Bladel and Van Bladel [1988] and Korn and Korn [2000]. Parts of the research documented in this dissertation have been presented in conferences and include Davis and Sitapati [2002], Davis and Sitapati [2003a], and Davis and Sitapati [2003b]. Some of the computer code developed during the research used algorithms and code presented in the book “Numerical Recipes in C” by Press et al. [1996].

### 1.3 Problem Statement

The initial part of the research on magnetostatics was to explore general methods in magnetostatics and justify the use of the magnetic-vector potential which has been used so successfully in finite-element analysis. This research led to the structural interaction between basis functions used for the different unknowns in a finite-element formulation. Application of the interaction between basis functions in dynamics became the next challenge along with the goal to eliminate spurious modes.
that are part of the solution when certain basis functions are used.

The problem statement can be split between the magnetostatic and dynamic parts. In magnetostatics the problem may be stated as: "to provide a justification for the use of the magnetic-vector potential in magnetostatic finite-element analysis and to seek general methods as alternatives". In dynamics the problem is stated as: "to demonstrate the effectiveness of the newly developed Direct Method in solutions to the propagating modes in waveguides and further the understanding of spurious mode solutions". The motivation is also to further the understanding of basis function behavior in finite-element analysis in electromagnetics.
Chapter 2

Mesh Generation and Basis Function Definitions
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2.1 Delaunay Mesh Generation

Two-dimensional finite-element analysis can be performed with either quadrilateral or triangular elements. In electromagnetics, triangular elements are typically used as triangles can easily approximate objects with arbitrary shapes. A triangular element or cell in a finite-element mesh consists of nodes or points that define the vertices of each triangle. It also consists of edges that make up the sides of each element. The storage data structure for a triangle and an edge in a finite-element mesh is shown in Fig. 2.1. In addition to the structure shown in Fig. 2.1, each element, edge, and node contain markers that determine the material and position on the boundary. Hence, it becomes fairly straightforward to impose boundary conditions and for example, to plot the field patterns by assigning the appropriate eigenvectors to the corresponding basis functions.

Delaunay mesh generation gets its name by satisfying the Delaunay criterion, which in two-dimensions states that no other point in space can lie inside the circum-circle of a triangle other than the three vertices of that triangle. If there is any other point that does lie inside the circum-center of the test...
triangle, the edge that forms a diagonal of the quadrilateral formed by the two neighboring triangles is swapped. This is demonstrated in Fig. 2.2. Typically one or two triangles that completely contain all the points and edges in the domain of interest are initially constructed before insertion starts. The process in Fig. 2.2 is repeated for each of the nodes in the domain of interest while inserting points into the triangulation one after another. At each insertion, the element with the newly inserted point is divided into three elements or the edge on which the point lies on is split and the entire mesh is checked again for the Delaunay condition. If an inserted point lies on an edge, the two elements on both sides of the edge are deleted and four new elements are created. Each node is checked with each element in the growing triangulation so that the Delaunay criterion is always honored.

Figure 2.2: Elements $e_1$ and $e_2$ represent two elements in a finite-element triangulation. The point $c$ is the circum-center of element $e_1$ and is shown in the figure on the left. A vertex of element $e_2$ lies inside the circum-circle of $e_1$. Hence the common edge is deleted and replaced with a new edge. The deleted edge is shown dotted in the figure on the right along with the newly inserted edge shown as a solid line. The two circles shown on the figure on the right are the circum-circles of $e_1$ and $e_2$ after the diagonal swap and no other point or vertex other than those belonging to their respective elements lie on or inside the circum-circles. The Delaunay criterion is now satisfied.

After all the points are inserted, the edges that define the input object are inserted one after another while maintaining the Delaunay or now constrained Delaunay criterion. The mesh density is increased by creating new nodes at the center of the circum-circles of the elements and by adding nodes on any input edges that intersect the circum-circle of cells. The mesh density can also be controlled by inserting special nodes that do not become a part of the mesh but specify the mesh density locally.
2.1. Delaunay Mesh Generation

Figure 2.3: Constrained Delaunay mesh generated on the cross section of a permanent-magnet magnetizing fixture. Multi-pole ring permanent-magnets are manufactured in a virgin state and have to be magnetized by applying a large controlled magnetic field which determines the field pattern of the magnetized permanent-magnet. Boundary conditions are applied to the bounding box. No angle of any triangle is below $20^\circ$ except if violated by the input geometry. Cross-section design of the magnetizing fixture is for display purposes only.

Fig. 2.3 shows an example of an engineering application of interest in industry of using constrained Delaunay mesh generation. Fig. 2.4 shows errors that may occur in triangular mesh generation which are unacceptable and will lead to inaccurate results. Triangular mesh generation results in an approximation of the actual boundary of the object to be meshed and this is shown in Fig. 2.5. A method commonly used to reduce the discretization error is to increase the mesh density at the boundary.

Rupert [1995], Frykestig [1994] and Shewchuk [2002] contain comprehensive details on Delaunay mesh generation as well as references that can be traced back to the origins of basic mesh generation. A crude mesh generation package was implemented for use in the research presented in this dissertation. But, it was extremely slow and lacked several features that are available in industrial strength mesh generation packages and hence was ultimately not used. All meshes that were generated for the finite-element research were generated by the excellent software package "Triangle"
written and made available by Shewchuk [2002].

Figure 2.4: Three commonly encountered errors in triangular mesh generation. Figure on the left shows an example of an edge being violated as a new point is defined between its start and end. Each edge must have only one element on either side. Figure in the center shows a void or hole created by improper triangulation. Figure on right shows overlapping elements. Generated meshes must be checked for errors before finite-element analysis can begin to eliminate errors in the results due to poor or improper mesh generation.

Figure 2.5: Mesh generation is an approximation to the actual object to be discretized. Errors typically arise where the boundary contains a large number of curves as seen in the figure on the left. The mesh density can be increased at the boundary to reduce this error as shown on the figure on the right. Higher order triangular elements solve this problem by being able to model curved boundaries.
2.2 Scalar Basis Functions

First and zero-order scalar basis functions that are used throughout this document are defined in this section. First-order bases are piecewise-linear whereas zero-order bases are piecewise-constant. Details on higher order scalar bases can be found in Peterson et al. [1998] and Jin [1993].

2.2.1 Piecewise-Linear Scalar Bases

First-order scalar basis functions for triangular elements are identical to the first-order Nedelec basis functions denoted commonly in literature by $L_1$, $L_2$, and $L_3$ or $N_1$, $N_2$ and $N_3$. These are linear basis functions that are defined at each node in the finite-element mesh. The basis function has an amplitude of unity at the node it has been defined at and linearly drops to zero at all surrounding nodes. The behavior in an element is shown in Fig. 2.6. Fig. 2.7 shows the behavior of the nodal basis function defined for a node in all surrounding elements and nodes it is connected to. Elements in the finite-element mesh that are expanded using nodal bases are often referred to as nodal elements.

Figure 2.6: Nodal basis function in element $e$ is shown for each of the nodes $i$, $j$, and $k$ in the element. The basis functions have an amplitude of 1 at the node it is defined at and linearly drops to zero at the other two nodes in the element.

The first-order, piecewise-linear nodal basis functions are denoted by $\Phi$. Within an element $\Phi$
Figure 2.7: Nodal basis function behavior in surrounding elements can be seen. The basis functions have an amplitude of 1 at the node it is defined at and linearly drops to zero at all surrounding nodes with a pyramid type behavior.

can be written for a node \( n \) as

\[
\Phi_n = \frac{\hat{z} \cdot (\bar{\rho}_{n+2} - \bar{\rho}_{n+1}) \times (\bar{\rho} - \bar{\rho}_{n+1})}{\hat{z} \cdot (\bar{\rho}_{n+2} - \bar{\rho}_{n+1}) \times (\bar{\rho}_n - \bar{\rho}_{n+1})} \tag{2.2.1}
\]

or

\[
\Phi_n = \frac{l_n \hat{z} \cdot \hat{\mathbf{e}}_n \times (\bar{\rho} - \bar{\rho}_{n+1})}{\hat{z} \cdot (\bar{\rho}_{n+2} - \bar{\rho}_{n+1}) \times (\bar{\rho}_n - \bar{\rho}_{n+1})} \tag{2.2.2}
\]

where

- \( n \) cycles between the nodes \( i \), \( j \), and \( k \) that make up the element
- \( \bar{\rho}_n \) is a vector from the origin to the location of node \( n \)
- \( l_n \) is the length of the edge opposite node \( n \)
- \( \hat{\mathbf{e}}_n \) is the direction of the edge opposite node \( n \)
- \( \bar{\rho} \) is a vector from the origin to any point \((x, y)\) inside the element.

Recognizing the denominator to be twice the area of the element, the nodal piecewise-linear basis
function is defined as

\[ \Phi_n = \frac{1_n}{2S^e} \hat{\mathbf{z}} \cdot \hat{\mathbf{l}}_n \times (\bar{\rho} - \bar{\rho}_{n+1}) \]  

(2.2.3)

where \( S^e \) is the area of element \( e \) and \( \hat{\mathbf{l}}_n = \bar{\rho}_{n+2} - \bar{\rho}_{n+1} \). This basis function is also called PWL due to its piecewise-linear behavior. Being first-order, \( \Phi_n \) can be differentiated and both the divergence and curl of vectors and the gradient of scalars expanded using this basis function exist inside the element.

### 2.2.2 Piecewise-Constant Scalar Bases

Zero-order basis functions have a constant amplitude of unity in the element that it has been defined in and zero at all other locations. Fig. 2.8 shows element \( e \) in a finite-element mesh having a constant amplitude only within \( e \) and zero everywhere else. Zero-order basis functions are defined as constant in each element in the mesh and are called piecewise-constant (PWC) bases. Eq. (2.2.4) defines the piecewise-constant basis functions.

\[ \Psi_e = \begin{cases} 
1 & (x, y) \in e \\
0 & (x, y) \notin e 
\end{cases} \]  

(2.2.4)

Figure 2.8: The piecewise-constant basis function has an amplitude of 1 in the element that it is defined in and zero everywhere else including surrounding elements.

Being a constant in each element, the divergence and curl of vectors and gradient of scalars expanded by this basis function are zero inside each element but undefined at the cell boundaries. If
the need does arise to operate on these bases with a differential operator, integration by parts must be used to transfer the derivative to a function that can be differentiated. These bases are identical to the RWG bases commonly used for surface currents (Rao et al. [1982]).

2.3 Vector Basis Functions

The most common first-order vector basis function is the edge basis function. Elements in the finite-element mesh expanded using edge bases are often referred to as edge elements. Additional information which is also useful regarding vector bases, higher order and three-dimensional vector bases, including details on the properties of edge bases, can be found in works by Cendes [1991], Gralia et al. [1997], Peterson et al. [1998], Webb [1993], and Mur [1994]. The two kinds of edge bases used in this research are the constant-tangential and the constant-normal first-order edge basis functions which are both linear in nature.

2.3.1 Constant-Tangential Vector Bases

Fig. 2.9 shows the behavior of first-order, tangential edge basis functions. The normal component at the edge follows a linear behavior. The constant-tangential, linear-normal edge basis function may be written as

\[
\mathbf{T}_n = \frac{l_n \mathbf{z} \times (\mathbf{\rho} - \mathbf{\rho}_n)}{\mathbf{z} \times (\mathbf{\rho}_{n+1} - \mathbf{\rho}_n) \cdot (\mathbf{\rho}_{n+2} - \mathbf{\rho}_{n+1})}
\]

(2.3.1)

The denominator is twice the area of the element that the basis function is being evaluated at, the tangential-edge basis function in an element may be defined as

\[
\mathbf{T}_n = \frac{l_n}{2S^e} \mathbf{z} \times (\mathbf{\rho} - \mathbf{\rho}_n)
\]

(2.3.2)

where

- \( n \) is the index of the edge opposite node \( n \) in the element
- \( S^e \) is the area of the element
2.3. Vector Basis Functions

Figure 2.9: Constant-tangential behavior of edge basis function on either side of edge $s$ can be seen. The two elements on either side contributes a constant-tangential component at the edge. Continuity of the tangential component of the field expanded by these bases at the edge is automatically ensured while the normal component at the edge is linear.

- $\hat{\rho}_n$ is a vector from the origin to node $n$
- $\hat{\rho}$ is a vector from the origin to any point $(x, y)$
- $l_n$ is the length of edge $n$

As these basis functions have a constant-tangential component at the edge and a linear-normal component, they are also called CT or constant-tangential bases. The divergence of the basis functions inside each element is zero while the curl is a constant in each element. It is important to note that the tangential continuity allows the curl to be defined everywhere, whereas the divergence is undefined at cell boundaries.

### 2.3.2 Constant-Normal Vector Bases

Fig. 2.10 shows the behavior of the constant-normal, linear-tangential edge basis function in the two elements of either side of edge $s$. This basis function is written as $-\hat{e}_z \times \hat{T}_n$ as it is just the tangential edge bases rotated in the clockwise direction by $90^\circ$. As these bases have a constant-
normal component at the edge and a linear-tangential component they are also called CN basis functions.

Figure 2.10: Constant-normal behavior of edge basis functions on either side of edge \( s \) can be seen. The two elements on either side of \( s \) each contribute a constant-normal component at the edge. Continuity of the normal component of the field expanded by these bases at the edge is automatically ensured while the tangential component at the edge is linear.

The CN basis function is defined as

\[
\vec{V}_n = -\hat{z} \times \frac{l_n}{2S_e} \hat{z} \times (\bar{\rho} - \bar{\rho}_n) = \frac{l_n}{2S_e} (\bar{\rho} - \bar{\rho}_n)
\]  

(2.3.3)

The curl of these basis functions is not defined at the element boundaries of each element while the divergence is piecewise-constant in each cell. This basis function may be thought of a vector complementary basis function to the constant-tangential (CT) basis function.

### 2.4 Chapter Summary

The basic method of generating Delaunay triangulations, which is a prerequisite for the finite-element analysis with triangular elements, was described. The data structures used to store information on the elements and edges in the mesh were described. These data structures help in applying proper boundary conditions and defining multiple media with ease. They also help in
traversing the mesh rapidly when required to identify the cell that contains an arbitrary point inside the domain.

Four basis functions that can be used in expansions of field variables were described and defined, two of these are scalar basis functions and the other two vector in nature. These basis functions are used throughout this document both in magnetostatics and dynamics. Only zero and first-order basis functions are considered in this research, though the principles can be extended to higher order bases. These functions are the

- Piecewise-linear scalar bases (PWL)
- Piecewise-constant scalar bases (PWC)
- Constant-tangential, linear-normal vector bases (CT)
- Constant-normal, linear-tangential vector bases (CN).

Traditional expansions for the PWL and CT bases functions can be found in the Appendix A.
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Chapter 3

The Magnetic-Vector Potential in Magnetostatic Finite-Element Analysis
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This chapter focuses on the magnetic-vector potential method in magnetostatic finite-element analysis. A functional is derived which has a variational form and includes the boundary conditions within the functional itself. Several three-dimensional problems that have a large axial or longitudinal length compared to the transverse dimensions may be reduced to two-dimensions and solved in a simpler manner. This is particularly so when the source component is only in the longitudinal direction. A proof that validates the assumption of the reduction from three-dimensions to two is displayed. The effect of different boundary conditions applied to the magnetic potential on the magnetic field is explored. Several examples including one with heterogeneous media are presented to demonstrate the effectiveness of the magnetic potential method.

3.1 A Brief Introduction to Variational Principles

As variational principles form an essential part of this dissertation, a brief introduction is provided in this section. Only information relevant to the methods used in the research is supplied. Several textbooks are available that are devoted exclusively to this subject including Mikhlin [1964] and Schechter [1967].

The inner product between two complex functions is defined by

\[
\langle \phi, \psi \rangle = \int_{\Omega} \phi \psi^* d\Omega
\]

(3.1.1)

where \( \ast \) denotes the complex conjugate. For magnetostatics, only real function will be used. Consider a boundary value problem which is defined by the linear differential equation

\[
\mathbb{L}\phi = f
\]

(3.1.2)

where the operator \( \mathbb{L} \) is self-adjoint and positive definite. A solution can often be obtained by minimizing the functional

\[
I(\phi) = \frac{1}{2} \langle \mathbb{L}\phi, \phi \rangle - \langle \phi, f \rangle
\]

(3.1.3)

To be self-adjoint the operator \( \mathbb{L} \) must satisfy \( \langle \mathbb{L}\phi, \psi \rangle = \langle \phi, \mathbb{L}\psi \rangle \), where \( \psi \) is any arbitrary function that satisfies the same boundary conditions that are imposed on \( \phi \). To be positive definite,
\[ \langle \mathcal{L}\phi, \phi \rangle > 0, \phi \neq 0. \] In magnetostatics, \( f \) is the source current density and \( \phi \) is the magnetic potential. Minimization is accomplished by the first variation being equated to zero as in

\[ \delta I(\phi) = \frac{1}{2} \langle \delta\phi, \mathcal{L}\phi - f \rangle + \frac{1}{2} \langle \mathcal{L}\phi - f, \delta\phi \rangle = 0 \]

The second variation can be used as a check to ensure that minimization or maximization has occurred by reaching an extremum.

### 3.2 Magnetostatics and Finite-Element Analysis

The magnetic-vector potential method is the most widely used in finite-element analysis for magnetostatic solutions. The magnetic-vector potential, \( \vec{A} \), is not a physical quantity and is only used as a mathematical tool to arrive at the solution. As in most finite-element methods, boundary conditions are required for uniqueness in the computed solutions. The two boundary conditions considered here are the Dirichlet and the Neumann type. Mixed boundary conditions which are a combination of the Dirichlet and the Neumann type are not considered for clarity.

The Dirichlet type boundary condition for the magnetic-vector potential is written as \( \vec{n} \times \vec{A} \), where \( \vec{n} \) is the outward normal at the boundary of the domain of interest. The Neumann type is written as \( \vec{n} \times \frac{1}{\mu} \nabla \times \vec{A} \). A new variational derivation of the required equations to solve magnetostatic problems using the finite-element method is completed in this chapter. Typical developments do not include the Dirichlet boundary condition within the functional and this boundary condition is usually enforced on the system matrix directly.

This derivation includes the Dirichlet boundary condition within the functional, even though the enforcement is identical to the standard method. A proof that the magnetic potential does not contain transverse components when the source term is only in the longitudinal (\( \hat{z} \)) direction is also show. This allows the use of the magnetic scalar potential for most two-dimensional problems. The effect of applying Dirichlet and Neumann boundary conditions on the magnetic potential is demonstrated as well as its influence on the computed magnetic fields. Several examples of using the magnetic potential in finite-element analysis to solve magnetostatic problems are presented, including an example with multiple materials.
3.3 Derivation of the Functional for Magnetostatics

A variational form is used to account for the stored energy, Maxwell’s equations, and the Dirichlet and the Neumann boundary conditions. The importance of this derivation is the inclusion of both Dirichlet and Neumann boundary conditions within the functional formulation itself. A variational formulation guarantees that the function under consideration has reached a minima. The required Maxwell’s equations for magnetostatic solutions are

\[ \nabla \times \vec{H} = \vec{J} \quad (3.3.1) \]

and

\[ \nabla \cdot \vec{B} = 0 \quad (3.3.2) \]

where \( \vec{H} \) is the magnetic field, \( \vec{J} \) is the source current, and \( \vec{B} \) is the magnetic flux density. Since the divergence of the flux density is zero, it can be written as a curl, \( \vec{B} = \mu \vec{H} = \nabla \times \vec{A} \), where \( \vec{A} \) is the magnetic-vector potential. Thus \( \nabla \times \vec{H} = \vec{J} \), also called Ampere’s law can be written as

\[ \nabla \times \left( \frac{1}{\mu} \nabla \times \vec{A} \right) - \vec{J} = 0 \quad (3.3.3) \]

which forms the basic equation required for the development.

Fig. 3.1 shows the domain of interest \( \Omega \) and the regions where the boundary conditions may be enforced. For a three-dimensional domain which is a volume, the boundary conditions are enforced over surfaces and/or contours at the surface of the boundary. For a two-dimensional case the domain reduces to an area and the boundary conditions are enforced on contours at the perimeter enclosing the area. The normal vector \( \hat{n} \) is the outward normal at the boundary.

3.3.1 Functional Definition

The functional can be written with the energy in the system as well as several constraints with Lagrange multipliers (\( \lambda_n \)). The constraints are Maxwell’s equation for magnetostatics, the boundary energy and the Dirichlet and Neumann boundary conditions.
Chapter 3. The Magnetic-Vector Potential in Magnetostatic Finite-Element Analysis

Figure 3.1: The domain of interest, $\Omega$, with regions on the boundary divided depending on the type of boundary conditions applied is shown. These include the Dirichlet, $\Gamma_D$, and the Neumann, $\Gamma_N$, regions. No boundary conditions are applied over region $\Gamma_{\text{Free}}$.

\[ I(\vec{A}) = \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} \, d\Omega + \int_{\Omega} \lambda_1 \vec{A} \cdot \left( \nabla \times \frac{1}{\mu} \nabla \times \vec{A} - \vec{f} \right) \, d\Omega + \oint_{\Gamma} \lambda_b \vec{A} \times \frac{1}{\mu} \nabla \times \vec{A} \cdot d\vec{n} \]
\[ + \int_{\Gamma_D} \lambda_2 (\vec{A}_b - \vec{A}) \times \frac{1}{\mu} \nabla \times \vec{A} \cdot d\vec{n} + \int_{\Gamma_N} \lambda_3 \vec{A} \times \frac{1}{\mu} \nabla \times (\vec{A}_b - \vec{A}) \cdot d\vec{n} \quad (3.3.4) \]

The coefficient $\lambda_b$ is not considered to be a Lagrange multiplier but an undetermined coefficient which helps to identify the required boundary terms and ensures that the system energy is accounted for. The subscript $b$ in $\vec{A}_b$ refers to the value of the magnetic potential constrained at the boundary. Note that the terms $\hat{n} \times \vec{A}$ for the Dirichlet and $\hat{n} \times \frac{1}{\mu} \nabla \times \vec{A}$ for the Neumann boundary conditions are present in the boundary integrals written above in their respective boundary integrals over $\Gamma_D$ and $\Gamma_N$.
3.3. Derivation of the Functional for Magnetostatics

Using the identity

$$\int_{\Omega} \vec{A} \cdot \left( \nabla \times \frac{1}{\mu} \nabla \times \vec{A} \right) d\Omega = \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} d\Omega - \oint_{\Gamma} \vec{A} \times \frac{1}{\mu} \nabla \times \vec{A} \cdot d\bar{\Gamma}$$  \hspace{1cm} (3.3.5)

the functional can be written as

$$I(\vec{A}) = \int_{\Omega} (\lambda_1 + 1) \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} d\Omega - \int_{\Omega} \lambda_1 \vec{A} \cdot \vec{J} d\Omega + \oint_{\Gamma} (\lambda_b - \lambda_1) \vec{A} \times \frac{1}{\mu} \nabla \times \vec{A} \cdot d\bar{\Gamma}$$

$$+ \int_{\Gamma_D} \lambda_2 (\vec{A}_b - \vec{A}) \times \frac{1}{\mu} \nabla \times \vec{A} \cdot d\bar{\Gamma} + \int_{\Gamma_N} \lambda_3 \vec{A} \times \frac{1}{\mu} \nabla \times (\vec{A}_b - \vec{A}) \cdot d\bar{\Gamma}$$  \hspace{1cm} (3.3.6)

The magnetic-vector potential \( \vec{A} \) will be expanded as

$$\vec{A} = \sum_i A_i \hat{\alpha}_i$$  \hspace{1cm} (3.3.7)

where \( \hat{\alpha}_i \) represents the basis functions for the magnetic potential and could be of any appropriate order or form. The coefficients \( A_i \) are the amplitudes of the potential for the respective expansion basis function. Here \( i \) ranges over the number of basis functions defined over the finite-element mesh of the domain \( \Omega \).

The first variation with respect to \( \lambda_n \) results in a weighted form of Amperes law,

$$\frac{\partial I}{\partial \lambda_1} = \int_{\Omega} \vec{A} \cdot \left( \nabla \times \frac{1}{\mu} \nabla \times \vec{A} - \vec{J} \right) d\Omega$$  \hspace{1cm} (3.3.8)

the Dirichlet boundary integral,

$$\frac{\partial I}{\partial \lambda_2} = \int_{\Gamma_D} (\vec{A}_b - \vec{A}) \times \frac{1}{\mu} \nabla \times \vec{A} \cdot d\bar{\Gamma}$$  \hspace{1cm} (3.3.9)
and the Neumann boundary integral
\[
\frac{\partial I}{\partial \lambda_3} = \int_{\Gamma_N} \vec{A} \times \frac{1}{\mu} \nabla \times (\vec{A}_b - \vec{A}) \cdot d\vec{r} \tag{3.3.10}
\]

### 3.3.2 Formulation with Dirichlet Boundary Conditions

Consider the case when the entire boundary is subjected to only the Dirichlet boundary condition. The first variation of the functional with respect to \( A_n \) is
\[
\frac{\partial I}{\partial A_n} = 2 \int_{\Omega} \left( \lambda_1 + 1 \right) \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{a}_n d\Omega - \int_{\Omega} \lambda_1 \vec{a}_n \cdot \vec{J} d\Omega \\
+ \int_{\Gamma} (\lambda_b - \lambda_1) \left( \vec{a}_n \times \frac{1}{\mu} \nabla \times \vec{A} + \vec{A} \times \frac{1}{\mu} \nabla \times \vec{a}_n \right) \cdot d\vec{r} \\
+ \int_{\Gamma_D} \lambda_2 \left( \vec{A}_b - \vec{A} \right) \times \frac{1}{\mu} \nabla \times \vec{a}_n - \vec{a}_n \times \frac{1}{\mu} \nabla \times \vec{A} \right) \cdot d\vec{r} = 0 \tag{3.3.11}
\]

To obtain the multiplier \( \lambda_1 \), consider only the interior where \( \vec{n} \times \vec{a}_n = 0 \) and \( \vec{n} \times \nabla \times \vec{a}_n = 0 \).
\[
\frac{\partial I_{int}}{\partial A_n} = 2 \int_{\Omega} \left( \lambda_1 + 1 \right) \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{a}_n d\Omega - \int_{\Omega} \lambda_1 \vec{a}_n \cdot \vec{J} d\Omega \tag{3.3.12}
\]

Integrating by parts, we obtain
\[
\frac{\partial I_{int}}{\partial \lambda_1} = 2 \int_{\Omega} (\lambda_1 + 1) \vec{a}_n \cdot \left( \nabla \times \frac{1}{\mu} \nabla \times \vec{A} \right) d\Omega - \int_{\Omega} \lambda_1 \vec{a}_n \cdot \vec{J} d\Omega = 0 \tag{3.3.13}
\]

To obtain the Ampere’s law form, we require
\[
2(\lambda_1 + 1) - \lambda_1 = 0 \quad \Rightarrow \lambda_1 = -2 \tag{3.3.14}
\]
Rewriting the first variation with respect to $A_n$, we obtain

$$
\frac{\partial I}{A_n} = -2 \int_{\Omega} \frac{1}{\mu} \nabla \times \bar{A} \cdot \nabla \times \bar{\alpha}_n d\Omega + 2 \int_{\Omega} \bar{\alpha}_n \cdot \bar{J} d\Omega \\
+ \oint_{\Gamma} (\lambda_b + 2) \left( \bar{\alpha}_n \times \frac{1}{\mu} \nabla \times \bar{A} + \bar{A} \times \frac{1}{\mu} \nabla \times \bar{\alpha}_n \right) \cdot d\bar{\Gamma} \\
+ \oint_{\Gamma_D} \lambda_2 \left( \bar{A}_b - \bar{A} \right) \times \frac{1}{\mu} \nabla \times \bar{\alpha}_n - \bar{\alpha}_n \times \frac{1}{\mu} \nabla \times \bar{A} \right) \cdot d\bar{\Gamma} = 0
$$

(3.3.15)

The second variation becomes

$$
\frac{\partial I^2}{A_n^2} = -2 \int_{\Omega} \frac{1}{\mu} \nabla \times \bar{\alpha}_n \cdot \nabla \times \bar{\alpha}_n d\Omega \\
+ 2 \oint_{\Gamma} (\lambda_b + 2) \bar{\alpha}_n \times \frac{1}{\mu} \nabla \times \bar{\alpha}_n \cdot d\bar{\Gamma} - 2 \oint_{\Gamma_D} \lambda_2 \bar{\alpha}_n \times \frac{1}{\mu} \nabla \times \bar{\alpha}_n \cdot d\bar{\Gamma}
$$

(3.3.16)

For the second variation to ensure an extremum of the functional $I(\bar{A})$ to within a gradient, we require

$$
\lambda_2 = 0
$$
$$
\lambda_b = -2
$$

Therefore, the functional $I(\bar{A})$ with only Dirichlet boundary conditions can be written as

$$
I(\bar{A}) = - \int_{\Omega} \frac{1}{\mu} \nabla \times \bar{A} \cdot \nabla \times \bar{A} d\Omega + 2 \int_{\Omega} \bar{A} \cdot \bar{J} d\Omega
$$

(3.3.17)

or

$$
I(\bar{A}) = \int_{\Omega} \frac{1}{\mu} \nabla \times \bar{A} \cdot \nabla \times \bar{A} d\Omega - 2 \int_{\Omega} \bar{A} \cdot \left( \nabla \times \frac{1}{\mu} \nabla \times \bar{A} - \bar{J} \right) d\Omega - 2 \oint_{\Gamma} \bar{A} \times \frac{1}{\mu} \nabla \times \bar{A} \cdot d\bar{\Gamma}
$$

giving

$$
\frac{\partial I}{A_n} = -2 \int_{\Omega} \frac{1}{\mu} \nabla \times \bar{A} \cdot \nabla \times \bar{\alpha}_n d\Omega + 2 \int_{\Omega} \bar{\alpha}_n \cdot \bar{J} d\Omega = 0
$$

(3.3.18)
with the explicit constraint of the Dirichlet boundary condition enforced separately as
\[
\int_{\Gamma} \left( \vec{A}_b - \vec{A} \right) \times \frac{1}{\mu} \nabla \times \vec{\alpha}_n \cdot d\vec{r} = 0
\]

### 3.3.3 Formulation with Neumann Boundary Conditions

When the entire boundary is subjected to only the Neumann boundary condition, the functional becomes
\[
I(\vec{A}) = -\int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} \, d\Omega + 2 \int_{\Omega} \vec{A} \cdot \vec{J} \, d\Omega + \oint_{\Gamma} \left( \lambda_b + 2 \right) \vec{A} \times \frac{1}{\mu} \nabla \times \vec{A} \cdot d\vec{r} \\
+ \int_{\Gamma_N} \lambda_3 \vec{A} \times \frac{1}{\mu} \nabla \times \left( \vec{A}_b - \vec{A} \right) \cdot d\vec{r} \tag{3.3.19}
\]

where \( \lambda_1 = -2 \), as with the Dirichlet formulation. The first variation with respect to \( A_n \) is
\[
\frac{\partial I}{\partial A_n} = -2 \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{\alpha}_n \, d\Omega + 2 \int_{\Omega} \vec{\alpha}_n \cdot \vec{J} \, d\Omega \\
+ \oint_{\Gamma} \left( \lambda_b + 2 \right) \left( \vec{\alpha}_n \times \frac{1}{\mu} \nabla \times \vec{A} + \vec{A} \times \frac{1}{\mu} \nabla \times \vec{\alpha}_n \right) \cdot d\vec{r} \\
+ \oint_{\Gamma_N} \lambda_3 \left( \vec{\alpha}_n \times \frac{1}{\mu} \nabla \times \left( \vec{A}_b - \vec{A} \right) - \vec{A} \times \frac{1}{\mu} \nabla \times \vec{\alpha}_n \right) \cdot d\vec{r} = 0 \tag{3.3.20}
\]

The second variation becomes
\[
\frac{\partial I^2}{A_n^2} = -2 \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{\alpha}_n \cdot \nabla \times \vec{\alpha}_n d\Omega \\
+ 2 \oint_{\Gamma} (\lambda_b + 2) \vec{\alpha}_n \times \frac{1}{\mu} \nabla \times \vec{\alpha}_n \cdot d\vec{r} - 2 \oint_{\Gamma_N} \lambda_3 \vec{\alpha}_n \times \frac{1}{\mu} \nabla \times \vec{\alpha}_n \cdot d\vec{r} \tag{3.3.21}
\]

For the second variation to ensure an extremum of the functional \( I(\vec{A}) \) to within a gradient, we
3.3. Derivation of the Functional for Magnetostatics

require
\[ \lambda_3 = 2 \]
\[ \lambda_b = 0 \]

Therefore, the functional \( I(\vec{A}) \) with only Neumann boundary conditions can be written as

\[
I(\vec{A}) = - \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} \, d\Omega + 2 \int_{\Omega} \vec{A} \cdot \vec{J} \, d\Omega + 2 \oint_{\Gamma_N} \vec{A} \times \frac{1}{\mu} \nabla \times \vec{A}_b \cdot d\vec{r} \tag{3.3.22}
\]

or

\[
I(\vec{A}) = \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} \, d\Omega - 2 \int_{\Omega} \vec{A} \left( \nabla \times \frac{1}{\mu} \nabla \times \vec{A} - \vec{J} \right) \, d\Omega + 2 \oint_{\Gamma_N} \vec{A} \times \frac{1}{\mu} \nabla \times (\vec{A}_b - \vec{A}) \cdot d\vec{r} \tag{3.3.23}
\]

The first variation is written as

\[
\frac{\partial I}{\partial A_n} = -2 \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A}_n \, d\Omega + 2 \int_{\Omega} \vec{A}_n \cdot \vec{J} \, d\Omega + 2 \oint_{\Gamma_N} \vec{A}_n \times \frac{1}{\mu} \nabla \times \vec{A}_b \cdot d\vec{r} = 0 \tag{3.3.24}
\]

When \( \nabla \times \vec{A}_b = 0 \), there are no Neumann boundary conditions that are explicitly applied, this case is commonly called the free boundary condition.

### 3.3.4 Combined Functional

The final functional can be written by taking into account both the Dirichlet and Neumann boundary conditions. The combined functional is

\[
I(\vec{A}) = - \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} \, d\Omega + 2 \int_{\Omega} \vec{A} \cdot \vec{J} \, d\Omega + 2 \oint_{\Gamma_N} \vec{A} \times \frac{1}{\mu} \nabla \times \vec{A}_b \cdot d\vec{r} \tag{3.3.25}
\]
Helmholtz equation as

or

\[
I(\vec{A}) = \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} \, d\Omega - 2 \int_{\Omega} \vec{A} \cdot \left( \nabla \times \frac{1}{\mu} \nabla \times \vec{A} - \vec{J} \right) \, d\Omega - 2 \oint_{\Gamma} \vec{A} \times \frac{1}{\mu} \nabla \times \vec{A} \cdot d\vec{\Gamma}
\]

\[
+ 2 \int_{\Gamma_N} \vec{A} \times \frac{1}{\mu} \nabla \times (\vec{A}_b - \vec{A}) \cdot d\vec{\Gamma}
\]

(3.3.26)

giving

\[
\frac{\partial I}{\partial A_n} = \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} \, d\Omega + \int_{\Omega} \vec{A}_n \cdot \vec{J} \, d\Omega + \int_{\Gamma_N} \alpha_n \times \frac{1}{\mu} \nabla \times \vec{A}_b \cdot d\vec{\Gamma} = 0
\]

(3.3.27)

with the explicit constraint of the Dirichlet boundary condition enforced separately as

\[
\int_{\Gamma_D} (\vec{A}_b - \vec{A}) \times \frac{1}{\mu} \nabla \times \vec{A}_n \cdot d\vec{\Gamma} = 0
\]

(3.3.28)

Note that the functional represents the various energy terms associated with the problem.

### 3.4 Two-Dimensional Behavior of the Magnetic-Vector Potential

For two-dimensional analysis, the results of the functional formulation are often reduced to a scalar form when the source component is only in the longitudinal \((\hat{z})\) direction. The field in this case lies in the transverse plane and the magnetic potential is assumed to have only a longitudinal component. A large number of magnetostatics problems are often reduced to two-dimensions and solved in this manner. It is typically assumed that there is no variation along the longitudinal axis in the longitudinal component of the magnetic potential \(A_z\). The validity of this assumption is explored in this section. The equation defined in (3.3.25) is used as the starting point and is rewritten below as

\[
I(\vec{A}) = - \int_{\Omega} \frac{1}{\mu} \nabla \times \vec{A} \cdot \nabla \times \vec{A} \, d\Omega + 2 \int_{\Omega} \vec{A} \cdot \vec{J} \, d\Omega + 2 \int_{\Gamma_N} \vec{A} \times \vec{H}_b \cdot d\vec{\Gamma}
\]

(3.4.1)
where the boundary term \( \frac{1}{\mu} \nabla \times \vec{A} \) is replaced by the specified boundary condition \( \vec{H}_b \), which is only in the transverse plane.

Figure 3.2: Figure shows an object of arbitrary cross-section and infinite length aligned with the \( \hat{z} \) axis. Outward normals are defined at the sides, the top, and bottom surfaces of a slice of arbitrary thickness \( h \). Boundary conditions are applied on the walls parallel to the \( \hat{z} \) axis.

Fig. 3.2 shows a body of arbitrary cross section and infinite length that lies along the \( \hat{z} \) axis and a slice of height \( h \) in the normal direction is considered. The outward normal at the sides is \( \hat{n} \) while it is \( \hat{z} \) and \( -\hat{z} \) at the top and bottom surfaces respectively. The magnetic-vector potential is expanded as

\[
\vec{A} = \sum_n A_n \vec{x}_n(x, y)
\]  

(3.4.2)

The considered height, \( h = z_1 - z_0 \), involves the interior, the top, and bottom surfaces of the slice as well as the entire surface on the sides or the wall of the cross section of the slice. The boundary conditions on the top and bottom faces of the body have to be treated as unknowns and solved for as the only locations that boundary conditions can be specified are on the walls of the object due to the infinite length.

A new functional \( I(\vec{A}) \) is written as the sum of three other functionals as \( I(\vec{A}) = I_1(\vec{A}) + I_2(\vec{A}) + I_3(\vec{A}) \). The three functionals represent the energy, the source component and the boundary terms.
respectively. All integrations are carried out over the cross sectional surface and the height $h$ of the slice considered.

\[
I_1(\vec{A}) = -\int_{z_0}^{z_1} \int_\Omega \frac{1}{\mu} (\nabla \times \vec{A}) \cdot (\nabla \times \vec{A}) \, d\Omega \, dz \tag{3.4.3}
\]

\[
I_2(\vec{A}) = 2 \int_{z_0}^{z_1} \vec{A} \cdot \vec{J} \, d\Omega \, dz \tag{3.4.4}
\]

\[
I_3(\vec{A}) = 2 \int_{z_0}^{z_1} \int_{\Gamma_{\text{side}}} (\vec{A} \times \vec{H}_b) \cdot d\vec{r} \, dz + 2 \int_{\Omega_{z_0}} (\vec{A} \times \vec{H}_{b0}) \cdot d\vec{\Gamma} + 2 \int_{\Omega_{z_1}} (\vec{A} \times \vec{H}_{b1}) \cdot d\vec{\Gamma} \tag{3.4.5}
\]

The boundary field $\vec{H}_b$ is the specified boundary condition along the sides whereas $\vec{H}_{b1}$ and $\vec{H}_{b0}$ are the unknown boundary conditions at the top and bottom surfaces and are numerically equal. This can be justified by reducing the height $h$ of the slice to an infinitesimal value.

The curl of $\vec{A}$ is expanded using Eq. (3.4.2).

\[
\nabla \times \vec{A} = \left( \sum_i A_i \nabla_t \times \vec{\alpha}_i \right) \tag{3.4.6}
\]

Substituting the expansion for the curl in the functionals and simplifying, we obtain

\[
I_1(\vec{A}) = -\int_\Omega \frac{1}{\mu} \left( \sum_n (A_n \nabla_t \times \vec{\alpha}_n) \right) \cdot \left( \sum_n (A_n \nabla_t \times \vec{\alpha}_n) \right) \, d\Omega \tag{3.4.7}
\]

\[
I_2(\vec{A}) = 2h \int_\Omega \sum_n (A_n \vec{\alpha}_n) \cdot \vec{J} \, d\Omega \tag{3.4.8}
\]

\[
I_3(\vec{A}) = 2h \int_{\Gamma_N} \sum_n (A_n \vec{\alpha}_n) \times \vec{H}_b \cdot d\vec{\Gamma} \tag{3.4.9}
\]

The boundary conditions on the top and bottom faces cancel due to the opposite normal directions which are $\hat{z}$ and $-\hat{z}$ respectively. The complete functional is the sum of the three functionals.
shown above. The first variation of the functional $I(\vec{A})$ is required for the stationary condition.

$$\frac{\partial (I(\vec{A}))}{\partial A_{mn}} = 0 \quad (3.4.10)$$

where $m$ represents each of the cartesian directions.

After simplification, the three equations that are the partial derivatives with respect to the three cartesian directional components of the magnetic-vector potential are written below in equations (3.4.11) to (3.4.13).

$$-\int_{\Omega} \frac{1}{\mu} \left( \sum_n A_n \nabla_t \times \vec{\alpha}_n \right) \cdot \nabla_t \times \vec{\alpha}_{jt} \, d\Omega + \int_{\Omega} \vec{\alpha}_{jt} \cdot \vec{J}_t \, d\Omega + \int_{\Gamma_N} \vec{\alpha}_{jt} \times \vec{H}_b \cdot d\vec{r} \quad (3.4.11)$$

$$-\int_{\Omega} \frac{1}{\mu} \left( \sum_n A_n \nabla_t \times \vec{\alpha}_n \right) \cdot \nabla_t \times \vec{\alpha}_{jy} \, d\Omega + \int_{\Omega} \vec{\alpha}_{jy} \cdot \vec{J}_y \, d\Omega + \int_{\Gamma_N} \vec{\alpha}_{jy} \times \vec{H}_b \cdot d\vec{r} \quad (3.4.12)$$

$$-\int_{\Omega} \frac{1}{\mu} \left( \sum_n A_n \nabla_t \times \vec{\alpha}_n \right) \cdot \nabla_t \times \vec{\alpha}_{jz} \, d\Omega + \int_{\Omega} \vec{\alpha}_{jz} \cdot \vec{J}_z \, d\Omega + \int_{\Gamma_N} \vec{\alpha}_{jz} \times \vec{H}_b \cdot d\vec{r} \quad (3.4.13)$$

The above equations confirm that when the source is only in the longitudinal or $\hat{z}$ direction, the magnetic-vector potential can be reduced to a scalar also in the longitudinal direction just as the source term and does not have any components in the transverse plane. The magnetic field only lies in the transverse plane in this case. It also implies that when the source is only in the transverse plane, there is no component of the magnetic potential in the longitudinal direction. The magnetic potential and the source are always in the same geometrical planes.

### 3.5 Implementation and Examples

For a two-dimensional case with the source term being in the longitudinal direction, the magnetic potential reduces to a scalar problem with $A_z$ being the unknown as shown in the previous section. Eq. (3.3.25) is the functional used for demonstration of the implementation without the Neumann
boundary conditions being enforced.

The magnetic potential is expanded of first-order piecewise linear basis functions, \( \Phi_m \), to demonstrate the solution as

\[
A_z = \sum_m A_m \Phi_m
\]

where \( m \) is the number of nodes in the finite-element mesh. Taking the first variation of Eq. (3.3.25), and assuming free boundary conditions at the Neumann region, we obtain

\[
\frac{\partial I(A_z)}{\partial A_n} = -2 \int_{\Omega} \left( \frac{1}{\mu_m} \left( \frac{\partial \Phi_m}{\partial x} \frac{\partial \Phi_n}{\partial x} + \frac{\partial \Phi_m}{\partial y} \frac{\partial \Phi_n}{\partial y} \right) A_m \right) d\Omega + 2 \int_{\Omega} \Phi_n J_z d\Omega = 0
\]

There are \( m \) such equations which can be written in matrix form as \( KA = Q \) where \( A \) is a vector of length \( m \) that contains all of the unknowns at each node. The unknowns are the magnetic potential at each node. \( Q \) is the source term and is also a vector of length \( m \).

\[
Q_m = 2 \int_{\Omega} \Phi_m J_z d\Omega
\]

\( K \) is a square matrix of size \( m \) which contain the coefficients defined by

\[
K_{ij} = 2 \int_{\Omega} \left( \frac{\partial \Phi_i}{\partial x} \frac{\partial \Phi_j}{\partial x} + \frac{\partial \Phi_i}{\partial y} \frac{\partial \Phi_j}{\partial y} \right) d\Omega
\]

The indices \( i \) and \( j \) span all the nodes and go from 1 to \( m \). If included, the Neumann boundary conditions are imposed by modifying the source vector \( Q \). Dirichlet boundary condition may be easily imposed by setting the entire row in \( K \) of the appropriate boundary node to zero and replacing the diagonal element with 1. The matrix \( K \) is singular if no Dirichlet boundary conditions are imposed. To arrive at a valid solution, a Dirichlet boundary condition must be enforced at a minimum of one point on the boundary and preferably along the entire boundary.

In several magnetostatic problems, a Dirichlet boundary condition of zero is applied to the entire boundary while moving the boundary position away from the region of interest such that the actual magnetic fields at the boundary are negligible. The last step for imposing the Dirichlet boundary
condition is to set the same appropriate boundary row in the source vector \( Q \) to the specific Dirichlet boundary value. This is repeated for all the boundary nodes that are subjected to the Dirichlet boundary constraint. The procedure described is identical with other bases.

After the scalar magnetic potential \( A_z \) has been solved, the magnetic flux density can be obtained by taking the numerical curl of the magnetic potential as

\[
B_x^e = \sum_{i=1}^{3} A_i \frac{\partial \Phi_i}{\partial y}
\]

(3.5.5)

\[
B_y^e = -\sum_{i=1}^{3} A_i \frac{\partial \Phi_i}{\partial x}
\]

(3.5.6)

where the superscript \( e \) denotes that the flux density is computed in each element and is piecewise-constant, \( i \) covers all the three nodes in each element where the magnetic potential has been defined at and has been solved for, \( \Phi_i \) is the basis function defined at node \( i \) and in element \( e \). The magnetic field can be easily obtained by scaling the flux density by a function of the permeability in each element. The magnetic field and the magnetic flux density are a constant in each element due to the curl operator on a piecewise-linear expansion for \( \vec{A} \).

The magnetic-vector potential is a mathematical tool and does not have any physical significance. The relationship between the boundary conditions on the magnetic potential and the actual magnetic field is explored. The Neumann relationship at the boundary can be written as \( (\hat{n} \times \vec{A}) \cdot \vec{H}_b \) which suggests that the Neumann boundary condition on the magnetic potential is indirectly imposing the boundary condition on the tangential magnetic field at the boundary. In a two-dimensional problem with the source only in the \( \hat{z} \) direction, the magnetic field is in the transverse or \( x - y \) plane and only the \( \hat{z} \) component of the vector-magnetic potential is required to be considered. In this case, Neumann boundary conditions on \( A_z \) indirectly sets the transverse magnetic field \( \vec{H}_t \) at the boundary. Similarly, the Dirichlet boundary condition on \( A_z \) sets the normal component of the magnetic flux density at the boundary.

If a two-dimensional problem with the source only in the transverse plane is considered, the magnetic field is only in the \( \hat{z} \) direction while the magnetic potential exists in the transverse plane. In this case, Neumann boundary conditions on \( \vec{A}_t \) indirectly affect \( H_z \) while the Dirichlet boundary
condition acts on $\mathbf{B}_t$ which does not exist. So, in this case the Dirichlet boundary value should not have any affect on the solution. Several three-dimensional problems can be reduced to two-dimensions if the variation of the fields along the relative axial length along the $\hat{z}$ direction is negligible. Examples of these include lifting electromagnets, electric motors, and conductors of infinite length carrying DC current. The following examples demonstrate the effects of these different boundary conditions applied to the magnetic potential on the magnetic field.

### 3.5.1 Square Conductor with Longitudinal Source

This example consists of a square conductor at the center of the bounding box carrying a current, $J_z$, only in the longitudinal direction. Dirichlet boundary conditions are applied to all four sides of the bounding square. The input geometry is shown in Fig. 3.3.

![Figure 3.3: A square conductor is located at the center of the bounding box with a longitudinal current source, $J_z$. Mesh generated contains 5678 nodes and 11081 elements or cells.](image)

From Fig. 3.4 it is seen that the boundary condition on $A_z$ on the top and bottom edges of the bounding box sets $H_y$ to zero and allows a natural boundary for $H_x$. Similarly, boundary conditions on $A_z$ on the vertical edges sets $H_x$ to zero while setting a natural boundary for $H_y$. This confirms that boundary conditions on $A_z$ affects the normal component of the transverse magnetic field at that boundary as described in the previous section.

Fig. 3.5 shows the same square conductor but the Dirichlet boundary condition is only applied to
Figure 3.4: Square conductor is located at the center of the bounding box with a longitudinal current source, \( J_z \). Dirichlet boundary conditions are applied at all four bounding edges. The transverse magnetic field \( H_x \) (left) and \( H_y \) (right) are shown. The magnetic field is piecewise-constant as the magnetic potential is piecewise-linear. See text for explanation of boundary conditions and effect on the magnetic field. Contours of \( A_z \) are also shown.

Figure 3.5: Dirichlet boundary conditions are applied only to the two vertical edges of the bounding box. Free boundary conditions exist at the horizontal edges of the bounding box. Transverse magnetic field \( H_x \) (left) and \( H_y \) (right) are shown. Contours of \( A_z \) are also shown.
the vertical sides of the bounding box. No boundary condition is applied at the top and bottom sides. Contours of the magnetic potential $A_z$ are also shown in the figure. It is once again seen that the application of the Dirichlet boundary condition on $A_z$ affects the normal component of $\vec{H}_t$ at the boundary. The free boundary condition which is a Neumann type on the top and bottom sides affects the transverse magnetic field which is $H_x$ in this case.

### 3.5.2 Lifting Electromagnet

The lifting electromagnet example demonstrates the ability of the vector-magnetic potential method to be able to provide accurate solutions when the domain of interest contains different materials. The example consists of a lifting magnet in vacuum and a steel bar as the load a short distance away. The coils are also assumed to have the same permeability as that of vacuum and carry current only in the $+/−z$ directions. The inverted ‘U’ shaped core has a $\mu_r = 100$ and $\mu_r = 50$ for the load arm. The developed finite-element method is used to solve for the magnetic field and flux density using the magnetic potential as the unknown. The domain was triangulated with 12699 Delaunay triangular elements, 6484 nodes and 19182 edges. The input geometry can be seen in Fig. 3.6.

![Problem geometry](image)

![Computed magnetic potential](image)

Figure 3.6: Problem geometry shown on left in (a). Computed magnetic potential, $A_z$, shown on right in (b) for the lifting electromagnet with first-order piecewise-linear bases. Note that the magnetic potential is not a physical quantity. Dirichlet boundary conditions of zero specified before initiation of computations at the bounding box is honored by the computed solution.
3.5. Implementation and Examples

Figure 3.7: Computed magnetic flux density magnitude, \( |B_t| \), for the lifting electromagnet results in being piecewise-constant. Contours of computed magnetic potential are also shown that indicates the presence of several leakage paths.

Figure 3.8: Computed magnetic field magnitude, \( |H_t| \), for the lifting electromagnet results in being piecewise-constant. PWC behavior of the magnetic field can be clearly seen. Note the dominance of the magnetic field in the air gap and very low magnitude as expected in the magnetic steel (or electrical grade steel used in transformers and motors) core and load arm.
Fig. 3.6 also shows the computed magnetic potential. As the potential is piecewise-linear, it can be linearly interpolated and the plot shows the actual interpolated distribution inside each element. Dirichlet type boundary conditions with a value of zero was applied to the entire boundary. That is, the magnetic potential at the four sides of the bounding box was set to zero and can be verified from Fig. 3.6.

The magnetic flux density is obtained by taking the curl of the potential and is shown in Fig. 3.7, it is piecewise-constant and is shown with a constant value in each element. Leakage paths in the vicinity of the magnet can also be observed. The steel is assumed to be linear and the maximum flux density is approximately 1.2 T. The contours of the magnetic potential can be retained as it is the actual path of the magnetic flux. Fig. 3.8 shows the magnetic field which is also piecewise-constant. Note the value of near zero in the steel and the dominance in the gap between the steel pieces.

3.6 Chapter Summary

A brief introduction into variational principles was followed by the detailed development of a functional for magnetostatics that includes the Dirichlet boundary condition. A weighted-residual approach results in the same set of equations. The functional has a variational form and includes the stored energy, Maxwell’s equation and both the types of boundary conditions considered. The assumption of reducing the magnetic-vector potential to the magnetic-scalar potential when the source term is only in the longitudinal or $\hat{z}$ direction was validated. This justifies the reduction of many three-dimensional problems to simpler two-dimensional problems which can be solved via the magnetic-scalar potential.

Implementation details of the magnetic potential in finite-element methods with nodal bases were described. The effects of imposing different Dirichlet and Neumann boundary conditions on the magnetic potential on the computed magnetic fields were discussed. Several results were shown with the use of different boundary conditions using the equations developed in the previous sections. Examples using nodal elements with source terms in the longitudinal directions including one with multiple materials were shown.
Examples using CT edge elements for solutions with only transverse source terms have been omitted for brevity even though this has been implemented. In this case, the magnetic field and flux density are only in the longitudinal directions. Edge bases are widely used in dynamics and will be back in the picture in the subsequent chapters.

All field plots presented in this chapter and those that follow in subsequent chapters are results of C++ code developed during the research and are outputs of the developed software. No results pictorially or otherwise from any commercial finite-element software that was used at times for certain validations have been presented in this document. Appendix B contains color and grayscale maps for all the field plots in this document.
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Chapter 4

General Methods in Magnetostatic Finite-Element Analysis
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This chapter addresses the development of equations that can be used to solve magnetostatic problems via finite-element methods without the use of the magnetic-vector potential. These techniques are termed as general methods. Most solution methods in dynamics do not make use of the magnetic-vector potential and this attempt is to be consistent with current dynamic solution methods. The required static Maxwell’s equations are

\[
\nabla \times \vec{H} = \vec{J} \tag{4.0.1}
\]

and

\[
\nabla \cdot \vec{B} = 0 \tag{4.0.2}
\]

For the two-dimensional case, the typical variational form or a weighted-residual approach results in the term \(\vec{H} \cdot \vec{J}\) that equals zero since the source term is only in the longitudinal direction while the field is in the transverse plane. To avoid this difficulty a least-squares approach is chosen to develop a form that is suited for two-dimensional finite-element analysis. Both piecewise-constant and piecewise-linear descriptions are used for the permeability of the material. The continuity at the boundaries of different media is also considered. Lastly, a different approach that does not use the divergence, but a curl equation, is developed. The curl formulation introduces some of the concepts that will be required in dynamics.

### 4.1 Least-Squares Approach

Using the two required Maxwell’s equations, a functional is written for each as

\[
I_1(\vec{H}) = \int_\Omega \left( \nabla \times \vec{H} - \vec{J} \right) \cdot \left( \nabla \times \vec{H} - \vec{J} \right) \, d\Omega \tag{4.1.1}
\]

and

\[
I_2(\vec{H}) = \int_\Omega \left( \nabla \cdot \left( \mu \vec{H} \right) \right)^2 \, d\Omega \tag{4.1.2}
\]

---

*Mixed Field Finite Element Computations*
The magnetic field is expanded as $\mathbf{H} = \sum_n \alpha_n \mathbf{H}_n$ to give a first variation of

$$\frac{\partial (I_1(\mathbf{H}))}{\partial \alpha_j} = 2 \int_{\Omega} \left( \nabla \times \mathbf{H}_j \right) \cdot \left( \nabla \times \mathbf{H} \right) \, d\Omega = 0 \quad (4.1.3)$$

and

$$\frac{\partial (I_2(\mathbf{H}))}{\partial \alpha_j} = 2 \int_{\Omega} \left( \nabla \cdot (\mu \mathbf{H}_j) \right) \cdot \left( \nabla \cdot (\mu \mathbf{H}) \right) \, d\Omega = 0 \quad (4.1.4)$$

The second variations that confirm the extrema for the least-squares approach are

$$\frac{\partial^2 (I_1(\mathbf{H}))}{\partial \alpha_j^2} = 2 \int_{\Omega} \left( \nabla \times \mathbf{H}_j \right) \cdot \left( \nabla \times \mathbf{H} \right) \, d\Omega > 0 \text{ if } \nabla \times \mathbf{H}_j \neq 0 \quad (4.1.5)$$

and

$$\frac{\partial^2 (I_2(\mathbf{H}))}{\partial \alpha_j^2} = 2 \int_{\Omega} \left( \nabla \cdot (\mu \mathbf{H}_j) \right) \left( \nabla \cdot (\mu \mathbf{H}) \right) \, d\Omega \geq 0 \text{ if } \nabla \cdot (\mu \mathbf{H}_j) \geq 0 \quad (4.1.6)$$

In this case the permeability of the material is treated with piecewise-constant (PWC) behavior. The discontinuity at cell boundaries creates an extra difficulty to be addressed. The expansion for the magnetic field is introduced into the equations for the first variations. The resulting equations for a two-dimensional problem with the source in the $\hat{z}$ direction are

$$\sum_n \left( \int_{\Omega} \left( -\frac{\partial \alpha_n}{\partial y} H_{nx} + \frac{\partial \alpha_n}{\partial x} H_{ny} \right) \cdot \left( \nabla_t \times \hat{m} \alpha_j \right) \, d\Omega + \int_{\Omega} \left( \nabla_t \cdot \hat{m} \alpha_j \right) \, d\Omega \right) = 0 \quad (4.1.7)$$

$$\sum_n \left( \int_{\Omega} \mu_n \left( \frac{\partial \alpha_n}{\partial x} H_{nx} + \frac{\partial \alpha_n}{\partial y} H_{ny} \right) \left( \mu_j \hat{m} \cdot \frac{\nabla \alpha_j}{\nabla \hat{m}} \right) \, d\Omega \right) = 0 \quad (4.1.8)$$

where $n$ spans all the basis functions defined in the finite-element mesh and $m = \hat{x}$ and $\hat{y}$. Each of the two above equations represent two separate equation sets.

The divergence term is a constraint on the curl equation. Thus we may add the divergence equation,
4.1. Least-Squares Approach

(4.1.8), to Eq. (4.1.7), which can be thought of as a constraint with a Lagrange multiplier $\lambda_m$. There are no common terms between the two equations as one contains the curl operator with the source term and the other the divergence operator. The process is identical to adding of the functionals $I_1(\vec{H})$ and $I_2(\vec{H})$ as shown below as

$$I(\vec{H}) = I_1(\vec{H}) + \lambda_m I_2(\vec{H}) \quad (4.1.9)$$

The two resulting equations can be arranged in matrix form as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \quad (4.1.10)$$

and solved for the unknowns, $H_x$ and $H_y$ after application of boundary conditions. $Q$ is the source vector.

### 4.1.1 Results using Least-Squares Method

Results shown in this section use the magnetic potential method as a benchmark and show the similarity of the solutions when the material is a constant in the domain of interest. Results that indicate the failure of the least-squares general method to produce accurate solutions when different media are introduced is also presented.

Fig. 4.1 shows a square current source carrying current in the longitudinal direction. The material has the same permeability everywhere in the domain of interest and is that of vacuum. As the general method formulation treats the magnetic field as piecewise-linear, it can be interpolated as shown in the left. The solution to the right is via the magnetic potential method and the field is shown piecewise-constant as it is obtained by taking the curl of a piecewise-linear function and is a constant in each cell. Identical boundary conditions were applied equivalently to both problems at the bounding box. The boundary conditions are observed to be satisfied as explained in the previous chapter. The results obtained via the least-squares method in homogeneous media is very accurate and compares well with the results from the magnetic potential method.

Fig. 4.2 shows a simple lifting electromagnet similar to the electromagnet in Fig. 3.6a. and the
Figure 4.1: Direct solution by least-squares method (a) and magnetic potential method (b) for longitudinal source current, $J_z$, in the square conductor shown in the center of the bounding box. The input geometry is shown in Fig. 3.3. Both plots show the transverse $H_x$ magnetic field. Identical equivalent boundary conditions are enforced on the bounding box. Results are plotted with the same color scale. As the magnetic field is piecewise-linear in the least-squares formulation it can be interpolated as shown on the left whereas the magnetic field is of an order lower and is piecewise-constant via the magnetic potential method as shown on the right. Mesh generated contains almost 4000 elements with 2051 nodes. ($\lambda_m = 1$)

computed flux density by the least-squares general method and by the magnetic-vector potential. The relative permeability $\mu_r$ was set to 10 in the magnetic steel core and load arm and is considered to be piecewise-constant. It is seen that the results from the least-squares method are not accurate when different media are present as the load arm appears to carry negligible flux. Several different values were assigned to the Lagrange coefficient without an improvement in the overall quality of the results.

It is possible to compute the divergence of the magnetic field as well as recompute the source term from the results. This is done to check the form of implementing the least-squares equations and the application of these equations. Fig. 4.3 indicates that while the curl of the magnetic field does result in the source term indicating that the result obtained is indeed due to the source, the divergence seems to have large magnitudes near the right angled corners of the coils. But the divergence conforms to the enforced value in homogeneous media. The curl of the computed
4.2 Piecewise Linear Expansion for the Permeability

In the second general method attempted, the permeability of the material is expanded using the same first-order piecewise-linear (PWL) bases as that of the magnetic field as $\sum_n \mu_n \alpha_n$. This results in the same curl equation as in the case with piecewise-constant permeability but the divergence equation is more involved. The divergence equation is rewritten as

$$\nabla \cdot (\mu \hat{H}) = \mu \nabla \cdot \hat{H} + \nabla \mu \cdot \hat{H} \quad (4.2.1)$$

**Figure 4.2**: Least-squares (a) and magnetic potential (b) solutions for $|\vec{B}_t|$. The constant $\mu_r = 10$ in the lifting magnet core and load arm. It is seen that the least-squares method fails to produce accurate results when different media are present as the flux density in the load arm appears to be negligible. This is a major shortcoming of the developed general method for magnetostatics via least-squares. Darker regions correspond to weaker (near zero) flux densities. ($\lambda_m = 1/\mu_0^2$)
Chapter 4. General Methods in Magnetostatic Finite-Element Analysis

Figure 4.3: Magnitude of the divergence $|\nabla \cdot (\mu \vec{H}_t)|$ (a) and magnitude of the curl $|\nabla \times \vec{H}_t|$ (b). Both the divergence and the curl can be back computed from the solution. The curl of the magnetic field computed from the solution should reproduce the source term in accordance with Maxwell’s equation $\nabla \times \vec{H} = \vec{J}$ and the plot of the right shows that this does occur to an acceptable accuracy. However the divergence is not zero at all locations in the domain of interest even though enforced as a constraint as seen in the plot on the left. A low density mesh has been intentionally used to highlight the weakness of this method. Solution accuracy does not increase with mesh density. Darker regions correspond to weaker (near zero) values. ($\lambda_m = 1/\mu_0^2$)

The first variation is taken on this new equation to form the second required equation. The possibility of adding a very thin layer on either side of a material interface with piecewise-linear permeability providing accurate results was the motivation behind this attempt. It was also considered that the piecewise-linear behavior of the permeability can be enforced only on the elements adjacent to each boundary interface in the case of heterogenous media.

It was found that the results obtained by this approach are also accurate only with a single medium. No results are shown for this reason and this method has since been completely abandoned due to the additional complexity involved in both the formulation and implementation with no gains.

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4.3 Forcing Continuity at Boundary Interfaces

To overcome the problem of erroneous results when the body of interest contained different materials, an attempt is made to force the boundary conditions at all boundary interfaces between materials. That is, the tangential magnetic field and the normal magnetic flux density are forced to be continuous at material interfaces. In the case of sharp corners or wedges, the continuity is enforced based on an average which is a function of the angle of the wedge.

Figure 4.4: Normal and tangential directions at boundary interfaces can be easily defined at interfaces along a straight line as shown on the left. When sharp angles and wedges from a boundary interface, the normal and tangential directions are defined based on an average of the angle formed by the boundary interfaces as shown on the figure on the right.

This is shown in Fig. 4.4 and the directions are adjusted depending on the angle of the wedge. Tangential continuity is forced on the magnetic field by

\[
\hat{n} \times (\vec{H}_2 - \vec{H}_1) = 0
\]

(4.3.1)

and normal continuity is forced on the magnetic flux density by

\[
\hat{n} \cdot (\mu_2 \vec{H}_2 - \mu_1 \vec{H}_1) = 0
\]

(4.3.2)

at the boundary interfaces. No charge or current is assumed to exist at these interfaces.

---

Mixed Field Finite Element Computations
4.3.1 Results of Forcing Continuity

The results obtained by forcing continuity are the best among the three methods considered via the general approach and can handle heterogenous media for small changes in the permeability of the materials present. Small changes means that the differences between the maximum and minimum permeability has to be less than 100 or so. Beyond this value, the accuracy starts to degrade and for large values of permeability such as those encountered in magnetic steel, this method proves to be unusable due to the poor results obtained.

\[
\mu_r = \begin{cases} 
1 & \text{in the lower half} \\
5 & \text{in the upper half of the rectangle} \\
1 & \text{in the entire current region} 
\end{cases}
\]

Figure 4.5: Input geometry with a vertical column current, \( J_z \), at the center of the rectangle. \( \mu_r = 1 \) in the lower half and \( \mu_r = 5 \) in the upper half of the rectangle. \( \mu_r = 1 \) in the entire current region.

Fig. 4.5 shows the input geometry with a vertical column current carrying a constant current \( J_z \) only in the longitudinal direction. The column of current is set up so that it extends above and below the horizontal boundaries in the \( \pm \hat{y} \) direction such that any field in the \( \pm \hat{x} \) direction is minimal. Figures (4.6) and (4.7) show the \( \hat{y} \) component of the magnetic field and magnetic flux density respectively with the input geometry shown in Fig. 4.5. The thin vertical column at the center carries the longitudinal current in both media.

The piecewise-constant behavior of the results obtained by the magnetic potential methods can be clearly seen as well as the piecewise-linear behavior of the field when obtained from the general method. It can be seen from the figures that the results agree with the magnetic potential method and there is tangential continuity in the magnetic field and normal continuity in the normal flux density at the material interfaces. As the field in the \( \hat{x} \) direction has a negligible magnitude, the plots have been omitted.
4.3. Forcing Continuity at Boundary Interfaces

Figure 4.6: Results of $|H_y|$ by forcing continuity (a) and results via magnetic potential (b). Results are accurate and show that the normal magnetic field is discontinuous across the boundary interface at the horizontal center. $\mu_r = 1$ in the lower half and $\mu_r = 5$ in the upper half. The layer of elements on either side of the boundary is intentionally large to demonstrate this method. In practice, this layer can be made extremely thin. Same color scale used, please refer to the Appendix for the color map. ($\lambda_m = 1$)

Fig. 4.8 shows that the source term that can be computed from the solution and does reproduce the vertical column of current. The divergence of the magnetic field shown on the right is near zero everywhere except at a few elements at the material interfaces.

Solutions that are identical with the magnetic potential method are obtained when the difference in the permeability is relatively small. The results obtained when sharp corners and wedges of different materials are present do not agree with the magnetic potential method. The results also start to change drastically as the relative permeability of any one of the materials is increased. A relative permeability of 5000 is used in the rectangular region at the bottom right in the plots shown in Fig. 4.9. A vertical column carrying current, $J_z$, is at the center of the region extending out in the $\pm z$ directions.
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Figure 4.7: Results of $|B_y|$ by forcing continuity (a) and results via magnetic potential (b). Results are accurate and show that the normal magnetic flux density is continuous across the boundary interface at the horizontal center. $\mu_r = 1$ in the lower half and $\mu_r = 5$ in the upper half. Same color scale used, please refer to the Appendix for the color map. Note that the results via the general method is plotted with piecewise-constant behavior. ($\lambda_m = 1$)

4.4 Complementary Bases and the Direct Method

With different materials present in the domain of interest, none of the general methods developed provide results that match reasonably well with the magnetic potential method. The difficulty associated with direct solution methods is the requirement that both the curl and divergence equations are required to be satisfied simultaneously. Alternative approaches that eliminate the divergence equation are sought. This is due to the fact that the curl equation can be implemented accurately as seen by the reproduction of the source term from the computed fields. But the divergence equation appears as though it has not been enforced when combined with the curl equation. As the divergence of the magnetic flux density is zero, it can be represented by the curl of another vector.

It is clear from the previous general developments that in all the solutions the curl of the magnetic field is a constant due to its expansion with first-order bases. The divergence relationship of the magnetic field is now represented as $\nabla \times \vec{A} = \mu \vec{H}$ to obtain $\nabla \cdot (\mu \vec{H}) = 0$. Now the two equations
4.4. Complementary Bases and the Direct Method

(a) Curl, $|\nabla \times \vec{H}|$

(b) Divergence, $|\nabla \cdot (\mu \vec{H})|$

Figure 4.8: Curl (a) and divergence (b) of $\vec{H}$ back computed from results obtained by forcing continuity. Divergence is seen to be mostly honored while the curl reproduces the source current very accurately. Dark regions represent very low magnitudes (near zero) of the parameters plotted. ($\lambda_m = 1$)

of interest become

$$\nabla \times \vec{H} - \vec{J} = 0 \quad (4.4.1)$$

and

$$\nabla \times \vec{A} - \mu \vec{H} = 0 \quad (4.4.2)$$

Eq. (4.4.2) replaces the divergence equation. The basis functions for the vector $\vec{A}$ and the magnetic field $\vec{H}$ are chosen to be piecewise-linear (PWL), $\Phi_i$, and piecewise-constant (PWC), $\Psi_k$, respectively. Eq. (4.4.2) indicates that the magnetic field will of an order lower than that of $\vec{A}$ as the magnetic field is obtained via the curl operator on $\vec{A}$.

$$\vec{A} = \sum_n \Phi_n \vec{A}_n \quad (4.4.3)$$

$$\vec{H} = \sum_k \vec{H}_k = \sum_k (\hat{x} H_{xk} \Psi_k + \hat{y} H_{yk} \Psi_k) \quad (4.4.4)$$

Mixed Field Finite Element Computations
Figure 4.9: $|\vec{B}_t|$ obtained by forcing continuity (b) and magnetic potential (c) methods. The input geometry is shown in (a). The relative permeability, $\mu_r = 5000$, in the rectangular region at the lower right corner. Plots demonstrate that forcing continuity fails to provide accurate results when the difference in permeability between heterogenous materials gets large. Brighter regions correspond to stronger flux densities. ($\lambda_m = 1$)

where $n$ and $k$ represent indices for the number of nodes $N$ and the number of elements $C$ in the finite-element mesh. A weighted-residual form using these equations results in

$$\int_{\Omega} (\nabla \times \vec{H} - \vec{J}) \cdot \vec{a}_i d\Omega = 0$$  \hspace{1cm} (4.4.5)

$$\int_{\Omega} (\nabla \times \vec{A} - \mu \vec{H}) \cdot \vec{\Psi}_j d\Omega = 0$$  \hspace{1cm} (4.4.6)

where $\vec{\Psi}_j = \hat{x}\Psi_j$ and $\hat{y}\Psi_j$, and $\vec{a}_i = \hat{z}\Phi_i$.

The weighting function for the magnetic field equation are the basis functions for the magnetic potential equation and vice-versa. As the bases complement each other in the two equations in a manner that follows the structure of the equations themselves, the two basis sets used are called complementary. They are also chosen in a way that is similar to the structure of the Maxwell’s equations used. The first equation which is for the magnetic field is broken down into two separate
equations, one each in the $\hat{x}$ and $\hat{y}$ directions. The PWC basis function is expanded locally in an element $j$ as

$$
\Psi_{mj} = \begin{cases} 
1 & \text{if } (x, y) \in j \\
0 & \text{if } (x, y) \notin j 
\end{cases}
$$

(4.4.7)

where $m = x, y$. The curl of the magnetic field has to be integrated by parts in the development as the magnetic field itself is a constant in each cell. Integrating by parts and applying Gauss’ theorem, we obtain

$$
\sum_{l=1}^{C} \left( \vec{z} \times \vec{H}_l \int_{\Gamma} \Phi_j \cdot \vec{n} \ d\Gamma - \left( \nabla \Phi_j \times \vec{H}_l \right) \cdot \vec{z} \int_{\Omega} d\Omega - \int_{\Omega} J_{zj} d\Omega \right) = 0
$$

(4.4.8)

The boundary integral is required to be evaluated only when the nodes are on the boundary of the domain and can be ignored for all internal nodes. Fig. 4.10 shows two nodes of the cell or element $e$ on the boundary $\Gamma$ of the domain $\Omega$, the line integral becomes

$$
\vec{H}_e \left( l_i \vec{l}_i(1) + l_k \vec{l}_k(0) \right)
$$

keeping in mind that edge $i$ is opposite to node $i$ and is on the boundary $\Gamma$ of the domain of interest.

Figure 4.10: Nodal basis function which are piecewise-linear (PWL) for a node $j$ on the boundary on the domain $\Omega$. Only the interaction with other nodes on the boundary is considered in the nodal boundary integration. Hence the interaction between node $j$ and $k$ is considered as node $k$ is also on the boundary, the interaction between $j$ and $i$ is ignored as node $i$ is an internal node.
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After simplifications, the two weighted-residual equations can be written as

\[ \sum_{i=1}^{N} \left( \frac{A_{zi}}{2} \hat{l}_i \cdot \hat{m} \right) = \sum_{i=1}^{C} (\mu_i H_{mi} S_i^e \psi_{mi}) \] (4.4.9)

and

\[ \sum_{i=1}^{C} \left( -\bar{H}_i \cdot \oint \Phi_j b_j d\Gamma + \frac{l_j \hat{l}_i}{2} \cdot \bar{H}_i \right) = \int \Omega J_{zj} d\Omega \] (4.4.10)

where \( b_j = \begin{cases} 0 & \text{if } j \notin \Gamma \\ 1 & \text{if } j \in \Gamma \end{cases} \) and \( S_i^e \) is the area of element \( i \).

Equations (4.4.9) and (4.4.10) can be written in matrix form as

\[
\begin{bmatrix}
M_{1(C+N)} \\
M_{2(C+N)}
\end{bmatrix}_{(2C+N)} \begin{bmatrix} A_{(N+1)} \end{bmatrix} = \begin{bmatrix}
H_{x(C+1)} \\
H_{y(C+1)}
\end{bmatrix}_{(2C+1)} = 0
\] (4.4.11)

\[
\begin{bmatrix}
M_{3(N+C)} \\
M_{4(N+C)}
\end{bmatrix}_{(N+2C)} \begin{bmatrix}
H_{x(C+1)} \\
H_{y(C+1)}
\end{bmatrix}_{(2C+1)} = Q_{(N+1)}
\] (4.4.12)

with the dimensions of the sub-matrices shown. To obtain the magnetic field, the vector of unknowns \( A \) is first solved for by

\[
\begin{bmatrix}
M_3 & M_4
\end{bmatrix} \begin{bmatrix} M_1 \\
M_2
\end{bmatrix} A = Q
\] (4.4.13)

\[ A = M^{-1} Q \] (4.4.14)

where \( M = \begin{bmatrix}
M_3 & M_4
\end{bmatrix} \begin{bmatrix} M_1 \\
M_2
\end{bmatrix} \) and \( Q \) is the source integral. The magnetic field is then solved for from

\[
\begin{bmatrix}
H_x \\
H_y
\end{bmatrix} = \begin{bmatrix} M_1 \\
M_2
\end{bmatrix} A
\] (4.4.15)

The boundary conditions are enforced by making appropriate modifications on matrix \( M \) and vector \( Q \). It is seen that this choice of equations and the bases that are used are identical to the
4.5. Chapter Summary

magnetic-vector potential method and the vector $\vec{A}$ is identical to the magnetic-vector potential. The elegant form of the traditionally used magnetic-vector potential reveals itself in this choice of solution. It must be noted that the above equations are used to reveal the structure of the traditionally used magnetic-vector potential method and are not suited for use in the same form in the general case due to the matrix inversions and additional storage required. However, the method does work with any geometry and heterogeneous media if it is used when the computational time and storage required are not of any concern.

It must be noted that the equations developed for the Direct method in magnetostatics can be shown to be the limiting case of the dynamic form developed for Maxwell’s equations in Chapter 6.

4.4.1 Results using complementary bases

Solutions obtained via the Direct method leads to the magnetic-vector potential form, thus the results obtained are identical to the variational results obtained by using the magnetic-vector potential method. Several problems were solved that provided identical results to eight decimal places. Field plots of the results are omitted here as they are identical to those presented in earlier sections and chapters as the same problems were also used as benchmarks. The Direct method provides a new perspective on the development of the magnetic-vector potential method. It also shows that the structure of complementary basis is hidden in the magnetic-vector potential method where $\vec{A}$ is piecewise-linear and $\vec{H}$ or $\vec{B}$ is piecewise-constant. The difficulties associated with the attempts at general solutions caused by enforcing the divergence equation along with the curl equation do not appear when only curl equations are used as there is a complementary nature to the problem variables.

4.5 Chapter Summary

New and unique developments for general methods of solutions from Maxwell’s equations to magnetostatic problems were formulated. It was shown that these methods can provide accurate results only in homogeneous media. The computed results for the magnetic field are of an order higher than those obtained via the magnetic potential method and may be useful for homogeneous prob-
lems. While the field plots of the magnetic field can be interpolated and are of a higher order than
the fields obtained from the magnetic-vector potential which are piecewise-constant, the solutions
obtained via the general methods are not necessarily better even though they may appear to be so
in the field plots. Additional research is required to arrive at a suitable conclusion in homogeneous
media. The complementary structure of the basis functions in the vector-magnetic potential was
demonstrated via a different solution method that actually results in the vector-magnetic potential
method. The new solution method was shown to be a Direct method from Maxwell’s equations.

Having provided a new perspective on and a complete derivation of a full variation form of the
vector-magnetic potential from fundamental principles in the previous chapter, identifying some
of the difficulties associated with general solutions in magnetostatics, providing an alternative per-
spective on the usually used methods which resulted in the use of complementary bases, the appli-
cation of complementary bases and Direct methods to dynamics is now considered.
Chapter 5

The Vector Helmholtz Method in Dynamics
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5.1 Test Waveguides

Solutions to the propagating modes in waveguides are typically solved by using the vector-Helmholtz equation via finite-element methods. The vector-Helmholtz method is considered to be reliable when used with vector basis sets and has been used to solve for the propagating modes via both the electric and magnetic field formulations. The Helmholtz equation method has also been used to solve for the modes in lossy waveguides of arbitrary shape. This method is used as a benchmark to compare the accuracy of results to those obtained from the newly developed Direct methods in dynamics to be introduced in the next chapter.

The application of the vector-Helmholtz equation to obtain solutions to the propagating modes via the finite-element method is described in detail including a weighted-residual development of the required functional. The penalty term which includes the divergence of either the magnetic or the electric field depending on the use of the magnetic or electric-field form of the Helmholtz equation is also described along with its derivation as well as details on the implementation. The application of scalar and vector bases via the vector-Helmholtz method is displayed with results including the penalty term on the electric-field form of the Helmholtz equation. A brief discussion regarding the origin of spurious modes with scalar basis sets is made.

This chapter introduces the test waveguides that are used with both the Helmholtz and Direct methods. The different finite-element meshes generated on them is defined. All waveguides considered here are perfect electric conductors (PEC) hence the requirement of enforcing the tangential electric field at the boundary to zero is mandatory for a well defined problem with a unique solution. The waveguides are also assumed to be lossless and homogeneous. Boundary-node normal vectors are required to be defined as the normals are necessary to enforce boundary conditions when the transverse fields are expanded using scalar, nodal basis functions.

5.1 Test Waveguides

Two waveguides of different geometry are chosen for benchmark comparisons. Both have analytical or exact solutions for the propagating modes. Some of the tessellations have also been made similar to those available in literature.

The first waveguide is the WR-90 which is rectangular in cross section. The internal width and
height of the WR-90 waveguide considered is 22.86 mm and 11.43 mm respectively. Four mesh densities are used in increasing mesh resolution and are shown in Fig. 5.1. Note that the tessellation with the lowest mesh resolution contains only one internal node.

![Meshes](image)

(a) 9 nodes, 8 cells, 16 edges  
(b) 13 nodes, 16 cells, 28 edges  
(c) 41 nodes, 64 cells, 104 edges  
(d) 145 nodes, 256 cells, 400 edges

Figure 5.1: Four different meshes on the WR-90 waveguide used as test cases for the propagating modes of electric and magnetic fields. Mesh with 9 nodes is used only for testing purposes during the development.

The second test waveguide is a circular waveguide of 1 m radius. Four mesh densities are used in increasing mesh resolution and are shown in Fig. 5.2. All the nodes on the boundary fall on the circumference of a circle of 1 m radius. The mesh with the highest mesh density has the maximum axes of symmetry.

The low order meshes shown for both the waveguides – the 9 node mesh for the WR-90 and the 12 node mesh for the circular waveguides were used for troubleshooting purposes during the development for both the Helmholtz and Direct methods. No results using these meshes are shown in this document.

The 149 node mesh on the circular waveguide uses narrow or thin triangles. While the accuracy of the modes obtained are higher that those with lower order meshes, the field plots with vector bases
5.1. Test Waveguides

(a) 12 nodes, 14 cells, 25 edges
(b) 31 nodes, 42 cells, 72 edges
(c) 85 nodes, 138 cells, 222 edges
(d) 149 nodes, 232 cells, 380 edges

Figure 5.2: Four different meshes on the circular waveguide of radius 1 m used as test cases for the propagating modes of electric and magnetic fields. Mesh with 12 nodes is used only for testing purposes during the development.

(edge bases) for the transverse fields are not smooth, as will be displayed in subsequent chapters. For smooth field plots using edge bases, the interpolation behaves better with near equilateral triangles and hence the mesh generated must consist of triangles that are nearly equilateral. Also the edges of each cell must be aligned with the \( \hat{x} \) or \( \hat{y} \) axes if possible so the continuity can be seen if cartesian components of any of the fields are being plotted. As this is not likely to happen for all three edges of any cell, the field plots appear to be distorted. This is a property of the edge bases themselves and is not a plotting error, nor does it represent the quality of the solution.
5.2 Boundary-Node Normal Definitions

Normal directional vectors are required to be defined at boundary nodes for scalar bases. There are three different ways that the normals can be defined. The first is to use the true normal of the waveguide itself before being approximated by the finite-element mesh at the boundary location. This method is easily implemented for rectangular and circular waveguides but the normals become ambiguous for arbitrarily shaped waveguides as there could be conflicts in the directions between adjacent boundary nodes after discretization. The second normal definition is to use the bisector of the angle between the two adjacent edges of the boundary node. The final definition is to use the length of the adjacent edges to the boundary nodes and determine the angle which is a weighted bisector of the angle between them. Significant differences in performance were not found between the second and third methods described, so the second is chosen as it is simpler and more intuitive.

Fig. 5.3 shows the definition of the node normal vector. It is the bisector of the angle created by the two outermost adjacent edges of a boundary node, which are also on the boundary of the finite-element mesh. For example, if the top, right corner of a rectangular waveguide is in the first quadrant, the nodal normal vector for that corner boundary node would be $0.707\hat{x} + 0.707\hat{y}$.

![Figure 5.3](image_url)  

Figure 5.3: The boundary-node normal vector for node $n$ on the boundary is defined as the bisector of the two edges adjacent to the boundary node. The node normal is defined to be outward to the boundary in direction.

Application of boundary conditions on the WR-90 or such rectangular waveguides when scalar...
basis functions are used is straightforward. Instead of using the node normal direction vectors, both the \( \hat{x} \) and \( \hat{y} \) components of the transverse electric fields are set to zero at the four corner nodes of the waveguide. At all the other boundary nodes, only the \( \hat{x} \) or \( \hat{y} \) components have to be set to zero depending on the node being on the horizontal or vertical boundary respectively. \( E_z \) is forced to zero at all boundary nodes. Enforcing boundary conditions with scalar bases on arbitrary shaped waveguide is more involved and will be described in detail in the Direct method implementation. The ease of enforcing boundary conditions on the vector-edge bases is one of the reasons for its popularity in dynamics.

5.3 The Vector-Helmholtz Method

The weighted-residual form of the vector-Helmholtz method is derived in this section. The electric-field form of the vector-Helmholtz equation is chosen as it can demonstrate the application of boundary conditions when tangential edge bases are used. The procedure for the magnetic-field form of the Helmholtz equation is identical. The equations required for the Matrix solutions for the propagation constants \( \gamma \) and the cutoff wavenumber \( k_c \) are derived. Results obtained by using scalar and vector bases as well as inclusion of the penalty term which includes the divergence of the electric field as a constraint is also shown.

5.3.1 Weighted-Residual Development for the Helmholtz Form

Maxwell’s equations for a source free, lossless and homogeneous region are

\[
\nabla \times \vec{E} = -j\omega \mu_r \mu_0 \vec{H}
\]  

(5.3.1)

and

\[
\nabla \times \vec{H} = j\omega \epsilon_r \epsilon_0 \vec{E}
\]  

(5.3.2)
Taking the curl on both sides of the first equation and substituting Maxwell’s equation for the right hand side with the curl of the magnetic field, we obtain

\[ \nabla \times \frac{1}{\mu_r} \nabla \times \vec{E} - \epsilon_r \omega^2 \varepsilon_0 \mu_0 \vec{E} = 0 \]  

(5.3.3)

which is the vector-Helmholtz equation for the electric field. The quantities \(\mu_r\) and \(\epsilon_r\) are retained in the equation as they can vary with position. Scaling the equations becomes relatively easy if it is required for numerical precision reasons. Writing the free space wavenumber as \(k = \omega \sqrt{\mu_0 \varepsilon_0}\), we obtain

\[ \nabla \times \frac{1}{\mu_r} \nabla \times \vec{E} - k^2 \epsilon_r \vec{E} = 0 \]  

(5.3.4)

The curl of the electric field can be written in terms of the transverse and longitudinal components as

\[ \nabla \times \vec{E} = \nabla_t \times \vec{E}_t - \hat{z} \times (\gamma \vec{E}_t + \nabla_t \vec{E}_z) \]  

(5.3.5)

The first term in the above equation is in the \(\hat{z}\) or longitudinal direction and the second is in the transverse plane. With the waveguide aligned in the \(\hat{z}\) direction the electric field is expanded as

\[ \vec{E} = \vec{E}(x, y, z) = \vec{E}_t(x, y) e^{-\gamma z} + \hat{z} \vec{E}_z(x, y) e^{-\gamma z} \]  

(5.3.6)

or

\[ \vec{E}(x, y, z) = \sum_n E_n \vec{e}_n(x, y) e^{-\gamma z} \]  

(5.3.7)

Taking the curl of Eq. (5.3.5), we obtain

\[ \nabla \times \left( \frac{1}{\mu_r} \nabla \times \vec{E} \right) = \nabla_t \times \left[ \frac{1}{\mu_r} (-\hat{z} \times (\gamma \vec{E}_t + \nabla_t \vec{E}_z)) \right] - \hat{z} \times \left[ \frac{\gamma}{\mu_r} (-\hat{z} \times (\gamma \vec{E}_t + \nabla_t \vec{E}_z)) \right] \]

\[ - \hat{z} \times \left[ \nabla_t \left( \frac{1}{\mu_r} \hat{z} \cdot \nabla_t \times \vec{E}_t \right) \right] \]

\[ = -\hat{z} \nabla_t \cdot \left( \frac{1}{\mu_r} (\gamma \vec{E}_t + \nabla_t \vec{E}_z) \right) - \frac{1}{\mu_r} (\gamma \vec{E}_t + \gamma \nabla_t \vec{E}_z) + \nabla_t \times \frac{1}{\mu_r} \nabla_t \times \vec{E}_t \]  

(5.3.8)

Using the above expansion, the two-dimensional, vector-Helmholtz Eq. in (5.3.4) can be written as
\[ \nabla \times \frac{1}{\mu_r} \nabla \times E_t - \frac{1}{\mu_r} \left( \gamma^2 E_t + \gamma \nabla_t E_z \right) - 2 \nabla_t \cdot \left( \frac{1}{\mu_r} \left( \gamma E_t + \nabla_t E_z \right) \right) - k^2 \varepsilon_r E_t - k^2 \varepsilon_r 2E_z = 0 \] (5.3.9)

It is easy to identify the transverse and longitudinal terms in the above equation just by inspection.

As the waveguide is aligned in the longitudinal direction the problem has been reduced to a two-dimensional form. The surface which is a cross section of the waveguide is represented by \( \Omega \) and the boundary of the surface is a contour represented by \( \Gamma \). Without loss of generality, the cross section can be defined to lie at the \( z = 0 \) plane. A weighted-residual approach results in the following equation where \( \tilde{e}_n^* \) are the testing functions for the electric field:

\[ \int_\Omega \tilde{e}_n^* \cdot \left( \nabla \times \frac{1}{\mu_r} \nabla \times \bar{E} - k^2 \varepsilon_r \bar{E} \right) d\Omega = 0 \] (5.3.10)

or in transverse and longitudinal components as

\[ \int_\Omega \tilde{e}_{tn}^* \cdot \left( \nabla_t \times \frac{1}{\mu_r} \nabla_t \times \bar{E}_t - \frac{1}{\mu_r} \left( \gamma^2 \bar{E}_t + \nabla_t E_z \right) - k^2 \varepsilon_r \bar{E}_t \right) d\Omega = 0 \] (5.3.11)

\[ \int_\Omega e_{zn}^* \left( - \nabla_t \cdot \left( \frac{1}{\mu_r} \left( \gamma E_t + \nabla_t E_z \right) \right) - k^2 \varepsilon_r E_z \right) d\Omega = 0 \] (5.3.12)

These two equations are the basic equations required for the electric-field form of the vector-Helmholtz equation method. Eq. (5.3.11) represents the equation obtained with the transverse basis functions and Eq. (5.3.12) represents the equation obtained with the longitudinal bases.

To reduce the order of the expansion in Eq. (5.3.11) and Eq. (5.3.12), the terms with the second derivative under the integrals are integrated by parts to obtain

\[ \int_\Omega \tilde{e}_{tn}^* \cdot \nabla_t \times \frac{1}{\mu_r} \nabla_t \times \bar{E}_t d\Omega = \int_\Omega \frac{1}{\mu_r} \nabla_t \times \tilde{e}_{tn}^* \cdot \nabla_t \times \bar{E}_t d\Omega - \int_\Gamma \frac{1}{\mu_r} \tilde{e}_{tn}^* \cdot \left( \nabla_t \times \bar{E}_t \times \hat{n} \right) d\Gamma \] (5.3.13)
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\[ \int_{\Omega} e_z^* \nabla_t \cdot \left( \frac{1}{\mu_r} \nabla_t E_z \right) \, d\Omega = - \int_{\Omega} \frac{1}{\mu_r} \nabla_t e_z^* \cdot \nabla_t E_z \, d\Omega + \oint_{\Gamma} \frac{1}{\mu_r} e_z^* \nabla_t E_z \cdot \hat{n} \, d\Gamma \quad (5.3.14) \]

\[ \int_{\Omega} e_z^* \nabla_t \cdot \left( \frac{\gamma}{\mu_r} E_t \right) \, d\Omega = - \int_{\Omega} \frac{\gamma}{\mu_r} \nabla_t e_z^* \cdot E_t \, d\Omega + \oint_{\Gamma} \frac{\gamma}{\mu_r} e_z^* E_t \cdot \hat{n} \, d\Gamma \quad (5.3.15) \]

Substituting the above three integrals into equations (5.3.11) and (5.3.12), we obtain

\[ \int_{\Omega} \frac{1}{\mu_r} \nabla_t \times \bar{e}_{tn}^* \cdot \nabla_t \times \bar{E}_t \, d\Omega - \int_{\Omega} \frac{1}{\mu_r} \bar{e}_{tn}^* \cdot \left( \gamma^2 \bar{E}_t + \gamma \nabla_t E_z \right) \, d\Omega \\
- \int_{\Omega} k^2 \bar{e}_{tn} \bar{e}_{tn}^* \cdot \bar{E}_t \, d\Omega - \oint_{\Gamma} \frac{1}{\mu_r} \bar{e}_{tn}^* \times \left( \nabla_t \times \bar{E}_t \right) \cdot d\bar{\Gamma} = 0 \quad (5.3.16) \]

and

\[ \int_{\Omega} \frac{1}{\mu_r} \nabla_t e_z^* \cdot \nabla_t E_z \, d\Omega + \int_{\Omega} \frac{\gamma}{\mu_r} \nabla_t e_z^* \cdot \bar{E}_t \, d\Omega - \int_{\Omega} k^2 \bar{e}_{tn} \bar{e}_{tn}^* \, E_z \, d\Omega \\
- \oint_{\Gamma} \frac{1}{\mu_r} e_z^* \nabla_t E_z \cdot d\bar{\Gamma} - \oint_{\Gamma} \frac{\gamma}{\mu_r} e_z^* \bar{E}_t \cdot d\bar{\Gamma} = 0 \quad (5.3.17) \]

Perfect-conducting boundary conditions are applied to the testing functions \( \bar{e}_{tn}^* \) and \( e_z^* \) at the boundary of the waveguide. The longitudinal electric field is set to zero at all boundary locations, the tangential component of the transverse electric field is also set to zero at all the boundary locations. Therefore the boundary integrals in the above equations are not explicitly used. Rewriting the equations, we obtain

\[ \int_{\Omega} \left[ \frac{1}{\mu_r} \left( \nabla_t \times \bar{e}_{tn}^* \cdot \nabla_t \times \bar{E}_t - \gamma^2 \bar{e}_{tn}^* \cdot \bar{E}_t - \gamma \bar{e}_{tn}^* \cdot \nabla_t E_z \right) - k^2 \bar{e}_{tn} \bar{e}_{tn}^* \cdot \bar{E}_t \right] \, d\Omega = 0 \quad (5.3.18) \]

\[ \int_{\Omega} \left[ \frac{1}{\mu_r} \left( \gamma \nabla_t e_z^* \cdot \bar{E}_t + \nabla_t e_z^* \cdot \nabla_t E_z \right) \, d\Omega - k^2 \bar{e}_{tn} \bar{e}_{tn}^* \, E_z \right] \, d\Omega = 0 \quad (5.3.19) \]

The above equations contain terms with 1, \( \gamma \), and \( \gamma^2 \) as coefficients that can be simplified by
defining a new vector as
\[ \bar{p}_t = \gamma \bar{E}_t \] (5.3.20)
and substituting in equations (5.3.18) and (5.3.19) to obtain
\[
\int_{\Omega} \left[ \frac{1}{\mu_r} \left( \nabla_t \times \bar{e}_{tn}^* \cdot \nabla_t \times \bar{p}_t - \gamma^2 \bar{e}_{tn}^* \cdot \bar{p}_t - \gamma^2 \bar{e}_{tn}^* \cdot \nabla_t E_z \right) - k^2 \epsilon_r \bar{e}_{tn}^* \cdot \bar{p}_t \right] d\Omega = 0 \tag{5.3.21}
\]
\[
\int_{\Omega} \left[ \frac{1}{\mu_r} \left( \nabla_t e_{zn}^* \cdot \bar{p}_t + \nabla_t e_{zn}^* \cdot \nabla_t E_z \right) - k^2 \epsilon_r e_{zn}^* E_z \right] d\Omega = 0 \tag{5.3.22}
\]

Equations (5.3.21) and (5.3.22) form the required equations for the vector-Helmholtz method using the electric-field form of Maxwell’s equations. A similar equation is obtained if the magnetic-field form is used. The equations can be written in matrix form as an eigenvalue problem, and the propagating modes or the cutoff wavenumber can be computed.

### 5.3.1.1 Vector Bases for the Transverse Fields

The equations derived in (5.3.21) and (5.3.22) take on different forms depending on the type of basis functions used for the expansion of the transverse fields. The two equations can be written in a simplified form as shown below.

\[
A_{11} \bar{p}_t - \gamma^2 A_{12} \bar{p}_t - \gamma^2 A_{13} E_z - k^2 A_{14} \bar{p}_t = 0 \tag{5.3.23}
\]

\[
B_{11} \bar{p}_t + B_{12} E_z - k^2 B_{13} E_z = 0 \tag{5.3.24}
\]

If a single basis function set is used to represent the transverse electric field, the above equations can be rearranged and solved as an eigenvalue problem with the unknown as \( k_c^2 \) as shown in Eq. (5.3.25) for cutoff when \( \gamma = 0 \). A single basis function for the transverse fields means that the basis functions are vectors.

\[
\begin{bmatrix}
A_{11} & 0 \\
0 & B_{12}
\end{bmatrix}
\begin{bmatrix}
\bar{E}_t \\
E_z
\end{bmatrix}
= k_c^2
\begin{bmatrix}
A_{14} & 0 \\
0 & B_{13}
\end{bmatrix}
\begin{bmatrix}
\bar{E}_t \\
E_z
\end{bmatrix} \tag{5.3.25}
\]
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Note that \( \bar{\rho}_t \) is written as \( \gamma \bar{E}_t \) in the eigenvalue matrix formulations to highlight the substitution and the unknowns. If the unknown is treated to be \( \gamma^2 \), equations (5.3.21) and (5.3.22) can be rearranged and solved as an eigenvalue problem as shown below:

\[
\begin{bmatrix}
    A_{11} - k^2 A_{14} & 0 \\
    0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
    \gamma \bar{E}_t \\
    E_z \\
\end{bmatrix} = \gamma^2
\begin{bmatrix}
    A_{12} & A_{13} \\
    B_{11} & B_{12} - k^2 B_{13} \\
\end{bmatrix}
\begin{bmatrix}
    \gamma \bar{E}_t \\
    E_z \\
\end{bmatrix}
\] (5.3.26)

The longitudinal Eq. (5.3.22) is scaled by \( \gamma^2 \) in Eq. (5.3.26). The sub-matrices are defined as follows:

- \( A_{11nj} = \int_{\Omega} \frac{1}{\mu_r} \nabla_t \times \bar{e}_{tn} \cdot \nabla_t \times \bar{e}_{tj} d\Omega \)
- \( B_{11nj} = \int_{\Omega} \frac{1}{\mu_r} \nabla_t e_{zn}^* \cdot \bar{e}_{tj} d\Omega \)
- \( A_{12nj} = \int_{\Omega} \frac{1}{\mu_r} \bar{e}_{tn}^* \cdot \bar{e}_{tj} d\Omega \)
- \( B_{12nj} = \int_{\Omega} \frac{1}{\mu_r} \nabla_t e_{zn}^* \cdot \nabla_t e_{zj} d\Omega \)
- \( A_{13nj} = \int_{\Omega} \frac{1}{\mu_r} \bar{e}_{tn}^* \cdot \nabla_t e_{zj}^* d\Omega \)
- \( B_{13nj} = \int_{\Omega} e_{r} e_{zn}^* e_{zj} d\Omega \)
- \( A_{14nj} = \int_{\Omega} \epsilon_r \bar{e}_{tn}^* \cdot \bar{e}_{tj} d\Omega \)
- \( B_{14nj} = \int_{\Omega} \epsilon_r e_{zn}^* e_{zj} d\Omega \)

If field plots of the electric field are required after the solution has been obtained, the eigenvectors that provide the transverse and longitudinal electric fields may be used.

### 5.3.1.2 Scalar Bases for the Transverse Fields

If scalar bases are used for the transverse electric field, two separate equations are required for each of the \( \hat{x} \) and \( \hat{y} \) components of the electric field. Eq. (5.3.21) is split into two equations corresponding to the transverse cartesian coordinates of the scalar basis functions used.

\[
\int_{\Omega} \left[ \frac{1}{\mu_r} \left( \nabla_t \times \hat{x} e_{xn}^* \cdot \nabla_t \times \hat{p}_t - \gamma^2 \hat{x} e_{xn}^* \cdot \hat{p}_t - \gamma^2 \hat{x} e_{xn}^* \cdot \nabla_t E_z \right) - k^2 \epsilon_r \hat{e}_{xn}^* \cdot \hat{p}_t \right] d\Omega = 0 \] (5.3.27)
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\[ \int_{\Omega} \left[ \frac{1}{\mu_r} \left( \nabla_t \times \hat{g} \hat{e}_{yn} \cdot \nabla_t \times \hat{p}_t - \gamma^2 \hat{g} \hat{e}_{yn} \cdot \hat{p}_t - \gamma^2 \hat{g} \hat{e}_{yn} \cdot \nabla_t \hat{E}_z \right) - k^2 \varepsilon_r \hat{g} \hat{e}_{yn} \cdot \hat{p}_t \right] d\Omega = 0 \] (5.3.28)

\[ \int_{\Omega} \left[ \frac{1}{\mu_r} \left( \nabla_t e_{zn}^* \cdot \nabla_t e_{zn} + \nabla_t e_{zn}^* \cdot \nabla_t \hat{E}_z \right) - k^2 \varepsilon_r e_{zn} \hat{E}_z \right] d\Omega = 0 \]

Eq. (5.3.22) remains the same as it represents the longitudinal bases and is repeated above. The three resulting equations for the scalar bases set used can be written in simplified form as shown below.

\[ A_{11x} p_x - \gamma^2 A_{12x} p_x - A_{11y} p_y - \gamma^2 A_{13x} \hat{E}_z - k^2 A_{14x} p_x = 0 \] (5.3.29)

\[ A_{21x} p_x + A_{21y} p_y - \gamma^2 A_{22y} p_y - \gamma^2 A_{23y} \hat{E}_z - k^2 A_{24y} p_y = 0 \] (5.3.30)

\[ B_{11x} p_x + B_{11y} p_y + B_{12} \hat{E}_z - k^2 B_{13} \hat{E}_z = 0 \] (5.3.31)

In addition to an extra equation for the transverse electric field, several additional matrices are created due to the scalar bases set used. The equations can be written in matrix form after again scaling the longitudinal equation by \( \gamma^2 \) as shown below. The scaling is to reduce numerical round off errors and may or may not be required. Note that \( \hat{p}_t \) is written as \( \gamma \hat{E}_x \) and \( \gamma \hat{E}_y \) in the eigenvalue matrix formulation. For a solution to \( \gamma^2 \), the following eigenvalue equation set can be solved:

\[
\begin{bmatrix}
A_{11x} - k^2 A_{14x} & A_{11y} & 0 \\
A_{21x} & A_{21y} - k^2 A_{24y} & 0 \\
0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
\gamma E_x \\
\gamma E_y \\
\hat{E}_z 
\end{bmatrix}
= -\gamma^2
\begin{bmatrix}
-A_{12x} & 0 & -A_{13x} \\
0 & -A_{22y} & -A_{23y} \\
B_{11x} & B_{11y} & B_{12} - k^2 B_{13} 
\end{bmatrix}
\begin{bmatrix}
\gamma E_x \\
\gamma E_y \\
\hat{E}_z 
\end{bmatrix}
\] (5.3.32)

Where the sub-matrices defined in the eigenvalue problems may be written as:

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Several of the integrals are identical and need to be computed only once and scaled by the material constants as and when required if the matrix assembly is based on an element by element method. If PWL basis function are used for the expansion of all three components of the electric field, $e_x = e_y = e_z$ or all the bases functions used for the electric field are identical. The different directional suffixes are retained for clarity and re-useability of the integrals for other scalar bases that may be used.

### 5.3.2 Results

Results include scalar and vector basis sets. Many other combination may be used. For example, piecewise-constant (PWC) bases for $E_z$ with constant-tangential (CT) edge bases for $\tilde{E}_t$ or with piecewise-linear (PWL) bases for $E_x$ and $E_y$. Some of these different combinations are used with the Direct method in the following chapters. The results displayed in this section only includes the
scalar and vector bases typically used with the vector-Helmholtz method.

It must be mentioned that the CN or constant-normal bases may appear to be well suited to be used for the transverse magnetic fields in the magnetic form of the Helmholtz equation as boundary conditions can be easily imposed on the normal component of the magnetic field. But as the curl is not defined, these bases are not suited for the magnetic-field form of the Helmholtz equation. All the plots of the modes shown for the vector-Helmholtz equation method have been mirrored such that solutions to both $\pm \gamma$ are displayed.

5.3.2.1 Results using Vector Bases

The transverse electric field $\vec{E}_t$ is expanded using constant-tangential edge bases (CT), $\hat{T}$, and the longitudinal electric field $E_z$ is expanded using first-order piecewise-linear nodal bases (PWL), $\Phi$. Results for both the rectangular and circular waveguide are shown.

It is seen from Fig. 5.4 and Fig. 5.5 that accurate results for both the rectangular and circular waveguides without any spurious modes are obtained as expected by using the vector basis functions for the transverse fields. There are only two equations to be solved in this case and the two components of the electric field are $\vec{E}_t$ and $E_z$.

The results shown contain modes computed at $0 + j0$. These modes correspond to the enforced boundary conditions since the equations are a part of the eigenvalue problem as the matrix size is not reduced. Field plots corresponding to these modes due to the boundary condition enforcement are not shown as they do not provide any useful information and appear to be random.

5.3.2.2 Results using Scalar Bases

The transverse electric field $\vec{E}_t$ is expanded using scalar first-order piecewise-linear nodal bases (PWL), $\Phi$, which results in two transverse fields $E_x$ and $E_y$. The longitudinal electric field, $E_z$, is also expanded using first-order nodal bases (PWL), $\Phi$. Three equations result in this case with the unknowns being $E_x$, $E_y$, and $E_z$. Results for the rectangular waveguide are shown. In Fig. 5.6 the results are corrupted with spurious modes for scalar bases.
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5.4 Spurious Modes and their Origin

The origin of spurious modes in the vector-Helmholtz equation method is often attributed to the curl-curl operator itself. Eigenfunction solutions to the vector-Helmholtz equation in the finite-element method can be separated into two categories. One that is a true and valid solution to the electromagnetic fields and the second that can be represented as a gradient of a scalar $S$. The gradient, $\nabla S$, is a valid solution but does not satisfy Maxwell's equation $\nabla \cdot \vec{E} = 0$. Peterson et al. [1998] suggests that solutions that have this characteristic form the null-space of the curl-curl operator and hence produces the spurious modes. The appearance of the spurious modes is also related to properties of the basis functions used. The tangential and normal continuity characteristics of the basis function affect the production of spurious modes. Peterson et al. [1998] and Collin [1991] contain detailed discussions and analysis of spurious modes and their origin.

Figure 5.4: Propagation constants ($\gamma$) computed for WR-90 with 41 node mesh of Fig. 5.1. Constant-tangential bases used for $\vec{E}_t$ - piecewise-linear bases used for $E_z$ (CT-PWL). The Helmholtz equation method is used with $k = 167.6676$ rad/m. Accurate results are obtained without any spurious modes. Eigenvalue results computed at $0 + j0$ have been retained in the plots. The propagating mode on the imaginary axis is the $TE_{10}$ mode.

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5.4. Spurious Modes and their Origin

Figure 5.5: Propagation constants ($\gamma$) computed for circular waveguide of 1 m radius with 85 node mesh of Fig. 5.2. Constant-tangential bases used for $\vec{E}_t$ - piecewise-linear bases use for $E_z$ (CT-PWL). The Helmholtz equation method with $k = 3.0$ rad/m. Accurate results are obtained without any spurious modes. Eigenvalue results computed at $0 + j0$ have been retained in the plots. Field plots corresponding to these modes that are due to the boundary condition enforcement do not have any symmetry and are random. The propagating modes on the imaginary axis are the $TE_{11}$ and $TM_{01}$ modes.

is not clear that this null-space argument is complete, but it does provide insight into the problem.

In summary, the origin of the spurious modes is that Maxwell’s equations $\nabla \cdot \vec{E}$ or $\nabla \cdot \vec{B}$ are not satisfied depending on the electric or magnetic form of the vector-Helmholtz equation being used. This produces several non physical solutions which do not represent the electromagnetic fields. Field plots of the spurious modes do not have symmetry in several cases and appear to be almost random but can also appear to be a real mode in a few instances. Often, the boundary conditions specified are violated in field plots of the eigenvectors corresponding to the spurious eigenvalues. These also occur mostly when scalar bases are used for the transverse fields.
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Figure 5.6: Propagation constants ($\gamma$) computed for WR-90 with 145 node mesh of Fig. 5.1. Piecewise-linear bases used for $E_x$, $E_y$, piecewise-linear bases used for $E_z$. The Helmholtz equation method with $k = 167.6676$ rad/m. Note the presence of large number of spurious modes with the scalar bases for the transverse fields. The propagating mode on the imaginary axis is the $TE_{10}$ mode.

The Direct method to be introduced is also a victim of the spurious modes when certain scalar bases are used. As the curl-curl operator is not directly used in the Direct method, the origin of the spurious modes is still though to be attributed to the divergence equations not being satisfied. This is discussed in detail in the following chapters.

5.5 The Penalty Term

When solutions to the propagating modes are corrupt with false modes that are called spurious modes, a technique that includes the divergence of the electric field as a constraint has been attempted to eliminate or reduce the number of spurious modes. Additional information regarding the penalty term can be found in Rahman and Davies [1984b] and Collin [1991], including an im-
5.5. The Penalty Term

Implementation for the magnetic-field form of the Helmholtz equation. The additional term which is the divergence of the electric field weighted by itself is called the penalty term. This section reviews the technique though it has been found to have limited success.

Spurious modes arise usually when scalar bases are used for the transverse electric field. Hence the penalty term discussed here is only applied to the formulation with scalar bases for the transverse electric field. In fact, the method cannot be used for vector bases that have infinite divergence at cell boundaries. The inclusion of the penalty term causes additional matrices to be created. The penalty term may be thought of as an additional constraint included in the Helmholtz equation form with a Lagrange multiplier $\lambda_e$ which can be chosen such that the computed results are free of the spurious modes or such that the spurious modes are easily identified.

The divergence of the electric field can be written from Eq. (5.3.6) as follows.

$$\nabla \cdot \vec{E} = \left( \nabla_t \cdot \vec{E}_t - \gamma E_z \right) e^{-\gamma z}$$  \hspace{1cm} (5.5.1)

The above constraint is added with a coefficient $\lambda_e$ to the equations developed for the Helmholtz method in an integral form as $\lambda_e \int_{\Omega} (\nabla \cdot \vec{e}_n) (\nabla \cdot \vec{E}) d\Omega$. Several additional matrices are created and can be seen in the following equations.

$$\int_{\Omega} \left[ \frac{1}{\mu_r} \left( \nabla_t \times \hat{\vec{e}}_{xn}^* \cdot \nabla_t \times \hat{\vec{p}}_t - \gamma^2 \hat{\vec{e}}_{xn}^* \cdot \hat{\vec{p}}_t - \gamma^2 \hat{\vec{e}}_{xn}^* \cdot \nabla_t E_z \right) - k^2 e_r \hat{\vec{e}}_{xn}^* \cdot \hat{\vec{p}}_t \right] d\Omega$$

$$+ \lambda_e \int_{\Omega} \nabla_t \cdot (\hat{\vec{e}}_{xn}^*) \nabla_t \cdot \hat{\vec{p}}_t d\Omega - \lambda_e \int_{\Omega} \gamma^2 \nabla_t \cdot (\hat{\vec{e}}_{xn}^*) E_z d\Omega = 0$$  \hspace{1cm} (5.5.2)

$$\int_{\Omega} \left[ \frac{1}{\mu_r} \left( \nabla_t \times \hat{\vec{e}}_{yn}^* \cdot \nabla_t \times \hat{\vec{p}}_t - \gamma^2 \hat{\vec{e}}_{yn}^* \cdot \hat{\vec{p}}_t - \gamma^2 \hat{\vec{e}}_{yn}^* \cdot \nabla_t E_z \right) - k^2 e_r \hat{\vec{e}}_{yn}^* \cdot \hat{\vec{p}}_t \right] d\Omega$$

$$+ \lambda_e \int_{\Omega} \nabla_t \cdot (\hat{\vec{e}}_{yn}^*) \nabla_t \cdot \hat{\vec{p}}_t d\Omega - \lambda_e \int_{\Omega} \gamma^2 \nabla_t \cdot (\hat{\vec{e}}_{yn}^*) E_z d\Omega = 0$$  \hspace{1cm} (5.5.3)
Chapter 5. The Vector Helmholtz Method in Dynamics

\[
\int_\Omega \left[ \frac{1}{\mu_r} (\nabla_t e_{zn}^* \cdot \vec{p}_t + \nabla_t e_{zn}^* \cdot \nabla_t E_z) - k^2 e_{zn}^* E_z \right] d\Omega \\
- \lambda e \int_\Omega e_{zn}^* \nabla_t \cdot \vec{p}_t d\Omega + \lambda e \int_\Omega \gamma^2 e_{zn}^* E_z d\Omega = 0 
\] (5.5.4)

The three equations can be written in matrix form and solved. Note that the longitudinal equation is not scaled by \( \gamma^2 \) in this case. Note that \( \vec{p}_t \) is written as \( \gamma E_x \) and \( \gamma E_y \) in the eigenvalue formulation. If the unknown is \( \gamma^2 \), the above equations can be rearranged as

\[
\begin{bmatrix}
A_{11x} - k^2 A_{14x} + \lambda e A_{15x} & A_{11y} + \lambda e A_{12y} & 0 \\
A_{21x} + \lambda e A_{22x} & A_{21y} - k^2 A_{24y} + \lambda e A_{25y} & 0 \\
B_{11x} - \lambda e B_{12x} & B_{11y} - \lambda e B_{12y} & B_{12} - k^2 B_{13}
\end{bmatrix}
\begin{bmatrix}
\gamma E_x \\
\gamma E_y \\
E_z
\end{bmatrix}

= \gamma^2 
\begin{bmatrix}
A_{12x} & 0 & A_{13x} + \lambda e A_{16x} \\
0 & A_{22y} & A_{23y} + \lambda e A_{26y} \\
0 & 0 & -\lambda e B_{14}
\end{bmatrix}
\begin{bmatrix}
\gamma E_x \\
\gamma E_y \\
E_z
\end{bmatrix} 
\] (5.5.5)

The additional matrices created by including the divergence of the electric field are defined as follows.

- **A\(_{15xnj}\)** = \( \int_\Omega \frac{\partial e_{zn}^*}{\partial x} \frac{\partial e_{xj}}{\partial x} d\Omega \)
- **A\(_{26ynj}\)** = \( \int_\Omega \frac{\partial e_{yn}^*}{\partial y} e_{zj} d\Omega \)
- **B\(_{12xnj}\)** = \( \int_\Omega e_{zn}^* \frac{\partial e_{xj}}{\partial x} d\Omega \)
- **B\(_{12ynj}\)** = \( \int_\Omega e_{zn}^* \frac{\partial e_{yj}}{\partial y} d\Omega \)
- **B\(_{14nj}\)** = \( \int_\Omega e_{zn}^* e_{zj} d\Omega \)
- **B\(_{14nj}\)** = \( \int_\Omega e_{zn}^* \frac{\partial e_{yj}}{\partial y} d\Omega \)
- **B\(_{14nj}\)** = \( \int_\Omega e_{zn}^* e_{zj} d\Omega \)
Figure 5.7: Propagation constants ($\gamma$) computed for WR-90 with 145 node mesh of Fig. 5.1. Piecewise-linear bases used for $\bar{E}_x$, $\bar{E}_y$ - piecewise-linear bases used for $E_z$ (PWL-PWL). Helmholtz equation method with $k = 167.6676$ rad/m. Divergence terms included with $\lambda_e = 0.065$. All spurious modes are moved away from the imaginary axis by including the divergence coefficient but several new spurious modes are produced on the real axis. The propagating mode on the imaginary axis is the $TE_{10}$ mode.

The divergence coefficient $\lambda_e$ was set equal to 0.065 for the 145 node mesh on the WR-90 for the results including the divergence term shown in Fig. 5.7. This value was arrived at by trial to provide results with reasonable accuracy for different mesh densities. But when this value was changed slightly, accurate results were not obtained with lower mesh densities. The difficulty in arriving at a coefficient without having to run several solutions with different values of $\lambda_e$ and observing the stationary modes that are the true modes is a weakness of the penalty-term method. While reducing the number of spurious modes on the imaginary axis, the penalty term simply moves a number of spurious modes to the real axis which is another drawback of this method. This movement is observed in Fig. 5.6 and Fig. 5.7.
5.6 Chapter Summary

A brief introduction to the vector-Helmholtz equation method applied to waveguide solutions via the finite-element method was presented. The test waveguides used in the solutions were defined. Four mesh densities were chosen for each of the waveguides.

Table 5.1: Comparison of $\vec{E}$ and $\vec{H}$ equation forms of the Helmholtz method. Results show exact values and the differences between the electric and magnetic-field formulation of the vector-Helmholtz method. Note that no boundary conditions have been applied in the magnetic-field formulation with the bases used and hence the results are inferior to the electric-field formulation. Boundary conditions cannot be applied to the normal component of the transverse magnetic field in a straightforward manner at the boundary with the constant-tangential (CT) expansion for the transverse magnetic field. The waveguide is a PEC.

<table>
<thead>
<tr>
<th>Vector/Waveguide</th>
<th>Exact $\vec{E}$ rad/m</th>
<th>Helmholtz E - Field CT-PWL rad/m</th>
<th>Helmholtz H - Field CT-PWL rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>WR-90</td>
<td>96.0526</td>
<td>92.5159</td>
<td>75.6514</td>
</tr>
<tr>
<td>13 Nodes</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41 Nodes</td>
<td>96.0526</td>
<td>95.1591</td>
<td>91.4983</td>
</tr>
<tr>
<td>145 Nodes</td>
<td>96.0526</td>
<td>95.8290</td>
<td>94.9469</td>
</tr>
<tr>
<td>Circular</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31 Nodes</td>
<td>2.3687</td>
<td>2.3558</td>
<td>2.3468</td>
</tr>
<tr>
<td></td>
<td>1.7933</td>
<td>1.7117</td>
<td>1.7297</td>
</tr>
<tr>
<td>85 Nodes</td>
<td>2.3687</td>
<td>2.3626</td>
<td>2.3604</td>
</tr>
<tr>
<td></td>
<td>1.7933</td>
<td>1.7684</td>
<td>1.7727</td>
</tr>
<tr>
<td>149 Nodes</td>
<td>2.3687</td>
<td>2.3664</td>
<td>2.3588</td>
</tr>
<tr>
<td></td>
<td>1.7933</td>
<td>1.7811</td>
<td>1.7658</td>
</tr>
</tbody>
</table>

The weighted-residual form of the vector-Helmholtz equation was derived. The electric-field form of the Helmholtz equation was used to derive matrices that define an eigenvalue problem from which the cutoff wavenumber or the propagation constant can be found depending on the desired unknown. Results using the vector-Helmholtz method were shown using vector bases for both the WR-90 and circular waveguides. Spurious modes corrupt the results with scalar bases. Table 5.1 shows the results obtained for both waveguides with the two possible vector-Helmholtz equation...
method formulations.

A solution that moves out all the spurious modes by including the divergence of the electric field as a penalty term was considered. It is straightforward to identify the spurious modes on the imaginary axis when compared to the solution obtained without the application of the divergence coefficient. The modes that have not changed location as the penalty parameter is changed are the true or desired modes. It was found that there is no consistent penalty parameter that works with different waveguide geometries. The difficulty in identifying the correct coefficient easily is a major drawback to the use of the penalty term with the electric-field form of the Helmholtz equation.
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Chapter 6

The Direct Method in Dynamics
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The Direct method uses Maxwell’s equations directly and is named due to this fact. Unlike the vector-Helmholtz method which either solves for only the electric or the magnetic field, the Direct method solves for both sets of fields simultaneously. The derivation of the equations that defines the Direct method is completed in a variational form, with extrema obtained. Identical equations are obtained with a weighted-residual approach.

Maxwell’s equations and energy form the initial set of equations required for the new functionals. The divergence of the electric and magnetic fields can be added as constraints to the functionals. This is similar to the application of the penalty term to the equations developed for the vector-Helmholtz equation method shown in the previous chapter. The developed functionals are applied to two-dimensional waveguide analysis as a demonstration of their use.

Other problems in electromagnetics that require finite-element analysis can also be solved by the Direct equation set. The number of resulting equations required to form the eigenvalue problem is a function of the behavior of the basis sets used for the fields. Equations are developed for vector and scalar bases. The resulting equations are arranged as an eigenvalue problem for $\gamma$ and all the required sub-matrices for implementation are defined.

### 6.1 Variational Development of the Direct Method

The required Maxwell’s equations are

\[
\nabla \times \vec{E} = -j\omega \mu \vec{H} - \sigma_m \vec{H} - \vec{M} \tag{6.1.1}
\]

\[
\nabla \times \vec{H} = j\omega \epsilon \vec{E} + \sigma \vec{E} + \vec{J} \tag{6.1.2}
\]

where $\mu = \mu' - j\mu''$ and $\epsilon = \epsilon' - j\epsilon''$. The variables $\mu'$, $\mu''$, $\epsilon'$, and $\epsilon''$ are all positive quantities to represent a finite loss when loss is considered. $\vec{J}$ is the electric source current and $\sigma$ is the electric conductivity of the medium.

In a volume $\Omega$ bounded by a surface $\Gamma$, the time averaged stored electric and magnetic energy can
be written as

\[ W_e = \frac{1}{4} \text{Re} \left( \int_{\Omega} \mathbf{D} \cdot \mathbf{E}^* \, d\Omega \right) = \frac{1}{4} \int_{\Omega} \varepsilon' \mathbf{E} \cdot \mathbf{E}^* \, d\Omega \quad (6.1.3) \]

and

\[ W_m = \frac{1}{4} \text{Re} \left( \int_{\Omega} \mathbf{B} \cdot \mathbf{H}^* \, d\Omega \right) = \frac{1}{4} \int_{\Omega} \mu' \mathbf{H} \cdot \mathbf{H}^* \, d\Omega \quad (6.1.4) \]

respectively. The additional \(1/2\) that makes up the coefficient \(1/4\) is due to the averaging over one time period. The imaginary parts of the integrals in \(W_e\) and \(W_m\) represent power loss. The above equations are valid only when \(\mu\) and \(\varepsilon\) are frequency independent and the media is without dissipation. Modifications become necessary in case of dissipative media and when \(\mu_r\) and/or \(\varepsilon_r\) are a function of the frequency and is not considered here for clarity and demonstration of the newly developed method.

Two functionals \(I_e\) and \(I_h\) are now defined, one to minimize the electric energy and the second to minimize the magnetic energy. Maxwell’s equations are included in the functional as constraints along with the use of Lagrange multiplies that are treated as unknowns. The Lagrange multipliers are written inside the integrals as they need not necessarily be constants.

\[ I_e = \frac{1}{4} \int_{\Omega} \varepsilon' \mathbf{E} \cdot \mathbf{E}^* \, d\Omega + \text{Re} \left( \int_{\Omega} \lambda_e \mathbf{E}^* \cdot \left( j \omega \varepsilon \mathbf{E} + \sigma \mathbf{E} + j \nabla \times \mathbf{H} \right) d\Omega \right) \quad (6.1.5) \]

\[ I_h = \frac{1}{4} \int_{\Omega} \mu' \mathbf{H} \cdot \mathbf{H}^* \, d\Omega + \text{Re} \left( \int_{\Omega} \lambda_h \mathbf{H}^* \cdot \left( j \omega \mu \mathbf{H} + \sigma_m \mathbf{H} + \mathbf{M} + \nabla \times \mathbf{E} \right) d\Omega \right) \quad (6.1.6) \]

The Lagrange multipliers \(\lambda_e\) and \(\lambda_h\) ensure that Maxwell’s equations are an additional constrain
and these parameters are to be determined. The functional for the electric field can be written as

\[ I_e = \frac{1}{4} \int_\Omega \bar{\varepsilon} \cdot \bar{E}^* \, d\Omega + \frac{1}{2} \int_\Omega \lambda_e \bar{\varepsilon}^* \cdot (j\omega \bar{\varepsilon} \bar{E} + \sigma \bar{E} + \bar{J} - \nabla \times \bar{H}) \, d\Omega + \frac{1}{2} \int_\Omega \lambda_e^* \bar{E} \cdot (j\omega \bar{\varepsilon} \bar{E} + \sigma \bar{E} + \bar{J} - \nabla \times \bar{H})^* \, d\Omega \]  

(6.1.7)

The electric and magnetic fields can be expanded as

\[ \bar{E} = \sum_n E_n \bar{e}_n = (E_{nr} + jE_{ni}) \bar{e}_n \]  

(6.1.8)

\[ \bar{H} = \sum_n H_n \bar{h}_n = (H_{nr} + jH_{ni}) \bar{h}_n \]  

(6.1.9)

and the conjugates \( \bar{E}_n^* \) and \( \bar{H}_n^* \) are written as

\[ \bar{E}_n^* = (E_{nr} - jE_{ni}) \bar{e}_n^* \]  

(6.1.10)

and

\[ \bar{H}_n^* = (H_{nr} - jH_{ni}) \bar{h}_n^* \]  

(6.1.11)

respectively. The required partial derivatives for the electric and magnetic fields are provided for clarity as follows:

\[ \frac{\partial \bar{E}}{\partial E_{nr}} = \bar{e}_n \]
\[ \frac{\partial \bar{E}}{\partial E_{ni}} = \bar{e}_n \]
\[ \frac{\partial \bar{E}^*}{\partial E_{nr}} = \bar{e}_n^* \]
\[ \frac{\partial \bar{E}^*}{\partial E_{ni}} = -\bar{e}_n^* \]
\[ \frac{\partial \bar{H}}{\partial H_{nr}} = \bar{h}_n \]
\[ \frac{\partial \bar{H}}{\partial H_{ni}} = \bar{h}_n \]
\[ \frac{\partial \bar{H}^*}{\partial H_{nr}} = \bar{h}_n^* \]
\[ \frac{\partial \bar{H}^*}{\partial H_{ni}} = -\bar{h}_n^* \]
Chapter 6. The Direct Method in Dynamics

The first variation with respect to the real part of the electric field is

$$\frac{\partial I_e}{\partial E_{nr}} = \frac{1}{4} \int_{\Omega} \epsilon' \left( \bar{e}_n \cdot \bar{E}^* + \bar{E} \cdot \bar{e}_n^* \right) d\Omega$$

$$+ \frac{1}{2} \int_{\Omega} \lambda_e \left( \bar{e}_n^* \cdot \left( j\omega \epsilon \bar{E} + \sigma \bar{E} + \bar{J} - \nabla \times \bar{H} \right) + \bar{E}^* \cdot \left( j\omega \epsilon \bar{e}_n + \sigma \bar{e}_n \right) \right) d\Omega$$

$$+ \frac{1}{2} \int_{\Omega} \lambda_e^* \left( \bar{e}_n^* \cdot \left( j\omega \epsilon \bar{E} + \sigma \bar{E} + \bar{J} - \nabla \times \bar{H} \right)^* + \bar{E} \cdot \left( -j\omega \epsilon \epsilon \bar{e}_n^* + \sigma \bar{e}_n^* \right) \right) d\Omega = 0 \quad (6.1.12)$$

The first variation with respect to the imaginary part of the electric field is

$$\frac{\partial I_e}{j \partial E_{ir}} = \frac{1}{4} \int_{\Omega} \epsilon' \left( \bar{e}_n \cdot \bar{E}^* - \bar{E} \cdot \bar{e}_n^* \right) d\Omega$$

$$+ \frac{1}{2} \int_{\Omega} \lambda_e \left( -\bar{e}_n^* \cdot \left( j\omega \epsilon \bar{E} + \sigma \bar{E} + \bar{J} - \nabla \times \bar{H} \right) + \bar{E}^* \cdot \left( j\omega \epsilon \bar{e}_n + \sigma \bar{e}_n \right) \right) d\Omega$$

$$+ \frac{1}{2} \int_{\Omega} \lambda_e^* \left( \bar{e}_n \cdot \left( j\omega \epsilon \bar{E} + \sigma \bar{E} + \bar{J} - \nabla \times \bar{H} \right)^* - \bar{E} \cdot \left( -j\omega \epsilon \epsilon \bar{e}_n^* + \sigma \bar{e}_n^* \right) \right) d\Omega = 0 \quad (6.1.13)$$

Subtracting Eq. (6.1.13) from Eq. (6.1.12) and simplifying, we obtain

$$\frac{\partial I_e}{\partial E_{nr}} - \frac{\partial I_e}{j \partial E_{ni}} = \frac{1}{2} \int_{\Omega} \bar{e}_n^* \cdot \bar{E} \left( \epsilon' - 2\lambda_e j\omega \epsilon \epsilon + 2\lambda_e^* \sigma \right) d\Omega$$

$$+ \int_{\Omega} \lambda_e \bar{e}_n^* \cdot \left( j\omega \epsilon \bar{E} + \sigma \bar{E} + \bar{J} - \nabla \times \bar{H} \right) d\Omega = 0 \quad (6.1.14)$$

The addition of the partial derivatives simply gives the complex conjugate of Eq. (6.1.14). From the above equation, we obtain

$$\lambda_e^* = -\frac{\epsilon'}{2(j\omega \epsilon + \sigma)^*} \quad (6.1.15)$$

which implies

$$\lambda_e = -\frac{\epsilon'}{2(j\omega \epsilon + \sigma)} \quad (6.1.16)$$
The second partial is used to ensure an extremum in variational methods. Substituting for \( \lambda_e \) and \( \lambda^*_e \) in (6.1.12) and taking the second variation with respect to the real part of the electric field, we obtain

\[
\frac{\partial^2 I_e}{\partial E_{nr}^2} = \frac{1}{2} \int_{\Omega} \epsilon' \bar{e}_n \cdot \bar{e}_n^* d\Omega - \int_{\Omega} \bar{e}_n \cdot \bar{e}_n^* (\lambda_e (j\omega \epsilon + \sigma) + \lambda_e^*(-j\omega \epsilon^* + \sigma)) d\Omega \quad (6.1.17)
\]

which is

\[
\frac{\partial^2 I_e}{\partial E_{nr}^2} = -\frac{1}{2} \int_{\Omega} \epsilon' \bar{e}_n \cdot \bar{e}_n^* d\Omega < 0 \quad (6.1.18)
\]

The second variation with respect to the imaginary part of the electric field is

\[
\frac{\partial^2 I_e}{(j\partial E_n)^2} = + \frac{1}{2} \int_{\Omega} \epsilon' \bar{e}_n \cdot \bar{e}_n^* d\Omega > 0 \quad (6.1.19)
\]

The basis functions \( \bar{e}_n \) and \( \bar{e}_n^* \) are non zero functions. An extremum has been reached for both the real and imaginary parts of the functional for the electric field. Rewriting (6.1.14), we obtain

\[
\int_{\Omega} \frac{\epsilon'}{2(j\omega \epsilon + \sigma)} \bar{e}_n^* \cdot \left( j\omega \bar{E} + \sigma \bar{E} + \bar{J} - \nabla \times \bar{H} \right) d\Omega = 0 \quad (6.1.20)
\]

By duality, the magnetic field functional results in

\[
\int_{\Omega} \frac{\mu'}{2(j\omega \mu + \sigma_m)} \bar{H}_n^* \cdot \left( j\omega \mu \bar{H} + \sigma_m \bar{H} + \bar{M} + \nabla \times \bar{E} \right) d\Omega = 0 \quad (6.1.21)
\]

In a lossless, source free region, with \( \mu \) and \( \epsilon \) being real (6.1.20) and (6.1.21) reduce to

\[
\frac{1}{j\omega} \int_{\Omega} \bar{e}_n^* \cdot \left( j\omega \bar{E} - \nabla \times \bar{H} \right) d\Omega = 0 \quad (6.1.22)
\]
The two equations shown above are just Maxwell’s equations in a weighted-residual form.

### 6.2 Application to Waveguides

The equations developed in the previous section require modifications to be used in the analysis of waveguides with finite-element methods. For two-dimensional applications to waveguides, equations (6.1.22) and (6.1.23) are split into their transverse and longitudinal components. In this case $\Omega$ is a surface and $\Gamma$ is a contour. The waveguide is aligned with the $\hat{z}$ axis and is parallel to the $x - y$ or transverse plane. Considering a lossless and source free case, and expanding the electric and magnetic fields by

$$\bar{E} = \sum_n E_n \bar{e}_n(x, y) e^{-\gamma z}$$

(6.2.1) and

$$\bar{H} = \sum_n H_n \bar{h}_n(x, y) e^{-\gamma z}$$

(6.2.2) respectively, Maxwell’s equations for the electric and magnetic fields can be written in terms of the transverse and longitudinal components as

$$\nabla \times \bar{E} = -j\omega \mu \bar{H} = -j\omega \mu \left( \bar{H}_t + \bar{H}_z \right) = -\hat{z} \times \left( \gamma \bar{E}_t + \nabla_t E_z \right) + \nabla_t \times \bar{E}_t$$

(6.2.3)

$$\nabla \times \bar{H} = +j\omega \epsilon \bar{E} = +j\omega \epsilon \left( \bar{E}_t + \bar{E}_z \right) = -\hat{z} \times \left( \gamma \bar{H}_t + \nabla_t H_z \right) + \nabla_t \times \bar{H}_t$$

(6.2.4)

The following equations result by splitting equations (6.1.22) and (6.1.23) in terms of the transverse and longitudinal components:

$$\frac{1}{j\omega} \int_{\Omega} \bar{h}_n^* \cdot \left( j\omega \mu \bar{H} + \nabla \times \bar{E} \right) d\Omega = 0$$

(6.1.23)
6.2. Application to Waveguides

\[ \int_{\Omega} \left[ j\omega \mu h_{zn}^* H_z + h_{zn}^* \hat{z} \cdot \nabla_t \times \vec{E}_t \right] d\Omega = 0 \quad (6.2.6) \]

\[ \int_{\Omega} \left[ j\omega e e_{zn}^* E_z - e_{zn}^* \hat{z} \cdot \nabla_t \times \vec{H}_t \right] d\Omega = 0 \quad (6.2.8) \]

These equations can now be used to solve for the propagating modes in waveguides after applying suitable basis functions. The following sections contain both vector and scalar bases applied to the above equations. The inclusion of the penalty terms which are the divergence of the electric and magnetic fields respectively is also derived.

6.2.1 Vector Bases for the Transverse Fields

Equations (6.2.5) through (6.2.8) are the equations for the electric and magnetic fields after splitting the fields into the transverse and longitudinal components. These equations are written below in simplified form. The equations for the electric fields are written ahead of the equations for the magnetic fields. The boundary conditions are applied on the electric field and this rearrangement simplifies the equations in a notational form to give

\[ j\omega m_{31} E_t + \gamma m_{32} H_t + m_{33} H_z = 0 \quad (6.2.9) \]

\[ j\omega m_{41} E_z - m_{42} H_t = 0 \quad (6.2.10) \]

\[ j\omega m_{11} H_t - \gamma m_{12} E_t - m_{13} E_z = 0 \quad (6.2.11) \]

\[ j\omega m_{21} H_z + m_{22} E_t = 0 \quad (6.2.12) \]
These equations can be collected and written as a matrix eigenvalue problem as

\[
\begin{bmatrix}
  j\omega m_{31} & 0 & 0 & m_{33} \\
  0 & j\omega m_{41} & -m_{42} & 0 \\
  0 & -m_{13} & j\omega m_{11} & 0 \\
  m_{22} & 0 & 0 & j\omega m_{21}
\end{bmatrix}
\begin{bmatrix}
  E_t \\
  E_z \\
  H_t \\
  H_z
\end{bmatrix}
= \gamma
\begin{bmatrix}
  0 & 0 & -m_{32} & 0 \\
  0 & 0 & 0 & 0 \\
  m_{12} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  E_t \\
  E_z \\
  H_t \\
  H_z
\end{bmatrix}
\] (6.2.13)

which can be solved using standard eigenvalue solution packages such as Lapack or via Matlab®.

If the wavenumber \( k_c \) is treated to be the unknown, the same matrix equations can be rearranged after extracting the cutoff wavenumber from the equations with \( \gamma = 0 \). The number of equations in (6.2.13) can change depending on the number of bases used for the transverse fields. For a PEC, boundary conditions are applied to matrices \( m_{31}, m_{33}, \) and \( m_{32} \) for the transverse electric fields. For the longitudinal electric fields, boundary conditions are applied to matrices \( m_{41} \) and \( m_{42} \).

The matrices defined in (6.2.13) are written as follows:

- \( m_{11nj} = \int_{\Omega} \mu \tilde{h}_{tn}^* \cdot \tilde{h}_{tj} d\Omega \)
- \( m_{12nj} = \int_{\Omega} \tilde{h}_{tn}^* \cdot \hat{z} \times \tilde{e}_{tj} d\Omega \)
- \( m_{13nj} = \int_{\Omega} \tilde{h}_{tn}^* \cdot \hat{z} \times \nabla_t e_{zj} d\Omega \)
- \( m_{21nj} = \int_{\Omega} \mu h_{2n}^* h_{zj} d\Omega \)
- \( m_{22nj} = \int_{\Omega} h_{2n}^* \hat{z} \cdot \nabla_t \tilde{e}_{tj} d\Omega \)
- \( m_{31nj} = \int_{\Omega} \epsilon \tilde{e}_{tn}^* \cdot \tilde{e}_{tj} d\Omega \)
- \( m_{32nj} = \int_{\Omega} \tilde{e}_{tn}^* \cdot \hat{z} \times \tilde{h}_{tj} d\Omega \)
- \( m_{33nj} = \int_{\Omega} \tilde{e}_{tn}^* \cdot \hat{z} \times \nabla_t h_{zj} d\Omega \)
- \( m_{41nj} = \int_{\Omega} \epsilon e_{2n}^* e_{zj} d\Omega \)
- \( m_{42nj} = \int_{\Omega} e_{2n}^* \hat{z} \cdot \nabla_t \tilde{h}_{tj} d\Omega \)

The above equations and matrices represent the eigenvalue problem when each of the transverse fields are expanded with a single vector bases set.
6.2.2 Scalar Bases for the Transverse Fields

In the case of scalar basis functions used for the transverse electric and magnetic fields, the following six equations result:

\[
\int_{\Omega} \left[ j\omega \mu \bar{h}_{xn}^* \cdot \bar{H}_t - \bar{h}_{xn}^* \cdot \hat{z} \times (\gamma \bar{E}_t + \nabla_t \bar{E}_z) \right] d\Omega = 0 \quad (6.2.14)
\]

\[
\int_{\Omega} \left[ j\omega \mu \bar{h}_{yn}^* \cdot \bar{H}_t - \bar{h}_{yn}^* \cdot \hat{z} \times (\gamma \bar{E}_t + \nabla_t \bar{E}_z) \right] d\Omega = 0 \quad (6.2.15)
\]

\[
\int_{\Omega} \left( j\omega \mu h_{zn}^* H_z + h_{zn}^* \hat{z} \cdot \nabla_t \bar{E}_t \right) d\Omega = 0 \quad (6.2.16)
\]

\[
\int_{\Omega} \left[ j\omega \epsilon \bar{e}_{xn}^* \cdot \bar{E}_t + \bar{e}_{xn}^* \cdot \hat{z} \times (\gamma \bar{H}_t + \nabla_t \bar{H}_z) \right] d\Omega = 0 \quad (6.2.17)
\]

\[
\int_{\Omega} \left[ j\omega \epsilon \bar{e}_{yn}^* \cdot \bar{E}_t + \bar{e}_{yn}^* \cdot \hat{z} \times (\gamma \bar{H}_t + \nabla_t \bar{H}_z) \right] d\Omega = 0 \quad (6.2.18)
\]

\[
\int_{\Omega} \left( j\omega \epsilon e_{zn}^* E_z - e_{zn}^* \hat{z} \cdot \nabla_t \bar{H}_t \right) d\Omega = 0 \quad (6.2.19)
\]

The above equations can be written in a simplified form as

\[
j\omega m_{11x} H_x - \gamma m_{12x} E_y - m_{13x} E_z = 0 \quad (6.2.20)
\]

\[
j\omega m_{11y} H_y - \gamma m_{12y} E_x - m_{13y} E_z = 0 \quad (6.2.21)
\]

\[
j\omega m_{21} H_z + \begin{bmatrix} m_{22x} & m_{22y} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0 \quad (6.2.22)
\]

\[
j\omega m_{31x} E_x + \gamma m_{32x} H_y + m_{33x} H_z = 0 \quad (6.2.23)
\]
Chapter 6. The Direct Method in Dynamics

\[ j\omega m_{31y} E_y + \gamma m_{32y} H_x + m_{33y} H_z = 0 \]  \hspace{1cm} (6.2.24)

\[ j\omega m_{41} E_z - \begin{bmatrix} m_{42x} & m_{42y} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} = 0 \]  \hspace{1cm} (6.2.25)

After collecting terms, the above equations can be written as a matrix eigenvalue problem given by

\[
\begin{bmatrix}
 j\omega m_{31x} & 0 & 0 & 0 & 0 & m_{33x} \\
 0 & j\omega m_{31y} & 0 & 0 & 0 & m_{33y} \\
 0 & 0 & j\omega m_{41} & -m_{42x} & -m_{42y} & 0 \\
 0 & 0 & -m_{13x} & j\omega m_{11x} & 0 & 0 \\
 0 & 0 & -m_{13y} & 0 & j\omega m_{11y} & 0 \\
 m_{22x} & m_{22y} & 0 & 0 & 0 & j\omega m_{21} \\
\end{bmatrix}
\begin{bmatrix}
 E_x \\
 E_y \\
 E_z \\
 H_x \\
 H_y \\
 H_z \\
\end{bmatrix} = \gamma
\begin{bmatrix}
 0 & 0 & 0 & 0 & -m_{32x} & 0 \\
 0 & 0 & 0 & -m_{32y} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & m_{12x} & 0 & 0 & 0 & 0 \\
 m_{12y} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
 E_x \\
 E_y \\
 E_z \\
 H_x \\
 H_y \\
 H_z \\
\end{bmatrix} \]  \hspace{1cm} (6.2.26)

There are eighteen sub-matrices that make up Eq. (6.2.26). The sub-matrix definitions follow:

- \( m_{11xnj} = \int_{\Omega} \mu h_{xn}^* h_{xj} d\Omega \)
- \( m_{11ynj} = \int_{\Omega} \mu h_{yn}^* h_{yj} d\Omega \)
- \( m_{12xnj} = -\int_{\Omega} h_{xn}^* e_{yj} d\Omega \)
- \( m_{12ynj} = \int_{\Omega} h_{yn}^* e_{xj} d\Omega \)
- \( m_{13xnj} = -\int_{\Omega} h_{xn}^* \frac{\partial e_{yj}}{\partial y} d\Omega \)
- \( m_{13ynj} = \int_{\Omega} h_{yn}^* \frac{\partial e_{xj}}{\partial x} d\Omega \)
6.2. Application to Waveguides

- \( m_{21nj} = \int_{\Omega} \mu h_{zn}^* h_{zj} d\Omega \)
- \( m_{22xnj} = - \int_{\Omega} h_{zn}^* \frac{\partial e_{xj}}{\partial y} d\Omega \)
- \( m_{22ynj} = \int_{\Omega} h_{zn}^* \frac{\partial e_{yj}}{\partial x} d\Omega \)
- \( m_{31xnj} = \int_{\Omega} e_{xn}^* e_{xj} d\Omega \)
- \( m_{32xnj} = - \int_{\Omega} e_{xn}^* \frac{\partial h_{xj}}{\partial y} d\Omega \)
- \( m_{33xnj} = - \int_{\Omega} e_{xn}^* \frac{\partial h_{xj}}{\partial x} d\Omega \)
- \( m_{31ynj} = \int_{\Omega} e_{yn}^* e_{yj} d\Omega \)
- \( m_{32ynj} = \int_{\Omega} e_{yn}^* h_{xj} d\Omega \)
- \( m_{33ynj} = \int_{\Omega} e_{yn}^* \frac{\partial h_{zj}}{\partial x} d\Omega \)
- \( m_{41nj} = \int_{\Omega} e_{zn}^* e_{zj} d\Omega \)
- \( m_{42xnj} = - \int_{\Omega} e_{zn}^* \frac{\partial h_{xj}}{\partial y} d\Omega \)
- \( m_{42ynj} = \int_{\Omega} e_{zn}^* \frac{\partial h_{yj}}{\partial x} d\Omega \)

Any scalar basis function can be applied to each component of the electric and magnetic fields. For example, if all the basis functions are chosen to be first-order piecewise-linear (PWL), the basis functions \( e_{xk}, e_{yk}, e_{zk}, h_{xk}, h_{yk}, \) and \( h_{zk} \) are identical in the above equations. Both the different bases and the subscript \( x, y, \) and \( z \) have been retained only to show the difference between the electric and magnetic field equations as well as to enable the same set of equations to be reused with different basis sets.

For piecewise-linear bases, the computation of \( m_{11x} \) provides ten out of the eighteen matrices to within a scale factor. There are only three sub-matrices that need to be computed in the entire set of eighteen as only PWL bases are present for all of the fields. These three computed matrices are then scaled with respect to the material constants as required.

Mixed Field Finite Element Computations
6.3 The Penalty Term

The divergence of the electric and magnetic fields can also be included in the Direct method. This section derives the equations required as well as displays the manipulations required to form the eigenvalue problem. The additional sub-matrices that are required are also derived. Including the transverse divergence terms for both the electric and magnetic fields will result in eight additional sub-matrices if vector bases are used for the transverse fields and eighteen additional sub-matrices if scalar bases are used for the transverse fields. The penalty term is applied to the Direct method to be consistent in the comparison with the vector-Helmholtz equation method. The electric and magnetic fields are expanded as

\[
\vec{E} = \sum_n \vec{E}_n(x, y, z) = \sum_j \vec{E}_{tj}(x, y)e^{-\gamma z} + \sum_k \hat{2E}_{zk}(x, y)e^{-\gamma z}
\]

\[
\vec{H} = \sum_n \vec{H}_n(x, y, z) = \sum_j \vec{H}_{tj}(x, y)e^{-\gamma z} + \sum_k \hat{2H}_{zk}(x, y)e^{-\gamma z}
\]

The divergence of the electric and magnetic fields can be written as

\[
\nabla \cdot \vec{E} = \left( \sum_j \nabla \cdot \vec{E}_{tj} + \sum_k (-\gamma) \vec{E}_{zk} \right) e^{-\gamma z}
\]

\[
\nabla \cdot \vec{H} = \left( \sum_j \nabla \cdot \vec{H}_{tj} + \sum_k (-\gamma) \vec{H}_{zk} \right) e^{-\gamma z}
\]

The procedure is identical to the penalty term method shown in the previous chapter when applied to the vector-Helmholtz equation method. However, the simplification process to obtain the eigenvalue formulation is different.

Including the divergence terms adds the following integrals to the transverse and longitudinal electric field equations developed for the Direct method:

\[
I_{et} = \lambda_e \int_{\Omega} \nabla \cdot \vec{e}_{tn} \nabla \cdot \vec{E}_t d\Omega - \gamma \lambda_e \int_{\Omega} \nabla \cdot \vec{e}_{tn} \vec{E}_z d\Omega
\]
6.3. The Penalty Term

The coefficients λ_e and λ_h are the divergence coefficients for the penalty method to be applied to the Direct method with scalar bases used for the transverse electric and magnetic fields. Even though equations are derived for both vector and scalar transverse basis functions, the vector bases used in this research provide accurate results in most cases and do not need the divergence terms to be included. The coefficients can be set to any value during the matrix assembly required for the eigenvalue problem $Ax = \lambda Bx$. It must be noted that when the divergence coefficients are included in the development, $\gamma^2$ is solved for instead of $\gamma$ which is the unknown in the development without the inclusion of the penalty terms.

This method cannot be applied to the vector bases due to the undefined divergence at cell boundaries.

### 6.3.1 Scalar Bases for Transverse Fields

The divergence equations defined in (6.3.5) through (6.3.8) are modified and additional equations become necessary for the equations for the transverse fields. These are defined below.

$$I_{ez} = -\gamma \lambda_e \int_{\Omega} e_{zn}^* \nabla \cdot \vec{E}_t d\Omega + \gamma^2 \lambda_e \int_{\Omega} e_{zn}^* E_z d\Omega \quad (6.3.6)$$

Expanding the magnetic field in a similar manner, we obtain the two addition integrals that are shown below.

$$I_{ht} = \lambda_h \int_{\Omega} \nabla \cdot \vec{h}_{tn}^* \nabla \cdot \vec{H}_t d\Omega - \gamma \lambda_h \int_{\Omega} \nabla \cdot \vec{h}_{tn}^* H_z d\Omega \quad (6.3.7)$$

$$I_{hz} = -\gamma \lambda_h \int_{\Omega} h_{zn}^* \nabla \cdot \vec{H}_t d\Omega + \gamma^2 \lambda_h \int_{\Omega} h_{zn}^* H_z d\Omega \quad (6.3.8)$$
\[ I_{ez} = -\lambda_h \int_{\Omega} \gamma h^*_m \nabla_t \cdot \vec{H}_t \, d\Omega + \lambda_h \int_{\Omega} \gamma^2 h^*_m H_z \, d\Omega \quad (6.3.11) \]

\[ I_{hx} = \lambda_e \int_{\Omega} \nabla_t \cdot (\vec{e}_{x_n}^*) \nabla_t \cdot \vec{E}_t \, d\Omega - \lambda_e \int_{\Omega} \gamma \nabla_t \cdot (\vec{e}_{x_n}^*) E_z \, d\Omega \quad (6.3.12) \]

\[ I_{hy} = \lambda_e \int_{\Omega} \nabla_t \cdot (\vec{e}_{y_n}^*) \nabla_t \cdot \vec{E}_t \, d\Omega - \lambda_e \int_{\Omega} \gamma \nabla_t \cdot (\vec{e}_{y_n}^*) E_z \, d\Omega \quad (6.3.13) \]

\[ I_{hz} = -\lambda_e \int_{\Omega} \gamma e^*_m \nabla_t \cdot \vec{E}_t \, d\Omega + \lambda_e \int_{\Omega} \gamma^2 e^*_m E_z \, d\Omega = 0 \quad (6.3.14) \]

The matrix equations in (6.2.26) are rewritten with the inclusion of all the sub-matrices created by the inclusion of the divergence of the electric and magnetic fields.

\[ j\omega m_{11x} H_x - \gamma m_{12x} E_y - m_{13x} E_z + \lambda_h m_{14x} H_x + \lambda_h m_{15x} H_y - \lambda_h \gamma m_{16x} H_z = 0 \quad (6.3.15) \]

\[ j\omega m_{11y} H_y - \gamma m_{12y} E_x - m_{13y} E_z + \lambda_h m_{14y} H_x + \lambda_h m_{15y} H_y - \lambda_h \gamma m_{16y} H_z = 0 \quad (6.3.16) \]

\[ j\omega m_{21} H_z + \begin{bmatrix} m_{22x} & m_{22y} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} - \lambda_h \gamma m_{23x} H_x - \lambda_h \gamma m_{23y} H_y + \lambda_e \gamma^2 m_{24} H_z = 0 \quad (6.3.17) \]

\[ j\omega m_{31x} E_x + \gamma m_{32x} H_y + m_{33x} H_z + \lambda_e m_{34x} E_x + \lambda_e m_{35x} E_y - \lambda_e \gamma m_{36x} E_z = 0 \quad (6.3.18) \]

\[ j\omega m_{31y} E_y + \gamma m_{32y} H_x + m_{33y} H_z + \lambda_e m_{34y} E_x + \lambda_e m_{35y} E_y - \lambda_e \gamma m_{36y} E_z = 0 \quad (6.3.19) \]
6.3. The Penalty Term

\[
\begin{align*}
\text{j} \omega m_{41} E_z - \begin{bmatrix} m_{42x} & m_{42y} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} - \lambda_e \gamma m_{43x} E_x - \lambda_e \gamma m_{43y} E_y + \lambda_e \gamma^2 m_{44} E_z &= 0 \quad (6.3.20) \\
\text{j} \omega m_{31x} Q_x + \gamma^2 m_{32x} H_y + m_{33x} P_z + \lambda_e m_{34x} Q_x + \lambda_e m_{35x} Q_y - \lambda_e \gamma^2 m_{36x} E_z &= 0 \quad (6.3.21) \\
\text{j} \omega m_{31y} Q_y + \gamma^2 m_{32y} H_x + m_{33y} P_z + \lambda_e m_{34y} Q_x + \lambda_e m_{35y} Q_y - \lambda_e \gamma^2 m_{36y} E_z &= 0 \quad (6.3.22) \\
\text{j} \omega m_{41} E_z - \begin{bmatrix} m_{42x} & m_{42y} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} - \lambda_e m_{43x} Q_x - \lambda_e m_{43y} Q_y + \lambda_e \gamma^2 m_{44} E_z &= 0 \quad (6.3.23) \\
\text{j} \omega m_{11x} H_x - m_{12x} Q_y - m_{13x} E_z + \lambda_h m_{14x} H_x + \lambda_h m_{15x} H_y - \lambda_h m_{16x} P_z &= 0 \quad (6.3.24) \\
\text{j} \omega m_{11y} H_y - m_{12y} Q_x - m_{13y} E_z + \lambda_h m_{14y} H_x + \lambda_h m_{15y} H_y - \lambda_h m_{16y} P_z &= 0 \quad (6.3.25) \\
\text{j} \omega m_{21} P_z + \begin{bmatrix} m_{22x} & m_{22y} \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \end{bmatrix} - \lambda_h \gamma^2 m_{23x} H_x - \lambda_h \gamma^2 m_{23y} H_y + \lambda_h \gamma^2 m_{24} P_z &= 0 \quad (6.3.26)
\end{align*}
\]

The above set of equations can be written as an eigenvalue problem with \( \gamma^2 \) as the unknown as
shown below.

\[
\begin{pmatrix}
  j\omega m_{31x} + \lambda e m_{34x} & \lambda e m_{35x} & 0 & 0 & 0 & m_{33x} \\
  \lambda e m_{34y} & j\omega m_{31y} + \lambda e m_{35y} & 0 & 0 & 0 & m_{33y} \\
  -\lambda e m_{43x} & -\lambda e m_{43y} & -m_{12x} & -m_{13x} & j\omega m_{11x} + \lambda h m_{14x} & \lambda h m_{15x} \\
  0 & 0 & -m_{12y} & -m_{13y} & \lambda h m_{14y} & j\omega m_{11y} + \lambda h m_{15y} \\
  -m_{12y} & m_{22y} & 0 & 0 & 0 & 0 \\
  m_{22x} & 0 & 0 & 0 & \lambda h m_{23x} & \lambda h m_{23y} \\
\end{pmatrix}
\begin{pmatrix}
  \gamma E_x \\
  \gamma E_y \\
  E_z \\
  H_x \\
  H_y \\
  \gamma H_z
\end{pmatrix}
= \gamma^2
\begin{pmatrix}
  0 & 0 & \lambda e m_{36x} & 0 & -m_{32x} & 0 \\
  0 & 0 & \lambda e m_{36y} & -m_{32y} & 0 & 0 \\
  0 & 0 & -\lambda e m_{44} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & \lambda h m_{23x} & \lambda h m_{23y} \\
  0 & 0 & 0 & 0 & -\lambda h m_{24} & 0
\end{pmatrix}
\begin{pmatrix}
  \gamma E_x \\
  \gamma E_y \\
  E_z \\
  H_x \\
  H_y \\
  \gamma H_z
\end{pmatrix}
\]

(6.3.27)

The eighteen additional sub-matrices in Eq. (6.3.27) are:

- \( m_{14xnj} = \int_{\Omega} \frac{\partial h_{xn}^*}{\partial x} \frac{\partial h_{xj}}{\partial x} \, d\Omega \)
- \( m_{23ynj} = \int_{\Omega} h_{zn}^* \frac{\partial h_{yj}}{\partial y} \, d\Omega \)
- \( m_{15xnj} = \int_{\Omega} \frac{\partial h_{xn}^*}{\partial x} \frac{\partial h_{yj}}{\partial y} \, d\Omega \)
- \( m_{24hnj} = \int_{\Omega} h_{zn}^* h_{zj} \, d\Omega \)
- \( m_{16xnj} = \int_{\Omega} \frac{\partial h_{xn}^*}{\partial x} h_{zj} \, d\Omega \)
- \( m_{34xnj} = \int_{\Omega} \frac{\partial e_{xn}^*}{\partial x} \frac{\partial e_{xj}}{\partial x} \, d\Omega \)
- \( m_{14ynj} = \int_{\Omega} \frac{\partial h_{yn}^*}{\partial y} \frac{\partial h_{xj}}{\partial x} \, d\Omega \)
- \( m_{35xnj} = \int_{\Omega} \frac{\partial e_{xn}^*}{\partial x} \frac{\partial e_{yj}}{\partial y} \, d\Omega \)
- \( m_{15ynj} = \int_{\Omega} \frac{\partial h_{yn}^*}{\partial y} \frac{\partial h_{yj}}{\partial y} \, d\Omega \)
- \( m_{36xnj} = \int_{\Omega} \frac{\partial e_{xn}^*}{\partial x} e_{zj} \, d\Omega \)
- \( m_{16ynj} = \int_{\Omega} \frac{\partial h_{yn}^*}{\partial y} h_{zj} \, d\Omega \)
- \( m_{34ynj} = \int_{\Omega} \frac{\partial e_{yn}^*}{\partial y} \frac{\partial e_{xj}}{\partial x} \, d\Omega \)
- \( m_{23xnj} = \int_{\Omega} h_{zn}^* \frac{\partial h_{xj}}{\partial y} \, d\Omega \)
- \( m_{35ynj} = \int_{\Omega} \frac{\partial e_{yn}^*}{\partial y} \frac{\partial e_{yj}}{\partial y} \, d\Omega \)
6.4 Chapter Summary

A new set of equations that solve for the propagating modes of waveguides was derived from Maxwell’s equations. The associated method is called the Direct method using Maxwell’s equations directly. A variational form that ensures minimization was shown. The equations were modified to be applied to two-dimensional waveguides. The matrix form that is used to solve for the modes or the wavenumber as an eigenvalue problem was derived. Equations were developed for both vector and scalar basis functions applied to the transverse fields. All the sub-matrices required for the solution were defined. An additional consideration of a penalty method form was also derived for comparison with the vector-Helmholtz method.

All the other matrices in Eq. (6.3.27) are identical to those defined in Eq. (6.2.26). Results using the divergence coefficients or the penalty terms are shown in the next chapter where scalar bases are applied to the Direct solution method.

\[
\begin{align*}
\mathbf{m}_{36ynj} &= \int_{\Omega} \frac{\partial e_{yn}^*}{\partial y} e_{zj} d\Omega \\
\mathbf{m}_{43ynj} &= \int_{\Omega} h_{zn}^* \frac{\partial h_{yj}}{\partial y} d\Omega \\
\mathbf{m}_{43xnj} &= \int_{\Omega} e_{zn}^* \frac{\partial e_{xj}}{\partial x} d\Omega \\
\mathbf{m}_{44nj} &= \int_{\Omega} e_{zn}^* e_{zj} d\Omega
\end{align*}
\]
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Chapter 7

Scalar Bases with the Direct Method
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This chapter focuses on applying scalar basis functions to the developed Direct method. Spurious modes usually occur when the transverse fields are expanded with these bases. Considerable attention is paid to the problem of spurious modes when piecewise-linear (PWL) bases are used for all three components of both the electric and magnetic fields. A filtering technique that can eliminate the spurious modes when these basis functions are used is described in detail. This technique is displayed on both the WR-90 waveguide as well as the circular waveguide.

Similar to the complementary nature observed in magnetostatics, basis functions can be chosen for the magnetic and electric fields based on the complementary nature of Maxwell’s equations themselves. It has been shown that scalar basis sets used in the vector-Helmholtz method retain the spurious modes with or without the divergence of the electric field being enforced. The divergence coefficient also appears to be sensitive to the mesh density. The goal with the Direct method and scalar bases is to obtain solutions when the traditionally used vector-Helmholtz method fails with scalar bases. The process of selecting basis functions for the transverse and longitudinal components of the electric and magnetic fields as well as results for several basis function sets with and without the divergence terms are displayed.

### 7.1 Selection of Scalar Bases Sets

Maxwell’s equations for the electric and magnetic fields are rewritten in terms of the transverse and longitudinal components as shown below. The waveguide is assumed to be aligned in the \( \hat{z} \) or longitudinal direction and perpendicular to the transverse or \( x - y \) plane.

\[
\nabla \times \vec{E} = -j \omega \mu \vec{H} = -j \omega \mu (\vec{H}_t + \vec{H}_z) = -\hat{z} \times (\gamma \vec{E}_t + \nabla_t \vec{E}_z) + \nabla_t \times \vec{E}_t \tag{7.1.1}
\]

\[
\nabla \times \vec{H} = +j \omega \epsilon \vec{E} = +j \omega \epsilon (\vec{E}_t + \vec{E}_z) = -\hat{z} \times (\gamma \vec{H}_t + \nabla_t \vec{H}_z) + \nabla_t \times \vec{H}_t \tag{7.1.2}
\]

It is observed that the transverse magnetic field is related to the transverse electric field and the gradient of the longitudinal electric field. The longitudinal magnetic field is only related to the curl of the transverse electric field. Similarly, the transverse electric field is related to both the transverse magnetic field and the gradient of the longitudinal magnetic field, and the longitudinal
electric field is only related to the curl of the transverse magnetic field. The scalar basis functions to be considered are first and zero-order. Combinations of just these two basis functions are applied to the six different components of the electric and magnetic field. It is straightforward to extend the first and zero-order scalar bases to higher orders.

The choice of the basis functions are made in a complementary manner where possible. For example, the transverse magnetic field contains a piecewise-constant part that arises from the gradient of the longitudinal electric field, so it can be expanded by piecewise-constant, PWC, bases. The longitudinal electric field is expanded with piecewise-linear bases to allow differentiation. Similarly, basis sets are chosen for the remaining field components. Another set, where the basis functions for all the field components are expanded using piecewise-linear basis functions is discussed.

Table 7.1 shows the different choices that were made. Some of these sets are complementary in nature while others are not. In the complementary basis sets, the basis functions are applied with respect to the order of the bases obtained from Maxwell’s equations.

Table 7.1: Scalar bases combinations used with the Direct method. Piecewise-linear (PWL) and piecewise-constant (PWC) bases are used. There are three components of the electric field and three components of the magnetic field for a total of six unknowns. The penalty term is applied to the bases shown in row 2.

<table>
<thead>
<tr>
<th>Set</th>
<th>$E_x$</th>
<th>$E_y$</th>
<th>$E_z$</th>
<th>$H_x$</th>
<th>$H_y$</th>
<th>$H_z$</th>
<th>Module number (internal use)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>PWC</td>
<td>PWC</td>
<td>PWL</td>
<td>PWC</td>
<td>PWL</td>
<td>PWL</td>
<td>23</td>
</tr>
<tr>
<td>2</td>
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<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
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<tr>
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<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWC</td>
<td>28</td>
</tr>
</tbody>
</table>

In addition to the combinations that are shown in the above table, PWC bases were also applied to the components of the electric field while the magnetic fields were expanded with PWL bases. Both types did not provide accurate results as the application of boundary conditions on $E_z$ expanded with PWC bases corrupt the solution. Hence these basis sets are not discussed.
7.2 PWCPWL-PWCPWL Bases

Set 1 from Table 7.1 is also referred to as the PWCPWL-PWCPWL set. The first PWC and PWL represent the transverse electric field and the longitudinal electric field respectively. The PWC and PWL after the hyphenation represents the transverse and longitudinal magnetic fields. In this case the basis functions, $e_x$, $e_y$, $h_x$, and $h_y$ are represented by piecewise-constant, $\Psi$, while $e_z$ and $h_z$ are represented by piecewise-linear, $\Phi$, using real bases.

The equation derived for scalar bases shown in Eq. (6.2.26) which forms the eigenvalue problem is used for the expansion. The equations are defined in (6.2.14) through (6.2.19). The divergence coefficients $\lambda_e$ and $\lambda_h$ for the transverse electric and magnetic fields do not apply in this case as the transverse bases functions cannot be differentiated. The coefficients are just set to zero and the divergence sub-matrices not computed or included in the matrix formulation.

It is to be noted that the transverse components are expanded using constant bases. Thus, a derivative involving a constant basis function has to be integrated by parts to transfer the derivative on to a field expanded with PWL bases. The identity $\nabla \times (A \bar{a}) = \nabla A \times \bar{a} + A \nabla \times \bar{a}$ and Gauss’ theorem in curl form $\int_{\Omega} \nabla \times \tilde{\vec{v}} d\Omega = \oint_{\Gamma} \tilde{\vec{n}} \times \tilde{\vec{v}} d\Gamma$ are used in the simplification. The integrals containing the curl of the transverse electric and magnetic fields are written as

$$\int_{\Omega} h^*_zn \cdot \nabla_t \times \tilde{E}_t d\Omega = -\int_{\Omega} \tilde{Z} \cdot \nabla_t h^*_zn \times \tilde{E}_t d\Omega + \oint_{\Gamma} h^*_zn \tilde{\vec{n}} \cdot \tilde{E}_t d\Gamma \quad (7.2.1)$$

and

$$\int_{\Omega} e^*_zn \tilde{Z} \cdot \nabla_t \times \tilde{H}_t d\Omega = -\int_{\Omega} \tilde{Z} \cdot \nabla_t e^*_zn \times \tilde{H}_t d\Omega + \oint_{\Gamma} e^*_zn \tilde{\vec{n}} \cdot \tilde{H}_t d\Gamma \quad (7.2.2)$$

Equations (6.2.16) and (6.2.19) become

$$\int_{\Omega} j\omega \mu h^*_zn H_z d\Omega - \int_{\Omega} \tilde{Z} \cdot \nabla_t h^*_zn \times \tilde{E}_t d\Omega + \oint_{\Gamma} h^*_zn \tilde{\vec{n}} \cdot \tilde{E}_t d\Gamma = 0 \quad (7.2.3)$$
and

\[ \int_{\Omega} j\omega e_{zn}^* E_z d\Omega + \int_{\Omega} \hat{z} \cdot \nabla_t e_{zn}^* \times \mathcal{H}_t d\Omega - \oint_{\Gamma} e_{zn}^* \hat{z} \cdot \hat{n} \times \mathcal{H}_t d\Gamma = 0 \]  \hspace{1cm} (7.2.4)

The contour integrals in equations (7.2.3) and (7.2.4) are applied only in the case of boundary elements and have no effect on internal nodes and cells. After integration by parts, it is seen that all the derivatives are transferred to the PWL bases which are used for the longitudinal fields.

The sub-matrices in the sixth equation in (6.2.26) for the longitudinal magnetic field are simplified as follows. Two of the three sub-matrices are modified to accommodate the integration by parts. Note that the transverse basis functions are PWC and can be removed from the integral.

\[ m_{22xnj} = - \int_{\Omega} e_{xj} \frac{\partial h_{zn}^*}{\partial y} d\Omega - l_{dnx} e_{xj} \oint_{\Gamma} h_{zn}^* d\Gamma = - e_{xj} \frac{\partial h_{zn}^*}{\partial y} S^e - l_{dnx} e_{xj} \oint_{\Gamma} h_{zn}^* d\Gamma \]  \hspace{1cm} (7.2.5)

\[ m_{22ynj} = \int_{\Omega} e_{yj} \frac{\partial h_{zn}^*}{\partial x} d\Omega - l_{dny} e_{yj} \oint_{\Gamma} h_{zn}^* d\Gamma = e_{yj} \frac{\partial h_{zn}^*}{\partial x} S^e - l_{dny} e_{yj} \oint_{\Gamma} h_{zn}^* d\Gamma \]  \hspace{1cm} (7.2.6)

The sub-matrices in the third equation in (6.2.26) for the longitudinal electric field which are also modified due to integration by parts are simplified in an identical manner to the two equations written above as follows:

\[ m_{42xnj} = - \int_{\Omega} h_{xj} \frac{\partial e_{zn}^*}{\partial y} d\Omega - l_{xdn} h_{xj} \oint_{\Gamma} e_{zn}^* d\Gamma = - h_{xj} \frac{\partial e_{zn}^*}{\partial y} S^e - l_{xdn} h_{xj} \oint_{\Gamma} e_{zn}^* d\Gamma \]  \hspace{1cm} (7.2.7)

\[ m_{42ynj} = \int_{\Omega} h_{yj} \frac{\partial e_{zn}^*}{\partial x} d\Omega - l_{ydn} h_{yj} \oint_{\Gamma} e_{zn}^* d\Gamma = h_{yj} \frac{\partial e_{zn}^*}{\partial x} S^e - l_{ydn} h_{yj} \oint_{\Gamma} e_{zn}^* d\Gamma \]  \hspace{1cm} (7.2.8)

The quantity \( S^e \) is the area of element \( j \), and \( l_{xdn} \) and \( l_{ydn} \) are the \( \hat{x} \) and \( \hat{y} \) directional components of edge \( n \) in the finite-element mesh. The process of integration by parts is identical to the Direct method with complementary bases in magnetostatics since the basis functions used for the longitudinal and transverse components are identical.

The matrices \( A \) and \( B \) that form the eigenvalue problem in Eq. (6.2.26) are highly sparse in this case. Both \( A \) and \( B \) have the highest sparsity among the scalar bases with several of the sub-matrices represented by a diagonal matrix. The sparsity for the PWCPWL-PWCPWL set is shown.
7.2. PWCPWL-PWCPWL Bases

in Fig. 7.1 for the 145 node mesh of Fig. 5.1.

7.2.1 Results via PWCPWL-PWCPWL Bases

The results obtained using the PWC bases for the transverse fields and PWL bases for the longitudinal fields are described in this section. For the transverse electric fields, boundary conditions are applied over the entire cell based on the edge that is on the boundary. This is straightforward for the WR-90. Two equations are derived for the transverse electric field in the case of arbitrary shaped waveguides, one which represents the combined field which is normal to the boundary and the second which is tangential. The boundary conditions replace this second equation.

The modes obtained for the WR-90 with the 145 node mesh of Fig. 5.1 is shown in Fig. 7.2. Results obtained with the other meshes vary with the mesh density, but are very similar in accuracy to the vector-Helmholtz method at all mesh resolutions. Figures (7.3) through (7.8) show the field plots for the TE_{10} mode for the WR-90 waveguide. It can be seen that the magnitude of \( E_z \) relative to the transverse fields is near zero.

It can be seen that applied boundary conditions on \( E_x \), \( E_y \) and \( E_z \) are honored by the solution. Note the magnitude of \( E_z \) that confirms the TE mode. The transverse fields are piecewise-constant and are plotted with a constant color in each cell. The longitudinal fields are piecewise-linear and
Figure 7.2: WR-90 waveguide. Computed propagation constants ($\gamma$) with 145 node mesh and $k = 167.6676$ rad/m via the Direct method with PWCPWL-PWCPWL basis sets. Very accurate results are obtained without any spurious modes. Note presence of computed modes at exactly $k$ called "k" modes. The propagating mode on the imaginary axis is the $\text{TE}_{10}$ mode.

Figure 7.3: WR-90, $\text{TE}_{10}$, $|E_x|$ 145 node mesh, PWCPWL-PWCPWL bases. Max = $5.96e-2$

the plots show the actual interpolated fields inside each element. The exact $\text{TE}_{10}$ propagation constant is at 96.0526 rad/m compared to the computed value of 95.5246 rad/m.
7.2. PWCPWL-PWCPWL Bases

Figure 7.4: WR-90, $\text{TE}_{10}$, $|E_y|$ 145 node mesh, PWCPWL-PWCPWL bases. Max = $8.98 \times 10^{-1}$

Figure 7.5: WR-90, $\text{TE}_{10}$, $|E_z|$ 145 node mesh, PWCPWL-PWCPWL bases. Max = $1.12 \times 10^{-12}$

Figure 7.6: WR-90, $\text{TE}_{10}$, $|H_x|$ 145 node mesh, PWCPWL-PWCPWL bases. Max = $1.35 \times 10^{-3}$
None of the field patterns of the modes at \( k \) offer any useful information and appear to be random. The origin of the modes at \( k \) is unknown as of now. By observation it was found that the multiplicity of the modes at \( k \) is

\[
M_k = M - \text{rank}(A) - N
\]

(7.2.9)

where \( M \) is the number of rows in the square matrix \( A \) and \( N \) is the number of nodes in the mesh. The number of computed eigenvalues at infinity was found to be \( M_\infty = M - \text{rank}(B) - N_z \) where \( N_z \) is the number of boundary nodes in the mesh where the longitudinal boundary condition is enforced. Note that while the matrices \( A \) and \( B \) of the eigenvalue problem are square, the size of the sub-matrices are dependant on the basis functions used for the expansion of the fields and may be rectangular.
7.2. PWCPWL-PWCPWL Bases

Fig. 7.9 shows the field pattern for the electric field for one of the "k" modes. Note that the fields appear to be random. The patterns change for the other "k" modes but offer no useful information and always appear random. The magnetic field also appears to be similar in the randomness.

Figure 7.9: Field plots of the electric field for one of the "k" modes for the WR-90 with the 145 node mesh shown in Fig. 5.1.

The computed modes for the circular waveguide with the 149 node mesh of Fig. 5.2 is shown in Fig. 7.10. It is seen that while accurate results for the propagating constants are obtained, the computed "k" modes are still present and can readily be identified. The field plots for the valid circular waveguide modes are also consistent with the vector-Helmholtz solution.

Tables 7.2 and 7.3 show the computed values for the propagating modes for both the waveguides considered.
Table 7.2: Propagating constant (\( \beta \)) via PWCPWL-PWCPWL bases and Direct method for the WR-90 waveguide. \( k = 167.6676 \) rad/m.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE(_{10})</td>
<td>13</td>
<td>104.8152</td>
<td>96.0526</td>
</tr>
<tr>
<td>TE(_{10})</td>
<td>41</td>
<td>95.9529</td>
<td>96.0526</td>
</tr>
<tr>
<td>TE(_{10})</td>
<td>145</td>
<td>95.5246</td>
<td>96.0526</td>
</tr>
</tbody>
</table>

Table 7.3: Propagating constant (\( \beta \)) via PWCPWL-PWCPWL bases and Direct method for the circular waveguide of radius 1 m. \( k = 3 \) rad/m.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE(<em>{11}) TM(</em>{01})</td>
<td>31</td>
<td>2.3622 1.7117</td>
<td>2.3687 1.7933</td>
</tr>
<tr>
<td>TE(<em>{11}) TM(</em>{01})</td>
<td>85</td>
<td>2.3620 1.7684</td>
<td>2.3687 1.7933</td>
</tr>
<tr>
<td>TE(<em>{11}) TM(</em>{01})</td>
<td>149</td>
<td>2.3744 1.7811</td>
<td>2.3687 1.7933</td>
</tr>
</tbody>
</table>
7.3 PWLPWL-PWLPWL Bases

This section demonstrates the use of scalar bases for all components of the electric and magnetic fields. The method of enforcing boundary conditions for the two waveguides is also explained.

7.3.1 Enforcing Boundary Conditions

The boundary conditions can be applied using different methods on the two test waveguides. The WR-90 has vertical and horizontal surfaces and four corners making the application of the boundary conditions can be made fairly simple. The tangential component of the electric field can be easily set to zero at all nodes on the boundary. All three components of the electric field are set to zero at
the four corners. The ease of this method of applying boundary conditions changes for any arbitrary shaped waveguide.

The boundary conditions pose a problem for arbitrarily shaped waveguides with the PWL-PWL set as there is no natural normal direction that can be associated with boundary nodes. Assuming that a normal direction can be defined for boundary nodes on arbitrarily shaped surfaces, a process of rearranging the equations such as to only solve for the normal component of the electric fields is sought. The first two equations in Eq. (6.2.26) are the equations that define the transverse electric fields. The first equation is replaced by

\[
\begin{align*}
\left[ j \omega m_{31} E_x + m_{33} H_z + \gamma m_{32} H_y \right] \hat{n}_n \cdot \hat{x} \\
+ \left[ j \omega m_{31} E_y + m_{33} H_z + \gamma m_{32} H_x \right] \hat{n}_n \cdot \hat{y} = 0
\end{align*}
\] (7.3.1)

where \( \hat{n}_n \) is the normal defined for boundary node \( n \). The subscript \( n \) for the matrices refers to the entire row corresponding to the node \( n \). The above equation solves only for the normal component of the transverse electric fields at all boundary nodes. The second equation is replaced by the tangential boundary condition which is enforced by

\[
\hat{t}_n \cdot \hat{x} E_x + \hat{t}_n \cdot \hat{y} E_y = 0
\] (7.3.2)

where \( \hat{t}_n \) is the tangential direction defined for boundary node \( n \). Once the normal direction \( \hat{n}_n \) has been determined for node \( n \), the tangential directions is defined as \( \hat{t} \times \hat{n}_n \). Fig. 5.3 shows the definition of the node normal vector. It is the bisector of the angle created by the two outermost adjacent edges of a boundary node which are also on the boundary of the finite-element mesh. For example, if the top, right edge of a rectangular waveguide is in the first quadrant, the nodal normal vector for that corner would be \( 0.707 \hat{x} + 0.707 \hat{y} \).

This method of enforcing the boundary conditions modifies the first two equations of (6.2.26). Additional coefficients matrices are required and the first two rows of matrix \( A \) and \( B \) are written below. It must be noted that these modification take place only for the rows of nodes corresponding to the boundary, all other rows are unaffected. The first set of rows corresponding to the boundary of \( A \) and \( B \) are modified as follows respectively to solve for the normal component of the electric
field.

\[
\begin{bmatrix}
j \omega m_{31x}\hat{n}_n \cdot \hat{x} & j \omega m_{31y}\hat{n}_n \cdot \hat{y} & 0 & 0 & 0 & (m_{33x}\hat{n}_n \cdot \hat{x} + m_{33y}\hat{n}_n \cdot \hat{y})
\end{bmatrix}
\] (7.3.3)

\[
\begin{bmatrix}
0 & 0 & 0 & -m_{32y}\hat{n}_n \cdot \hat{y} & -m_{32x}\hat{n}_n \cdot \hat{x} & 0
\end{bmatrix}
\] (7.3.4)

The second set of rows of \(A\) and \(B\) are modified as follows respectively to enforce the tangential boundary condition.

\[
\begin{bmatrix}
\hat{t}_n \cdot \hat{x} & \hat{t}_n \cdot \hat{y} & 0 & 0 & 0 & 0
\end{bmatrix}
\] (7.3.5)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] (7.3.6)

Note that the sub-matrices that are part of the divergence of the fields have been omitted in the boundary condition enforcement illustration for clarity.

### 7.3.2 Equations

Set 2 from Table 7.1 is also referred to as the PWLPWL-PWLPWL set. Both the transverse and longitudinal fields are expanded using piecewise-linear basis functions. In this case, \(e_x\), \(e_y\), \(e_z\), \(h_x\), \(h_y\) and \(h_z\) are all represented by \(\Phi\).

As before, the equation derived for scalar bases in Eq. (6.2.26) forming the eigenvalue problem is used for the expansion along with all the sub-matrices. The equations required are the same as those that are defined in (6.2.14) through (6.2.19). All the sub-matrices are used in an identical manner with the PWL basis functions. The divergence of the basis function exist in this case and the divergence coefficients \(\lambda_e\) and \(\lambda_h\) for the transverse electric and magnetic fields apply. Hence the divergence sub-matrices also have to be computed if it is desired to use the divergence coefficients. Boundary conditions on the rectangular waveguide are easily applied while the more involved process of applying the boundary conditions using the node normal vectors is used for the circular waveguide. The matrices \(A\) and \(B\) that form the eigenvalue problem are highly sparse in this case also. The sparsity for the PWLPWL-PWLPWL set is shown in Fig. 7.11 for the 145 node
mesh of Fig. 5.1.

Figure 7.11: Sparse nature of the PWLPWL-PWLPWL set, Direct method. Figure on left shows matrix $A$ and on right is matrix $B$ of the eigenvalue problem defined by $Ax = \lambda Bx$. All the sub-matrices are square and are identical in structure with this bases set.

### 7.3.3 Results for the WR-90 Waveguide and Elimination of Spurious Modes

Results without and those that include the divergence coefficients for both the electric and magnetic fields are displayed. It is to be noted that scalar piecewise-linear basis functions are well known for producing spurious modes. The PWL bases allow for both the divergence and curl of vectors expanded with these basis functions to exist as well as the gradient of scalars expanded to exist inside each element. Hence the transverse curl and divergence of the electric and magnetic fields exist and are defined inside each element in the finite-element mesh. The curl and divergence are piecewise-constant since the basis functions are piecewise-linear.

Fig. 7.12 shows results obtained for the WR-90 waveguide with the 41 node mesh shown in Fig. 5.1. It is seen that the results are corrupt with spurious modes. Only on closer observations, it can be noticed that the true modes are reproduced with acceptable accuracy which improves with mesh density and that "k" modes are also present. As the "k" modes appear to be a characteristic of the Direct method with scalar bases they can be readily identified with any arbitrary waveguide. But it is extremely difficult if not impossible to identify the true modes for an arbitrary shaped waveguide that does not have an exact solution.

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*Kartik Sitapati*
7.3. PWLPWL-PWLPWL Bases

Figure 7.12: WR-90 waveguide. Computed propagation constants (\(\gamma\)) with 41 node mesh and \(k = 167.6676\) rad/m via the Direct method with PWLPWL-PWLPWL basis sets. True modes are reproduced but are corrupt with spurious modes. Computed modes at exactly \(k\) called "k" modes are present among the spurious modes. \(\lambda_e = 0\) and \(\lambda_h = 0\) in this case. The propagating mode on the imaginary axis is the TE\(_{10}\) mode.

The divergence coefficients or the penalty terms were not used in the results shown in Fig. (7.12). The transverse divergence of the magnetic and electric fields can be computed from the results similar to the process of back computing the divergence and curl of the magnetic field in magnetostatics. The transverse divergence of the fields should not exist in the case of the electric field for the WR-90 for the propagating TE\(_{10}\) mode. The divergence is expanded as \(\nabla \cdot \bar{E} = \nabla_t \cdot \bar{E}_t - \gamma E_z\) and \(E_z\) is known to be zero for the TE mode.

The divergence of the transverse electric field for the TE\(_{10}\) mode that can be identified in this case due to the availability of an exact solution is shown in Fig. 7.13. The amplitudes of the divergence of other modes including the spurious ones followed a similar trend where the transverse divergence of the electric field was an order of three to five greater than the transverse divergence of the magnetic field even though \(E_z\) was negligible. It was thought that the boundary conditions
Chapter 7. Scalar Bases with the Direct Method

Figure 7.13: WR-90 waveguide. Divergence of the transverse electric field magnitude $|\nabla \cdot \vec{E}|$ for the TE$_{10}$ mode. Refer to Fig. 7.12 for details. Large magnitude can be observed in the piecewise-constant divergence which should be zero according to Maxwell’s equations. $\lambda_e = 0$ and $\lambda_h = 0$.

which are enforced on the electric field were responsible for this but similar results were obtained for the divergence when no boundary conditions were enforced. This was a check to verify if the large divergence of the electric field was related to the boundary conditions.

An attempt is now made to reduce or eliminate the spurious modes completely and the use of the penalty divergence coefficients is made. For the WR-90 waveguide rectangular boundary conditions are applied. The tangential electric fields are set to zero along the boundary and additionally both the transverse components are set to zero at the four corners. The longitudinal electric field is also set to zero along the boundary.

It was found that an effective method to arrive at a minimum value of the divergence of the transverse electric field is to set $\lambda_e = 1$ and $\lambda_h = 0$ for the WR-90 waveguide with rectangular boundary conditions. This effectively includes only the divergence coefficients for the electric field in the matrix eigenvalue problem. Also Eq. (6.3.27) is used with $\gamma^2$ as the unknown. Boundary conditions are enforced on the transverse electric fields, the divergence sub-matrices that are now part of the formulation have to also be included in the boundary enforcement.

Fig. 7.14 shows the divergence of the transverse electric field after application of the penalty divergence coefficient of $\lambda_e = 1$ and $\lambda_h = 0$. It is seen that the divergence of the electric field is brought down significantly by an order of two. The transverse divergence of the magnetic field before and after the application of $\lambda_e$ is of the same order.
7.3. PWLPWL-PWLPWL Bases

Figure 7.14: WR-90 waveguide. Divergence of the transverse electric field magnitude \( |\nabla \cdot \vec{E}| \) for the TE\(_{10}\) mode. \( \lambda_e = 1 \) and \( \lambda_h = 0 \). Amplitude is reduced significantly and can be observed when compared to Fig. 7.13.

Fig. 7.15 shows the computed modes after applying the divergence coefficients. It is seen that most of the spurious modes have been removed from the solution by the inclusion of the divergence term. It appears that there are multiple eigenvalues computed for the TE\(_{10}\) mode but not all provide accurate field patterns and all of them except the one that provides accurate field patterns are spurious modes. Multiple eigenvalues exist for the "k" modes. Even a single spurious mode is unacceptable in solutions to waveguides with no analytical or exact solution.

Table 7.4 shows a partial list of the computed modes including all the "k" modes generated. It also shows the approximation to the TE\(_{10}\) modes as well as the other spurious mode on the \( \beta \) axis. It can also be seen that the real part of the spurious modes are larger than the real part of the "k" modes closest to the actual numerical value of the wavenumber but less than \( k \) and in this case is obtained from row 4. This is used as a filter to eliminate all modes that have a larger real part than the filter tolerance obtained from row 4.

Fig. 7.16 shows the filtered results with out any spurious modes. The field pattern for the propagating modes is consistent with the Helmholtz method and to the results shown via the PWCPWL-PWCPWL bases set in the previous section. It must be noted that there are several different values of the divergence coefficients that appear to help to reduce the computed spurious modes. These include solution with both \( \lambda_e \) and \( \lambda_h \) being applied or just \( \lambda_e \) being applied. While the plots of the computed modes appear to contain fewer spurious modes, the field plots of the propagating
Figure 7.15: WR-90 waveguide. Computed propagation constants $\gamma$ with 41 node mesh and $k = 167.6676$ rad/m via the Direct method with PWLPWL-PWLPWL basis sets. $\lambda_e = 1$ and $\lambda_h = 0$ in this case. The spurious modes are reduced dramatically and can be readily noticed when compared to Fig. 7.12. But some spurious modes still exist. Note the existence of the "k" modes consistent with all Direct methods with scalar bases. The "k" mode is critical to eliminate all spurious modes completely. The propagating mode on the imaginary axis is the $TE_{10}$ mode.

mode do not match to the field plots obtained with the Helmholtz method. Neither is the transverse divergence of the electric field reduced in any way.

It is mandatory for any method attempting to reduce or eliminate spurious modes to satisfy the following conditions-

- Divergence of the fields satisfy Maxwell’s equations. The numerical divergence of eigenvectors corresponding to the computed eigenvalue of interest must be numerically small. The order has to be no more than that of the computed fields themselves.

- Field plots of modes obtained must correspond to the exact solution when known.
Table 7.4: Partial list of computed modes ($\gamma$) for the WR-90 waveguide with $\lambda_e = 1$ and $\lambda_h = 0$ via PWLPWL-PWLPWL bases for the 41 node mesh. $k = 167.6676$ rad/m and multiple "k" modes are generated via the Direct method. The absolute value of the real part of the "k" mode closest to $k$ and less than $k$ is used as the tolerance for the filter and this value is found by inspection as 0.000112716224 corresponding to row 4. As the mesh density is increased, the imaginary part of this mode approaches $k$ rapidly.

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>real ($\alpha$)</th>
<th>imaginary ($\beta$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>208.134120825170</td>
<td>0.000113197614</td>
</tr>
<tr>
<td>2</td>
<td>204.621794601791</td>
<td>0.0000000006207</td>
</tr>
<tr>
<td>3</td>
<td>130.145729787430</td>
<td>0.001187792835</td>
</tr>
<tr>
<td>4</td>
<td>0.000112716224</td>
<td>137.525873825289</td>
</tr>
<tr>
<td>5</td>
<td>0.000175027946</td>
<td>103.913377859360</td>
</tr>
<tr>
<td>6</td>
<td>0.0000000001423</td>
<td>95.952880668220</td>
</tr>
<tr>
<td>7</td>
<td>0.000008404388</td>
<td>96.380946401442</td>
</tr>
<tr>
<td>8</td>
<td>0.000000001943</td>
<td>-167.667601086067</td>
</tr>
<tr>
<td>9</td>
<td>0.000000003127</td>
<td>-167.667601086637</td>
</tr>
<tr>
<td>10</td>
<td>0.000000000427</td>
<td>-167.667601087266</td>
</tr>
</tbody>
</table>

- All enforced boundary conditions are honored by the solution. For a PEC, the transverse electric fields that are tangential to the boundary and the longitudinal component of the electric field at the boundary must be zero.

Any method that violates any of the above listed points is unsuitable to be used practically. This is emphasized here due to encounters in literature of reduction in spurious modes by inclusion of the divergence terms in the vector-Helmholtz method without any mention of the resulting field plots or the divergence itself. As mentioned before, different coefficients can be found that appear to reduce the spurious modes by just looking at the numerical values of the computed modes in the Direct method itself but the field plots are not accurate and neither is there any reduction in the transverse divergence of the electric field. It is important to analyze the field plots as a post processing step as the field plots are obtained from the eigenvectors that also specify the quality of the results.

The divergence coefficients used of $\lambda_e = 1$ and $\lambda_h = 0$ work well with any mesh resolution on the WR-90 waveguide via the Direct method unlike the vector-Helmholtz method where the coefficient appeared to be dependent on mesh density. Further proof of this is provided in Fig. 7.17 and Fig.
Figure 7.16: WR-90 waveguide. Filtered propagation constants (γ) with 41 node mesh and \( k = 167.6676 \) rad/m via the Direct method with PWLPWL-PWLPWL basis sets. \( \lambda_e = 1 \) and \( \lambda_h = 0 \). The spurious modes on the imaginary axis are completely filtered out. The propagating mode on the imaginary axis is the \( \text{TE}_{10} \) mode. Some of the modes on the real axis have also been filtered out. This can be corrected by increasing the filter tolerance for the real axis and this problem is sorted out at higher mesh densities. The "k" modes are also shown.

7.18 that shows the transverse divergence of the electric field on the 145 node mesh shown of Fig. 5.1 without and with the application of the same divergence coefficients.

A low density mesh which was the 41 node mesh used on the WR-90 was intentionally used to illustrate the implementation of this method as several spurious modes are obtained that are not easy to identify and eliminate. The process described in this section becomes easier as the mesh density is increased. Fig. 7.19 shows the filtered modes obtained with the 145 node mesh for the WR-90 waveguide with the same coefficients used for the 41 node mesh. A reduction of order 2 can once again be seen in the transverse divergence of the electric field in figures (7.17) and (7.18). The computed modes are also more accurate as the mesh density is higher.
Figure 7.17: WR-90, $|\nabla \cdot \bar{E}|$, PWLPWL-PWLPWL set, 145 node mesh, $\lambda_e = 0$ and $\lambda_h = 0$.

Figure 7.18: WR-90, $|\nabla \cdot \bar{E}|$, PWLPWL-PWLPWL set, 145 node mesh, $\lambda_e = 1$ and $\lambda_h = 0$.

### 7.3.4 Results for the Circular Waveguide and Elimination of Spurious Modes

This section demonstrates the Direct method with piecewise-linear bases for all the fields applied to circular waveguides. The most important difference between the process of obtaining solutions for the WR-90 and the circular waveguide is the process of enforcing the boundary conditions. The node normal vectors are used for the circular waveguide and the tangential electric fields are set to zero at the boundary while the normal component is solved for. This is the method of boundary condition enforcement for any arbitrary shaped waveguide that is not rectangular in cross section.

Fig. 7.20 shows the modes computed for the circular waveguide of 1 m radius with the 85 node
Chapter 7. Scalar Bases with the Direct Method

Figure 7.19: WR-90 waveguide. Filtered propagation constants ($\gamma$) with 145 node mesh and $k = 167.6676$ rad/m via the Direct method with PWLPWL-PWLPWL basis sets. $\lambda_e = 1$ and $\lambda_h = 0$. The spurious modes on the imaginary axis are completely filtered out. The propagating mode on the imaginary axis is the $TE_{10}$ mode. Some of the modes on the real axis have also been filtered out but the results are more accurate than those obtained with the 41 node mesh. The "k" modes are also shown.

mesh shown in Fig. 5.2 with both the divergence coefficients set to zero. There are spurious modes in the solution but the true modes as well as the "k" modes are present. Fig. 7.21 shows the computed modes for the same waveguide and mesh with $\lambda_e = 10\mu_0$ and $\lambda_h = 0$.

It is seen that all the spurious modes are perturbed and have shifted. The coefficient $\lambda_e$ takes on a different value for the circular waveguide as the application of boundary conditions are performed in a different manner. $\lambda_e$ was set to unity for the rectangular WR-90 waveguide. The process of enforcing boundary conditions on the WR-90 does not create any new sub-matrices and is fairly simple. As the boundary conditions are enforced differently for the circular waveguide, the structure of the matrices in the eigenvalue problem change and it becomes necessary to suitably modify the applied divergence coefficient for the electric field. In general, a set requires to be defined
7.3. PWLPWL-PWLPWL Bases

Figure 7.20: Circular waveguide of 1 m radius. Computed propagation constants ($\gamma$) with 85 node mesh of Fig. 5.2 and $k = 3$ rad/m via the Direct method with PWLPWL-PWLPWL basis sets. $\lambda_e = 0$ and $\lambda_h = 0$. Note presence of several spurious modes. Also present among the spurious modes are approximations to the true modes as well as "k" modes. The propagating modes on the imaginary axis are the $TE_{11}$ and $TM_{01}$ modes.

when rectangular boundary conditions are enforced and another when node normal based boundary conditions are enforced. In either case, the method of reducing or eliminating the spurious modes is the same. The first stage is to use the electric field divergence coefficient $\lambda_e$ to eliminate most of the spurious modes or to displace them. The second stage is to introduce the magnetic field divergence coefficient $\lambda_h$ if required to further perturb the spurious modes and then use a filter whose tolerance is found from the "k" modes.

After applying the filter in an identical manner as described earlier, it is seen from Fig. 7.22 that all the spurious modes have been identified and eliminated. It can also be seen that some of the higher order evanescent modes have been falsely eliminated. This can be rectified by increasing the filter tolerance for the computed modes on the real axis. The coefficients remain identical while changing the mesh density and provide accurate results. The divergence of the electric field was

Mixed Field Finite Element Computations
Figure 7.21: Circular waveguide of 1 m radius. Computed propagation constants (\( \gamma \)) with 85 node mesh of Fig. 5.2 and \( k = 3 \) rad/m via the Direct method with PWLPWL-PWLPWL basis sets. \( \lambda_e = 10\mu_0 \) and \( \lambda_h = 0 \). The spurious modes are all perturbed to a greater extent than the true modes making the identification process easy. The propagating modes on the imaginary axis are the TE\(_{11}\) and TM\(_{01}\) modes.

The higher order modes on the real axis that had been falsely eliminated as seen with the lower order mesh shown in Fig. 7.22 are all retained with the higher order mesh as seen in Fig. 7.23 without having to adjust the tolerance for the axis. Field plots for the propagating transverse magnetic TM\(_{01}\) mode are shown in Fig. 7.24 for the 149 node mesh shown in Fig. 5.2. The field plots agree with results obtained with vector bases and the results for the electric field from the vector-Helmholtz equation method. As the piecewise-linear bases can be interpolated inside each element in the finite-element mesh, all the field plots show the interpolated values.

It must be noted here that the process of identifying the filter and eliminating the spurious modes becomes easier as the mesh density is increased. Highly accurate results are obtained without any
7.3. PWLPWL-PWLPWL Bases

Figure 7.22: Circular waveguide of 1 m radius. Filtered propagation constants ($\gamma$) with 85 node mesh of Fig. 5.2 and $k = 3$ rad/m via the Direct method with PWLPWL-PWLPWL basis sets. $\lambda_e = 10\mu_0$ and $\lambda_h = 0$. The spurious modes are all filtered out. Some of the higher order evanescent modes on the real axis have also been eliminated. This can be rectified by increasing the filter tolerance for the real axis. The propagating modes on the imaginary axis are the TE$_{11}$ and TM$_{01}$ modes. The "k" modes are also shown.

Spurious modes after applying the filter but with a mesh density that is higher than the results obtained with vector bases on a lower density mesh for a given accuracy.

7.3.5 Behavior of Spurious Modes

The two coefficients $\lambda_e$ and $\lambda_h$ affect the "k" modes differently. Rather than use large values for the divergence coefficients that push all the spurious modes away from the region of interest in the solution, small perturbations are sought. This ensures that the computed true modes are most accurate as using the divergence coefficients also affect the accuracy of the computed modes. After the filter has been applied, only approximations to the true modes remain. The computed
Figure 7.23: Circular waveguide of 1 m radius. Filtered propagation constants ($\gamma$) with 149 node mesh of Fig. 5.2 and $k = 3$ rad/m via the Direct method with PWLPWL-PWLPWL basis sets. $\lambda_e = 10\mu_0$ and $\lambda_h = 0$. The spurious modes are all filtered out. All of the higher order evanescent modes on the real axis are retained with the higher order mesh. The filter tolerance for the real axis has not been increased. The propagating modes on the imaginary axis are the $\text{TE}_{11}$ and $\text{TM}_{01}$ modes. The "$k$" modes are also shown.

true modes become sensitive to the perturbation only after the spurious modes so it is fairly easy to identify the spurious modes for any waveguide. The availability of two divergence coefficients is an advantage over the vector-Helmholtz method where the "$k$" modes do not appear and only one coefficient is present.

An observation made in the application of the divergence coefficients is that the spurious solutions all attain a more 'complex' characteristic. That is, the application of this divergence coefficient makes the real part of the computed spurious modes on the imaginary axis larger. It also makes the imaginary part of spurious modes on the real axis larger. The "$k$" modes that are obtained as part of the solution in Direct methods using scalar bases is used to determine a filter for further reduction or total elimination of computed spurious modes. Perturbing the spurious modes is made possible by
7.3. PWLPWL-PWLPWL Bases

Figure 7.24: $|\bar{E}|$ and $|\bar{H}|$ for the circular waveguide via piecewise-linear PWLPWL-PWLPWL bases and the Direct method. Field plots are for the TM$_{01}$ propagating mode. $\lambda_e = 10\mu_0$ and $\lambda_h = 0$. Note low magnitude computed for $|H_z|$. Electric field boundary conditions are satisfied.
the divergence coefficients. Beyond a certain value true modes also begin to shift and it becomes impossible to identify the desired modes. The following example demonstrates the perturbation with a range of values used for the magnetic field divergence coefficient $\lambda_h$ until the true modes also get affected.

Fig. 7.25 and Fig. 7.26 show the effects of varying just one of the divergence coefficient at a time for the circular waveguide. It is seen that both coefficients can displace the spurious modes making it possible to identify and eliminate them. The process involves perturbing the spurious modes just enough and reducing the perturbation of the "k" modes to a minimum as they are used to find the filter tolerance. Once the filter tolerance is identified, all the spurious modes are easily eliminated.

While the process of using piecewise-linear bases with the divergence coefficients has been demonstrated with detailed results and field plots, it is not the best suited method due to the additional complexity involved with the filtering to remove the spurious modes. Also, there is no guarantee that the use of the divergence coefficients with scalar PWL bases will provide accurate solutions to arbitrarily shaped waveguides with no analytical or exact solutions. Using the divergence coefficients to perturb the spurious modes also perturb the true modes though it is to a lesser extent. The application of the boundary conditions is ambiguous as the node normal vectors do not define the boundary perfectly and is only a crude approximation.

### 7.4 Other Scalar Bases Combinations

Other scalar basis sets shown in Table 7.1 are implemented in a similar manner to the two methods described in the previous sections. Eq. (6.2.26) is used for the matrix eigenvalue problem definition. The equations defined in (6.2.14) through (6.2.19) are used after the required sub-matrices have been computed by using the appropriate basis function expansion for each of the components of the electric and magnetic fields. Boundary conditions are then enforced on the electric field appropriately. If PWC bases are used, integration by parts is essential to transfer the derivative to a PWL basis function. The divergence coefficients or the penalty terms cannot be applied if the fields are chosen to be expanded with piecewise-constant bases.

Table 7.5 shows a summary of some of the basis sets used via the Direct method. Poor results were
7.4. Other Scalar Bases Combinations

Figure 7.25: Propagation constants with varying $\lambda_e$ and constant $\lambda_h = 0$, PWLPWL-PWLPWL set. It is seen the increasing $\lambda_e$ effectively perturbs most spurious modes including the "k" modes that are used to find the filter tolerance. After $10\mu_0$, the true modes are also affected and are rapidly displaced with increasing $\lambda_e$. The accuracy of all the desired modes are reduced.
Figure 7.26: Propagation constants with varying $\lambda_h$ and constant $\lambda_e = 0$, PWLPWL-PWLPWL set. Increasing $\lambda_h$ has a slower effect of displacing the spurious modes. $\lambda_h$ has a smaller effect on the "k" modes and together with $\lambda_e$ can be used to find the required filter to eliminate all displaced spurious modes.
7.5 Chapter Summary

A detailed description of the use of different scalar bases with the developed Direct method was described. A set of scalar bases was described that provides results that are equal in quality to vector bases and this is the PWCPWL-PWCPWL bases set. Several results and field plots obtained through the use of these bases were described.

Scalar piecewise-linear bases applied to the Direct method as well as the attempt to eliminate spurious modes through the use of the divergence coefficients was described in detail. It was shown that in most cases the spurious modes can be eliminated completely with the proper choice of the coefficients and the appropriate filter. Details regarding the requirements of proper solutions

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Table 7.5: Summary of scalar bases used via the Direct method. The PWCPWL-PWCPWL basis set in row 1 produce excellent results without any filtering and without the use of the penalty terms.

<table>
<thead>
<tr>
<th></th>
<th>$E_x$</th>
<th>$E_y$</th>
<th>$E_z$</th>
<th>$H_x$</th>
<th>$H_y$</th>
<th>$H_z$</th>
<th>Spurious modes</th>
<th>Divergence coefficients applied</th>
<th>Spurious modes with divergence coefficients?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>PWC</td>
<td>PWC</td>
<td>PWL</td>
<td>PWC</td>
<td>PWL</td>
<td>PWL</td>
<td>No</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWC</td>
<td>PWC</td>
<td>PWL</td>
<td>Yes</td>
<td>No</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWC</td>
<td>PWC</td>
<td>PWC</td>
<td>Yes</td>
<td>No</td>
<td>–</td>
</tr>
<tr>
<td>5</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWL</td>
<td>PWC</td>
<td>Yes</td>
<td>No</td>
<td>–</td>
</tr>
</tbody>
</table>

obtained when the longitudinal electric field was expanded with piecewise-constant bases irrespective of the expansions of the other fields. It is likely that the improper representation of boundary conditions that are imposed on $E_z$ is responsible for this. Results with some other combinations that are possible are not displayed or discussed due to the poor quality of results obtained as well as the lack of useful response to the divergence coefficients when applied. The penalty term can also be applied to some of the other basis sets and it is felt that spurious free solutions can be obtained with a similar process to that introduced in this chapter.
as well as implementation of enforcing boundary conditions on arbitrary shaped waveguides were provided. Several different scalar bases combinations were attempted, some that performed well and some that fail to provide accurate, spurious mode free solutions.

The highlight of this chapter is the use of complementary bases with the PWCPWL-PWCPWL bases set that provides extremely accurate solutions and field plots using first and zero-order scalar bases. The solutions are free of spurious modes without the use of the divergence coefficients. This scalar bases set will be used for a detailed comparison with other Direct methods and the vector-Helmholtz method is the following chapters.
Chapter 8

Vector and Mixed Bases with the Direct Method
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8.1 Vector Bases

The previous chapter focuses on using scalar basis functions with the equations that form the Direct method to obtain solutions to the propagating modes in waveguides. This chapter addresses the use of vector edge bases that were described in Chapters 2 for the transverse fields. The process of applying boundary conditions on the transverse electric field is straightforward when it is expanded by constant-tangential (CT) edge bases. The edge basis functions at all the boundary edges are simply set to zero and this satisfies the condition that the tangential component of the electric field is zero at the boundary for a perfect electric conductor (PEC). The longitudinal component of the fields are expanded both by piecewise-linear (PWL) and piecewise-constant (PWC) bases and boundary conditions are applied to the longitudinal component of the electric field as well. Different combinations of vector bases are used and the ability of the use of complimentary bases is also demonstrated. The process of mixing scalar and vector basis functions for the transverse fields is explored. Edge bases are known to not generally produce spurious modes and none of the methods produce spurious modes, compared to some of scalar bases shown in the previous chapter.

8.1 Vector Bases

Though the number of unknown fields are reduced when the transverse fields are expanded using vector bases, the size of the matrices that define the eigenvalue problem remains more or less the same. Let \( N, E, \) and \( S \) be the number of nodes, edges and elements respectively in the convex finite-element mesh. Let \( N_b \) be the number of boundary nodes. The following relationships hold for any valid Delaunay triangulation of a closed convex hull and has been proven by induction (Carey [1997]).

\[
S = 2N - N_b - 2 \quad (8.1.1)
\]

\[
E = S + N - 1 = 3N - N_b - 3 \quad (8.1.2)
\]

\[
N - 2 \leq S \leq 2N - 5 \quad (8.1.3)
\]

\[
2N - 3 \leq E \leq 3N - 6 \quad (8.1.4)
\]
It can be seen from the second equation above that although a single vector basis function set is used to represent a transverse field, the number of edges are much higher than the number of nodes in the mesh. Hence the number of basis functions required for the transverse fields with vector bases almost equals the number of bases required for the two components of the transverse fields when expanded with scalar bases. Only the quality of the solution determine which basis sets are used.

The sparse nature of the matrices in the eigenvalue problem with vector are shown in Fig. 8.1 with CTPWL-CTPWL bases for the 149 node mesh shown in Fig. 5.2. The sparse matrices are similar for the other vector basis sets that are described in this chapter.

![Figure 8.1: Sparse nature of the CTPWL-CTPWL set, Direct method for the mesh shown in Fig. 5.2. Figure on left shows matrix $A$ and on right is matrix $B$ of the eigenvalue problem defined by $Ax = \lambda Bx$.](image)

Table 8.1 shows the different basis sets used for the electric and magnetic field components that are described in this chapter. Note that the set defined in row 4 uses mixed complementary bases where the transverse electric field is expanded with vector bases and the transverse magnetic field is expanded with scalar bases.

The vector bases are applied only to the transverse electric and magnetic fields and the longitudinal fields are expanded using scalar bases. The vector-Helmholtz method has been used successfully with tangential edge bases for the transverse fields and scalar piecewise-linear bases for the longitudinal fields. This set is the most commonly used bases with the Helmholtz method. Results
8.1. Vector Bases

Table 8.1: Vector bases set combinations used with the Direct method. Constant-tangential (CT), constant-normal (CN), piecewise-constant (PWC) and piecewise-linear (PWL) bases are used. There are two components of the electric field and two components of the magnetic field for a total of four unknowns for all cases except row 4 which mixes vector and scalar bases for a total of 5 unknowns.

<table>
<thead>
<tr>
<th>Set</th>
<th>$\bar{E}_t$</th>
<th>$E_z$</th>
<th>$\bar{H}_t$</th>
<th>$H_z$</th>
<th>Module number (internal use)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>CT</td>
<td>PWL</td>
<td>CT</td>
<td>PWL</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>CT</td>
<td>PWC</td>
<td>CT</td>
<td>PWC</td>
<td>32</td>
</tr>
<tr>
<td>3</td>
<td>CT</td>
<td>PWL</td>
<td>CN</td>
<td>PWC</td>
<td>24</td>
</tr>
<tr>
<td>4</td>
<td>CT</td>
<td>PWL</td>
<td>PWL, PWC</td>
<td>PWC</td>
<td>33</td>
</tr>
</tbody>
</table>

using the electric-field form of the vector-Helmholtz equation with these bases have been described earlier. The first set of bases in the Direct method consists of using the same combination as that of the Helmholtz method for both the magnetic and electric fields. These are the constant-tangential (CT) edge elements for the transverse fields and piecewise-linear (PWL) bases for the longitudinal fields. The second set consists of using the constant-tangential edge bases for the transverse field and scalar piecewise-linear bases for the longitudinal fields. It was shown in the previous chapter that piecewise-constant bases for the longitudinal electric field did not work well with scalar bases for the transverse fields.

The third set uses bases that are complimentary in nature similar to the complementary nature of Maxwell’s equations. While the transverse electric field is expanded using CT edge base, the transverse magnetic field is expanded using constant-normal (CN) edge bases. The longitudinal component of the electric field is expanded using PWL bases while the longitudinal component of the magnetic field is expanded using PWC bases.

Eq. (6.2.13) is used to form the matrices required for the eigenvalue problem for all of the vector bases. The divergence coefficients do not apply in this case as the results do not require them and neither can they be used as the constant-tangential (CT) edge bases are divergence free inside each element and undefined at cell boundaries. The divergence coefficient for the magnetic field $\lambda_h$ can be applied when the transverse magnetic field is expanded with constant-normal (CN) edge bases.
but it is shown that there is no need to do so as the results are excellent.

### 8.1.1 CTPWL-CTPWL Bases Set

The basis function required to form the sub-matrices are $\tilde{e}_t$, $\tilde{h}_t$, $e_z$, and $h_z$ using real basis functions. For the basis functions chosen in the CTPWL-CTPWL set, $\tilde{e}_t$ and $\tilde{h}_t$ are constant-tangential edge bases $\tilde{T}$. The longitudinal bases, $e_z$, and $h_z$, are piecewise-linear bases, $\Phi$, in equations (6.2.5) through (6.2.8). After substituting for the bases, the sub-matrices are evaluated and assembled to form the eigenvalue problem. Several sub-matrices are identical and must only be computed once and scaled by the appropriate material constant.

Table 8.2: Propagating constant ($\beta$) via CTPWL-CTPWL bases and Direct method for the WR-90 waveguide. $k = 167.6676$ rad/m.

<table>
<thead>
<tr>
<th>Mesh Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{10}$</td>
<td>13</td>
<td>98.8080</td>
</tr>
<tr>
<td>$TE_{10}$</td>
<td>41</td>
<td>96.4411</td>
</tr>
<tr>
<td>$TE_{10}$</td>
<td>145</td>
<td>96.1341</td>
</tr>
</tbody>
</table>

Table 8.3: Propagating constant ($\beta$) via CTPWL-CTPWL bases and Direct method for the circular waveguide of radius 1 m. $k = 3$ rad/m.

<table>
<thead>
<tr>
<th>Mesh Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td>$TM_{01}$</td>
<td>2.4288</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.8646</td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>$TM_{01}$</td>
<td>2.3797</td>
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<tr>
<td></td>
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<td>1.8017</td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>$TM_{01}$</td>
<td>2.3871</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.8040</td>
</tr>
</tbody>
</table>

Tables (8.2) and (8.3) show the computed results for the propagating modes for the WR-90 and the circular waveguide respectively. It is seen that fairly accurate results are obtained and the accuracy
increases with mesh density for the WR-90 waveguide. The accuracy obtained for the circular waveguide does improve with higher mesh densities than that shown in Table 8.3. The results for both waveguides are free of spurious modes as shown in Fig. 8.2.

Note that boundary conditions cannot be enforced on the normal component of the transverse magnetic field due to tangential edge bases that are used to expand the transverse magnetic field. Neither can the divergence of the transverse fields be computed as a check, since the divergence of transverse edge bases are not defined at the cell boundaries. However, it can be argued that the average divergence is zero.

### 8.1.2 CTPWC-CTPWC Bases Set

The longitudinal components of both the magnetic and electric fields are expanded using piecewise-constant (PWC) bases. Results obtained are more accurate that those used with scalar bases with the longitudinal electric field expanded with PWC bases. Table 8.4 shows the results obtained for the WR-90 waveguide and several spurious modes that are greater than twice the value of $k$ appear exactly on the imaginary axis as seen in Fig. 8.3. These can be easily identified. For the basis function chosen in the CTPWC-CTPWC set, $\tilde{e}_t$ and $\tilde{h}_t$ are substituted by constant-tangential edge bases $\tilde{T}$. The bases $e_z$ and $h_z$ are substituted by piecewise-constant bases $\Psi$ in equations (6.2.5) through (6.2.8).

Table 8.4: Propagating constant ($\beta$) via CTPWC-CTPWC bases and Direct method for the WR-90 waveguide. Results are for the $TE_{10}$ mode. $k = 167.6676$ rad/m. Accurate results are obtained. Several spurious modes appear on the imaginary axis that are all greater than $2k$ in magnitude and are easily identified.

<table>
<thead>
<tr>
<th>Mesh Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{10}$</td>
<td>13 103.7386</td>
<td>96.0526</td>
</tr>
<tr>
<td>$TE_{10}$</td>
<td>41 97.7398</td>
<td>96.0526</td>
</tr>
<tr>
<td>$TE_{10}$</td>
<td>145 96.4636</td>
<td>96.0526</td>
</tr>
</tbody>
</table>

Table 8.5 shows the results obtained for the circular waveguide. The spurious modes on the imaginary axis are greater than $2k$ but the results are not accurate. The results appear to deteriorate with
Figure 8.2: Result shown are via the CTPWL-CTPW set. Propagating modes are the $\text{TE}_{10}$ for the WR-90 and $\text{TE}_{01}, \text{TE}_{11}$ for the circular waveguide. 45 and 31 node meshes were used for the WR-90 and circular waveguides respectively. Note the swapped mode for the circular guide, this is corrected with higher order meshes.
increasing mesh density for the circular waveguide. The bases chosen here do not provide accurate results with high mesh densities for the circular guide.

Table 8.5: Propagating constant ($\beta$) via CTPWC-CTPWC bases and Direct method for the circular waveguide of radius 1 m. $k = 3$ rad/m. Accuracy appears to reduce with increasing mesh density.

<table>
<thead>
<tr>
<th>Mesh Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE$_{11}$</td>
<td>31 2.4172</td>
<td>2.3687</td>
</tr>
<tr>
<td>TM$_{01}$</td>
<td>1.7381</td>
<td>1.7933</td>
</tr>
<tr>
<td>TE$_{11}$</td>
<td>85 2.3762</td>
<td>2.3687</td>
</tr>
<tr>
<td>TM$_{01}$</td>
<td>1.7364</td>
<td>1.7933</td>
</tr>
<tr>
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<td>2.3687</td>
</tr>
<tr>
<td>TM$_{01}$</td>
<td>1.4245</td>
<td>1.7933</td>
</tr>
</tbody>
</table>

8.1.3 CTPWL-CNPWC Complementary Bases Set

The complementary bases are chosen such that they are natural complements in Maxwell’s equations. Constant-tangential edge bases are used for the transverse electric field, while constant-normal edge bases are used for the transverse magnetic field. The longitudinal electric field is expanded using piecewise-linear bases while the longitudinal magnetic field is expanded using piecewise-constant bases. For the basis function chosen in the CTPWL-CNPWC set, $\vec{e}_t$ and $\vec{h}_t$ are represented by constant-tangential $\vec{T}$ and constant-normal bases $\vec{V}$ respectively. The bases $e_z$ and $h_z$ are substituted by piecewise-linear $\Phi$ and piecewise-constant $\Psi$ bases in equations (6.2.5) through (6.2.8).

It was pointed out that the constant-normal (CN) basis functions are unsuitable for use in the vector-Helmholtz method due to the integral involving a dot product of two curls of the form $\int_{\Omega} \nabla \times \vec{e}_{tn} \cdot \nabla \times \vec{e}_{tj} d\Omega$ and the curl of the CN bases is not defined at the cell boundaries of each element. However, in the Direct method the curl of the constant-normal basis function is present in the sub-matrix defined in

$$m_{42nj} = \int_{\Omega} e_{zn}^* \vec{e}_{tn} \cdot \nabla \times \vec{h}_{tj} d\Omega$$

(8.1.5)
Figure 8.3: Result shown are via the CTPWC-CTPWC set. Propagating modes are the TE$_{10}$ for the WR-90 and TE$_{01}$, TE$_{11}$ for the circular waveguide. 45 and 31 node meshes were used for the WR-90 and circular waveguides respectively.
Using the identity \( \nabla_t \cdot \left( \mathbf{a} \times \mathbf{b} \right) = \mathbf{b} \cdot \nabla_t \times \mathbf{a} - \mathbf{a} \cdot \nabla_t \times \mathbf{b} \), we obtain

\[
m_{42nj} = \int_{\Omega} e_{zn}^* \nabla_t \cdot \left( \mathbf{h}_{tj} \times \hat{z} \right) d\Omega = - \int_{\Omega} e_{zn}^* \nabla_t \cdot \mathbf{T}_j d\Omega \quad (8.1.6)
\]

as \( \mathbf{h}_{tj} \times \hat{z} = -\mathbf{T}_j \) by definition and \( \mathbf{h}_{tj} = \hat{V}_j \) for this bases set. The Direct method allows for the use of these bases that complement the constant-tangential bases and allows for the expansion of the magnetic field appropriately in a PEC.

Neither is the divergence of the CT bases are defined at the cell boundaries of each element. Using the identity \( \nabla_t \cdot (A \mathbf{a}) = A \nabla_t \cdot \mathbf{a} + \nabla_t A \cdot \mathbf{a} \), simplifying and using Gauss’ theorem we obtain

\[
m_{42nj} = - \int_{\Omega} \nabla_t e_{zn}^* \cdot \mathbf{T}_j d\Omega + \oint_{\Gamma} e_{zn}^* \mathbf{T}_j \cdot d\mathbf{\Gamma} \quad (8.1.7)
\]

The first integral in the above equation is used for \( m_{42nj} \) as the second boundary integral contains the dot product between two orthogonal vectors and is zero as \( \mathbf{T}_j \cdot \hat{n} = 0 \) where \( \hat{n} \) is the normal vector at the boundary.

Table 8.6: Propagating constant (\( \beta \)) via CTPWL-CNPWC bases and Direct method for the WR-90 waveguide. \( k = 167.6676 \) rad/m. Accurate results without any spurious modes are obtained.

<table>
<thead>
<tr>
<th>Mesh Nodes</th>
<th>Mesh Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE_{10}</td>
<td>13</td>
<td>92.5159</td>
<td>96.0526</td>
</tr>
<tr>
<td>TE_{10}</td>
<td>41</td>
<td>95.1591</td>
<td>96.0526</td>
</tr>
<tr>
<td>TE_{10}</td>
<td>145</td>
<td>95.8290</td>
<td>96.0526</td>
</tr>
</tbody>
</table>

Tables (8.6) and (8.7) show the results obtained for the WR-90 and the circular waveguides. Excellent results are obtained without any spurious modes present at any mesh resolution for both waveguides. The results are also accurate. The field patterns obtained for all waveguides match very well with the Helmholtz method. The computed constants for both waveguides are shown in Fig. 8.4 to illustrate the quality of the results. No spurious modes are present and accurate results are obtained.
Figure 8.4: Result shown are via the CTPWL-CNPWC set. Propagating modes are the TE$_{10}$ for the WR-90 and TE$_{01},$ TE$_{11}$ for the circular waveguide. 45 and 31 node meshes were used for the WR-90 and circular waveguides respectively.
Field patterns for the TE_{11} mode for the circular waveguide is shown in Fig. 8.5. Note the magnitude of the longitudinal electric field E_z approaches zero as would be expected in a transverse electric or TE mode. The boundary condition enforced on E_z can also be observed in the plot for E_z. Boundary condition enforced on \( \vec{E}_t \) can also be seen in the field plots for E_x and E_y. All the basis functions are interpolated in accordance to their definitions in the field plots.

The zigzag pattern observed in Fig. 8.5 is partially due to thin triangles used for the 149 node mesh on the circular waveguide seen in Fig. 5.2. Better field plots with edge bases can be observed on the WR-90 waveguide since several of the edges are aligned parallel to the \( \hat{x} \) and \( \hat{y} \) axes. It must also be mentioned that all the field plots presented in this document are the computed solutions themselves without any modifications that may improve the appearance of the fields. It is also possible to plot the field patterns with a constant value in each cell after computing the interpolated value at the in-center of each triangle when any basis function that is higher than zero-order is used.

### 8.2 Vector and Scalar Mixed Bases

This section demonstrates the ability of the Direct method to provide solutions when using a mixture of vector and scalar bases for the transverse fields. The electric field is expanded using constant-tangential edge bases for the transverse and piecewise-linear bases for the longitudinal component. The magnetic field is expanded using piecewise-constant bases for all three compo-
Figure 8.5: $|\vec{E}|$ and $|\vec{H}|$ for circular waveguide via CTPWL-CNPWC bases and the Direct method with the 149 node mesh. Field plots are for the $\text{TE}_{11}$ propagating mode. Excellent results are obtained. Plots show actual interpolation of the respective basis functions used. $\vec{E}_t$ and $\vec{H}_t$ are split into the transverse cartesian components for display purposes.
nents. There are a total of five unknown fields in this case which are \( \mathbf{E}_t \), \( \mathbf{E}_z \), \( \mathbf{H}_x \), \( \mathbf{H}_y \), and \( \mathbf{H}_z \). These bases set are chosen in a complementary sense. The vectorial nature for the transverse electric field is retained but the vectorial nature for the transverse magnetic field is replaced by a scalar form. Eq. (8.2.1) shows the construction of the matrix eigenvalue problem. The divergence coefficients cannot be used for the electric field.

\[
\begin{bmatrix}
  j\omega m_{31} & 0 & 0 & 0 & m_{33} \\
  0 & j\omega m_{41} & -m_{42x} & -m_{42y} & 0 \\
  0 & -m_{13x} & j\omega m_{11x} & 0 & 0 \\
  0 & -m_{13y} & 0 & j\omega m_{11y} & 0 \\
  m_{22} & 0 & 0 & 0 & j\omega m_{21}
\end{bmatrix}
\begin{bmatrix}
  \mathbf{E}_t \\
  \mathbf{E}_z \\
  \mathbf{H}_x \\
  \mathbf{H}_y \\
  \mathbf{H}_z
\end{bmatrix} = \gamma
\begin{bmatrix}
  0 & 0 & -m_{32x} & -m_{32y} & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  m_{12x} & 0 & 0 & 0 & 0 \\
  m_{12y} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  \mathbf{E}_t \\
  \mathbf{E}_z \\
  \mathbf{H}_x \\
  \mathbf{H}_y \\
  \mathbf{H}_z
\end{bmatrix}
\]

The sub-matrices are defined by

- \( m_{31nj} = \int_{\Omega} e^{*} t_n \cdot \bar{e} t_j d\Omega \)
- \( m_{32nj} = \int_{\Omega} e^{*} t_n \cdot \vec{g} h_{xj} d\Omega \)
- \( m_{32ynj} = -\int_{\Omega} e^{*} t_n \cdot \vec{h} u_{yj} d\Omega \)
- \( m_{41nj} = \int_{\Omega} e^{*} z_n e_{zj} d\Omega \)
- \( m_{21nj} = \int_{\Omega} \mu \bar{h}^*_{zj} h_{zj} d\Omega \)

- \( m_{22nj} = \int_{\Omega} h^*_{zn} \bar{\nabla} t \times \bar{e} t_j d\Omega \)
- \( m_{12nj} = \int_{\Omega} \muf^*_{xn} h_{xj} d\Omega \)
- \( m_{13nj} = -\int_{\Omega} h^*_{xn} \partial_{yj} e_{zj} d\Omega \)
- \( m_{11nj} = \int_{\Omega} \mu h^*_{yn} h_{yj} d\Omega \)
- \( m_{13ynj} = \int_{\Omega} h^*_{yn} \partial_{x} e_{zj} d\Omega \)

Mixed Field Finite Element Computations
\[ m_{12xnj} = -\int_{\Omega} h_{xn}^* \hat{e}_{tj} \cdot \hat{y} \, d\Omega \]
\[ m_{12ynj} = -\int_{\Omega} h_{yn}^* \hat{e}_{tj} \cdot \hat{x} \, d\Omega \]

and

\[ m_{33nj} = -\int_{\Omega} h_{zj} \hat{z} \cdot \nabla_t \times \hat{e}_{tn}^* \, d\Omega + \oint_{\Gamma} h_{zj} \hat{e}_{tn}^* \cdot \hat{t} \, d\Gamma \]
\[ m_{42xnj} = \int_{\Omega} h_{xj} \frac{\partial e_{zn}^*}{\partial y} \, d\Omega + \oint_{\Gamma} e_{zn}^* h_{xj} \hat{x} \cdot \hat{t} \, d\Gamma \]
\[ m_{42ynj} = -\int_{\Omega} h_{yj} \frac{\partial e_{zn}^*}{\partial x} \, d\Omega + \oint_{\Gamma} e_{zn}^* h_{yj} \hat{y} \cdot \hat{t} \, d\Gamma \]

As piecewise-constant bases are used, integration by parts is required to transfer the derivative to a function that can be differentiated. The boundary condition application is identical to the vector bases described in the previous section. For the basis functions chosen in the CTPWL-PWCPWC set, \( \hat{e}_t \) is substituted by constant-tangential vector edge bases, \( \hat{T} \). The transverse magnetic field, \( h_x \), and \( h_y \), are replaced by piecewise-constant scalar bases, \( \Psi \). The longitudinal bases, \( e_z \), and \( h_z \), are substituted by the scalar piecewise-linear bases, \( \Phi \), and scalar piecewise-constant bases \( \Psi \) respectively in the matrix equations.

The sparse nature of the matrices in the eigenvalue problem with vector bases for the transverse fields are shown in Fig. 8.6 with CTPWL-PWCPWC bases for the 149 node mesh of Fig. 5.2. The matrices are highly sparse in this case and the size of the matrices is comparable to other basis sets used with the Direct method. This is due to the relationship between the number of nodes, elements, and edges in the finite-element mesh highlighted earlier. The results obtained using these basis sets are shown in Fig. 8.7 for both the waveguides.

### 8.3 Chapter Summary

This chapter demonstrated the use of vector bases with the Direct method. Several sets of vector bases were used to obtain solutions to the propagating modes in waveguides. The use of comple-
Figure 8.6: Sparse nature of the CTPWL-PWCPWC set, Direct method. Figure on left shows matrix $A$ and on right is matrix $B$ of the eigenvalue problem defined by $Ax = \lambda Bx$. The transverse electric field is expanded with vector bases while the transverse magnetic field is expanded with scalar bases. There are a total of five unknown fields that are represented in the matrices. Note the purely diagonal sub-matrices for all three components of the magnetic field in matrix $A$ as they are expanded with piecewise-constant bases.

Table 8.8: Propagating constant ($\beta$) via CTPWL-PWCPWC bases and Direct method for the WR-90 waveguide. Accurate results without spurious modes are obtained. $k = 167.6676$ rad/m. Accuracy of computed modes increases with increasing mesh density.

<table>
<thead>
<tr>
<th></th>
<th>Mesh Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE$_{10}$</td>
<td>13</td>
<td>93.5520</td>
<td>96.0526</td>
</tr>
<tr>
<td>TE$_{10}$</td>
<td>41</td>
<td>95.4166</td>
<td>96.0526</td>
</tr>
<tr>
<td>TE$_{10}$</td>
<td>145</td>
<td>95.8933</td>
<td>96.0526</td>
</tr>
</tbody>
</table>

Complementary bases for the electric and magnetic fields was demonstrated with excellent results. The complementary bases are chosen similar to the complementary nature of Maxwell’s equations. The use of mixed bases was also demonstrated, with both vector and scalar bases used for the transverse electric and magnetic fields. High quality results were obtained using this method. Field plots that accurately model the TE propagating modes were shown. In summary, the complementary vector bases set and the mixed bases set provide accurate and spurious-mode free solutions.
Figure 8.7: Result shown are via the CTPWL-PWCPWC set. Propagating modes are the TE_{10} for the WR-90 and TE_{01}, TE_{11} for the circular waveguide. 45 and 31 node meshes were used for the WR-90 and circular waveguides respectively.
Table 8.9: Propagating constant ($\beta$) via CTPWL-PWCPWC bases and Direct method for the circular waveguide of radius 1 m. $k = 3$ rad/m. Accurate results without any spurious modes are obtained. Overall accuracy of computed modes increases with increasing mesh density.

<table>
<thead>
<tr>
<th></th>
<th>Mesh Nodes</th>
<th>Computed rad/m</th>
<th>Exact rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE$<em>{11}$ TM$</em>{01}$</td>
<td>31</td>
<td>2.3752 1.7117</td>
<td>2.3687 1.7933</td>
</tr>
<tr>
<td>TE$<em>{11}$ TM$</em>{01}$</td>
<td>85</td>
<td>2.3673 1.7684</td>
<td>2.3687 1.7933</td>
</tr>
<tr>
<td>TE$<em>{11}$ TM$</em>{01}$</td>
<td>149</td>
<td>2.3704 1.7811</td>
<td>2.3687 1.7933</td>
</tr>
</tbody>
</table>

Table 8.10: Summary of vector bases used via the Direct method.

<table>
<thead>
<tr>
<th>Set</th>
<th>$\vec{E}_t$</th>
<th>$E_z$</th>
<th>$\vec{H}_t$</th>
<th>$H_z$</th>
<th>Spurious modes</th>
<th>Quality of solutions obtained</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>CT</td>
<td>PWL</td>
<td>CT</td>
<td>PWL</td>
<td>No</td>
<td>Average</td>
</tr>
<tr>
<td>2</td>
<td>CT</td>
<td>PWC</td>
<td>CT</td>
<td>PWC</td>
<td>No</td>
<td>Poor</td>
</tr>
<tr>
<td>3</td>
<td>CT</td>
<td>PWL</td>
<td>CN</td>
<td>PWC</td>
<td>No</td>
<td>Good</td>
</tr>
<tr>
<td>4</td>
<td>CT</td>
<td>PWL</td>
<td>PWC, PWC</td>
<td>PWC</td>
<td>No</td>
<td>Good</td>
</tr>
</tbody>
</table>

Table 8.10 shows a summary of some of the basis sets used via the Direct method. Poor results were obtained when the longitudinal electric field was expanded with PWC bases irrespective of the expansions of the transverse fields with scalar or vector bases. Similar to the scalar bases used, it is again likely that the improper representation of boundary conditions that are imposed on $E_z$ is responsible for this behavior. The results with the complementary and mixed complementary basis sets were very accurate.
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Chapter 9

Comparison of Results Obtained by the Direct Method
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The previous two chapters presented a variety of bases combinations which included both scalar and vector basis sets that are used in the Direct method. It was shown that several of the basis sets chosen provided accurate solutions. Actual comparisons of the results obtained via the Direct method with different basis sets are made to the traditionally used vector-Helmholtz equation method. Four different basis sets used on the Direct method are chosen for the comparison. These include the scalar PWCPWL-PWCPWL, the vector CTPWL-CTPWL, the vector complementary CTPWL-CNPWC, and the mixed CTPWL-PWCPWC bases combinations. Constant-tangential edge bases (CT) are used for the expansion of the transverse electric field in the vector-Helmholtz method and piecewise-linear (PWL) bases are used for $E_z$. The comparison of results consists of:

- Accuracy of the results obtained
- Computer memory usage
- Sparsity of matrices $A$ and $B$
- Required time for computations for the solution to the eigenvalue problem
- Propagation constants ($\beta$) vs. Frequency

None of the cases chosen for the comparison include the divergence. This is for clarity and understanding of the results obtained without the enforcement of the divergence. It was mentioned that different divergence coefficients were required for the two test waveguides in the Direct method for boundary conditions differ. It was also pointed out that the divergence coefficients change with the mesh density in the vector-Helmholtz method. The divergence is zero inside each cell and undefined at cell boundaries for all of the bases used for the transverse fields in the comparison except for the constant-normal edge elements.

The time for assembly is not compared since the time is similar for all the cases. The vector-Helmholtz method requires only a fourth of the time required by the Direct methods for assembly as it only solves for the electric field and the size of the required matrices are roughly halved. The assembly times for all the methods described were less than 3 s.

Modes considered for the comparison for both the rectangular and circular waveguides include only the modes computed for the imaginary part of $\gamma$ which is $\beta$ or the propagating modes. This comparison also neglects the lowest order meshes for both waveguides which are the nine node mesh for the WR-90 and the twelve node mesh for the circular waveguide shown in Fig. 5.1 and
Fig. 5.2 respectively. The accuracy obtained with these low density meshes is low (within 15-20\% of the exact values) and some spurious modes are generated. In practice higher mesh densities are usually generated. All solutions were obtained on a PC with a 2 GHz AMD® Athlon XP processor, a 133 MHz system bus, and 1 Gb of DDR-RAM.

### 9.1 Accuracy of Results

The results are compared to the exact analytical values. Error values are also displayed. Table 9.1 shows the actual computed results for the propagating modes for the WR-90 and the circular waveguide. Table 9.2 shows the corresponding error values. The propagating mode for the WR-90 is the $\text{TE}_{10}$ mode. For the circular waveguide, the first number corresponds to the $\text{TE}_{11}$ while the second corresponds to the $\text{TM}_{01}$ mode.

Table 9.1: Comparison of propagation constants from the Direct method using different basis sets and the Helmholtz method. Exact solutions are also shown.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Set/Waveguide</th>
<th>PWCPWL-CTPWL-rad/m</th>
<th>CTPWL-PWCPWL-rad/m</th>
<th>CTPWL-CNPWC-rad/m</th>
<th>Exact-rad/m</th>
<th>Helmholtz-CN-PWL-rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>WR-90</td>
<td>$\text{TE}_{10}$ 13 Nodes</td>
<td>104.8152</td>
<td>98.8080</td>
<td>92.5159</td>
<td>93.5519</td>
<td>96.0526</td>
</tr>
<tr>
<td></td>
<td>$\text{TE}_{10}$ 41 Nodes</td>
<td>95.9529</td>
<td>96.4411</td>
<td>95.1591</td>
<td>95.4166</td>
<td>96.0526</td>
</tr>
<tr>
<td></td>
<td>$\text{TE}_{10}$ 145 Nodes</td>
<td>95.5246</td>
<td>96.1341</td>
<td>95.8290</td>
<td>95.8932</td>
<td>96.0526</td>
</tr>
<tr>
<td></td>
<td>$\text{TE}_{11}$ 31 Nodes</td>
<td>2.3622</td>
<td>2.4288</td>
<td>2.3558</td>
<td>2.3752</td>
<td>2.3687</td>
</tr>
<tr>
<td></td>
<td>$\text{TE}_{11}$ 85 Nodes</td>
<td>2.3620</td>
<td>2.3797</td>
<td>2.3626</td>
<td>2.3673</td>
<td>2.3687</td>
</tr>
<tr>
<td></td>
<td>$\text{TM}_{01}$ 149 Nodes</td>
<td>2.3744</td>
<td>2.3871</td>
<td>2.3664</td>
<td>2.3704</td>
<td>2.3687</td>
</tr>
</tbody>
</table>

Note that the Direct method with the complementary basis set CTPWL-CNPWC produces modes that are exactly the same as the vector-Helmholtz method for both waveguides. It is seen that the mixed complementary basis set CTPWL-PWCPWC provides results with the least error among

Kartik Sitapati
Table 9.2: Comparison of error values from the Direct method using different basis sets and the Helmholtz method. Numbers represent percentage errors. Lower absolute values are better.

<table>
<thead>
<tr>
<th>Modes</th>
<th>Set/ Waveguide</th>
<th>Direct Method</th>
<th>Helmholtz Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>PWCPWL-CTPWL-CTPWL-CTPWL-CTPWL-PWCPWL</td>
<td>CTPWL-CNPWC-CTPWL-PWCPWC-CT-PWL</td>
</tr>
<tr>
<td>WR-90</td>
<td>13 Nodes</td>
<td>-9.1227%</td>
<td>3.682%</td>
</tr>
<tr>
<td>TE10</td>
<td>41 Nodes</td>
<td>0.1038%</td>
<td>0.9302%</td>
</tr>
<tr>
<td>TE10</td>
<td>145 Nodes</td>
<td>0.5497%</td>
<td>0.2328%</td>
</tr>
<tr>
<td>Circular</td>
<td>31 Nodes</td>
<td>0.2744%</td>
<td>0.5446%</td>
</tr>
<tr>
<td>TM01</td>
<td>4.5503%</td>
<td>4.5503%</td>
<td>4.5503%</td>
</tr>
<tr>
<td>TE11</td>
<td>85 Nodes</td>
<td>0.2829%</td>
<td>0.2575%</td>
</tr>
<tr>
<td>TM01</td>
<td>1.3885%</td>
<td>1.3885%</td>
<td>1.3885%</td>
</tr>
<tr>
<td>TE11</td>
<td>149 Nodes</td>
<td>-0.2406%</td>
<td>0.0971%</td>
</tr>
<tr>
<td>TM01</td>
<td>0.6803%</td>
<td>0.6803%</td>
<td>0.6803%</td>
</tr>
</tbody>
</table>

the bases for the Direct method. Results at all mesh resolutions of the CTPWL-PWCPWC set for the circular waveguide are superior to the Helmholtz method for the TE11 mode. All the basis sets used in the Direct method, except the CTPWL-CTPWLC, reproduces the TM01 mode for the circular waveguide exactly as the vector-Helmholtz method.

A further study into the circular waveguide is made with respect to the radius of the waveguide during the mesh generation. As the cells of the finite-element mesh are triangular first-order elements, the perimeter of the waveguide is approximated by straight lines connecting the boundary nodes. The radius of the numerical circular waveguide (equivalently, the radius of the boundary nodes) is modified slightly to obtain four cases. These are

- Radius $a = a_0$, resulting in an inscribed structure (default)
- Radius $a = a_1$, resulting in an equal perimeter structure
- Radius $a = a_2$, resulting in an equal area structure
- Radius $a = a_3$, resulting in a circumscribed structure

Significant difference in the accuracy were not encountered and most of the computed results follow
the trends shown in tables 9.1 and 9.2. The computed propagating constants are shown in Table 9.3 for all the different cases considered.

Table 9.3: Propagation constant (β) computed with different radii. Results were obtained by using the 85 node mesh shown in Fig. 5.2.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Set/Radius</th>
<th>Direct</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE_{11}</td>
<td>a_0 = 1</td>
<td>2.3622</td>
<td>2.3558</td>
</tr>
<tr>
<td>TM_{01}</td>
<td></td>
<td>1.7117</td>
<td>1.7117</td>
</tr>
<tr>
<td>TE_{11}</td>
<td>a_1 = 1.0018</td>
<td>2.3620</td>
<td>2.3626</td>
</tr>
<tr>
<td>TM_{01}</td>
<td></td>
<td>1.7684</td>
<td>1.7684</td>
</tr>
<tr>
<td>TE_{11}</td>
<td>a_2 = 1.0037</td>
<td>2.3646</td>
<td>2.3652</td>
</tr>
<tr>
<td>TM_{01}</td>
<td></td>
<td>1.7750</td>
<td>1.7744</td>
</tr>
<tr>
<td>TE_{11}</td>
<td>a_3 = 1.0055</td>
<td>2.3672</td>
<td>2.3678</td>
</tr>
<tr>
<td>TM_{01}</td>
<td></td>
<td>1.7805</td>
<td>1.7804</td>
</tr>
</tbody>
</table>

9.2 Computer Memory Requirements

The amount of memory required for the matrices A and B that define the eigenvalue problem for each basis set is compared in this section. The amount of memory required depends on the basis sets defining the number of nodes, edges, and elements in the finite-element mesh in the Direct method. Both the matrices A and B that form the eigenvalue problem are complex, double precision, using 16 bytes of storage space for every entry in each matrix. The size of the matrices are shown in Table 9.4 rather than the actual memory required. The matrix sizes represent the full size required via non-sparse eigenvalue solution techniques.

For example, using double precision arithmetic for the Direct method with the PWCPWL-PWCPWL basis set and for the 145 node mesh for the WR-90, approximately 55 Mb of storage space will be required for the matrices A and B. It is seen that the dimensions of the matrices in the vector-Helmholtz method is roughly half the size of any of the Direct methods. Only the electric field is solved for in the vector-Helmholtz method, whereas both the electric and magnetic fields are solved.
9.3. Sparsity

The sparsity of the matrices required for the eigenvalue problem are shown in Table 9.5. The sparse nature of the matrices for several basis sets were shown graphically in previous chapters at the appropriate section that defined the basis sets. It is seen that the large matrices created by the Direct methods are highly sparse. If the solution process included a sparse eigenvalue algorithm, the eigenvalue and eigenvector solution times would drastically reduce as well as the required memory requirements compared to the requirements for the full matrices. This is relevant to take full advantage of the Direct method using both the magnetic and electric fields simultaneously. For instance, a 1000 x 1000 matrix may only require 1000 x 5 entries of storage space.

The Direct method that use piecewise-constant bases for the transverse fields (PWCPWL-PWCPWL) has the highest sparsity since several of the sub-matrices are reduced to just a diagonal form. Refer to Fig. (7.1) for the sparsity of the PWCPWL-PWCPWL bases in the Direct method. All of the
Table 9.5: Sparsity of matrices $A$ and $B$ that define the eigenvalue problem $Ax = \lambda Bx$ where $\lambda$ is $\gamma$ for the Direct methods and $\gamma^2$ for the Helmholtz method. Numbers indicate percentage of non-zero entries.

<table>
<thead>
<tr>
<th>Set/Waveguide</th>
<th>PWCPWL-CTPWL</th>
<th>CTPWL-PWCPWC</th>
<th>CTPWL-CNPWC</th>
<th>CTPWL-CT-PWL</th>
<th>Helmholtz CT-PWL</th>
</tr>
</thead>
<tbody>
<tr>
<td>WR-90</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13 Nodes</td>
<td>5.16 0.69</td>
<td>8.83 2.61</td>
<td>7.01 3.10</td>
<td>4.96 2.05</td>
<td>20.52 6.42</td>
</tr>
<tr>
<td>41 Nodes</td>
<td>1.57 0.21</td>
<td>2.91 0.87</td>
<td>2.19 0.94</td>
<td>1.51 0.59</td>
<td>6.43 2.16</td>
</tr>
<tr>
<td>145 Nodes</td>
<td>0.43 0.05</td>
<td>0.84 0.25</td>
<td>0.62 0.26</td>
<td>0.42 0.15</td>
<td>1.83 0.63</td>
</tr>
<tr>
<td>Circular</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31 Nodes</td>
<td>2.32 0.31</td>
<td>3.78 1.10</td>
<td>2.00 1.26</td>
<td>2.12 0.87</td>
<td>9.05 2.71</td>
</tr>
<tr>
<td>85 Nodes</td>
<td>0.80 0.10</td>
<td>1.42 0.42</td>
<td>1.07 0.45</td>
<td>0.75 0.28</td>
<td>3.24 1.05</td>
</tr>
<tr>
<td>149 Nodes</td>
<td>0.49 0.06</td>
<td>0.80 0.23</td>
<td>0.62 0.25</td>
<td>0.44 0.17</td>
<td>1.90 0.58</td>
</tr>
</tbody>
</table>

Direct methods produce matrices that are more spare than the Helmholtz equation set.

9.4 Time Required for Solutions

The time required for the solution of the eigenvalue problem, $Ax = \lambda Bx$, is shown in Table 9.6. All the eigenvalues and eigenvectors corresponding to the size of the matrices were computed. Matlab® was used to compute the complex eigenvalues and eigenvectors by using the command $[V, D] = \text{eig}(A, B)$ where $V$ and $D$ are the matrices containing the eigenvectors and eigenvalues respectively (Inc. [1995]). Internally, Matlab® uses complex math Lapack routines are used for computing the solution. If $n$ is the dimension of matrix $A$, the Direct method has $4n^2$ entries while the Helmholtz method has just $n^2$ entries.

It is seen that results obtained with the Direct method have much higher solution times. Sparse eigenvalue and eigenvector solution routines can be used to dramatically reduce the solution time. Also, it is not required to compute all the eigenvalues and their eigenvectors, including the eigen-
9.5. *Propagation constants* \( (\beta) \) *vs. Frequency*

Table 9.6: Eigenvalue and eigenvector solution time for different Direct methods and the vector-Helmholtz method. All the eigenvalues and eigenvectors are computed. See Table 9.4 for dimensions of the matrices. The number of eigenvalues computed equal the dimension of the square matrix \( A \) or \( B \). The length of each eigenvector also equals the dimension of the matrices.

| Set/Waveguide | Direct Method |  |
|---------------|---------------|---------------|---------------|---------------|
|               | PWCPWL-CTPWLS | CTPWL-CTPWLS | CTPWL-CNPWCS  | CTPWL-PWCPWCS |
| **WR-90**     |               |               |               |               |
| 13 Nodes      | 0.06          | 0.06          | 0.07          | 0.06          |
| 41 Nodes      | 5.20          | 3.84          | 5.29          | 6.29          |
| 145 Nodes     | 447.70        | 238.50        | 431.42        | 577.70        |
| **Circular**  |               |               |               |               |
| 31 Nodes      | 1.84          | 1.21          | 1.54          | 1.76          |
| 85 Nodes      | 56.12         | 36.81         | 51.89         | 61.90         |
| 149 Nodes     | 345.04        | 203.45        | 287.03        | 380.65        |

Values computed at infinity. To reduce the computational time of the Direct methods, sparse techniques may be used along with selective eigenvalue and eigenvector computations. The application of these techniques is left for future work and the data presented here represents the worst case solution times.

9.5. *Propagation constants* \( (\beta) \) *vs. Frequency*

Further analysis of the Direct methods involve computing the modes at different frequencies. These values are compared with the exact values obtained by analytical means as well as the Helmholtz equation method. Results for both waveguides are shown with focus on the \( \text{TE}_{10} \) mode for the rectangular and the \( \text{TE}_{11}, \text{TM}_{01} \) modes for the circular waveguide.

---

*Mixed Field Finite Element Computations*
9.5.1 WR-90 Waveguide

Table 9.7 shows the results obtained by Direct methods compared to the exact solution as well as the vector-Helmholtz method.

Table 9.7: Propagation constant ($\beta$) vs. frequency for the TE$_{10}$ mode for WR-90. The frequency is varied from 8 GHz to 13 GHz. The usual operating range of the WR-90 waveguide is 8.2 to 12.4 GHz. Note that the Direct method using scalar bases PWCPWL-PWCPWL provides more accurate results than the Helmholtz method throughout the frequency range for the rectangular waveguide. Results were obtained by using the 41 node mesh shown in Fig. 5.1.

<table>
<thead>
<tr>
<th>Frequency (GHz)</th>
<th>k PWCPWL</th>
<th>CTPWL</th>
<th>CTPWL-CNWC</th>
<th>CTPWL-PWCPWC</th>
<th>Exact</th>
<th>Helmholtz CT-PWL</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.0000</td>
<td>167.6676</td>
<td>95.9529</td>
<td>96.4411</td>
<td>95.1591</td>
<td>95.4166</td>
<td>96.0526</td>
</tr>
<tr>
<td>8.5556</td>
<td>179.3112</td>
<td>115.0957</td>
<td>116.0870</td>
<td>114.4348</td>
<td>114.7446</td>
<td>115.1789</td>
</tr>
<tr>
<td>9.1111</td>
<td>190.9548</td>
<td>132.5076</td>
<td>133.9129</td>
<td>131.9339</td>
<td>132.2910</td>
<td>132.5798</td>
</tr>
<tr>
<td>10.2222</td>
<td>214.2419</td>
<td>164.2989</td>
<td>166.4048</td>
<td>163.8366</td>
<td>164.2802</td>
<td>164.3572</td>
</tr>
<tr>
<td>10.7778</td>
<td>225.8855</td>
<td>179.2172</td>
<td>181.6406</td>
<td>178.7934</td>
<td>179.2776</td>
<td>179.2706</td>
</tr>
<tr>
<td>11.3333</td>
<td>237.5291</td>
<td>193.6869</td>
<td>196.4177</td>
<td>193.2948</td>
<td>193.8184</td>
<td>193.7363</td>
</tr>
<tr>
<td>11.8889</td>
<td>249.1727</td>
<td>207.8017</td>
<td>210.8368</td>
<td>207.4364</td>
<td>207.9983</td>
<td>207.8478</td>
</tr>
<tr>
<td>12.4444</td>
<td>260.8163</td>
<td>221.6295</td>
<td>224.9732</td>
<td>221.2870</td>
<td>221.8866</td>
<td>221.6727</td>
</tr>
<tr>
<td>13.0000</td>
<td>272.4599</td>
<td>235.2210</td>
<td>238.8881</td>
<td>234.8983</td>
<td>235.5348</td>
<td>235.2617</td>
</tr>
</tbody>
</table>

It is seen once again that the Direct method with the vector complementary basis set CTPWL-CNWC are exactly identical to the results obtained by the vector-Helmholtz method. The results using the scalar PWCPWL-PWCPWL bases are more accurate than the vector-Helmholtz method. The Direct method is sensitive to the basis set used but provides results that are reasonably accurate. No spurious modes are present in any of the four solutions methods described. The 41-node mesh shown in Fig. 5.1 was used to generate the data shown in the above table.

The percentage error values for the computed propagating constant with the Direct method and the Helmholtz method is shown in Fig. 9.1.

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9.5. Propagation constants ($\beta$) vs. Frequency

Figure 9.1: Error in $\beta$ vs. frequency for WR-90, $\text{TE}_{10}$ mode. The Direct method with the scalar bases set PWCPWL-PWCPWL and the mixed complementary set CTPWL-PWCPWC provides more accurate data than the vector-Helmholtz method with CT-PWL bases for the WR-90.

9.5.2 Circular Waveguide

The 31-node mesh shown in Fig. 5.2 is used for the comparison for the circular waveguide. Tables (9.8) and (9.9) show the results for the two propagating modes, the $\text{TE}_{11}$ and $\text{TM}_{01}$. Fig. 9.2 shows the error values for the $\text{TE}_{11}$ mode. It is seen that the scalar complementary set PWCPWL-PWCPWL and the mixed complementary set CTPWL-PWCPWC with the Direct method is more accurate than the vector-Helmholtz method with CT-PWL bases. No spurious modes are present in any of the solutions. It is interesting to note that almost all the basis sets used with the Direct method provide identical values to the vector-Helmholtz method for the $\text{TM}_{01}$ mode. The basis set CTPWL-CTPWL proves to be the most inaccurate for both the rectangular and circular waveguides. All results for the circular waveguide were obtained by using the 31 node mesh shown in Fig. 5.2. The Direct method, with the scalar PWCPWL-PWCPWL basis set shown in column 3, provides the most accurate solution for the $\text{TE}_{11}$. 

Mixed Field Finite Element Computations
Chapter 9. Comparison of Results Obtained by the Direct Method

Figure 9.2: Error in $\beta$ vs. frequency for the circular waveguide, TE$_{11}$ mode. The Direct method with the scalar bases set PWCPWL-PWCPWL (throughout the range) and the mixed complementary set CTPWL-PWCPWC (partially through the range) provides more accurate data than the vector-Helmholtz method with CT-PWL bases for the circular waveguide.

9.6 Chapter Summary

Detailed comparisons of different basis sets used with the Direct method were displayed as well as comparisons made with the vector-Helmholtz method. It was shown that the Direct method with complementary bases can be more accurate than the vector-Helmholtz method. The main disadvantage of the Direct method appears to be the slow computation time for the solution when using full matrix solution techniques. The memory requirements also exceed the vector-Helmholtz method as the Direct method provides solutions to both the electric and magnetic fields.

Several of the sub-matrices can be identical and can just be scaled enabling reuse of smaller sub-matrices that can reduce the memory requirements. If the several identical sub-matrices are reused, the sparsity is increased even further.

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Table 9.8: Propagation constant ($\beta$) vs. frequency for the $TE_{11}$ mode for the circular waveguide. The frequency is varied from 0.1312 GHz to 0.1909 GHz. Results were obtained by using the 31 node mesh shown in Fig. 5.2.

| Frequency (GHz) | $k$ (rad/m) | Direct Method | | | Exact Method | | | Helmholtz CT-PWL (rad/m) |
|----------------|-------------|---------------|---------------|---------------|-----------------|---------------|------------------|
| 0.1312         | 2.7497      | 2.0350        | 2.0931        | 2.0275        | 2.0442          | 2.0428        | 2.0275           |
| 0.1365         | 2.8608      | 2.1827        | 2.2446        | 2.1758        | 2.1937          | 2.1901        | 2.1758           |
| 0.1418         | 2.9719      | 2.3265        | 2.3921        | 2.3199        | 2.3390          | 2.3334        | 2.3199           |
| 0.1471         | 3.0830      | 2.4667        | 2.5362        | 2.4606        | 2.4809          | 2.4734        | 2.4606           |
| 0.1577         | 3.3051      | 2.7393        | 2.8169        | 2.7338        | 2.7563          | 2.7454        | 2.7338           |
| 0.1630         | 3.4162      | 2.8724        | 2.9541        | 2.8671        | 2.8907          | 2.8783        | 2.8671           |
| 0.1683         | 3.5273      | 3.0037        | 3.0898        | 2.9986        | 3.0233          | 3.0093        | 2.9986           |
| 0.1736         | 3.6384      | 3.1333        | 3.2241        | 3.1285        | 3.1543          | 3.1388        | 3.1285           |
| 0.1789         | 3.7495      | 3.2617        | 3.3574        | 3.2570        | 3.2839          | 3.2670        | 3.2570           |
| 0.1909         | 4.0010      | 3.5479        | 3.6570        | 3.5437        | 3.5729          | 3.5512        | 3.5437           |

Table 9.9: Propagation constant ($\beta$) vs. frequency for the $TM_{01}$ mode for the circular waveguide. The frequency is varied from 0.1312 GHz to 0.1909 GHz. Results were obtained by using the 31 node mesh shown in Fig. 5.2.

| Frequency (GHz) | $k$ (rad/m) | Direct Method | | | Exact Method | | | Helmholtz CT-PWL (rad/m) |
|----------------|-------------|---------------|---------------|---------------|-----------------|---------------|------------------|
| 0.1312         | 2.7497      | 1.2210        | 1.3860        | 1.2210        | 1.2210          | 1.3336        | 1.2210           |
| 0.1365         | 2.8608      | 1.4540        | 1.6107        | 1.4540        | 1.4540          | 1.5498        | 1.4540           |
| 0.1418         | 2.9719      | 1.6619        | 1.8153        | 1.6619        | 1.6619          | 1.7464        | 1.6619           |
| 0.1471         | 3.0830      | 1.8534        | 2.0060        | 1.8533        | 1.8533          | 1.9295        | 1.8533           |
| 0.1577         | 3.3051      | 2.2032        | 2.3591        | 2.2031        | 2.2031          | 2.2677        | 2.2031           |
| 0.1630         | 3.4162      | 2.3665        | 2.5257        | 2.3665        | 2.3665          | 2.4268        | 2.3665           |
| 0.1683         | 3.5273      | 2.5242        | 2.6874        | 2.5242        | 2.5242          | 2.5809        | 2.5242           |
| 0.1736         | 3.6384      | 2.6773        | 2.8450        | 2.6773        | 2.6773          | 2.7308        | 2.6773           |
| 0.1789         | 3.7495      | 2.8264        | 2.9995        | 2.8264        | 2.8264          | 2.8772        | 2.8264           |
Appendix C contains the exact propagation constants for the WR-90 and the circular waveguide of 1 m radius.
Chapter 10

Dielectric Loaded Waveguides
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10.1. Rectangular Waveguide Partially Filled with Dielectric

The examples considered in previous chapters did not contain any media other than vacuum or air. In this chapter waveguides with multiple dielectrics are considered. The electric-field form of the vector-Helmholtz method is used as the benchmark and a comparison is made with respect to the accuracy of the results obtained with Direct methods. A rectangular waveguide with the same dimensions of the WR-90 is partially loaded with dielectric material and the same basis sets used with the Direct method for detailed comparisons for a vacuum shown in the previous chapter.

10.1 Rectangular Waveguide Partially Filled with Dielectric

Fig. 10.1 shows a rectangular waveguide which is loaded with a dielectric material in the lower portion of the cross section. For this waveguide we used, $b = a/2$. The height $h$ can be varied in value between zero and $b$ to include free space and a completely filled dielectric waveguide. Attempts to find modes that are either purely transverse electric or transverse magnetic to $\hat{z}$ will generally be unsuccessful except for the $TE_{0n}$ case. Most of the modes obtained are hybrid modes that have both $E_z$ and $H_z$. However, modes that are transverse to $\hat{y}$ can be obtained. The tangential components of $\bar{E}$ and $\bar{H}$ have to be continuous at the boundary interface of the dielectric and vacuum in accordance to Maxwell’s equations and this fact is used in the process of obtaining results through analytical means.

Figure 10.1: Rectangular waveguide partially loaded with dielectric material.

Results for the $TE_{01}^y$ mode are displayed along with the exact results that can be obtained analytically. Solutions to the transcendental equations of the exact problem are required for comparison.
The cutoff frequency for a mode is first computed and then the transcendental equation is solved at a frequency higher than the cutoff. The transverse electric and magnetic fields are also called longitudinal section electric (LSE\textsuperscript{y}) and longitudinal section magnetic (LSM\textsuperscript{y}) modes. The relative permittivity of the dielectric material for comparison is chosen to be 2.56 which represents polystyrene. The cutoff frequency, \( f_c \), and propagating modes, \( \text{TE}^y \) and \( \text{TM}^y \), are computed via analytical methods for reference. The material constants for the dielectric are represented by \( \varepsilon_d \) and \( \mu_d \).

Table 10.1: Analytical results (\( \beta \)) for \( \text{TE}^y \) and \( \text{TM}^y \) modes for the WR-90 waveguide which is partially filled with dielectric material at fixed frequency of 14.9 GHz. \( h = b/2 \) and \( \varepsilon_{rd} = 2.56 \).

Table 10.1 shows the modes computed analytically at a fixed frequency with \( h = b/2 \). Only the \( \text{TE}_{01}^y \) and \( \text{TM}_{11}^y \) modes are used for detailed comparisons.

For other waveguides which are arbitrarily shaped and contain heterogeneous media, where an analytical or an exact solution is not possible, the identification of transverse electric or transverse magnetic fields can be only achieved by inspection of the fields. The Direct method is extremely useful in such cases as it provides accurate approximation to the propagating modes as well as provides accurate representations of all components of the magnetic and electric fields without any post-processing involved. The additional storage and time required for computations is due to the simultaneously solution of both the electric and magnetic fields.

The vector-Helmholtz equation solves for either the magnetic or electric field, not both. To identify both the magnetic and electric fields, the vector-Helmholtz equation method must be computed twice or the curl operator may be used to obtain the other field. The curl reduces the order of the resulting field to less than those of the basis functions of the original field. Problems associated with the magnetic-field form of the vector-Helmholtz equation method applied to waveguides include
10.2 Results of WR-90 Partially Filled with Dielectric

the difficulty of applying boundary conditions to first-order vector bases.

The waveguide is a PEC and the magnetic-field form of the vector-Helmholtz equation method requires boundary conditions to be applied to the normal component of the transverse magnetic field at the boundary. It is not straightforward to apply boundary conditions with tangential-edge bases on the normal component of the magnetic field. Attempts were made to apply boundary conditions by using the adjacent edges of boundary edges in each cell but the result were not accurate. Typically, boundary conditions are not applied when the magnetic-field form of the vector-Helmholtz equation is used. As no boundary conditions can be applied, the magnetic-field form of the vector-Helmholtz equation proves to be less accurate than the electric-field form of the Helmholtz equation and the Direct method. The application of boundary conditions is essential to make the finite-element eigenvalue problem well defined with an unique solution.

Comparisons made earlier were with the best possible results typically used in the Helmholtz equation method based on the electric-field form. Neither can the transverse magnetic field be expanded with constant-normal (CN) vector edge bases. In this case the boundary conditions can be easily applied, but these bases do not allow for the curl to be properly defined. Refer to Table 5.1 for a comparison of the results obtained by the electric and magnetic-field form of the vector-Helmholtz equation for the WR-90 and circular waveguides in a vacuum.

10.2 Results of WR-90 Partially Filled with Dielectric

Table 10.2 shows the results obtained via direct methods and the vector-Helmholtz method. The first propagating modes for both transverse electric and transverse magnetic, \( \text{TE}_{01} \) and \( \text{TM}_{11} \), are shown. The 41 node mesh shown in Fig. 5.1 is used to obtain the results. The PWCPWL-PWCPWL bases set again produces modes that are exactly equal to the free space wave number \( k_0 \) and the wave number corresponding to the dielectric \( k_d \). Also the Direct method with the CTPWL-CNPWC bases set produces modes that are exactly equal to the results obtained via the vector-Helmholtz method. Most of the basis sets with the Direct method as well as the vector-Helmholtz method have acceptable accuracy for both the modes compared. This trend continues at different frequencies.
Table 10.2: Computed propagation constants for WR-90 partially loaded with dielectric $\epsilon_{rd} = 2.56$. Lower half of the waveguide is filled with the dielectric as shown in Fig. 10.1. Computed modes is for the 41 node mesh shown in Fig. 5.1. Frequency is 14.9377 GHz which is higher than the cutoff frequency. Modes are the propagating, $TE_{01}^y$ and $TM_{11}^y$, respectively.

<table>
<thead>
<tr>
<th>WR-90 41 Nodes</th>
<th>Direct PWCPWL-CTPWL rad/m</th>
<th>Method CTPWL-CNPWC rad/m</th>
<th>CTPWL-PWCPWC rad/m</th>
<th>Exact rad/m</th>
<th>Helmholtz CT-PWL rad/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{01}^y$</td>
<td>343.8660</td>
<td>357.5336</td>
<td>342.6231</td>
<td>347.1608</td>
<td>342.3237</td>
</tr>
<tr>
<td>$TM_{11}^y$</td>
<td>424.5812</td>
<td>443.4444</td>
<td>425.2180</td>
<td>426.6835</td>
<td>425.9260</td>
</tr>
</tbody>
</table>

Field patterns for the propagating $TE_{01}^y$ and $TM_{11}^y$ modes obtained using CTPWL-CNPWC bases are shown in figures (10.2) and (10.3). The $TE_{01}^y$ mode is also transverse to $\hat{z}$ and can be written as $TE_{01}^z$. The magnitude of $E_y$ is of an order lower than $E_x$ while $E_z$ is essentially zero.

The tangential components of the electric and magnetic fields are continuous across the boundary interface and are verified by analyzing $E_x$ and $H_x$. Actual interpolation of the respective bases inside each element is shown in field plots. The applied boundary conditions on $\vec{E}$ are satisfied. There is a higher concentration of the field in the dielectric. The field patterns obtained are consistent with transverse electric and transverse magnetic forms. No spurious modes are present in any of the solutions.

Table 10.3 shows the computed modes using the different methods across a range of frequencies.

Fig. 10.4 shows the corresponding error values obtained from the data in Table 10.3. It is seen that the Direct method with the CTPWL-CNPWC bases provides exactly the same results as the vector-Helmholtz method. The other basis sets used with the Direct method also have acceptable accuracy.

Table 10.4 contains data for the field plots shown in Fig. 10.5. Here the value of the relative permittivity of the dielectric is increased and the resulting electric field component $|E_x|$ is shown for each case. The field is observed to be stronger in the dielectric.
10.2. Results of WR-90 Partially Filled with Dielectric

Figure 10.2: Field plots of the propagating $\text{TE}_{01}^y$ mode for the WR-90 partially loaded with dielectric at the lower half of the cross section with CTPWL-CNPWC bases. Plots show actual interpolation of the respective basis functions used. Note the very low magnitude of $E_z$ and the fields being stronger in the dielectric. Tangential fields are continuous across the boundary interface of the dielectric and vacuum. The 145 node mesh in Fig. 5.1 is shown superimposed on the fields for the cartesian components of the transverse fields.
Figure 10.3: Field plots of the propagating $\text{TM}_{11}^y$ mode for the WR-90 partially loaded with dielectric at the lower half of the cross section with CTPWL-CNPWC bases. Plots show actual interpolation of the respective basis functions used. The fields are stronger in the dielectric. Note the very low magnitude of $H_y$ confirming that this is a $\text{TM}_{11}^y$ mode. Tangential fields are continuous across the boundary interface of the dielectric and vacuum. This can be observed from the plots (a) and (b) for $E_x$ and $H_x$ respectively. Enforced boundary conditions being satisfied can also be seen. The 145 node mesh in Fig. 5.1 is shown superimposed on the fields for the cartesian components of the transverse fields.
10.2. Results of WR-90 Partially Filled with Dielectric

Table 10.3: Computed propagation constants ($\beta_z$) for a rectangular waveguide with dimensions equal to the WR-90 partially loaded with dielectric $\varepsilon_{rd} = 2.56$ at different frequencies. Lower half of the waveguide is filled with the dielectric. Computed modes is for the 41 node mesh shown in Fig. 5.1. Modes are for the propagating $TE_{01}^V$ mode.

<table>
<thead>
<tr>
<th>Frequency (GHz)</th>
<th>$k_0$ (rad/m)</th>
<th>$k_d$ (rad/m)</th>
<th>PWCPWL</th>
<th>CTWL</th>
<th>Exact</th>
<th>Helmholz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>PWCPWL</td>
<td>CTWL</td>
<td>CNPWC</td>
<td>PWCPWC</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>rad/m</td>
<td>rad/m</td>
<td>rad/m</td>
<td>rad/m</td>
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<tr>
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<td>299.63</td>
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<td>507.61</td>
<td>529.73</td>
<td>504.58</td>
<td>512.76</td>
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</table>

Table 10.4: Results ($\beta_z$) for $TE_{01}$ mode with increasing $\varepsilon_{dr}$ in partially filled WR-90. For each value of $\varepsilon_{dr}$, the cutoff frequency is first computed analytically. The values shown in the first column are higher than the cutoff frequencies. Exact values and results by the Direct method with CTPWL-CNPWC basis sets are also shown. $E_z$ is zero in all cases. Refer to Fig. 10.5 for field plots.

<table>
<thead>
<tr>
<th>Frequency (GHz)</th>
<th>$\varepsilon_{dr}$</th>
<th>Exact (rad/m)</th>
<th>CTPWL-CNPWC (rad/m)</th>
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<tr>
<td></td>
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<tr>
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<td>327.0845</td>
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<td>16.42</td>
<td>2</td>
<td>336.6826</td>
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<tr>
<td>14.01</td>
<td>3</td>
<td>345.8533</td>
<td>345.9442</td>
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<tr>
<td>28.37</td>
<td>4</td>
<td>1093.6141</td>
<td>1092.5338</td>
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<tr>
<td>26.13</td>
<td>5</td>
<td>1130.3029</td>
<td>1129.1254</td>
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<tr>
<td>24.29</td>
<td>6</td>
<td>1153.5835</td>
<td>1152.3465</td>
</tr>
</tbody>
</table>
Chapter 10. Dielectric Loaded Waveguides

Figure 10.4: Error in $\beta_z$ vs. Frequency for WR-90 partially loaded with dielectric, $\text{TE}_{01}^y$ mode. Raw data is in Table 10.3.

10.3 Chapter Summary

The use of the Direct method to obtain modes of dielectrically loaded waveguides was demonstrated on a rectangular waveguide partially filled with dielectric. Accurate results are obtained with the Direct method and the CTPWL-CNPWC bases combination again provided exactly the same results as the vector-Helmholtz method as seen earlier in the previous chapter in vacuum. Accurate field plots are obtained that satisfy the boundary conditions as well as the requirements for transverse electric and transverse magnetic fields. The effects of varying the dielectric constant on the transverse fields was shown as well as detailed field plots for two of the modes.
10.3. Chapter Summary

Figure 10.5: Field plots of $E_x$ of the $\text{TE}_{01}^y$ mode for WR-90 partially filled dielectric waveguide with increasing $\epsilon_{rd}$. The waveguide is partially loaded with dielectric at the lower half of the cross section. Plots show actual interpolation of the respective basis functions used. The fields are stronger in the dielectric steadily increasing with $\epsilon_{rd}$ and at higher values of $\epsilon_{rd}$ are completely absent in vacuum. The field starts to weaken in the dielectric as $\epsilon_{rd}$ is increased further as observed from the magnitudes. Results are obtained using the Direct method with CTPWL-CNPWC complementary vector basis sets. $E_z$ is zero in all cases.
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Chapter 11

Conclusion
The Direct method is a robust method to obtain accurate solutions to the propagating modes in waveguides. It provides an alternative solution technique to the commonly used vector Helmholtz method. The Direct method solves for both the electric and magnetic field simultaneously and provides solutions that can be more accurate than the Helmholtz equation method. Both scalar, vector, and mixed bases were used to obtain accurate solutions for homogeneous and dielectrically loaded waveguides.

Initially, Delaunay mesh generation was described as well as the definitions of the required scalar and vector basis functions for finite-element analysis. To provide a foundation for dynamics, solution techniques for magnetostatics were first considered. A new approach to the magnetic-vector potential method which includes the boundary conditions to solve problems with finite elements was described. The equations required for the magnetostatic solutions were derived in a variational form with both the Dirichlet and Neumann boundary conditions being included in the functional. Several examples including effects of differently applied boundary conditions on the results were displayed. The next chapter focused on least-squares methods applied in magnetostatics. Several techniques were used to attempt to solve for the magnetic fields without the use of the magnetic-vector potential. It was shown that these techniques are well suited only in homogeneous media where the divergence of the magnetic field is easily defined. The fields were accurate in a homogeneous medium and the order of the basis functions are higher than in a equivalent magnetic-vector potential solution. The complementary structure of the magnetic-vector potential was introduced and the hidden complementary nature in the basis functions was revealed for magnetostatics. The application of the complementary bases to Direct methods in dynamics was considered next.

A full chapter was devoted to the popular vector-Helmholtz equation method for solutions to the propagating modes in waveguides. The normal node vectors for scalar basis functions were defined to address the application of boundary conditions. The weighted-residual form of the vector-Helmholtz equation was used for the solution. The equations, matrices, and sub-matrices required for the eigenvalue problem of the propagating modes were defined and field plots shown. Examples with scalar and vector bases were shown for the rectangular and circular waveguides used. Comparisons were made with respect to accuracy between the electric and magnetic-field form of the vector-Helmholtz equations. The application of the divergence coefficient was also considered in the form of a penalty term.
The equations that define the Direct method were derived from Maxwell’s equations in a general form applicable to other problems of interest in electromagnetics via the finite-element method. The equations were modified for application to waveguides. The Direct method is variationally based, to ensure that the equations provide minimization.

Several different scalar basis sets were applied to the Direct method to obtain the propagating modes. The excellent results obtained with complementary basis sets was demonstrated as well as the poor results with spurious modes obtained with other sets. The use of penalty terms to eliminate spurious modes when the transverse fields are expanded with scalar bases was described in detail, with little improvement. It is often found that fields due to the spurious modes are approximately divergence free with null fields, arising from dependant bases for the specific spurious propagation constant. For most methods that solve for the propagating modes in waveguides, vector bases are used for the transverse fields as they avoid problems associated with the scalar bases such as generation of spurious modes and enforcing boundary conditions. The use of scalar bases and techniques to reduce or eliminate the spurious modes is more of academic interest for identifying the root cause of spurious modes and methods of elimination. The quality of the results change with the basis sets, some being acceptable while other are not. The application of mixed bases to the transverse electric and magnetic fields was illustrated where the transverse electric field was expanded with vector edge bases and the transverse magnetic field was expanded with piecewise-constant scalar bases. Results as well as detailed field plots were displayed to illustrate the ability of the Direct method to produce accurate results.

It was also shown that the best results are obtained when the bases are chosen in a complimentary manner similar to Maxwell’s equations. It is also seen in the several field plots of results via the Direct method solutions that the behavior of the magnetic field at the boundary satisfies the boundary condition behavior of perfect electric conductors. This occurs even though the boundary conditions are only applied on the electric field.

A detailed analysis of the results obtained via the Direct method and comparisons made with the vector-Helmholtz equation method was displayed. The sparsity of the matrices that form the Direct method was shown and the need for sparse eigenvalue techniques was stressed. It was shown that some of the basis sets used with the Direct method produce more accurate results than the electric-field form of the vector-Helmholtz equation method. The CTPWL-CNPWC complementary vector...
bases produced results that are exactly the same as the vector-Helmholtz method with edge bases, suggesting that the Helmholtz method also has a variational form.

11.1 Recommendations for Future Work

Recommendations for future work in magnetostatics include extending the general method to work in heterogeneous media and three dimensions. The general method in magnetostatics presented in this document used the divergence of the magnetic field as a constraint with a Lagrange coefficient. This equation set works well only in homogeneous media. Part of the extension will include modifying the presented equations or developing a new mechanism that may provide accurate results in heterogeneous media. The Direct method was found to have identical results to the vector-potential method when an appropriate complementary basis set is used.

In dynamics, the equations derived in the chapter on Direct methods can be applied to solve other problems in electromagnetics such as in scattering and to find resonant frequencies of homogeneous and dielectric filled cavities. The resonant frequency of cavities would be a natural extension of waveguides to three-dimensions along with the use of a three-dimensional mesh generator. The entire demonstration of the Direct method ignores loss. Hence the inclusion of loss in the derived equations will also be a natural extension and will help in making the Direct method more mature, particularly for lossy side-walls in waveguides. The development of sparse eigenvalue and eigenvector computational techniques will help in reducing the solution time and storage requirements for the Direct method.

The application of different combinations of basis functions to the Helmholtz equation method itself could prove to be a useful exercise. However, the number of combinations are limited as the number of equations are one half of those present in the Direct methods.
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Appendix A

Traditional Expansion for Bases

The PWL or piecewise-linear basis functions can also be expanded for every node in each element defined by the three points \((x_i, y_i), (x_j, y_j),\) and \((x_k, y_k)\) in a more traditional way as shown below.

\[
\Phi_n = L_n = N_n = \frac{1}{2S} (a_n + xb_n + yc_n)
\]

where

- \(a_n = x_{n+1}y_{n+2} - x_{n+2}y_{n+1}\)
- \(b_n = y_{n+1} - y_{n+2}\)
- \(c_n = x_{n+2} - c_{n+1}\)

The CT or constant-tangential basis functions can also be expanded for every edge in each element in a more traditional way using the first-order PWL basis functions as shown below.

\[
\bar{T}_n = L_{n+1} \nabla L_{n+2} - L_{n+2} \nabla L_{n+1}
\]

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Appendix B

Color Maps

Figure B.1: Color maps of field plots in color and gray scale. If magnitudes are plotted, the range is from zero to the maximum value otherwise the color scale ranges from -maximum value to +maximum value. Gray-scale color maps are also shown for convenience if plots are printed in gray-scale. All field plots in this document use the same color maps shown here.
Appendix C

Exact Modes for WR-90 and Circular Waveguides

Table C.1: Exact TE modes for the WR-90 waveguide with $k = 167.6676$ rad/m.

<table>
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<tr>
<th>n</th>
<th>m</th>
<th>$TE_{nm}$ γ rad/m</th>
</tr>
</thead>
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<td>217.7908</td>
</tr>
<tr>
<td>0</td>
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<td>523.5157</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
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<tr>
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Table C.2: Exact TE and TM modes for circular waveguide of radius 1 m with $k = 3$ rad/m.

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<th>p</th>
<th>$\text{TE}_{np} \gamma$ rad/ m</th>
<th>$\text{TM}_{np} \gamma$ rad/ m</th>
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Kartik Sitapati


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