Multivariable Interpolation Problems

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Multivariable interpolation problems

Quanlei Fang

( Abstract )

In this dissertation, we solve multivariable Nevanlinna-Pick type interpolation problems. Particularly, we consider the left tangential interpolation problems on the commutative or noncommutative unit ball. For the commutative setting, we discuss left-tangential operator-argument interpolation problems for Schur-class multipliers on the Drury-Arveson space and for the noncommutative setting, we discuss interpolation problems for Schur-class multipliers on Fock space. We apply the Krein-space geometry approach (also known as the Grassmannian Approach). To implement this approach $J$-versions of Beurling-Lax representers for shift-invariant subspaces are required. Here we obtain these $J$-Beurling-Lax theorems by the state-space method for both settings. We see that the Krein-space geometry method is particularly simple in solving the interpolation problems when the Beurling-Lax representer is bounded. The Potapov approach applies equally well whether the representer is bounded or not.
Dedication

This dissertation is dedicated to my loving family.
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Chapter 1

Introduction

This dissertation is about multivariable Nevanlinna-Pick interpolation problems on the unit ball. To illustrate the motivation of this study, we will recall the classical interpolation theory first. Following a presentation of historical facts, we give a brief overview of related recent research including the main goal of this dissertation. A roadmap of the dissertation is given at the end of this chapter.

1.1 Classical Nevanlinna-Pick interpolation problem

The story begins with a problem that Georg Pick posed in 1916:

Problem 1.1.1 Given $n$ distinct points $z_1, \ldots, z_n$ in the unit disk $D$ and $n$ complex numbers $w_1, \ldots, w_n$, does there exist holomorphic $f : D \to \overline{D}$ (a classical Schur-class function) that interpolates the sequences $\{z_i\}$ and $\{w_i\}$ ($f(z_i) = w_i$, $1 \leq i \leq n$)?

Here $\overline{D}$ is the closed unit disk.

Note that if we consider $H^\infty(D)$ (the space of bounded holomorphic functions on the unit disk $D$) with the sup norm, by a standard compactness argument and rescaling analysis, it is not hard to see that Problem 1.1.1 is equivalent to the problem of finding the infimum of the $H^\infty(D)$-norm among all bounded functions $f$ that interpolate the data set.
G. Pick discovered a necessary and sufficient condition to solve this problem, and proved that in the extremal case, the solution is unique and given by a Blaschke product. The necessary and sufficient condition for the existence of a solution is the positive-semidefiniteness of the so-called Pick matrix $P$ as shown below. The necessity of the condition was obtained by using a variant of the Schwarz-Pick Lemma. The sufficiency of this condition is more elegant and can now be formulated as the assertion that the Szegő kernel is a complete Pick kernel; we do not discuss complete Pick kernels here but refer to [2, 4, 43, 77, 98] for more on this topic.

**Theorem 1.1.2** Problem 1.1.1 has a solution if and only if the Pick matrix

$$P := \left[ \frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j} \right]_{i,j=1}^n$$

is positive semidefinite. Moreover, the solution is unique if and only if in addition the matrix is singular.

Rolf Nevanlinna considered and solved this problem independently in 1919 since he was unaware of Pick’s result due to the first world war. (Actually, Pick gave necessary and sufficient condition for solvability when the data set is finite and R. Nevanlinna considered the problem for a countable data set. R. Nevanlinna also gave a parametrization of all solutions in the nonunique case in 1929. Schur considered almost the same problem, but with all the interpolation points coinciding at the origin). So the problem has been called Nevanlinna-Pick interpolation problem. Since then, this type of interpolation problem, despite having been seemingly completely solved, has drawn much attention. The reason is that the problem not only has deep meaning in mathematics but also comes up in engineering, say control theory, model matching, cascade synthesis, network and semiconductor modeling etc. ([79, 91, 127]). Several different approaches to the problem are known and a number of new types of interpolation problems have been considered. Obviously one can consider the interpolation problems for various function classes. For example, a natural matrix extension of the Nevanlinna-Pick problem consists in finding a matrix-valued function, holomorphic with values having norm at most 1 the unit disk (the Schur class of matrix valued functions), interpolating given values at certain points. Not only matricial, but also operator versions of classical interpolation
problems are crucial because they too are connected with applications in the field of electrical circuits (see [72]) and problems in system theory (see [14,15]). Interest in the operator-valued version of the Nevanlinna-Pick problem has grown since the important discovery of Sarason (see [120]) that many of the results on interpolation by holomorphic functions can be obtained by operator-theoretic approaches. Sz.-Nagy and Foias generalized the results of Sarason to the celebrated commutant lifting theorem which can treat not only matrix-valued versions of the Nevanlinna-Pick problem, but also various other problems in engineering (see [73]). After more connections with operator theory and linear system theory were discovered around mid 1970s and early 1980s, the research on the variants of this type of interpolation problem became very intensive in recent years. So we can say that the Nevanlinna-Pick type of interpolation problem is highly multifaceted and has been inspiring both pure mathematics and applied mathematics for nearly a century (see [4,35,96]).

1.2 Variants of classical Nevanlinna-Pick interpolation problem

In this section we give a closer look at the variants of the classical interpolation problem. As we mentioned above, the classical Schur class is the set of holomorphic functions defined on the unit disk with modulus no more than 1. In the last few decades there have emerged a variety of formulations for matrix- or operator-valued Schur class functions. Particularly we focus on the operator-valued case here, as the matrix-valued situation is similar.

Let $\mathcal{U}$ and $\mathcal{Y}$ be two Hilbert spaces and let $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ be the space of all bounded linear operators between $\mathcal{U}$ and $\mathcal{Y}$. We also let $H^2_{\mathcal{U}}$ be the standard Hardy space of the $\mathcal{U}$-valued holomorphic functions on the unit disk $\mathbb{D}$. The operator-valued version of the classical Schur-class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ is defined to be the set of all holomorphic, contractive $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions on $\mathbb{D}$ with values of norm at most 1 (i.e, $S: \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ with $\|S(z)\| \leq 1$ for $z \in \mathbb{D}$). It is natural to consider the related interpolation problems for this type of Schur class functions. We list some of the most popular ones in the following:
1. (Left-tangential Nevanlinna-Pick Interpolation Problem:
Given two $N$-tuples of operators $x_1, x_2, \ldots, x_N$ in $\mathcal{L}(\mathcal{Y}, \mathcal{H}_L)$, $y_1, y_2, \ldots, y_N$ in $\mathcal{L}(\mathcal{U}, \mathcal{H}_L)$, where $\mathcal{U}, \mathcal{Y}, \mathcal{H}_L$ are Hilbert spaces, and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{D}$, find $S \in S(\mathcal{U}, \mathcal{Y})$ such that
\[ x_i S(\lambda_i) = y_i, \quad i = 1, 2, \ldots, N \] (1.2.1)

2. Left-tangential Carathéodory-Fejér Interpolation Problem:
Given $N$-tuples of polynomials $x_1(\lambda), x_2(\lambda), \ldots, x_N(\lambda)$ with coefficients in $\mathcal{L}(\mathcal{Y}, \mathcal{H}_L)$, $y_1(\lambda), y_2(\lambda), \ldots, y_N(\lambda)$ with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{H}_L)$ where $\mathcal{U}, \mathcal{Y}, \mathcal{H}_L$ are Hilbert spaces, points $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{D}$ and nonnegative integers $n_1, n_2, \ldots, n_N$, find $S \in S(\mathcal{U}, \mathcal{Y})$ such that
\[ x_i(\lambda) S(\lambda) = y_i(\lambda) + o((\lambda - \lambda_i)^{n_i}), \quad i = 1, 2, \ldots, N \] (1.2.2)

Note that the particular case with $N = 1$ and $\lambda_1 = 0$ is called (Left) Tangential
Carathéodory-Schur Interpolation:

Given a polynomial $x(\lambda)$ with coefficients in $\mathcal{L}(\mathcal{Y}, \mathcal{H}_L)$, a polynomial $y(\lambda)$ with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{H}_L)$ and a nonnegative integer $n$, find $S \in S(\mathcal{U}, \mathcal{Y})$ such that
\[ x(\lambda) S(\lambda) = y(\lambda) + o((\lambda)^n). \] (1.2.3)

3. Leech Interpolation Problem:
Given $x \in H^\infty_{\mathcal{L}(\mathcal{Y}, \mathcal{H}_L)}(\mathbb{D})$ and $y \in H^\infty_{\mathcal{L}(\mathcal{U}, \mathcal{H}_L)}(\mathbb{D})$, find $S \in S(\mathcal{U}, \mathcal{Y})$ such that
\[ x(\lambda) S(\lambda) = y(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{D}. \] (1.2.4)

Note that this problem can be considered as infinity order Carathéodory-Fejér interpolation in (2) with set of interpolation nodes \{\lambda_1, \lambda_2, \ldots, \lambda_N\} taken to be all of $\mathbb{D}$.

4. Left-tangential Operator-argument Interpolation Problem:
Given $(Z, X, Y)$ with $Z \in \mathcal{L}(\mathcal{H}_L)$, $X \in \mathcal{L}(\mathcal{Y}, \mathcal{H}_L)$, $Y \in \mathcal{L}(\mathcal{U}, \mathcal{H}_L)$ such that
$(Z, X)$ is a stable input pair, i.e, $\sum_{n=0}^\infty Z^n X^* Z^{\ast n}$ bounded above in $\mathcal{L}(\mathcal{H}_L)$
as $n \rightarrow \infty$, find $S \in S(\mathcal{U}, \mathcal{Y})$ such that
\[ (XS)^{\Lambda L}(Z) := \sum_{n=0}^\infty Z^n X S_n = Y. \quad \text{where} \quad S(\lambda) = \sum_{n=0}^\infty S_n \lambda^n. \] (1.2.5)
It can be checked that this problem contains all the previous examples as special cases.

5. Sarason Interpolation Problem:
Given \( F \in H_\infty^{\infty}(\mathbb{D}) \) and an inner function \( \psi \in H_\infty^{\infty}(\mathbb{U}'(\mathbb{U}), \mathbb{Y}) \) of the form \( S = F + \psi G \) for some \( G \in H_\infty^{\infty}(\mathbb{U}(\mathbb{X})) \).

Actually (4) and (5) are dual formulation of each other; which form of the given data is more convenient is determined by the given application. For more details, we refer to [41] and the references therein.

For completeness, we also mention here two other interpolation problems for special Schur-class functions which generalize the classical Nevanlinna-Pick interpolation problem. One is the boundary interpolation problem: Given a set \( \{t_i, w_i, \gamma_i\}_{i=1}^n \), which consists of unimodular \( t_i, w_i \in \mathbb{T} = \partial \mathbb{D} \) and real \( \gamma_i \in \mathbb{R}, i = 1, \ldots, n \), find all functions \( w \in S_\kappa \) such that

\[
 w(t_i) = w_i, \quad \lim_{z \to z_i} \frac{1 - |w(z)|^2}{1 - |z|^2} = \gamma_i \quad (i = 1, \ldots, n).
\]

where \( S_\kappa \) (\( \kappa \in \mathbb{N} \)) is the generalized Schur class which was introduced by Krein and Langer (see [89]). By definition \( S_\kappa \) (\( \kappa \in \mathbb{N} \)) is the set of functions \( w \) meromorphic in the unit disk \( \mathbb{D} \) and represented as a ratio \( w = S/B \), where \( S \) belongs to the Schur class \( S \) and \( B \) is a Blaschke product of degree \( \kappa \), which has no common zeroes with \( S \). Another one is Abstract Interpolation Problem (AIP) introduced by Katsnelson, Kheifets and Yuditskili (see [84]) in late 1980s. This encodes the most general bitangential Nevanlinna-Pick interpolation problem and also contains commutant lifting as a special case. There is also an elegant coordinate-free formulation of the AIP given by Ball and Trent in [41], replacing the de Branges-Rovnyak model space with an abstract scattering system (ASS): A tuple \( (U, K, G_1, G_2) \) is called an ASS if, among other things, \( G_1 \) and \( G_2 \) are subspaces of \( K \) orthogonal to one another, \( U \) is a unitary operator on \( K \), \( G_1 \) is invariant for \( U \) and \( G_2 \) is invariant for \( U^* \) (i.e., \( \cap_{n=0}^\infty U^n G_1 = \{0\}, \cap_{n=0}^\infty U^{*n} G_2 = \{0\} \)). A data set \( (T_1, T_2, D(\cdot), M_1, M_2, X, E_1, E_2) \) is called an admissible AIP data set provided \( X \) is a linear space, \( T_j \) are operators on \( X \), \( D(\cdot) \) is a positive-semidefinite Hermitian form on \( X \times X \), \( M_j \) are operators
from $X$ to $E_j$ satisfying $D(T_1 x, T_1 x) + \|M_1 x\|^2 = D(T_2 x, T_2 x) + \|M_2 x\|^2$. Let $X_D$ denote the Hilbert space determined by the form $D$. The coordinate-free AIP is the following: given an AIP admissible data set, find a minimal ASS such that $E_1 = G_1 \ominus U G_1$, $E_2 = U G_2 \ominus G_2$ and a contractive linear operator $F: X_D \to H := K \ominus \{G_1 + G_2\}$ such that $FT_1 + M_1 = U FT_2 + M_2$.

1.3 Different approaches for interpolation problems

Since the interpolation problem is multifaceted, different people looking at it think and solve it in different ways. It is not easy to write this section since there are now so many different approaches treating the interpolation problem. We sketch the important ones from our point of view.

1. **Schur Algorithm**: Given a Schur-class function $S$ on the open unit disk $\mathbb{D}$, for any $a \in \mathbb{D}$, the function $\frac{S(z) - S(a)}{1 - \overline{z}a} (1 - S(z)S(a)^*)$ is still a Schur function. This is the crucial step in the algorithm (called Schur algorithm) which was applied by Nevanlinna to solve interpolation problems. It implies that the problem can be solved iteratively by the algorithm. The Schur algorithm has extensions and applications to various settings. The book [6] serves as a very good reference for this method.

2. **Reproducing kernel method**: The reproducing kernel method was introduced by Dym. This method can be applied to solve interpolation problems by means of the de Branges theory of Hilbert spaces of holomorphic functions. We refer to [71] for more on Dym’s work on this.

3. **State space approach**: Ball, Gohberg and Rodman (see [35]) have developed a state space approach to solve the interpolation problem. This approach is based on the connection of operator model theory, systems theory, factorization and realization. Beginning with the Kalman theory of minimal state space realizations of rational, matrix-valued transfer functions, there have been
several different and significant problems in factorization and realization theory. The links between these problems, interpolation theory and invariant subspaces are indicated explicitly.

4. **Potapov’s approach:** A powerful approach to matricial interpolation problems was developed by Potapov. Instead of the original interpolation problem, Potapov’s approach to interpolation problems research considered an inequality (the Fundamental Matrix Inequality or FMI for the corresponding interpolation problem) for analytic functions in an appropriate domain. We need to figure out how to solve this inequality and how to see that this inequality is equivalent to the original interpolation problem. Potapov’s algorithm for the solution of matricial interpolation problems was generalized to operator theoretical interpolation problems by Ivanchenko and Sakhnovich (see [83]). The main tool in their approach is based on dilation theory. The study of the second problem consists of two parts. First, we have to prove that any function which is a solution of the original problem is also a solution of the FMI. Usually this part is not difficult. Secondly, we have to extract the full interpolation information from the FMI. This means that we have to prove that any analytic function which satisfies the FMI is also a solution of the original interpolation problem. This approach has been applied to solve Caratheodory-Fejer interpolation problems, boundary interpolation problems and the abstract interpolation problem.

5. **Lurking isometry:** The so-called “lurking isometry” method is based on the relationships between interpolation problems and operator extensions (studied by Koranyi, Sz-Nagy, Kreïn, see [105]). One constructs an isometric operator based on interpolation data and considers the one-to-one correspondence between certain unitary extensions of that operator and the solutions of the interpolation problem. This method has been used in a number of closely related realization and interpolation problems.

6. **Commutant lifting approach:** As we mentioned in the first section, the fundamental paper by Sarason and the later work by Sz-Nazy and Foias moti-
vated the operator-theoretic research of the interpolation problem (see [104]).

7. **Grassmannian Approach:** A further operator theoretical approach for interpolation problems was created by Ball and Helton (see [21, 36, 37]). They had the original idea to embed interpolation problems into a context of Krein spaces. Hereby the associated operator extension problems were transformed into extension problems of subspaces of the Krein space. The method of Ball and Helton can also be used to treat boundary value interpolation problems and indefinite interpolation problems. An essential part of their approach is a useful indefinite generalization of the theorem of Beurling/Lax/Halmos/Masani about shift-invariant subspaces. Furthermore Ball and Helton got the most far-reaching results concerning the treatment of one- and two-sided tangential interpolation problems which were considered earlier by Fedcina in connection with the papers of Adamjan, Arov and Krein. As we mentioned before, such tangential interpolation problems are characterized by the fact that not the interpolation data themselves but their projection into certain given directions are prescribed.

### 1.4 Multivariable Nevanlinna Pick interpolation problems

So far we have stayed in the unit disk. But now if we move from the single-variable case to the multivariable case, i.e., move from the unit disk to a multivariable domain, say, the unit ball or polydisk, what happens? It is well known that there are various phenomena for the multivariable case not present in the single-variable case. Hence it is interesting to ask which kind of interpolation problems can be considered and whether we can solve these multivariable problems following the classical approaches. The purpose of this dissertation is to try to answer some of these questions. Particularly, we shall solve the following two types of problems: interpolation for the Drury-Arveson space on the commutative unit ball and interpolation for the Fock space on the noncommutative ball.
1.4.1 Interpolation for the Drury-Arveson space on the unit ball

A multivariable generalization of the Szegő kernel \( k(\lambda, \zeta) = (1 - \lambda \overline{\zeta})^{-1} \) much studied of late is the positive kernel \( k_d(\lambda, \zeta) = \frac{1}{1 - \langle \lambda, \zeta \rangle} \) over the unit ball of \( \mathbb{C}^d \). The reproducing kernel Hilbert space \( \mathcal{H}(k_d) \) associated with \( k_d \) (via Aronszajn’s construction [13]) is called the Drury-Arveson space and acts as a natural multivariable analogue of the Hardy space \( H^2 \) of the unit disk: in case \( d = 1 \), the spaces \( \mathcal{H}(k_d) \) and \( H^2 \) coincide. The generalization of the classical Schur class, which is the class of contractive operator-valued multipliers \( S(\lambda) \) for \( \mathcal{H}(k_d) \), has also been much studied. The reproducing kernel space \( \mathcal{H}(K_S) \) associated with the positive kernel \( K_S(\lambda, \zeta) = (I - S(\lambda)S(\zeta)^*) \cdot k_d(\lambda, \zeta) \) is a multivariable generalization of the classical de Branges-Rovnyak canonical model space. A special feature appearing in the multivariable case is that the space \( \mathcal{H}(K_S) \) in general may not be invariant under the adjoints \( M_{\lambda_j}^* \) of the multiplication operators \( M_{\lambda_j} : f(\lambda) \mapsto \lambda_j f(\lambda) \) on \( \mathcal{H}(k_d) \). In the early work with Ball and Bolotnikov ([27, 29, 30]), we showed that invariance of \( \mathcal{H}(K_S) \) under \( M_{\lambda_j}^* \) for each \( j = 1, \ldots, d \) is equivalent to \( S(\lambda) \) being of the form \( S(\lambda) = D + C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}(\lambda_1 B_1 + \cdots + \lambda_d B_d) \) as the transfer function of a Fornasini-Marchesini-type linear system such that connecting operator \( U = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \end{bmatrix} \) is “weakly coisometric”, i.e., the adjoint \( U^* \) is isometric on a certain natural subspace and the state operators \( A_1, \ldots, A_d \) pairwise commute. In this case one can take the state space to be the functional-model space \( \mathcal{H}(K_S) \) and the state operators \( A_1, \ldots, A_d \) to be given by \( A_j = M_{\lambda_j}^* |_{\mathcal{H}(K_S)} \) (a de Branges-Rovnyak functional-model realization). We also showed that this special situation always occurs for the case of inner functions \( S \) (where the associated multiplication operator \( M_S \) is a partial isometry), and that inner multipliers are characterized by the existence of a realization such that the state operators \( A_1, \ldots, A_d \) satisfy an additional stability property.

We solve the Left-tangential Operator-argument Interpolation Problem for Schur-class multipliers on the Drury-Arveson space. This problem contains the classical Nevanlinna-Pick interpolation problem and the Carathéodory-Fejér problem as spe-
cial cases and has been treated in various forms by a number of authors (see [26]). We show how to obtain necessary and sufficient conditions for the existence of a solution following a simple way by using a factorization lemma due to Douglas with a commutant lifting theorem for Drury-Arveson space setting discovered in [43]. The more difficult problem is to parametrize the set of all solutions. This has been discussed in [25]. Our contribution here is to show how the Kreǐn-space geometry from [36] can be adapted to this multivariable setting to obtain such parametrization results: we translate the interpolation problem to finding subspaces of an associated Kreǐn space which satisfy some specific conditions: containment in a prescribed interpolation subspace (corresponding to the interpolation conditions), being maximal negative in the ambient Hilbert space, and invariance under shifts $M_{\lambda_j}$. A $J$-Beurling-Lax representer for the interpolation subspace then leads to a parametrization for the set of contractive solutions of the interpolation conditions. We arrive at the $J$-Beurling-Lax theorem by the state-space technique and discuss the more delicate case where the Beurling-Lax representer may be unbounded.

### 1.4.2 Interpolation problems on the noncommutative ball

Recently there has been much interest in and an evolving theory of noncommutative function theory and associated multivariable operator theory and multidimensional Fornasini-Marchesini-type system theory with evolution along a free semigroup; we mention [10, 25, 33, 34, 61, 82, 103, 116]. The noncommutative Schur multiplier class consisting of formal power series in a set of noncommuting indeterminates which define contractive multipliers between (unsymmetrized) vector-valued Fock spaces plays a central role. Here we introduce and study a Fock-space noncommutative analogue of reproducing kernel Hilbert spaces of de Branges-Rovnyak type. Unlike the methods in current literature, we deal with formal power series with operator coefficients as parts of some formal structure (e.g., as inducing multiplication operators between two Hilbert spaces whose elements are formal power series with vector coefficients) rather than as themselves functions on some collection of noncommutative operator-tuples. We use the de Branges-Rovnyak space $H(K_S)$ as the state space for the unique (up to unitary equivalence) observable, coisometric transfer-function
realization of the Schur-class multiplier \( S \), realization-theoretic characterization of inner Schur-class multipliers, and a calculus for obtaining a realization for an inner multiplier with prescribed left zero-structure. In contrast with the parallel theory for the Drury-Arveson space on the unit ball (which can be viewed as the symmetrized version of the Fock space used here), we obtain results which are canonical generalizations of the classical univariate case. The noncommutative left-tangential operator-argument interpolation problem (including the parametrization of the set of all solutions) on the noncommutative ball is treated here via the Kreın-space geometry approach. Other work (see [25, 109–111, 113–115, 117]) has treated the same problem via other approaches (lurking isometry and commutant lifting),

1.5 Dissertation Outline

This dissertation is organized as follows. After the present Introduction, in Chapter 2 we review known material on reproducing kernel spaces and Kreın spaces which will be needed throughout the dissertation. In Chapter 3 we discuss Nevanlinna-Pick-type interpolation problems for Schur-class multipliers on the Drury-Arveson space; the new feature here is the adaptation of the Ball-Helton Grassmannian approach to the multivariable setting. Chapter 4 obtains parallel results for the noncommutative ball; the new feature here again is the approach via Kreın-space geometry. Chapter 5 gives a conclusion and discusses directions for future research.
Chapter 2

Preliminaries and Notation

In this chapter, we give some of the standard definitions and basic facts that will be used throughout this dissertation. In Section 2.1, definitions and properties of reproducing kernel Hilbert spaces is presented. In Section 2.2, we give the definition of operator-valued Schur-class functions and discuss the characterization of this class. In Section 2.3, we provide some fundamental facts of Kreĭn spaces since Kreĭn space geometry is crucial in the approach we apply to solve interpolation problems later on. We generally follow the conventional notation that points and variables are lower-case, whilst matrices, operators and spaces are upper-case.

2.1 Reproducing kernel Hilbert spaces

Roughly speaking, a reproducing kernel Hilbert space is a Hilbert space of functions in which pointwise evaluation is a continuous linear functional. In other words, it is a space that can be defined by a reproducing kernel. Searching “reproducing kernel” at Encyclopedia online, one can see that the idea of reproducing kernel was used by S. Zaremba early last century for his work on boundary value problems for harmonic and biharmonic functions. He did not develop any theory or give any particular name to the kernels he introduced though. In 1909, J. Mercer introduced the concept of “positive definite kernels” and showed that these positive definite kernels have nice properties among all continuous kernels of integral equations. After some years of silence on this topic, G. Szegö, S. Bergman (1922) and S. Bochner
(1922) investigated the idea of reproducing kernels in their dissertations. In 1935, E.H. Moore examined the positive definite kernels in his general analysis under the name of positive Hermitian matrix. The subject was developed and systematized by N. Aronszajn and S. Bergman in the 1950s. An important subset of the reproducing kernel Hilbert spaces are the reproducing kernel Hilbert spaces associated to a continuous kernel. A nice application of these spaces is machine learning theory (see [62]). Now let us recall the basic concepts and some fundamental facts about reproducing kernel Hilbert space.

A Hilbert space $\mathcal{H}$ of $E$-valued functions (where $E$ is a coefficient Hilbert space) defined on an abstract set $\Omega$ is called a reproducing kernel Hilbert space if there exists a function $K$, the reproducing kernel, defined on $\Omega \times \Omega$ with values in $\mathcal{L}(E)$ such that

1. $K(\cdot, y)e \in \mathcal{H}$ for any $y \in \Omega$ and for all $e \in E$;

2. $\langle f(\cdot), K(\cdot, y)e \rangle = \langle f(y), e \rangle_E$ for all $f \in \mathcal{H}$ and $e \in E$ (the reproducing property).

The existence of a kernel is guaranteed by the Riesz theorem, and the uniqueness of the kernel can be shown by a simple computation. The construction of the Hilbert space given the kernel is related to the GNS construction, and it is also known as Kolmogorov theorem (see [60]).

An important example of reproducing kernel Hilbert space is the de Branges-Rovnyak space. It is defined as the Hilbert function space $\mathcal{H}(s)$ with the kernel $k(\lambda, \zeta) = \frac{1 - s(\lambda)\overline{s(\zeta)}}{1 - \lambda\zeta}$, where $s$ is a function in the unit ball of $H^\infty(\mathbb{D})$.

Another example is the famous Hardy space $H^2$. By the relation between an orthonormal basis for a Hilbert space of functions $\mathcal{H}$ and its kernel $K(x, y)$ (i.e, $K(x, y) = \sum_{i \in I} \overline{e_i(x)} e_i(y)$ where $\{e_i\}_{i \in I}$ form the orthonormal basis for $\mathcal{H}$), using $e_i(\lambda) = \lambda^i, i = 0, 1, 2, \cdots$ as orthonormal basis for the Hardy space $H^2$, we see that the kernel for $H^2$ is the Szegö kernel $k(\lambda, \zeta) = \frac{1}{1 - \lambda\zeta}$ and it is easy to see by geometric-series expansion that $k(\lambda, \zeta)$ is positive definite. Similarly one can compute the kernels for the Bergman space and the Dirichlet space, but we have no need for these here. More details can be found in the books [4,88].
The following well-known proposition gives several equivalent definitions for the term “positive kernel” which turns out to be very useful. Here we focus on the operator-valued version.

**Proposition 2.1.1** Let \( K : \Omega \times \Omega \to \mathcal{L}(\mathcal{Y}) \) be a given function where \( \mathcal{Y} \) is a Hilbert space. Then the following conditions are equivalent:

1. For any finite collection of points \( \omega_1, \ldots, \omega_N \in \Omega \) and of vectors \( y_1, \ldots, y_N \in \mathcal{Y} \) \((N = 1, 2, \ldots)\) it holds that
   \[
   \sum_{i,j=1,\ldots,N} \langle K(\omega_i, \omega_j) y_j, y_i \rangle_\mathcal{Y} \geq 0.
   \]  
   (2.1.1)

2. (Kolmogorov decomposition) There exists an operator-valued function \( H : \Omega \to \mathcal{L}(\mathcal{H}, \mathcal{Y}) \) for some auxiliary Hilbert space \( \mathcal{H} \) so that
   \[
   K(\omega', \omega) = H(\omega') H(\omega)^*.
   \]  
   (2.1.2)

3. (Reproducing-kernel property) There exists a Hilbert space \( \mathcal{H}(K) \) of \( \mathcal{Y} \)-valued functions \( f \) so that the function \( K(\cdot, \omega)y \) is in \( \mathcal{H}(K) \) for each \( \omega \in \Omega \) and \( y \in \mathcal{Y} \) and has the reproducing property
   \[
   \langle f, K(\cdot, \omega)y \rangle_{\mathcal{H}(K)} = \langle f(\omega), y \rangle_\mathcal{Y}.
   \]

When any (and hence all) of these equivalent conditions hold, we say that \( K \) is a **positive kernel on** \( \Omega \times \Omega \).

### 2.2 Schur-class functions

To formulate the definition of the operator-valued Schur-class, we let \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) denote the space of bounded linear operators acting between Hilbert spaces \( \mathcal{U} \) and \( \mathcal{Y} \). We also let \( H^2_\mathcal{U}(\mathbb{D}) \) and \( H^2_\mathcal{Y}(\mathbb{D}) \) be the standard Hardy spaces of \( \mathcal{U} \)-valued (respectively \( \mathcal{Y} \)-valued) holomorphic functions on the unit disk \( \mathbb{D} \). By the Schur-class \( \mathcal{S}(\mathcal{U}, \mathcal{Y}) \) we mean the set of \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued functions holomorphic on the unit disk \( \mathbb{D} \) with values \( \mathcal{S}(z) \) having norm at most 1 for each \( z \in \mathbb{D} \). The class \( \mathcal{S}(\mathcal{U}, \mathcal{Y}) \) admits several remarkable characterizations. The following result is well known and is formulated
as the prototype for the multivariable generalizations to follow (see more details in [23]).

**Theorem 2.2.1** Let $S$ be an $\mathcal{L}(U, Y)$-valued function defined on $\mathbb{D}$. The following are equivalent:

1. $S \in \mathcal{S}(U, Y)$, i.e., $S$ is holomorphic on $\mathbb{D}$ with contractive values in $\mathcal{L}(U, Y)$.

   (1') The multiplication operator $M_S : f(z) \mapsto S(z) \cdot f(z)$ defines a contraction from $H^2_\mathbb{D}(\mathbb{D})$ into $H^2_\mathbb{Y}(\mathbb{D})$.

   (1'') $S$ is holomorphic and satisfies the von Neumann’s inequality: if $S \in \mathcal{L}(U, Y)$ and $T$ is any strictly contractive operator on a Hilbert space $\mathcal{K}$, i.e., $\|T\| < 1$, then $S(T)$ is a contraction operator ($\|S(T)\| \leq 1$), where $S(T)$ is the operator defined by

   $$S(T) = \sum_{n=0}^{\infty} S_n \otimes T^n \in \mathcal{L}(U \otimes \mathcal{K}, Y \otimes \mathcal{K}) \quad \text{if} \quad S(z) = \sum_{n=0}^{\infty} S_n z^n.$$

2. The function $K_S : \mathbb{D} \times \mathbb{D} \to \mathcal{L}(Y)$ given by

   $$K_S(z, w) = \frac{I_Y - S(z)S(w)^*}{1 - z\bar{w}}$$

   is a positive kernel on $\mathbb{D} \times \mathbb{D}$.

3. There exists a Hilbert space $\mathcal{H}$ and a coisometric (or even unitary or contractive) connecting operator (or colligation) $U$ of the form

   $$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ U \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ Y \end{bmatrix}$$

   so that $S(z)$ can be realized in the form

   $$S(z) = D + zC(I_H - zA)^{-1}B.$$  \hspace{1cm} (2.2.1)

From the point of view of systems theory, the function (2.2.1) is the *transfer function* of the linear system

$$\Sigma = \Sigma(U) : \begin{cases} 
   x(n + 1) = A x(n) + B u(n) \\
   y(n) = C x(n) + D u(n)
\end{cases}.$$
We provide a sketch of the proof of Theorem 2.2.1 as a model for how extensions to more general settings may proceed in later chapters.

**Proof:** [Sketch of the proof of Theorem 2.2.1] The easy part is (3) $\implies$ (2) $\implies$ (1'') $\implies$ (1') $\implies$ (1):

(3) $\implies$ (2): Assume that $S(z)$ is as in (2.2.1) with $U$ unitary, and hence, in particular, coisometric. From the relations arising from the coisometric property of $U$:

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
A^* & C^* \\
B^* & D^*
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
\]

one can verify that

\[
I - S(z)S(w)^* = I - [D + zC(I - zA)^{-1}B][D + wC(I - wA)^{-1}B]^*
\]

\[
= C(I - zA)^{-1}[(1 - z\overline{w})I_H](I - \overline{w}A^*)^{-1}C^*.
\]

This implies that $H(z) = C(I - zA)^{-1}$ satisfies (2.1.2).

(2) $\implies$ (1''): Due to $I - S(z)S(w)^* = H(z)[(1 - z\overline{w})I_H]H(w)^*$, we can see that for any $\|T\| < 1$

\[
I - S(T)S(T)^* = H(T)[(1 - TT^*) \otimes I_H]H(T)^* \geq 0.
\]

(1'') $\implies$ (1'): Observe that $M_S = S(S) = s - \lim_{r \uparrow 1} S(rS)$ where $S$ is the shift operator $M_z$ on $H^2(\mathbb{D})$. Thus the fact that $\|S(rS)\| \leq 1$ for any $r < 1$ implies $\|M_S\| \leq 1$.

(1') $\implies$ (1): Note that since $S(z)u = M_S \cdot u$ for any $u \in \mathcal{U}$, we have $\|M_S\|_{op} = \|S\|_{\infty}$. So $\|M_S\| \leq 1$ implies that $S \in S(\mathcal{U}, \mathcal{V})$.

The harder part is (1) $\implies$ (1') $\implies$ (1'') $\implies$ (2) $\implies$ (3):

(1) $\implies$ (1'): We can view $H^2(\mathbb{D}) \subset L^2(\mathbb{T})$. Thus $\|M_S u\|_{L^2(\mathbb{T})} \leq \|S\|_{\infty} \cdot \|u\|_{L^2(\mathbb{T})}$.

(1') $\implies$ (1''): According to the Sz.-Nagy dilation theorem, any contraction operator $T$ has a unitary dilation $U$. In the strictly contractive case $\|T\| < 1$, one can show that in fact the unitary dilation is the bilateral shift with some multiplicity $N: U = S \otimes I_N$ (if $N = \infty$, we interpret $I_N$ as the identity operator on $l^2$). We then have $T^n = P_K(S \otimes I_N)^n|_K$. Therefore $\|S(T)\| = \|P_{\mathcal{Y} \otimes K}S(S \otimes I_N)|_{U \otimes K}\| \leq \|M_S\| \leq 1$.

(1'') $\implies$ (2): A direct proof of this implication can be done via a rather long, intricate argument using a Gelfand-Naimark-Segal construction in conjunction with
a Hahn-Banach separation argument—we refer to this as a GNS/HB argument. This argument was used by Agler (see [1]) in the polydisk setting.

Alternatively, one can avoid the GNS/HB argument via the following shortcut:

\((1'') \implies (1') \implies (2)\): We have seen that \((1'') \implies (1')\) is easy. For \((1') \implies (2)\), we assume \(\|MS\| \leq 1\). View \(H^2(\mathbb{D})\) as the reproducing kernel Hilbert space \(\mathcal{H}(k_{Sz})\), where \(k_{Sz}(z, w) = \frac{1}{1-z\overline{w}}\) is the Szegö kernel. Since \(M^*_S k_{Sz}(\cdot, w)y = k_{Sz}(\cdot, w)S(w)^*y\), we see that

\[
\sum_{i,j=1, ..., N} \langle K_S(z_i, z_j)y_j, y_i \rangle_Y = \|\sum_j k_{Sz}(\cdot, z_j)y_j\|^2 - \|(MS)^* \sum_j k_{Sz}(\cdot, z_j)y_j\|^2 \geq 0
\]

and it follows (via criterion (2.1.1)) that \(K_S\) is a positive kernel on \(\mathbb{D} \times \mathbb{D}\).

\((2) \implies (3)\): This implication can be done by the now standard lurking isometry argument—see [22] where this coinage was introduced.

Our goal is to study recent extensions of the Schur class and the associated analogues of Theorem 2.2.1 to more general multivariable settings and use this type of results to solve the Nevanlinna-Pick type interpolation problems.

### 2.3 Facts about Kreın spaces

A Kreın space \(\mathcal{K}\) is an inner-product space which is isomorphic to the direct sum \(\mathcal{K}_+ \oplus \mathcal{K}_-\) of two Hilbert spaces \(\mathcal{K}_+\) and \(\mathcal{K}_-\) with an indefinite inner product given by

\[
[k_+ \oplus k_-, k'_+ \oplus k'_-]_{\mathcal{K}} = \langle k_+, k'_+ \rangle_{\mathcal{K}_+} - \langle k_-, k'_- \rangle_{\mathcal{K}_-}
\]

As a direct sum of Hilbert spaces, \(\mathcal{K}\) can be made a Hilbert space with norm

\[
\|k_+ \oplus k_-\|^2 = \|k_+\|^2 + \|k_-\|^2
\]

We will denote this Hilbert space as \(\mathcal{K}_H\) and always use \([\cdot, \cdot]\) to represent a Kreın space inner product and use \(\langle \cdot, \cdot \rangle\) to represent a Hilbert space inner product.

The decomposition \(\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-\) is called a fundamental decomposition. Generally, fundamental decompositions are often not given and are unique only in trivial cases. While fundamental decompositions are not unique, the associated Hilbert
norms are all topologically equivalent. All topological notions in a Kreın space, if not stated explicitly otherwise, refer to this topology (also called the Mackey topology of $K$, i.e., the strongest topology for a topological vector space which still preserves the continuous dual, see [95] or [122]).

Given a Kreın space $K = K_+ \oplus K_-$, the orthogonal projections onto $K_+$ and $K_-$ are denoted by $P_+$ and $P_-$, respectively. The operator $J = P_+ - P_-$ is defined and called the fundamental symmetry or signature operator for the given fundamental decomposition. We have

$$\langle [x, y]_K = \langle Jx, y \rangle_{KH}, \text{ for any } x, y \in K \tag{2.3.1}$$

and $J$ has the properties $J^2 = I, J = J^*$. Conversely, given a Hilbert space $K$ with inner product $\langle \cdot, \cdot \rangle$ and in it an operator $J$ with these properties, then an indefinite inner product is defined on $K$ by (2.3.1) and $K$ with $\langle \cdot, \cdot \rangle$ is a Kreın space. So Kreın spaces are sometimes called $J$-spaces due to this construction.

A subspace of a Kreın space $K$ is a linear submanifold in $K$. For $M$ a subspace of a Kreın space $K$, the $\langle \cdot, \cdot \rangle_K$-orthogonal complement of $M$ is defined as

$$M^{\perp} = \{ f \in K : \langle f, g \rangle_K = 0, \text{ for all } g \in M \}$$

The Kreın space orthogonal complement coincides with the Hilbert space orthogonal complement of $JM$ in $KH$. The subspace $M$ of $K$ is dense in $K$ if and only if $M^{\perp} = \{ 0 \}$. The subspace $M$ is said to be regular (or ortho-complemented) if $K = M + M^{\perp}$. Since no nonzero vector of $K$ can be orthogonal to all of $K$, this leads to $M \cap M^{\perp} = \{ 0 \}$. Thus the sum $K = M + M^{\perp}$ is a $\langle \cdot, \cdot \rangle_K$-orthogonal direct sum. Equivalently, $M$ is a regular subspace if and only if the restriction of $\langle \cdot, \cdot \rangle_K$ to $M$ makes $M$ a Kreın space (see [55]). In particular, the Hilbert and anti-Hilbert subspaces are regular. Note that, different from the Hilbert space case, the relation $M + M^{\perp} = K$ may fail for a closed subspace $M$ of a Kreın space $K$ and a closed subspace $M$ of $K$ need not itself be a Kreın space in the inner product of $K$.

The indefinite inner product on $K$ gives rise to a classification of the elements of $K$: $f \in K$ is called positive, negative, neutral if $\langle f, f \rangle_K \geq 0, \langle f, f \rangle_K = 0, \langle f, f \rangle_K \leq 0$ respectively. We say a subspace of a Kreın space $M \subset K$ is
• **positive** if \([f, f]_K \geq 0\) for all \(f\) in \(\mathcal{M}\)

• **uniformly positive** if for some (and hence any) fundamental symmetry \(J\) on \(K\), there is a \(\delta > 0\) such that \([f, f]_K \geq \delta \|f\|^2\) for \(f\) in \(\mathcal{M}\)

• **\(K\)-maximal positive** if \(\mathcal{M}\) is itself positive and it is not contained in any larger positive subspace of \(K\).

Applying all these notations to the inner product \(-[\cdot, \cdot]_K\) in place of \([\cdot, \cdot]_K\), we get the definitions for (uniformly, maximal) negative subspace. Moreover, a subspace which is either positive or negative is said to be **definite**. It is immediate that maximal definite subspaces are closed and a closed positive subspace of a Kreın space \(K\) is a Hilbert space if and only if it is uniformly positive. An example is \(K_+\) in any fundamental decomposition \(K = K_+ \oplus K_-\).

A typical example of real Kreın space is 3-dimensional Minkowski space shown in Figure 1 (see [80]) which is defined by the inner product \([v_1, v_2] = x_1x_2 + y_1y_2 - t_1t_2\) for \(v_i = (x_i, y_i, t_i), i = 1, 2\) on \(\mathbb{R}^3\). The indefinite square norm of each vector \(v = (x, y, t)\) is \([v, v] = x^2 + y^2 - t^2\).

Let \(\mathcal{M}\) be a negative subspace of a Kreın space \(K\) with a fixed fundamental decomposition \(K = K_+ \oplus K_-\). So if \(h = g + f\) with \(g \in K_+, f \in K_-\), we can write

\[
h = \begin{bmatrix} g \\ f \end{bmatrix}
\]
In this representation, no nonzero element of $\mathcal{M}$ has the form $\begin{bmatrix} g \\ 0 \end{bmatrix}$, so $\mathcal{M}$ is the graph

$$G(A) = \left\{ \begin{bmatrix} Af \\ f \end{bmatrix} : f \in D(A) \right\}$$

of a Hilbert space contraction operator $A$, with domain $D(A) \subset \mathcal{K}_-$ and range $R(A) \subset \mathcal{K}_+$, which is called the angle operator for $\mathcal{M}$. And it is easy to see that $\mathcal{M}$ is maximal negative if $D(A) = \mathcal{K}_-$, which means

$$\mathcal{M} = \left\{ \begin{bmatrix} Af \\ f \end{bmatrix}, f \in \mathcal{K}_- \right\} \text{ with } \|A\| \leq 1 \tag{2.3.2}$$

We have a parallel graph-representation result for positive subspaces. And the following statements are also immediate:

- A closed subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{K}$ is maximal negative (positive) if and only if $\mathcal{M}^\perp$ is maximal positive (negative) in $\mathcal{K}$.

- Every negative (positive) subspace of $\mathcal{M}$ is contained in a maximal negative (positive) subspace of $\mathcal{K}$.

Furthermore, we have the following general result of characterizing which maximal negative subspaces $\mathcal{G}$ are contained in a given regular subspace $\mathcal{M}$.

**Lemma 2.3.1** Let $\mathcal{K}$ be a Kreĭn space and $\mathcal{M}$ be a regular subspace of $\mathcal{K}$ such that $\mathcal{M}^{[\perp]}$ is a positive subspace. If $\mathcal{G}$ is a maximal negative subspace of $\mathcal{K}$, then the following are equivalent:

(a) $\mathcal{G} \subset \mathcal{M}$.

(b) $P_{\mathcal{M}}\mathcal{G}^{[\perp]}$ is a positive subspace, where $P_{\mathcal{M}}$ is the $J$-orthogonal projection onto $\mathcal{M}$.

**Proof**: (a)$\Rightarrow$(b): If $\mathcal{G}$ is maximal negative, we know $\mathcal{G}^{[\perp]}$ is maximal positive. Also if $\mathcal{G} \subset \mathcal{M}$, then $\mathcal{M}^{[\perp]} \subset \mathcal{G}^{[\perp]}$. It follows that $P_{\mathcal{M}}\mathcal{G}^{[\perp]} = \mathcal{G}^{[\perp]} \cap \mathcal{M}$ and hence $P_{\mathcal{M}}\mathcal{G}^{[\perp]}$ is a positive subspace.
Let $K = K_+ \oplus K_-$ be a fundamental decomposition. Suppose $M$ is regular with $M^{[\|]}$ positive and set $\begin{bmatrix} D_+ \\ 0 \end{bmatrix} = M \cap \begin{bmatrix} K_+ \\ 0 \end{bmatrix}$, then we have

$$M = \begin{bmatrix} X \\ I \end{bmatrix} K_- \oplus \begin{bmatrix} D_+ \\ 0 \end{bmatrix}$$

where $X : K_- \to K_+ \oplus D_+$. In fact we have $I - XX^*$ invertible because $M$ is regular. In this case $M^{[\|]} = \begin{bmatrix} I \\ X^* \end{bmatrix} D_+^\perp$ is positive and this implies $\|X\| \leq 1$. Then $I - XX^*$ invertible gives $\|X\| < 1$ Since $\mathcal{G}$ is a maximal negative subspace, then $\mathcal{G} = \begin{bmatrix} Y \\ I \end{bmatrix} K_-$ with $\|Y\| \leq 1$, hence $\mathcal{G}^{[\|]} = \begin{bmatrix} I \\ Y^* \end{bmatrix} K_+$. An easy computation shows that

$$P_M = \begin{bmatrix} X \\ I \end{bmatrix} (X^*X - I)^{-1} \begin{bmatrix} X^* \\ -I \end{bmatrix} \oplus \begin{bmatrix} P_{D_+} \\ 0 \\ 0 \end{bmatrix}$$

Then for $k_+ \in D_+^\perp$,

$$P_M \begin{bmatrix} I \\ Y^* \end{bmatrix} k_+ = \begin{bmatrix} X \\ I \end{bmatrix} (X^*X - I)^{-1} (X^* - Y^*) k_+$$

All such vectors are positive means

$$(X - Y)|_{D_+^\perp} (X^*X - I)^{-1} \begin{bmatrix} X^* \\ I \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \\ -I \end{bmatrix} \begin{bmatrix} X \\ I \end{bmatrix} (X^*X - I)^{-1} (X^* - Y^*)|_{D_+}$$

$$= (X - Y)|_{D_+^\perp} (X^*X - I)^{-1} (X^* - Y^*)|_{D_+^\perp} \geq 0$$

Since $\|X\| < 1$, this forces $(X^* - Y^*)|_{D_+^\perp} = 0$, i.e, $X^* = Y^*|_{D_+^\perp}$. Therefore

$$M^{[\|]} = \begin{bmatrix} I \\ X^* \end{bmatrix} D_+^\perp = \begin{bmatrix} I \\ Y^* \end{bmatrix} D_+^\perp \subset \begin{bmatrix} I \\ Y^* \end{bmatrix} K_+ = \mathcal{G}^{[\|]}$$

which is equivalent to $\mathcal{G} \subset M$. 

Now we will explore the operators on Krein spaces. We will use $C(K)$ to denote the set of closed linear operators on a Krein space $K$. Similarly, if $K_1$ and $K_2$ are
Kreĭn spaces, $C(K_1, K_2)$ will be used to denote the set of closed linear operators from $K_1$ to $K_2$. Every densely defined operator $A \in C(K_1, K_2)$ has a unique Kreĭn space adjoint $A^\ast \in C(K_2, K_1)$ satisfying

$$[Af, g]_{K_1} = [f, A^\ast g]_{K_2}, f \in D(A), g \in D(A^\ast) \subset K_2.$$  \hspace{1cm} (2.3.3)

Note that Kreĭn space adjoints and Hilbert space adjoints should be distinguished.

Let $K_1$ and $K_2$ be Kreĭn spaces with signature operators $J_{K_1}$ and $J_{K_2}$, then the Kreĭn space adjoint $A^\ast$ of an operator $A \in C(K_1, K_2)$ and the Hilbert space adjoint $A^* \in C(K_{1H}, K_{2H})$ are connected via $A^\ast = J_{K_1}A^*J_{K_2}$.

We would like to generalize the special operators in the Hilbert space setting to the Kreĭn space setting. The proof of the following result is not difficult, we refer to [69].

**Lemma 2.3.2** Let $K_1$ and $K_2$ be Kreĭn spaces and $A \in C(K_1, K_2)$. The following statements are equivalent:

1. There exist regular subspaces $\mathcal{M}$ of $K_1$ and $\mathcal{N}$ of $K_2$ such that $A|_{D(A) \cap \mathcal{M}}$ maps $D(A) \cap \mathcal{M}$ isometrically onto a dense subspace of $\mathcal{N}$ and $\ker A = D(A) \cap \mathcal{M}^\perp$.

2. $AA^\ast A = A$.

3. $A^\ast A$ extends to a bounded projection operator onto $\mathcal{M}$ and $\ker A^\ast A = \ker A$.

4. $AA^\ast$ extends to a bounded projection operator onto $\mathcal{N}$ and $\ker AA^\ast = \ker A^\ast$.

**Definition 2.3.3** Let $K_1$ and $K_2$ be Kreĭn spaces and $A \in C(K_1, K_2)$. We say $A$ is a Kreĭn space partial isometry (sometimes called $J$-partial isometry) if any one of the four statements in Lemma 2.3.2 is satisfied. We then say that $\mathcal{M}$ is the initial space of $A$ and $\mathcal{N}$ is the final space of $A$.

It is clear that a partial isometry $A$ has the following properties.

- For any $f$ and $g$ in $K_1$, the identity

$$[Af, Ag]_{K_2} = [f, g]_{K_1}$$

holds if at least one of $f$ and $g$ is in the initial space $\mathcal{M}$ of $A$. 

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• The adjoint $A^*\in C(K_1,K_2)$ is a partial isometry with initial space $\mathcal{N}$ and final space of $\mathcal{M}$.

• If $P_1$ is the $J$-orthogonal projection of $\mathcal{K}_1$ on $\mathcal{M}$ and $P_2$ is the $J$-orthogonal projection of $\mathcal{K}_2$ on $\mathcal{N}$, then $A^*A = P_1$, $\text{Ker} A = \text{Ker} P_1$, and $AA^*A = P_2$, $\text{Ker} A^* = \text{Ker} P_2$.

**Definition 2.3.4** Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Krein spaces and $A \in C(\mathcal{K}_1, \mathcal{K}_2)$. We say $A$ is

1. **Krein space isometric** (sometimes we say $A$ is $J$-isometric) if $A^*A = I_D(A)$. Equivalently, $J_{\mathcal{K}_1}|_{D(A)} - A^*J_{\mathcal{K}_2}A = 0|_{D(A)}$ or $[Af, Af]_{\mathcal{K}_2} = [f, f]_{\mathcal{K}_1}$ for all $f \in D(A)$.

2. **Krein space contractive** (sometimes we say $A$ is $J$-contractive) if $A^*A \leq I_D(A)$. Equivalently, $J_{\mathcal{K}_1}|_{D(A)} - A^*J_{\mathcal{K}_2}A \geq 0|_{D(A)}$ or $[Af, Af]_{\mathcal{K}_2} \leq [f, f]_{\mathcal{K}_1}$ for all $f \in D(A)$.

3. **Krein space bicontractive** (or $A$ is $J$-bicontractive) if both $A$ and $A^*$ are $J$-contractive, equivalently, both $J_{\mathcal{K}_1}|_{D(A)} - A^*J_{\mathcal{K}_2}A \geq 0$ and $J_{\mathcal{K}_2}|_{D(A^*)} - AJ_{\mathcal{K}_1}A^* \geq 0$ are satisfied.

The following result shows how $J$-contractions or $J$-bicontractions map maximal negative subspaces.

**Lemma 2.3.5** Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Krein spaces. If $A \in C(\mathcal{K}_1, \mathcal{K}_2)$ is $J$-contractive, then $\text{Ker} A$ is a uniformly positive space. Moreover if $A$ is $J$-bicontractive and $N$ is a maximal negative space in $\mathcal{K}_1$ then $A \cdot (N \cap D(A))$ is a maximal negative space in $\mathcal{K}_2$.

**Proof:** Suppose $f \in \text{Ker} A$, i.e., $Af = 0$. If we denote $E = J_{\mathcal{K}_1}|_{D(A)} - A^*J_{\mathcal{K}_2}A$, then $E \geq 0$ because $A$ is a $J$-contraction. We can see that $Ef = J_{\mathcal{K}_1}f$ since $Af = 0$. And $\|f\|^2 = \|J_{\mathcal{K}_1}f\|^2 = \|Ef\|^2 \leq \|E\| \cdot \langle Ef, f \rangle_{\mathcal{K}_1}$, while we note that $\langle Ef, f \rangle_{\mathcal{K}_1} = [f, f]_{\mathcal{K}_1} - [Af, Af]_{\mathcal{K}_2} = [f, f]_{\mathcal{K}_1}$, thus we get $\|f\|^2 \leq \|E\|[f, f]_{\mathcal{K}_1}$. If $\|E\| = 0$ the result is trivial. If $\|E\| > 0$ then $[f, f]_{\mathcal{K}_1} \geq \frac{1}{\|E\|}\|f\|^2$, i.e., $\text{Ker} A$ is a uniformly positive space in $\mathcal{K}_1$. 

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Now let $N \cap D(A)$ be a maximal negative space in $\mathcal{K}_1$. Notice that $J$-contractions map negative subspaces to negative subspaces. Actually if $A$ is $J$-contractive, we have $J_{\mathcal{K}_1}D(A) - A^*J_{\mathcal{K}_2}A \geq 0$, equivalently, $[Af, Af]_{\mathcal{K}_2} \leq [f, f]_{\mathcal{K}_1} \leq 0$ for any $f \in N \cap D(A)$. Thus $A \cdot (N \cap D(A))$ is a closed negative space in $\mathcal{K}_2$, it is sufficient to prove by the graph representation of negative subspaces that it is maximal if $A$ is $J$-bicontractive. Suppose there is a nonzero vector $h \in \mathcal{K}_2_-$ such that $\langle h, Ag \rangle_{\mathcal{K}_2} = 0$ for any $g \in N \cap D(A)$, then we have

$$[A^*[h], g]_{\mathcal{K}_1} = [h, Ag]_{\mathcal{K}_2} = 0$$

This means $A^*[h] \in (N \cap D(A))^{\perp_1}$, which is maximal positive since $N \cap D(A)$ is maximal negative. Thus $[A^*[h], A^*[h]] \geq 0 \geq [h, h]_{\mathcal{K}_2}$. But since $A^*$ is $J$-contractive, we can get $[A^*[h], A^*[h]] \leq [h, h]_{\mathcal{K}_2}$, which forces $[h, h]_{\mathcal{K}_2} = 0$ and thus $h = 0$ since it is in $\mathcal{K}_2_-$. This contradicts that $h$ is nonzero. Therefore there is no nonzero vector in $\mathcal{K}_2_-$ which is orthogonal to $A \cdot (N \cap D(A))$. By the reduction above, we conclude that $A \cdot (N \cap D(A))$ is a maximal negative subspace of $\mathcal{K}_2$.

$\Box$
Chapter 3

Interpolation Problems in the Drury-Arveson Space

In this chapter we solve the left tangential operator argument interpolation problem in the Drury-Arveson space. We give the definition and properties of Drury-Arveson space in Section 3.1. In Section 3.2, we form and solve the left tangential operator argument interpolation problem. We discuss Beurling-Lax theorem in more detail.

3.1 Drury-Arveson space

Recall from Section 1.4.1 the multivariable generalization of the Szegö kernel

\[ k_d(\lambda, \zeta) = \frac{1}{1 - \langle \lambda, \zeta \rangle} \]

on \( B^d \times B^d \) where \( B^d = \{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \langle \lambda, \lambda \rangle < 1 \} \) is the unit ball of the \( d \)-dimensional Euclidean space \( \mathbb{C}^d \). By

\[ \langle \lambda, \zeta \rangle = \langle \lambda, \zeta \rangle_{\mathbb{C}^d} = \sum_{j=1}^{d} \lambda_j \overline{\zeta}_j \quad \text{for} \quad \lambda, \zeta \in \mathbb{C}^d \]

we mean the standard inner product in \( \mathbb{C}^d \). The associated RKHS \( \mathcal{H}(k_d) \) obtained via Aronszajn’s construction (called the Drury-Arveson space, see Arveson [18] and Drury [70]) is a natural multivariable analogue of the Hardy space \( H^2 \) of the unit disk and coincides with \( H^2 \) if \( d = 1 \). Note that here \( k_d \) is a complete Pick kernel as mentioned in Section 1.1. Let \( k \) be the reproducing kernel for a Hilbert space \( \mathcal{H}(k) \)
of holomorphic functions on $B_d$, the open unit ball in $\mathbb{C}^d$, $d \geq 1$. The kernel $k$ is called a complete Pick kernel, if (after a normalization) $k_0 \equiv 1$ and if $1 - 1/k_\lambda(z)$ is positive definite on $B_d \times B_d$.

For $\mathcal{Y}$ an auxiliary Hilbert space, we consider the tensor product Hilbert space $\mathcal{H}_\mathcal{Y}(k_d) := \mathcal{H}(k_d) \otimes \mathcal{Y}$ whose elements can be viewed as $\mathcal{Y}$-valued functions in $\mathcal{H}(k_d)$. Then $\mathcal{H}_\mathcal{Y}(k_d)$ can be characterized as follows:

$$\mathcal{H}_\mathcal{Y}(k_d) = \left\{ f(\lambda) = \sum_{n \in \mathbb{Z}^d_+} f_n \lambda^n : \|f\|^2 = \sum_{n \in \mathbb{Z}^d_+} \frac{n!}{|n|!} \cdot \|f_n\|^2_\mathcal{Y} < \infty \right\}.$$  \hspace{1em} (3.1.1)

Here and in what follows, we use standard multivariable notations: for multi-integers $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d_+$ and points $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ we set

$$|n| = n_1 + n_2 + \ldots + n_d, \quad n! = n_1! n_2! \ldots n_d!, \quad \lambda^n = \lambda_1^{n_1} \lambda_2^{n_2} \ldots \lambda_d^{n_d}. \hspace{1em} (3.1.2)$$

The space of multipliers $\mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ is defined as the space of all $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued holomorphic functions $S$ on $\mathbb{B}^d$ such that the induced multiplication operator

$$M_S : f(\lambda) \mapsto S(\lambda) \cdot f(\lambda) \hspace{1em} (3.1.3)$$

maps $\mathcal{H}_\mathcal{U}(k_d)$ into $\mathcal{H}_\mathcal{Y}(k_d)$. It follows by the closed graph theorem that for every $S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})$, the operator $M_S$ is bounded. We shall pay particular attention to the unit ball of $\mathcal{M}_d(\mathcal{U}, \mathcal{Y})$, denoted by

$$S_d(\mathcal{U}, \mathcal{Y}) = \{ S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y}) : \|M_S\|_{\text{op}} \leq 1 \}.$$  

Since $S_1(\mathcal{U}, \mathcal{Y})$ collapses to the classical Schur class, we refer to $S_d(\mathcal{U}, \mathcal{Y})$ as a $d$-variable Schur class. The following result appears in [2, 43] and is the precise analogue of the classical result Theorem 2.2.1 for the multivariable case.

**Theorem 3.1.1** Let $S$ be an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function defined on $\mathbb{B}^d$. The following are equivalent:

1. $S$ belongs to $S_d(\mathcal{U}, \mathcal{Y})$.

2. The kernel

$$K_S(\lambda, \zeta) = \frac{I_\mathcal{Y} - S(\lambda)S(\zeta)^*}{1 - \langle \lambda, \zeta \rangle} \hspace{1em} (3.1.4)$$

is positive on $\mathbb{B}^d \times \mathbb{B}^d$. 

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3. There exists a Hilbert space $\mathcal{X}$ and a unitary connecting operator (or colligation) $U$ of the form

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \end{bmatrix} : \mathcal{X} \to \mathcal{X}^d$$

so that $S(\lambda)$ can be realized in the form

$$S(\lambda) = D + C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}(\lambda_1 B_1 + \cdots + \lambda_d B_d)$$

where we set

$$Z(\lambda) = \begin{bmatrix} \lambda_1 I_X & \cdots & \lambda_d I_X \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}$$

(3.1.5)

4. There exists a Hilbert space $\mathcal{X}$ and a contractive connecting operator $U$ of the form (3.1.5) so that $S(\lambda)$ can be realized in the form (3.1.6).

Associated with any $S \in \mathcal{S}_d(U, Y)$ is the de Branges-Rovnyak space $\mathcal{H}(K_S)$, the reproducing kernel Hilbert space with reproducing kernel $K_S$ (which is positive). The original characterization of $\mathcal{H}(K_S)$, as the space of all functions $f \in \mathcal{H}_Y(k_d)$ such that

$$\|f\|_{\mathcal{H}(K_S)} := \sup_{g \in \mathcal{H}_U(k_d)} \left\{ \|f + Sg\|_{\mathcal{H}_Y(k_d)}^2 - \|g\|_{\mathcal{H}_U(k_d)}^2 \right\} < \infty,$$

(3.1.8)

is due to de Branges and Rovnyak [58]. In particular, it follows from (3.1.8) that $\|f\|_{\mathcal{H}(K_S)} \geq \|f\|_{\mathcal{H}_Y(k_d)}$ for every $f \in \mathcal{H}(K_S)$, i.e., that $\mathcal{H}(K_S)$ is contained in $\mathcal{H}_Y(k_d)$ contractively. On the other hand, the general complementation theory applied to the contractive operator $M_S$ provides the characterization of $\mathcal{H}(K_S)$ as the operator range

$$\mathcal{H}(K_S) = \text{Ran}(I - M_S M_S^*)^{1/2}$$

(3.1.9)

with the lifted norm

$$\|(I - M_S M_S^*)^{1/2} h\|_{\mathcal{H}(K_S)} = \|(I - Q) h\|_{\mathcal{H}_Y(k_d)},$$

(3.1.10)
for all $f \in \mathcal{H}_Y(k_d)$ where $Q$ here is the orthogonal projection onto $\text{Ker}(I - M_SM^*_S)^{\frac{1}{2}}$. Upon setting $f = (I - M_SM^*_S)^{\frac{1}{2}}h$ in (3.1.10) we get

$$\|(I - M_SM^*_S)h\|_{\mathcal{H}(K_S)} = \langle (I - M_SM^*_S)h, h \rangle_{\mathcal{H}(k_d)}. \quad (3.1.11)$$

The range space characterization (3.1.9), (3.1.10) of $\mathcal{H}(K_S)$ can be expressed in terms of the unitary realization of $S$:

$$\mathcal{H}(K_S) = \{ f(z) = C(I - Z(\lambda)A)^{-1}h : h \in \mathcal{H}\}$$

with the lifted norm $\|f\|_{\mathcal{H}(K_S)} = \|\pi h\|_{\mathcal{H}}$ where $\pi$ is the orthogonal projection onto the subspace $\{ x \in \mathcal{H} : C(I - Z(\lambda)A)^{-1}x \equiv 0 \}$. More complete details concerning the spaces $\mathcal{H}(K_S)$ and related matters of realization and the model theory for commutative row contractions can be found in the recent series of papers [27,29,30].

### 3.2 Left-tangential operator-argument interpolation problem

Let $A = (A_1, \ldots, A_d)$ be a commutative $d$-tuple of bounded, linear operators on the Hilbert space $\mathcal{X}$. If $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then the pair $(C, A)$ is said to be an commutative output pair. Such an output pair is said to be contractive if

$$A_1^*A_1 + \cdots + A_d^*A_d + C^*C \leq I_{\mathcal{X}},$$

to be isometric if equality holds in the above relation, and to be output-stable if the associated observability operator

$$O_{C,A} : x \mapsto C(I - \lambda_1A_1 - \cdots - \lambda_dA_d)^{-1}x \quad (3.2.1)$$

maps $\mathcal{X}$ into $\mathcal{H}_Y(k_d)$, or equivalently (by the closed graph theorem), the observability operator is bounded. In this case it makes sense to introduce the observability gramian:

$$G_{C,A} := O_{C,A}^*O_{C,A}.$$  

We next observe that the operators $M_{\lambda_j}$ of multiplication by the coordinate functions of $\mathbb{C}^d$ for $j = 1, \ldots, d$ act as contractions on the Arveson space $\mathcal{H}_Y(k_d)$. 28
We call the commuting $d$-tuple $M_{\lambda} := (M_{\lambda_1}, \ldots, M_{\lambda_d})$ the shift of $\mathcal{H}(k_d)$, whereas the commuting $d$-tuple $M^*_\lambda := (M^*_\lambda_1, \ldots, M^*_\lambda_d)$ consisting of the adjoints of $M_{\lambda_j}$'s (adjoint taken in the metric of $\mathcal{H}(k_d)$) are referred to as to the backward shift. Now the space $\mathcal{M}_d(U, \mathcal{Y})$ can be characterized as those elements $R$ of $\mathcal{L}(\mathcal{H}_U(k_d), \mathcal{H}_Y(k_d))$ which intertwine the shifts of $\mathcal{H}_U(k_d)$ and $\mathcal{H}_Y(k_d)$ (i.e., such that $M_{\lambda_j} R = R M_{\lambda_j}$ for $j = 1, \ldots, d$); if such an $R$ is a contraction, then $R = M_S$ for some $S \in \mathcal{S}_d(U, \mathcal{Y})$.

Recall that monomials $\lambda^n$ form an orthogonal basis for $\mathcal{H}(k_d)$. As we have seen, 

$$
\langle \lambda^n, \lambda^m \rangle_{\mathcal{H}(k_d)} = \begin{cases} 
n! \frac{n!}{|n|!} & \text{if } n = m \\
0 & \text{otherwise.} \end{cases} \tag{3.2.2}
$$

A simple calculation based on (3.2.2) gives

$$
M^*_\lambda \lambda^m = \frac{m_j}{|m|} \lambda^{m-e_j} \quad (m_j \geq 1) \quad \text{and} \quad M^*_\lambda \lambda^m = 0 \quad (m_j = 0) \tag{3.2.3}
$$

where $m = (m_1, \ldots, m_d)$ and $e_j \in \mathbb{Z}_+^d$ as defined in the previous section. More generally,

$$
(M^*_\lambda)^n \lambda^m = \begin{cases} 
n! m! m-n! |m-n|! \lambda^{m-n} & \text{if } m_j \geq n_j \text{ for } j = 1, \ldots, d, \\
0, & \text{otherwise,} \end{cases} \tag{3.2.4}
$$

where

$$(M^*_\lambda)^n := (M^*_\lambda_1)^{n_1} (M^*_\lambda_2)^{n_2} \cdots (M^*_\lambda_d)^{n_d}.$$ 

And notice that

$$(M_{\lambda_j})^* \mathcal{O}_{C,A} x = (M_{\lambda_j})^* \left( \sum_{n \in \mathbb{Z}_+^d} \frac{|n|!}{n!} C A^n x \cdot \lambda^n \right) = \sum_{n \in \mathbb{Z}_+^d} \frac{|n|!}{n!} n_j C A^n x \cdot \lambda^{n-e_j} = \sum_{n \in \mathbb{Z}_+^d} \frac{|n-e_j|!}{(n-e_j)!} C A^{n-e_j} \cdot \lambda^{n-e_j} A_j x = \left( \sum_{n \in \mathbb{Z}_+^d} \frac{|n|!}{n!} C A^n \lambda^n \right) \cdot A_j x = \mathcal{O}_{C,A} \cdot A_j x$$
so we have the intertwining relations

\[ M_j^* \mathcal{O}_{C,A} = \mathcal{O}_{C,A} A_j \quad \text{for } j = 1, \ldots, d \] (3.2.5)

and hence the linear submanifold \( \text{Ran} \mathcal{O}_{C,A} \) of \( \mathcal{H}_Y(k_d) \) is \( M_j^* \)-invariant.

For an output stable pair \((C, A)\), we define a left-tangential functional calculus \( f \mapsto (C^* f)^{\wedge L}(A^*) \) on \( \mathcal{H}_Y(k_d) \) by

\[ (C^* f)^{\wedge L}(A^*) = \sum_{n \in \mathbb{Z}^d_+} A^n C^* f_n \quad \text{if } f = \sum_{n \in \mathbb{Z}^d_+} f_n \lambda^n \in \mathcal{H}_Y(k_d). \] (3.2.6)

The computation

\[
\left\langle \sum_{n \in \mathbb{Z}^d_+} A^n C^* f_n, x \right\rangle_X = \sum_{n \in \mathbb{Z}^d_+} \left\langle f_n, C A^n x \right\rangle_Y \\
= \sum_{n \in \mathbb{Z}^d_+} \frac{n!}{|n|!} \left\langle f_n, \frac{|n|!}{n!} C A^n x \right\rangle_Y \\
= \left\langle f, \mathcal{O}_{C,A} x \right\rangle_{\mathcal{H}_Y(k_d)}
\]

shows that the output-stability of the pair \((C, A)\) is exactly what is needed to verify that the infinite series in the definition (3.2.6) of \( (C^* f)^{\wedge L}(A^*) \) converges in the weak topology on \( X \). In fact the left-tangential evaluation with operator argument \( f \mapsto (C^* f)^{\wedge L}(A^*) \) amounts to the adjoint of the observability operator:

\[ (C^* f)^{\wedge L}(A^*) = (\mathcal{O}_{C,A})^* f \quad \text{for } f \in \mathcal{H}_Y(k_d). \] (3.2.7)

This evaluation map extends to multipliers \( S \in \mathcal{S}_d(U, Y) \) by

\[ (C^* S)^{\wedge L}(A^*) = (\mathcal{O}_{C,A})^* M_S|_U \quad \text{for } S \in \mathcal{M}_d(U, Y). \]

and raises the following interpolation problem with operator argument.

The LTOA interpolation problem: Let \( U, Y \) and \( X \) be Hilbert spaces. Given \((Z, X, Y)\) with \( Z = (Z_1, \ldots, Z_d) \in \mathcal{L}(X, \oplus_d^d Z) \), \( X \in \mathcal{L}(Y, X) \), \( Y \in \mathcal{L}(U, X) \) such that \((Z, X)\) is an input stable pair, or, \((X^*, Z^*)\) is an output stable pair with \( Z^* \) commutative. Find \( S \in \mathcal{S}_d(U, Y) \) such that

\[ (X S)^{\wedge L}(Z) := \mathcal{O}_{X^*, Z}^* M_S|_U = Y. \] (3.2.8)
Just like any other interpolation problem, the LTOA interpolation problem has two aspects: 1) to specify the necessary and sufficient conditions for the existence of a solution, 2) to characterize the set of all solutions under the conditions for existence.

To motivate the approach, we return to the following special case: take $U = \mathbb{C}$, $X = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$, $Y = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$, $Z = (Z_1, \ldots, Z_d)$ with $Z_j = \begin{bmatrix} \lambda_j^{(1)} \\ \lambda_j^{(2)} \\ \vdots \\ \lambda_j^{(N)} \end{bmatrix}$, where $j = 1, \ldots, d$, $\lambda^{(i)} \in \mathbb{B}^d, i = 1, \ldots, N$. Then the LTOA interpolation problem becomes: find $S \in S_d$ so that

$$S(\lambda^{(i)}) = w_i \quad \text{for} \quad i = 1, \ldots, N$$  \hspace{1cm} (3.2.9)

The idea is to identify $S \in S_d$ with its graph $G_S = [M_S] \mathcal{H}(k_d) \subset \left[ \frac{\mathcal{H}(k_d)}{\mathcal{H}(k_d)} \right]$ to convert the nonhomogeneous interpolation conditions (3.2.9) to homogenous interpolation conditions

(Hom) $G_S \subset \mathcal{M} = \{ f \in \left[ \frac{\mathcal{H}(k_d)}{\mathcal{H}(k_d)} \right] : [1 - w_i] f(\lambda^i) = 0 \quad \text{for} \quad i = 1, \ldots, N \}$

The condition that $\|M_S\| \leq 1$ translates to the condition:

(MN) $G_S$ is maximal negative in the Kre˘ın space $\left[ \frac{\mathcal{H}(k_d)}{\mathcal{H}(k_d)} \right]$, $J = \left[ I_{\mathcal{H}(k_d)} - I_{\mathcal{H}(k_d)} \right]$.

The fact that $G_S$ is the graph space of a multiplication operator translates to the condition:

(Inv) $G_S$ is $M_{\lambda_k}$-invariant for $k = 1, \ldots, d$.

Conversely, one can use the results from Chapter 2 to see that $G$ satisfying the conditions (Hom),(MN),(Inv) has the form $G = G_S$ where $S \in S_d$ satisfies (3.2.9).

To describe the set of all solutions we suppose that $\mathcal{M}$ has a representation as $\mathcal{M} = \Theta \cdot \left[ \frac{\mathcal{H}(k_d)}{\mathcal{H}(k_d)} \right]$ where $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ and $M_{\Theta}$ is a Kre˘ın-space isomorphism. Then using results in chapter 2, one can see that subspace $G$ satisfying (Hom),(MN),(Inv) are characterized as $G = \Theta \cdot G'$ where $G'$ satisfies

(MN') $G'$ is maximal negative in the Kre˘ın space $\left[ \frac{\mathcal{H}(k_d)}{\mathcal{H}(k_d)} \right]$, $J = \left[ I_{\mathcal{H}(k_d)} - I_{\mathcal{H}(k_d)} \right]$.

(Inv') $G'$ is $M_{\lambda_k}$-invariant for $k = 1, \ldots, d$.  \hspace{1cm} (31)
If we reverse the angle-operator-graph correspondence, we see that such $G'$ are characterized as being of the form $G' = M_{S'}$ where $S' \in S_d$ with no interpolation constraints, i.e., $S'$ is a free Schur-class parameter. This immediately leads to a parametrization for the set of all solutions: $S \in S_d$ solves (3.2.9) if and only if there exists $S' \in S_d$ so that

$$[egin{bmatrix} M_S \\ I \end{bmatrix} \mathcal{H}(k_d) = G_S = \Theta \cdot G_{S'} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} M_{S'} \\ I \end{bmatrix} \mathcal{H}(k_d) = \begin{bmatrix} \Theta_{11}S' + \Theta_{12} \\ \Theta_{21}S' + \Theta_{22} \end{bmatrix} \mathcal{H}(k_d)$$

or

$$S = (\Theta_{11}S' + \Theta_{12})(\Theta_{21}S' + \Theta_{22})^{-1}$$

The practical bottleneck in this approach is the construction of $\Theta$. The discussion here is oversimplified in two aspects:

1. In general $M_\Theta$ is only a $J$-partial isometry rather than $J$-isometry (for $d = 1$ this issue can be bypassed)

2. $M_\Theta$, while being a partial isometry in the Krein space, may be unbounded in the Hilbert-space sense (this phenomenon occurs even for $d = 1$).

We now provide the details of the approach for the general LTOA problem.

We discuss the easier part, i.e., existence of a solution first, then move to characterize the set of all solutions. The following lemma which goes back to [65] (can also be found in [7]) will be used to get the necessary and sufficient condition for the existence of a solution.

**Lemma 3.2.1** If $A$ and $B$ are bounded operators with final space $X$. A necessary and sufficient condition that $A = BC$ for some contraction $C$ is that $AA^* \leq BB^*$.

**Remark 3.2.2** Note that the contraction $C$ in the factorization $A = BC$ may not be unique. The uniqueness of this factorization will determine the uniqueness of the solution of the interpolation problem.

**Lemma 3.2.3**

$$\mathcal{O}_{X^*,Z^*} M_S f = (XSf)^{^L}(Z) = ((XS)^{^L}(Z)f)^{^L}(Z) = (Yf)^{^L}(Z) = \mathcal{O}_{Y^*,Z^*} f$$

(3.2.10)
Proof: This is a simple computation using the definition 3.2.6. □

**Theorem 3.2.4** Let \((Z, X, Y)\) be given as above. The LTOAIP has a solution if and only if \((X^*, Z^*)\) is an output stable pair and \(O_{Y^*, Z^*} O_{Y^*, Z^*} \leq O_{X^*, Z^*} O_{X^*, Z^*}\).

Proof: Suppose \(S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})\) is a solution of the problem. For \(f = \sum_{n \in \mathbb{Z}^d} f_n \lambda_n \in \mathcal{H}_d(k_d)\), by Lemma 3.2.3 we get

\[
O_{X^*, Z^*} M_S f = O_{Y^*, Z^*} f
\]

(3.2.11)

since \(f\) is arbitrary in \(H_d(k_d)\), we have \(O_{X^*, Z^*} M_S = O_{Y^*, Z^*}\). \(O_{Y^*, Z^*}\) is bounded because \(O_{X^*, Z^*}\) is bounded and \(S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})\). Moreover, if \(S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})\), then \(M_S\) is a contraction, from Lemma 3.2.1, we have \(O_{Y^*, Z^*} O_{Y^*, Z^*} \leq O_{X^*, Z^*} O_{X^*, Z^*}\).

Conversely, suppose we have \((X^*, Z^*)\) as an output stable pair and \(O_{Y^*, Z^*} O_{Y^*, Z^*} \leq O_{X^*, Z^*} O_{X^*, Z^*}\), from Lemma 3.2.1, we know \(O_{Y^*, Z^*} = O_{X^*, Z^*} F\) with a unique contraction \(F \in \mathcal{L}(\mathcal{H}_d(k_d), (\text{Ker} O_{X^*, Z^*})^\perp)\). It suffices to show that \(F\) can be lifted to a multiplier. By (3.2.5),

\[
M_{\lambda_j} O_{X, Z} = O_{X, Z} A_j \quad M_{\lambda_j} O_{Y, Z} = O_{Y, Z} A_j \quad \text{for } j = 1, \ldots, d
\]

and \(\text{Ran} O_{X, Z}\) of \(\mathcal{H}_d(k_d)\) and \(\text{Ran} O_{Y, Z}\) of \(\mathcal{H}_d(k_d)\) are \(M_{\lambda_j^{-1}}\)-invariant, we see that

\[
F^* M_{\lambda_j} O_{X, Z} = F^* O_{X, Z} A_j = O_{X, Z} A_j = M_{\lambda_j} F^* O_{X, Z}
\]

So from Theorem 5.1 in [43], there exists \(S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})\) such that \(F^* = M_S\), hence \(O_{Y^*, Z^*} = M_S O_{X^*, Z^*}\) or \(O_{Y^*, Z^*} = O_{X^*, Z^*} M_S\). □

Now we apply the Kreîn space geometry method to the problem aiming to parametrize the set of all solutions under some proper assumption. This gives a second proof of existence in the presence of the extra assumption.

Consider the direct sum \(\mathcal{K} = \begin{bmatrix} \mathcal{K}_+ \\ \mathcal{K}_- \end{bmatrix} = \begin{bmatrix} \mathcal{H}_y(k_d) \\ \mathcal{H}_d(k_d) \end{bmatrix} = \mathcal{H}_{\mathcal{E}^*}(k_d) \) with \(\mathcal{E}^* = \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}\) and let \(J_*\) be the operator with block matrix form

\[
J_* = \begin{bmatrix} I_{\mathcal{H}_y(k_d)} & 0 \\ 0 & -I_{\mathcal{H}_d(k_d)} \end{bmatrix}
\]
$\mathcal{H}_{\mathcal{E}}(k_d)$ becomes a Krein space in the inner product $\langle k, k' \rangle_{k_d} = \langle J_s k, k' \rangle_{k_d}$

Suppose $S \in S_d(U, Y)$ solves $(XS)^{\wedge L}(Z) = Y$. Consider the graph

$$\mathcal{G} = \mathcal{G}_S = \left\{ \begin{bmatrix} S(\lambda) \\ I \end{bmatrix} h(\lambda) : h \in \mathcal{H}_d(k_d) \right\} \subset \mathcal{H}_{\mathcal{E}_s}(k_d)$$

If $\mathcal{G}$ is of this form, we have $\left( \begin{bmatrix} X & -Y \end{bmatrix} f \right)^{\wedge L}(Z) = 0$ for any $f \in \mathcal{G}$. Actually we have

$$\left( XSh \right)^{\wedge L}(Z) = ((XS)^{\wedge L}(Z)h)^{\wedge L}(Z) = (Yh)^{\wedge L}(Z)$$

thus $f(\lambda) = \begin{bmatrix} S(\lambda)h(\lambda) \\ h(\lambda) \end{bmatrix}$ implies $\left( \begin{bmatrix} X & -Y \end{bmatrix} f \right)^{\wedge L}(Z) = 0$.

Therefore if we define

$$\mathcal{M} = \left\{ f \in \mathcal{H}_{\mathcal{E}_s}(k_d) : \left( \begin{bmatrix} X & -Y \end{bmatrix} f \right)^{\wedge L}(Z) = 0 \right\}$$

then $\mathcal{G} \subset \mathcal{M}$. Since $S$ is a contraction, due to the graph representation of definite subspaces of Krein spaces (see (2.3.2)) and the definition of $\mathcal{G}$, we see that $\mathcal{G}$ is $J$-maximal negative subspace of $\mathcal{H}_{\mathcal{E}_s}(k_d)$. Moreover, graph spaces $\mathcal{G}$ of the form $\mathcal{G}_{M_F}$ for a function $F \in \mathcal{M}_d(U, Y)$ are characterized by the condition that $\mathcal{G}$ is invariant under the shift $M_\lambda$. This is natural since $\mathcal{M}_d(U, Y)$ is characterized as the elements which intertwine the shifts. The subspace $\mathcal{G}$ is invariant under $M_{\lambda_k}, k = 1, 2, \ldots, d$. So if we sort out the conditions, we know $\mathcal{G}$ satisfies the following conditions:

1. $\mathcal{G} \subset \mathcal{M} = \left\{ f \in \mathcal{H}_{\mathcal{E}_s}(k_d) : \left( \begin{bmatrix} X & -Y \end{bmatrix} f \right)^{\wedge L}(Z) = 0 \right\}$

2. $\mathcal{G}$ is $J$-maximal negative subspace of $\mathcal{H}_{\mathcal{E}_s}(k_d)$.

3. $\mathcal{G}$ is invariant under $M_{\lambda_k}, k = 1, 2, \ldots, d$

And conversely, if $\mathcal{G}$ as a subspace of $\mathcal{H}_{\mathcal{E}_s}(k_d)$ satisfies (1), (2), (3), then $\mathcal{G}$ is in the form of $\begin{bmatrix} M_S \\ I \end{bmatrix} H_d(k_d)$ for a solution $S$ of the interpolation problem (see the detail for classical case in [21]). Thus the interpolation problem translates to finding subspaces $\mathcal{G}$ of $\mathcal{H}_{\mathcal{E}_s}(k_d)$ which satisfy the above conditions.
For convenience, we denote \( C^* = \begin{bmatrix} X & Y \end{bmatrix} \) and \( A^* = Z \) from now on. Hence \((C, A)\) is an output-stable pair, and the subspace \( \mathcal{M} \) above can be denoted as

\[
\mathcal{M} = \mathcal{M}_{A^*, C^*} = \{ f \in H_{E^*}(k_d) : (C^*Jf)^L(A^*) = 0 \}. \tag{3.2.13}
\]

Moreover, we will assume that \( \mathcal{M} \) is regular, which makes \( \mathcal{M} \) a Krein space itself in the induced inner product when we do the parametrization of all solutions. Notice that if we define the \( J \)-gramian:

\[
G^J_{X^*, Y^*}[Z^*] := O^*_{X^*, Y^*}J O^*_{X^*, Y^*}Z^* = O^*_{X^*, Y^*}O^*_{X^*, Y^*}Z^* - O^*_{Y^*, Y^*}O^*_{Y^*, Y^*}Z^*. \tag{3.2.14}
\]

An important property of the gramian is that it satisfies the Stein equation

\[
P - \sum_{j=1}^{d} Z_j P Z_j^* = [Y^*, X^*]^* J [X^*, Y^*]
\]
as follows from the fact that \( G_{X^*, Z^*} \) and \( G_{Y^*, Z^*} \) satisfy Hilbert space version Stein equations. The necessary and sufficient condition for the existence of a solution to the interpolation problem becomes that the \( J \)-gramian \( G^J_{X^*, Y^*}[Z^*] > 0 \). Note that we can derive Pick matrix as \( J \)-gramian for \( M^\perp \), in fact, given the similar data set in Problem 1.1.1, take the bases \( \{ k_{\lambda^{(i)}} \left[ \begin{array}{c} 1 \\ \overline{w_i} \end{array} \right] : i = 1, \ldots, N \} \) for \( M^\perp \), we see that

\[
[k_{\lambda^{(i)}} \left[ \begin{array}{c} 1 \\ \overline{w_i} \end{array} \right], k_{\lambda^{(j)}} \left[ \begin{array}{c} 1 \\ \overline{w_j} \end{array} \right]] = \frac{1 - \overline{w_i}w_j}{1 - \langle \lambda^{(i)}, \lambda^{(j)} \rangle} \geq 0
\]
if and only if \( M^\perp \) is a positive space.

Recall that for any output stable pair \((C, A)\)

\[
(C^* J[M_{\lambda^j}, f])^L(A^*) = A^*_j (C^* J f)^L(A^*).
\]

So any subspace \( \mathcal{M} \subset H_{E^*}(k_d) \) of the form in (3.2.13) is \( M_{A^*} \)-invariant.

For Krein spaces \( E \) and \( E^* \), a \( C(E, E^*) \)-valued function \( S \) is said to be a multiplier if the respective multiplication operators \( M_S : f(\lambda) \mapsto S(\lambda) \cdot f(\lambda) \) is closed from \( H_E(k_d) \cap D(S) \) into \( H_{E^*}(k_d) \). A multiplier \( \Theta \) in \( C(E, E^*) \) is said to be \( J \)-phase if the associated multiplication operator \( M_\Theta : H_E(k_d) \to H_{E^*}(k_d) \) is a \((J_E, J_{E^*})\)-partial isometry (or \( J \)-partial isometry). If moreover \( (\text{Ran } M_\Theta)^\perp \) is a positive subspace,
we say $M_{\Theta}$ is $J$-inner. In this case, the space $\mathcal{H}(K^J_{\Theta})$ defined as the reproducing kernel Hilbert space with reproducing kernel

$$K^J_{\Theta}(\lambda, \zeta) = \frac{J_{\Theta} - \Theta(\lambda)J_{\Theta}(\zeta)}{1 - \langle \lambda, \zeta \rangle},$$

is $J$-isometrically included in $\mathcal{H}_{E_\cdot}(K_{\Theta})$ and

$$\mathcal{H}(K_{\Theta}) = \mathcal{H}_{E_\cdot}(k_d) \ominus J_{\Theta}\mathcal{H}_{E_\cdot}(k_d) \quad (3.2.15)$$

Moreover, the $J$-orthogonal projection $P_{\mathcal{H}(K_{\Theta})}$ of $\mathcal{H}_{E_\cdot}(k_d)$ onto $\mathcal{H}_{E_\cdot}(K_{\Theta})$ is given by the continuous extension of

$$P_{\mathcal{H}(K_{\Theta})} = P_{\mathcal{H}_{E_\cdot}(k_d)} - M_{\Theta}M_{\Theta}^*.$$ 

We are interested in finding corresponding $J$-case Beurling-Lax representer such that any closed subspace $\mathcal{M}$ of $\mathcal{H}_{E_\cdot}(K_{\Theta})$ invariant under the operators $M_{\lambda_j}$ of multiplication by the coordinate functions necessarily has the form $\Theta \cdot \mathcal{H}_{E_\cdot}(k_d)$ for some $J$-inner multiplier $\Theta \in \mathcal{C}(E, E_*)$. We will give more information on the Beurling-Lax theorem in next section. We also delay the proof of the version we need here to the next section to keep the focus here on the interpolation.

Suppose that we are given a bi-($J, J_e$)-contractive $\Theta(\lambda)$ for each $\lambda \in \mathbb{B}^d$. If we decompose $\Theta$ as a block $2 \times 2$ matrix

$$\begin{bmatrix} \Theta_{11}(\lambda) & \Theta_{12}(\lambda) \\ \Theta_{21}(\lambda) & \Theta_{22}(\lambda) \end{bmatrix} : \begin{bmatrix} \tilde{Y} \\ U \end{bmatrix} \rightarrow \begin{bmatrix} Y \\ U \end{bmatrix}$$

and follow the bicontraction inequalities, a simple computation shows that $\Theta_{22}(\lambda)$ is invertible and $\|\Theta_{22}(\lambda)^{-1}\Theta_{21}(\lambda)\| < 1$ for each $\lambda \in \mathbb{B}^d$. Thus

$$\Theta_{11}(\lambda)S_0(\lambda) + \Theta_{22}(\lambda) = \Theta_{22}(\lambda)(\Theta_{22}(\lambda)^{-1}\Theta_{21}(\lambda) + I)$$

is invertible for all $\lambda \in \mathbb{B}^d$ and $S_0 \in S_d(U, \tilde{Y})$ and hence the linear fractional transform of $S_0$

$$T_\Theta[S_0](\lambda) = (\Theta_{11}S_0 + \Theta_{12})(\Theta_{21}S_0 + \Theta_{22})^{-1}$$

is a well-defined holomorphic $\mathcal{L}(U, Y)$-valued function on $\mathbb{B}^d$.

**Theorem 3.2.5** Let $\Theta$ be $J$-Beurling Lax representer for

$$\mathcal{M} = \left\{ h \in \mathcal{H}_{E_\cdot}(k_d) = \begin{bmatrix} \mathcal{H}_{E_\cdot}(k_d) \\ \mathcal{H}_{E_\cdot}(k_d) \end{bmatrix} : \left( \begin{bmatrix} X & -Y \end{bmatrix} h \right)^v_\mathcal{L}(Z) = 0 \right\}$$
Then $S$ solves the interpolation problem LTOAIP if and only if there exists $S_0 \in S_d(U, \tilde{Y})$ (where $\tilde{Y}$ is a bigger Hilbert space containing $Y$) such that

$$S = (\Theta_{11}S_0 + \Theta_{12})(\Theta_{21}S_0 + \Theta_{22})^{-1}$$

**Proof:**

Suppose $S$ solves the interpolation problem. From Theorem 3.3.2, we can write $\mathcal{M}$ as $\mathcal{M} = \text{clos.}\{M_{\Theta}D(M_{\Theta})\}$. Suppose $G = \begin{bmatrix} S(\lambda) & I \\ I & -M_S^* \end{bmatrix}$ for some $S \in S_d(U, Y)$ and $P_M = M_{\Theta}J\mathcal{M}^*J_k$. From Lemma 2.3.1, we know $G \subset \mathcal{M}$ if only if $P_MG^\perp > 0$, and the latter condition turns to

$$\begin{bmatrix} I & -M_S^* \end{bmatrix} M_{\Theta}J\mathcal{M}^* \begin{bmatrix} I \\ -M_S^* \end{bmatrix} \succeq 0$$

which translates to

$$\begin{bmatrix} I & -S(\lambda) \end{bmatrix} \frac{\Theta(\lambda)J\Theta(\zeta)^*}{1 - \langle \lambda, \zeta \rangle} \begin{bmatrix} I \\ -S^*(\zeta) \end{bmatrix} \succeq 0$$

If we set

$$\begin{bmatrix} u(\lambda) & -v(\lambda) \end{bmatrix} := \begin{bmatrix} I & -S(\lambda) \end{bmatrix} \Theta(\lambda),$$

then we get

$$\frac{u(\lambda)u(\zeta)^* - v(\lambda)v(\zeta)^*}{1 - \langle \lambda, \zeta \rangle} \succeq 0$$

where

$$u(\lambda) = \Theta_{11}(\lambda) - S(\lambda)\Theta_{21}(\lambda), \quad -v(\lambda) = \Theta_{12}(\lambda) - S(\lambda)\Theta_{22}(\lambda)$$

By the Leech theorem for Drury-Arveson space multipliers (see [24]), it follows that there exists $S_0 \in S(U, \tilde{Y})$ so that $v(\lambda) = u(\lambda)S_0(\lambda)$, i.e,

$$-(\Theta_{12}(\lambda) - S(\lambda)\Theta_{22}(\lambda)) = (\Theta_{11}(\lambda) - S(\lambda)\Theta_{21}(\lambda))S_0(\lambda)$$

or

$$S = (\Theta_{11}S_0 + \Theta_{12})(\Theta_{21}S_0 + \Theta_{22})^{-1}$$

In this case, we have

$$G = \begin{bmatrix} S(\lambda) & I \\ I & -M_S^* \end{bmatrix} \mathcal{H}_U(k_d) = M_{\Theta}(\begin{bmatrix} S(\lambda) & I \\ I & -M_S^* \end{bmatrix} \mathcal{H}_U(k_d) \cap D(M_{\Theta}))$$
Conversely, if we have \( S \) in the form
\[
S = (\Theta_{11}S_0 + \Theta_{12})(\Theta_{21}S_0 + \Theta_{22})^{-1}
\]
we can see that \( G = \begin{bmatrix} S(\lambda) \\ I \end{bmatrix} \mathcal{H}_d(k_d) = M_\Theta \begin{bmatrix} S(\lambda) \\ I \end{bmatrix} \mathcal{H}_d(k_d) \cap \mathcal{D}(M_\Theta) \subset \mathcal{M} \) and \( G \) is maximal negative and \( M_\mathcal{X} \)-invariant, which means \( S \) is a solution of the interpolation problem.

**Remark 3.2.6** Notice that here we combined the Krein space geometry approach and reproducing kernel approach (similar to Potapov’s approach) for general \( \Theta \). If we suppose \( \Theta \) is bounded, we have a more direct Krein space geometry argument. In fact, in this case, \( \mathcal{M} = \Theta \begin{bmatrix} \mathcal{H}_y(k_d) \\ \mathcal{H}_d(k_d) \end{bmatrix} \) and \( M_\Theta \) is bicontraction and \( J \)-partial isometry. From Lemma 2.3.5, we know \( \ker M_\Theta \) is uniformly positive, and \( M_\Theta \) maps isometrically \( \mathcal{H}_\mathcal{E}(k_d)/\ker M_\Theta \) onto \( \text{Ran} M_\Theta \). Thus if \( G \) is in the form \( G = \Theta G' \) with \( G' \) maximal negative, then \( G \) is \( \mathcal{M} \)-maximal negative, hence \( \mathcal{H}_\mathcal{E}(k_d) \)-maximal negative. Conversely, if a subspace \( G \) is \( \mathcal{M} \)-maximal negative subspace (hence \( \mathcal{H}_\mathcal{E}(k_d) \)-maximal negative subspace) then we can write any \( G \) satisfying the three conditions as \( G = \Theta G' \), where \( G' \) is \( \mathcal{H}_\mathcal{E}(k_d)/\ker M_\Theta \)-maximal negative subspace which is also \( \mathcal{H}_\mathcal{E}(k_d) \)-maximal negative subspace since \( \ker M_\Theta \) is uniformly positive. While we know that \( \mathcal{H}_\mathcal{E}(k_d) \)-maximal negative subspace is easily characterized as subspaces of the graph representation
\[
\begin{bmatrix} S_0(\lambda) \\ I \end{bmatrix} \mathcal{H}_d(k_d)
\]
So we have
\[
G = \Theta \begin{bmatrix} S_0(\lambda) \\ I \end{bmatrix} \mathcal{H}_d(k_d)
\]
thus for a solution \( S \) of the interpolation problem, we have
\[
G = \begin{bmatrix} S(\lambda) \\ I \end{bmatrix} \mathcal{H}_d(k_d) = \Theta \begin{bmatrix} S_0(\lambda) \\ I \end{bmatrix} \mathcal{H}_d(k_d)
\]
If we write \( \Theta \) in the block matrix form
\[
\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}
\]
Then we obtain
\[
S = (\Theta_{11}S_0 + \Theta_{12})(\Theta_{21}S_0 + \Theta_{22})^{-1}
\]
for free \( S_0 \) in Drury-Arveson Schur multiplier class \( S(\mathcal{U}, \tilde{\mathcal{Y}}) \). This is the approach of [21, 36] for single variable case: there this approach is adapted to handle the unbounded case via special single-variable techniques for functions on the unit circle.

Thus, we can summarize this section with the following theorem.

**Theorem 3.2.7** Suppose we are given the admissible data in the LTOAIP. Then solutions of LTOAIP exist if and only if \( J \)-gramian \( G_{[X^*]}\mathcal{Z} \geq 0 \). Moreover, if we assume that \( J \)-gramian \( G_{[X^*]}\mathcal{Z} \) is strictly positive, then the class of all solutions \( S \) of the problem coincides with the class of all \( S \) with the representation
\[
S = (\Theta_{11}S_0 + \Theta_{12})(\Theta_{21}S_0 + \Theta_{22})^{-1}
\]
for some free \( S_0 \) in Drury-Arveson Schur multiplier class \( S(\mathcal{U}, \tilde{\mathcal{Y}}) \), where \( \Theta \) is the Beurling-Lax representer.

### 3.3 Beurling-Lax theorems

We have seen from the last section that the \( J \)-version of Beurling-Lax theorem plays an important role in parametrizing all the solutions of our interpolation problem. Since the topic of Beurling-Lax theorems is itself very active, we would like to give it a brief review here and then prove the version we used for the interpolation problem we consider.

The Beurling-Lax theorem gives a characterization of all invariant subspaces for the shift operator on the Hardy space \( H^2 \) in terms of inner functions. We know that the space \( H^2 \) can be characterized as the space of holomorphic functions \( f(z) \) on the unit disk having Taylor series representation \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with square-summable coefficients. The operator \( S \) of multiplication by the coordinate function \( z \), i.e., \( S : f(z) \rightarrow zf(z) \) is called the shift operator on \( H^2 \). A closed linear subspace \( \mathcal{M} \) of \( H^2 \) is said to be invariant for \( S \) if \( Sf \in \mathcal{M} \) whenever \( f \in \mathcal{M} \).

Beurling characterized all invariant subspaces for the shift operator on the Hardy space \( H^2 \) in terms of inner functions in 1949 (see [49]).
Theorem 3.3.1  Any shift-invariant subspace as above has the form
\[ \theta H^2 = \{ \theta(z) f(z) : f \in H^2 \} \]
where \( \theta(z) \) is an inner function, i.e., a holomorphic function on the unit disk with contractive values such that its boundary values have modulus 1 (i.e., \(|\theta(e^{it})| = | \lim_{r \to 1} \theta(re^{it})| = 1 \) for almost all \( e^{it} \).

P.D. Lax extended the result to finite-dimensional vector-valued \( H^2 \) in [93] (where of the unit disk is replaced by the right half-plane). Later, the proof for the infinite-dimensional case was given by P.R. Halmos in [78].

There have been plenty of generalizations and applications of this result since then. Beurling himself observed that the detailed parametrization of inner functions leads to a complete characterization of the lattice structure of the lattice of invariant subspaces for the shift operator. In the 1960s, operator theorists found that the compression of the shift operator to the orthogonal complement of a shift-invariant subspace \( \theta H^2 \) serves as a model for a rich class of Hilbert-space operators. The model theories of L. de Branges and J. Rovnyak ([58]) and that of B. Sz.-Nagy and C. Foias ([104]) both give generalizations of the result leading to a model for an arbitrary, completely non-unitary contraction operator on a Hilbert space. The function \( \theta \) involved is called the characteristic function of the associated operator. In the case where \( \theta \) is inner, the model space reduces to the Beurling-Lax form.

There also has evolved a theory of so-called \( C_0 \) operators, for which the operators (with a scalar inner function) are the building blocks in an analogue of a canonical Jordan form which classifies operators up to quasi-similarity (see [48] and [104]).

In 1980s, a series of papers of Ball and Helton uncovered a new type of Beurling-Lax representation for the vector-valued case where one demands that an indefinite rather than Hilbert-space inner product be preserved. The results lead directly to the linear-fractional parametrization for the solution set of the classical Nevanlinna Pick interpolation problem and can be applied to systems control theory, electrical circuits, etc. We already mentioned this in the introduction. The Hilbert space Beurling-Lax theorem has been extended to Drury-Arveson space (see [18, 19, 27, 30, 98]). Here our goal is to extend the Beurling-Lax theorem for
the Drury-Arveson space to a Kre

in-space version, i.e., the \( J \)-version which we need to solve the interpolation problem in Drury-Arveson space.

**Theorem 3.3.2** Suppose that \( \mathcal{M} \) is a regular subspace of \( \mathcal{H}_{E^*}(k_d) \) (hence \( \mathcal{H}_{E^*}(k_d) = \mathcal{M} + \mathcal{M}^{[1]} \) and \( \mathcal{M} \) is a Kre

in space) with \( \mathcal{M}^{[1]} \) positive, where \( E^* = \begin{bmatrix} Y \\ U \end{bmatrix} \) with the \( J^* \)-inner product, then there is a coefficient Kre

in space \( E \) and a \( J \)-inner multiplier \( \Theta \) so that

\[ \mathcal{M} = \text{Ran} M_{\Theta} = \text{clos.} \{ \Theta \cdot D(M_{\Theta}) \}. \]

**Proof:** Suppose \( \mathcal{M} \) is a Kre

in space and \( \mathcal{M}^{[1]} \) positive. Define a \( d \)-tuple of operators \( A = (A_1, \ldots, A_d) \) on \( \mathcal{M}^{[1]} \) by

\[ A_j = M_{\lambda_j}^{[1]}|_{\mathcal{M}^{[1]}} \quad \text{for} \quad j = 1, \ldots, d. \]

Define an operator \( C \) on \( \mathcal{M}^{[1]} \) by

\[ C : f \to f(0) \quad \text{for} \quad f \in \mathcal{M}^{[1]}. \]

We denote \( \mathcal{X} = \mathcal{M}^{[1]} \). Note that \( \mathcal{M}^{[1]} \) is contained in \( \mathcal{H}_{E^*}(k_d) \) isometrically and that

\[ C = G|_{\mathcal{M}^{[1]}}, \quad A_j = M_{\lambda_j}^{[1]}|_{\mathcal{M}^{[1]}} \]

where \( C : f \to f(0) \) and \( O_{G,M_{\lambda_j}^{[1]}} \) is the identity on \( \mathcal{H}_{E^*}(k_d) \) (see more details in [30] for the Hilbert space case). Hence in particular \( \text{Ran} O_{C,A} = \mathcal{M}^{[1]} \). Since \( \mathcal{M} \) is regular, then the \( J \)-gramian \( P := G_{C,A}^d = O_{C,A}^* J \) is invertible. From the construction in Theorem 2.3 in [26], we know \( \mathcal{M}^{[1]} \) is isometrically equal to the reproducing kernel Hilbert space with reproducing kernel \( K_{C,A}^P \) given by

\[ K_{C,A}^P(\lambda, \zeta) = C(I - Z(\lambda)A)^{-1}P^{-1}(1 - A^*Z(\zeta))^{-1}C^* \]

If we can take \( \tilde{Y} = \mathcal{X}^{d-1} \oplus \mathcal{Y} \), we can construct an operator

\[ \begin{bmatrix} B \\ D \end{bmatrix} : \begin{bmatrix} \tilde{Y} \\ U \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y}^{d-1} \end{bmatrix} \]

such that the operator

\[ U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y}^{d-1} \oplus \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \oplus \mathcal{U} \end{bmatrix} \]
where \( \Theta \) and both
\[
\begin{bmatrix} P^{-1} & 0 \\ 0 & J \end{bmatrix}
\]
and
\[
\begin{bmatrix} P \otimes I_d & 0 \\ 0 & J \end{bmatrix}
\]
where \( J = \begin{bmatrix} I_{\lambda^d-1} \ominus \gamma & 0 \\ 0 & -I_{\delta} \end{bmatrix} \) then the kernel \( K_{C,A}^P(\lambda, \zeta) \) appearing above can be expressed as
\[
K_{C,A}^P(\lambda, \zeta) = K_{\Theta_0}^J(\lambda, \zeta) = \frac{J_* - \Theta_0(\lambda)J\Theta_0(\zeta)^*}{1 - \langle \lambda, \zeta \rangle},
\]
where \( \Theta_0(\lambda) \) is the characteristic function of the colligation \( U \)
\[
\Theta_0(\lambda) = D + C(I - Z(\lambda)A)^{-1}Z(\lambda)B
\]
In this case, \( \Theta_0(\lambda) \) is bi-(\( J, J_* \))-contractive for each \( \lambda \in B^d \). Now we can define \( M_{\Theta_0} \) on the set \( \{ \text{span}_{\zeta \in B^d} J\Theta_0(\zeta)^*k_d(\cdot, \zeta) \} \) + Ker \( M_{\Theta_0} \) by
\[
M_{\Theta_0}J\Theta_0(\zeta)^*k_d,\zeta(\lambda)e_* = \frac{\Theta_0(\lambda)J\Theta_0(\zeta)^*}{1 - \langle \lambda, \zeta \rangle} \in H_{E_*(k_d)}
\]
(This is because
\[
K_{\Theta_0}^J(\lambda, \zeta)e_* = \frac{J_* - \Theta_0(\lambda)J\Theta_0(\zeta)^*}{1 - \langle \lambda, \zeta \rangle} = \frac{J_*e_*}{1 - \langle \lambda, \zeta \rangle} - \Theta_0(\lambda)J\Theta_0(\zeta)^*
\]
and both \( K_{\Theta_0}^J(\lambda, \zeta)e_* \) and \( \frac{J_*e_*}{1 - \langle \lambda, \zeta \rangle} \) are in \( H_{E_*(k_d)} \) as functions of \( \lambda \).

We know there exist a bounded \( J \)-orthogonal projection \( P_M : H_{E_*(k_d)} \to \mathcal{M} \) and \( P_M^{[\lambda]} = I - P_M : H_{E_*(k_d)} \to \mathcal{M}^{[\lambda]} \). The \( C(\mathcal{Y} \oplus \mathcal{U}, \mathcal{Y} \oplus \mathcal{U}) \)-valued function \( \Theta_0 \) maps constant functions in \( H_{\mathcal{Y} \oplus \mathcal{U}}(k_d)(\text{and hence also polynomials}) \) into \( H_{\mathcal{Y} \oplus \mathcal{U}}(k_d) \). We can check that \( J_*k_d(\cdot, \zeta)e_* \in \mathcal{D}(M_{\Theta_0}^*) \) and \( M_{\Theta_0}^*J_*k_d(\cdot, \zeta) = \Theta_0(\zeta)^*J_*k_d(\cdot, \zeta) \). So in this case the operator \( I - M_{\Theta_0}M_{\Theta_0}^{[\lambda]} \) initially only defined on linear combinations of kernel functions via
\[
I - M_{\Theta_0}M_{\Theta_0}^{[\lambda]} = I - M_{\Theta_0}JM_{\Theta_0}J_* : k_d(\cdot, \zeta)e_* \to k_d(\cdot, \zeta)e_* - \Theta_0(\cdot)k_d(\cdot, \zeta)J\Theta_0(\zeta)^*J_*e_*
\]
extends continuously to the \( J \)-orthogonal projection operator mapping \( H_{\mathcal{Y} \oplus \mathcal{U}}(k_d) \) onto \( \mathcal{M}^{[\lambda]} \), or we can say that \( M_\Theta \) is \( J \)-inner multiplier (maybe unbounded). If
we denote $D_0 = \{ M_{\Theta_0} J, k \xi \in \varepsilon \} + \text{Ker} M_{\Theta_0} J$, it can be verified that $D_0$ is dense in $H_{\mathcal{E}}(k_d)$ and $M_{\Theta_0} J |_{D_0}$ is closable, so if we define $M_\Theta = \text{closure of } M_{\Theta_0} J |_{D_0}$, then $\mathcal{M} = \text{Ran} M_{\Theta}$. Notice that $M_\Theta$ is $J$-inner multiplier since $M_{\Theta_0}$ is.

\[ \square \]

Remark 3.3.3 Note that if $\Theta$ is a bounded multiplier between $H_{\mathcal{Y} \oplus \mathcal{U}}(k_d)$ and $H_{\mathcal{Y} \oplus \mathcal{U}}(k_d)$, it follows that $\mathcal{M} = M_{\Theta} H_{\mathcal{Y} \oplus \mathcal{U}}(k_d)$ where the multiplication operator $M_{\Theta}$ is a $J$-partial isometry. This is the Kreĭn space version of the Beurling-Lax theorem.

Remark 3.3.4 For more information about unbounded $J$-unitary operators, we refer to [63]. In the single-variable case $(d = 1)$, we have $J = J_*$ and these functions coincide with the strongly regular $J$-inner functions in the sense of Arov-Dym [16] (see also [39]) which are holomorphic on the unit disk. There exist examples of $J$-inner functions which are unbounded. In [16], using the equivalence between Muckenhoupt condition and the boundedness of orthogonal projection given in [126], the authors give a family of matrix-valued functions which are $J$-inner but not bounded.

Remark 3.3.5 Notice that the example in the previous remark can also serve as the example for strictly contractive Hankel matrix $\Gamma$ associated with $\{c_n\}_{n \leq -1}$ such that the central contractive extension $f_c$ of $\Gamma$ satisfies $\|f_c\|_\infty = 1$, which appeared in [20]. Since it is proved in [75] that $\|\Gamma\| < 1$ and $\sum_{n=-\infty}^{-1} |c_n| < \infty$ imply $\|f_c\|_\infty < 1$, so if we take any unbounded strongly regular $J$-inner function $f$, then the associated Nehari problem is strictly completely indeterminate, thus the corresponding $\Gamma$ satisfies $\|\Gamma\| < 1$ but $\|f_c\|_\infty = 1$. More information on this can be found in [53].

There have been also extensions of Beurling-Lax theorem from the point of view of function-theoretic operator theory since 1960s. Halmos’ wandering subspace construction in other directions has been considered too. For example, there is a Beurling-like representation theorem for invariant subspaces of the Dirichlet shift (see [119]), a notion of inner divisor for the Bergman space (see [81]) and a Beurling-type theorem for Bergman space (see [5]). Recently Beurling-like representation theorem about some kinds of quasi-inner functions has come up too (see [86]).
Since these topics are not quite related to this dissertation, we won’t develop further discussion here.
Chapter 4

Interpolation Problems on the Noncommutative Ball

In this chapter we present noncommutative analogues of the results in the previous chapter, where the Schur class is replaced by the noncommutative Schur class of contractive multipliers between Fock spaces of formal power series in noncommuting indeterminates and where the reproducing kernel Hilbert spaces become the noncommutative formal reproducing kernel Hilbert spaces introduced in [44]. In Section 4.1 we recall the main facts from [44] which are needed in the sequel. The noncommutative Schur class of contractive Fock-space multipliers $S$ and the associated noncommutative positive kernel $K_S(z,w)$ are introduced and developed in Section 4.2. Using the results concerning functional models we arrive at the Fock-space version of Beurling-Lax Theorem in Section 4.3. Then we solve the left-tangential operator-argument interpolation problem for the Fock space on the noncommutative ball in the final section 4.4. In contrast with the parallel theory for the Arveson space on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$ (which can be viewed as the symmetrized version of the Fock space used here), the results here are much more in line with the classical univariate case, with the extra ingredient of the existence of all results having both a “left” and a “right” version.
4.1 Noncommutative reproducing kernel Hilbert spaces

Noncommutative formal reproducing kernel Hilbert spaces were introduced in [44]. We recall the basic ideas that we will use here.

Let \( z = (z_1, \ldots, z_d) \) and \( w = (w_1, \ldots, w_d) \) be two sets of noncommuting indeterminates. We let \( \mathcal{F}_d \) denote the free semigroup generated by the \( d \) letters \( \{1, \ldots, d\} \). A generic element of \( \mathcal{F}_d \) is a word \( w \) equal to a string of letters

\[
\alpha = i_N \cdots i_1 \quad \text{where} \quad i_k \in \{1, \ldots, d\} \quad \text{for} \quad k = 1, \ldots, N. \quad (4.1.1)
\]

Given two words \( \alpha \) and \( \beta \) with \( \alpha \) as in (4.1.1) and \( \beta \) of the form \( \beta = j_N \cdots j_1 \), say, the product \( \alpha \beta \) is defined by concatenation:

\[
\alpha \beta = i_N \cdots i_1 j_N \cdots j_1 \in \mathcal{F}_d.
\]

The unit element of \( \mathcal{F}_d \) is the empty word denoted by \( \emptyset \). For \( \alpha \) a word of the form (4.1.1), we let \( z^\alpha \) denote the monomial in noncommuting indeterminates

\[
z^\alpha = z_{i_N} \cdots z_{i_1}
\]

and we let \( z^\emptyset = 1 \). We extend this noncommutative functional calculus to a \( d \)-tuple of operators \( \mathbf{A} = (A_1, \ldots, A_d) \) on a Hilbert space \( \mathcal{X} \):

\[
\mathbf{A}^v = A_{i_N} \cdots A_{i_1} \quad \text{if} \quad v = i_N \cdots i_1 \in \mathcal{F}_d \setminus \{\emptyset\}; \quad \mathbf{A}^\emptyset = I_{\mathcal{X}}. \quad (4.1.2)
\]

We will also have need of the transpose operation on \( \mathcal{F}_d \):

\[
\alpha^\top = i_1 \cdots i_N \quad \text{if} \quad \alpha = i_N \cdots i_1. \quad (4.1.3)
\]

Given a coefficient Hilbert space \( \mathcal{Y} \) we let \( \mathcal{Y}(z) \) denote the set of all polynomials in \( z = (z_1, \ldots, z_d) \) with coefficients in \( \mathcal{Y} \):

\[
\mathcal{Y}(z) = \left\{ p(z) = \sum_{\alpha \in \mathcal{F}_d} p_\alpha z^\alpha : p_\alpha \in \mathcal{Y} \text{ and } p_\alpha = 0 \text{ for all but finitely many } \alpha \right\},
\]

while \( \mathcal{Y}(\langle z \rangle) \) denotes the set of all formal power series in the indeterminates \( z \) with coefficients in \( \mathcal{Y} \):

\[
\mathcal{Y}(\langle z \rangle) = \left\{ f(z) = \sum_{\alpha \in \mathcal{F}_d} f_\alpha z^\alpha : f_\alpha \in \mathcal{Y} \right\}.
\]
Note that vectors in $\mathcal{Y}$ can be considered as Hilbert space operators between $\mathcal{C}$ and $\mathcal{Y}$. More generally, if $\mathcal{U}$ and $\mathcal{Y}$ are two Hilbert spaces, we let $\mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z \rangle\rangle$ denote the space of polynomials (respectively, formal power series) in the noncommuting indeterminates $z = (z_1, \ldots, z_d)$ with coefficients in $\mathcal{L}(\mathcal{U},\mathcal{Y})$.

Given $S = \sum_{\alpha \in \mathcal{F}_d} s_{\alpha} z^\alpha \in \mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z \rangle\rangle$ and $f = \sum_{\beta \in \mathcal{F}_d} f_{\beta} z^\beta \in \mathcal{U}\langle\langle z \rangle\rangle$, the product $S(z) \cdot f(z) \in \mathcal{Y}\langle\langle z \rangle\rangle$ is defined as an element of $\mathcal{Y}\langle\langle z \rangle\rangle$ via the noncommutative convolution:

$$ S(z) \cdot f(z) = \sum_{\alpha,\beta \in \mathcal{F}_d} s_{\alpha} f_{\beta} z^{\alpha \beta} = \sum_{v \in \mathcal{F}_d} \left( \sum_{\alpha,\beta : \alpha \cdot \beta = v} s_{\alpha} f_{\beta} \right) z^v. \quad (4.1.4) $$

Note that the coefficient of $z^v$ in (4.1.4) is well defined since any given word $v \in \mathcal{F}_d$ can be decomposed as a product $v = \alpha \cdot \beta$ in only finitely many distinct ways.

In general, given a coefficient Hilbert space $\mathcal{C}$, we use the $\mathcal{C}$ inner product to generate a pairing $\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C}} : \mathcal{C} \times \mathcal{C}\langle\langle w \rangle\rangle \rightarrow \mathcal{C}\langle\langle w \rangle\rangle$ via

$$ \left\langle c, \sum_{\beta \in \mathcal{F}_d} f_{\beta} w^{\beta} \right\rangle_{\mathcal{C} \times \mathcal{C}\langle\langle w \rangle\rangle} = \sum_{\beta \in \mathcal{F}_d} \langle c, f_{\beta} \rangle_{\mathcal{C}} w^{\beta T} \in \mathcal{C}\langle\langle w \rangle\rangle. $$

We also may use the pairing in the reverse order

$$ \langle \sum_{\alpha \in \mathcal{F}_d} f_{\alpha} w^{\alpha}, c \rangle_{\mathcal{C}\langle\langle w \rangle\rangle \times \mathcal{C}} = \sum_{\alpha \in \mathcal{F}_d} \langle f_{\alpha}, c \rangle_{\mathcal{C}} w^{\alpha} \in \mathcal{C}\langle\langle w \rangle\rangle. $$

These are both special cases of the more general pairing

$$ \left\langle \sum_{\alpha \in \mathcal{F}_d} f_{\alpha} w^{\alpha}, \sum_{\beta \in \mathcal{F}_d} g_{\beta} w^{\beta} \right\rangle_{\mathcal{C}\langle\langle w' \rangle\rangle \times \mathcal{C}\langle\langle w \rangle\rangle} = \sum_{\alpha,\beta \in \mathcal{F}_d} \langle f_{\alpha}, g_{\beta} \rangle_{\mathcal{C}} w^{\beta T} w^{\alpha} \in \mathcal{C}\langle\langle w, w' \rangle\rangle. $$

Suppose that $\mathcal{H}$ is a Hilbert space whose elements are formal power series in $\mathcal{Y}\langle\langle z \rangle\rangle$ and that $K(z, w) = \sum_{\alpha,\beta \in \mathcal{F}_d} K_{\alpha,\beta} z^{\alpha} w^{\beta T}$ is a formal power series in the two sets of $d$ noncommuting indeterminates $z = (z_1, \ldots, z_d)$ and $w = (w_1, \ldots, w_d)$. We say that $K(z, w)$ is a reproducing kernel for $\mathcal{H}$ if, for each $\beta \in \mathcal{F}_d$ the formal power series $K_{\beta}(z) := \sum_{\alpha \in \mathcal{F}_d} K_{\alpha,\beta} z^{\alpha}$ belongs to $\mathcal{H}$ and we have the reproducing property

$$ \langle f, K(\cdot, w) y \rangle_{\mathcal{H} \times \mathcal{H}\langle\langle w \rangle\rangle} = \langle f(w), y \rangle_{\mathcal{Y}\langle\langle w \rangle\rangle \times \mathcal{Y}} \quad \text{for every } f \in \mathcal{H}. $$
As a consequence we then also have
\[ \langle K(\cdot, w')y', K(\cdot, w)y \rangle_{H(\langle w' \rangle) \times H(\langle w \rangle)} = \langle K(w, w')y', y \rangle_{Y(\langle w, w' \rangle) \times Y}. \]

It is not difficult to see that a reproducing kernel for a given \( H \) is necessarily unique.

Let us now suppose that \( H \) is a Hilbert space whose elements are formal power series \( f(z) = \sum_{\alpha \in \mathcal{F}_d} f_\alpha z^\alpha \in \mathcal{Y}(\langle z \rangle) \) for a coefficient Hilbert space \( \mathcal{Y} \). We say that \( H \) is a NFRKHS (noncommutative formal reproducing kernel Hilbert space) if, for each \( \alpha \in \mathcal{F}_d \), the linear operator \( \Phi_\alpha : H \to \mathcal{Y} \) defined by \( f(z) = \sum_{\alpha \in \mathcal{F}_d} f_\alpha z^\alpha \mapsto f_\alpha \) is continuous. Define \( K(z, w) \in \mathcal{L}(\mathcal{Y}(\langle z, w \rangle)) \) by
\[
K(z, w) = \sum_{\beta \in \mathcal{F}_d} \Phi^*_\beta w^{\beta T} =: \sum_{\alpha, \beta \in \mathcal{F}_d} K_{\alpha, \beta} z^\alpha w^{\beta T}.
\]

Then one can check that \( K(z, w) \) is a reproducing kernel for \( H \) in the sense defined above. Conversely (see [44, Theorem 3.1]), a given formal kernel \( K(z, w) = \sum_{\alpha, \beta \in \mathcal{F}_d} K_{\alpha, \beta} z^\alpha w^{\beta T} \in \mathcal{L}(\mathcal{Y}(\langle z, w \rangle)) \) is the reproducing kernel for some NFRKHS \( H \) if and only if \( K \) is positive definite in either one of the equivalent senses:

1. \( K(z, w) \) has a factorization
\[
K(z, w) = H(z)H(w)^* \quad (4.1.5)
\]
for some \( H \in \mathcal{L}(\mathcal{X}, \mathcal{Y})(\langle z \rangle) \) for some auxiliary Hilbert space \( \mathcal{X} \). Here
\[
H(w)^* = \sum_{\beta \in \mathcal{F}_d} H^*_\beta w^{\beta T} = \sum_{\beta \in \mathcal{F}_d} H^*_\beta w^{\beta T} \quad \text{if} \quad H(z) = \sum_{\alpha \in \mathcal{F}_d} H_\alpha z^\alpha.
\]

2. For all finitely supported \( \mathcal{Y} \)-valued functions \( \alpha \mapsto y_\alpha \) it holds that
\[
\sum_{\alpha, \alpha' \in \mathcal{F}_d} \langle K_{\alpha, \alpha'} y_{\alpha'}, y_\alpha \rangle \geq 0. \quad (4.1.6)
\]

If \( K \) is such a positive kernel, we denote by \( \mathcal{H}(K) \) the associated NFRKHS consisting of elements of \( \mathcal{Y}(\langle z \rangle) \).

### 4.2 Noncommutative Fock space and Schur-class

We let \( z = (z_1, \ldots, z_d) \) to be a collection of \( d \) formal noncommuting variables and let \( \mathcal{Y}(\langle z \rangle) \) denote the set of formal noncommutative series \( \sum_{v \in \mathcal{F}_d} f_v z^v \) where \( f_v \in \mathcal{Y} \)
and where
\[ z^v = z_{i_N} z_{i_{N-1}} \cdots z_{i_1} \text{ if } v = i_N i_{N-1} \cdots i_1. \]  
(4.2.1)

The Fock space \( \ell^2_F(\mathcal{F}_d) \) is defined as
\[ \ell^2_F(\mathcal{F}_d) := \left\{ \{f_v\}_{v \in \mathcal{F}_d} : \sum_{v \in \mathcal{F}_d} \|f_v\|_{\mathcal{Y}}^2 < \infty \right\}. \]  
(4.2.2)

If we let \( \chi_v \) be the characteristic function of the word \( v \), so
\[ \chi_v = \{\chi_v(v')\}_{v' \in \mathcal{F}_d} \text{ where } \chi_v(v') = \begin{cases} 1 & \text{if } v' = v, \\ 0 & \text{otherwise}, \end{cases} \]
and we let \( \mathcal{B}_\mathcal{Y} \) be an orthonormal basis for \( \mathcal{Y} \), then \( \{\chi_v y_i : v \in \mathcal{F}_d, y_i \in \mathcal{B}_\mathcal{Y}\} \) is an orthonormal basis for \( \ell^2_F(\mathcal{F}_d) \). The space \( \ell^2_F(\mathcal{F}_d) \) can be identified as the tensor product \( \ell^2(\mathcal{F}_d) \otimes \mathcal{Y} \) and is mapped unitarily onto the space
\[ H^2_\mathcal{Y}(\mathcal{F}_d) = \left\{ \sum_{v \in \mathcal{F}_d} f_v z^v \in \mathcal{Y}(\langle z \rangle) : \sum_{v \in \mathcal{F}_d} \|f_v\|_{\mathcal{Y}}^2 < \infty \right\} \]  
(4.2.3)

by the noncommutative \( Z \)-transform
\[ \{f_v\}_{v \in \mathcal{F}_d} \mapsto f^\wedge(z) = \sum_{v \in \mathcal{F}_d} f_v z^v \]  
(4.2.4)

with the monomials \( z^v \) playing the role of the basis vectors \( \chi_v \).

This Fock space \( H^2_\mathcal{Y}(\mathcal{F}_d) \) is a natural analogue of the vector-valued Hardy space over the unit disk (see e.g. [110]) with coefficients in \( \mathcal{Y} \). When \( \mathcal{Y} = \mathbb{C} \) we write simply \( H^2(\mathcal{F}_d) \). As explained in [44], \( H^2(\mathcal{F}_d) \) is a NFRKHS with reproducing kernel equal the following noncommutative analogue of the classical Szegö kernel:
\[ k_{S_k}(z, w) = \sum_{\alpha \in \mathcal{F}_d} z^\alpha w^{\alpha^\top}. \]  
(4.2.5)

Thus we have in general \( H^2_\mathcal{Y}(\mathcal{F}_d) = \mathcal{H}(k_{S_k} \otimes I_\mathcal{Y}) \). We let \( S_j \) denote the shift operator
\[ S_j : f(z) = \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto f(z) \cdot z_j = \sum_{v \in \mathcal{F}_d} f_v z^{v+j} \text{ for } j = 1, \ldots, d \]  
(4.2.6)
on \( H^2_\mathcal{Y}(\mathcal{F}_d) \); when we wish to specify the coefficient space \( \mathcal{Y} \) explicitly, we write \( S_j \otimes I_\mathcal{Y} \) rather than only \( S_j \). The adjoint of \( S_j : H^2_\mathcal{Y}(\mathcal{F}_d) \rightarrow H^2_\mathcal{Y}(\mathcal{F}_d) \) is then given by
\[ S_j^* : \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto \sum_{v \in \mathcal{F}_d} f_{v-j} z^v \text{ for } j = 1, \ldots, d. \]  
(4.2.7)
We let $\mathcal{M}_{nc,d}(\mathcal{U}, \mathcal{Y})$ denote the set of formal power series $S(z) = \sum_{\alpha \in F_d} s_{\alpha} z^\alpha$ with coefficients $s_{\alpha} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ such that the associated multiplication operator $M_S: f(z) \mapsto S(z) \cdot f(z)$ (see (4.1.4)) defines a bounded operator from $H^2_U(F_d)$ to $H^2_Y(F_d)$. It is not difficult to show that $\mathcal{M}_{nc,d}(\mathcal{U}, \mathcal{Y})$ is the intertwining space for the two tuples $\mathbf{S} \otimes I_{\mathcal{U}} = (S_1 \otimes I_{\mathcal{U}}, \ldots, S_d \otimes I_{\mathcal{U}})$ and $\mathbf{S} \otimes I_{\mathcal{Y}} = (S_1 \otimes I_{\mathcal{Y}}, \ldots, S_d \otimes I_{\mathcal{Y}})$: an operator $X \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ equals $M_S$ for some $S \in \mathcal{M}_{nc,d}(\mathcal{U}, \mathcal{Y})$ whenever $(S_j \otimes I_{\mathcal{Y}})X = X(S_j \otimes I_{\mathcal{U}})$ for $j = 1, \ldots, d$ (see e.g. [111] with somewhat different conventions). We define the noncommutative Schur-class $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ to consist of such multipliers $S$ for which $M_S$ has operator norm at most 1:

$$\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y}) = \{ S \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) : M_S : H^2_U(F_d) \to H^2_Y(F_d) \text{ with } \|M_S\|_{op} \leq 1 \}. \quad (4.2.8)$$

The following is the noncommutative analogue of Theorem 3.1.1 for this setting.

**Theorem 4.2.1** Let $S(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ be a formal power series in $z = (z_1, \ldots, z_d)$ with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then the following are equivalent:

1. $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$, i.e., $M_S : \mathcal{U}\langle\langle z \rangle\rangle \to \mathcal{Y}\langle\langle z \rangle\rangle$ given by $M_S : p(z) \to S(z)p(z)$ extends to define a contraction operator from $H^2_U(F_d)$ into $H^2_Y(F_d)$.

2. The kernel

$$K_S(z, w) := k_{Sz}(z, w) - S(z)k_{Sz}(z, w)S(w)^* \quad (4.2.9)$$

is a noncommutative positive kernel (see (4.1.5) and (4.1.6)).

3. There exists a Hilbert space $\mathcal{X}$ and a unitary connection operator $\mathbf{U}$ of the form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \quad (4.2.10)$$

so that $S(z)$ can be realized as a formal power series in the form

$$S(z) = D + \sum_{j=1}^{d} \sum_{v \in F_d} C A^v B_j z^v \cdot z_j = D + C(I - Z(z)A)^{-1}Z(z)B \quad (4.2.11)$$

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where we have set

\[ Z(z) = \begin{bmatrix} z_1 I_X & \ldots & z_d I_X \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}. \quad (4.2.12) \]

4. There exists a Hilbert space \( X \) and a contractive block operator matrix \( U \) as in (4.2.10) such that \( S(z) \) is given as in (4.2.11)

**Proof:** (1) \( \implies \) (2) is Theorem 3.15 in [44]. A proof of (2) \( \implies \) (3) is done in [45, Theorem 5.4.1] as an application of the Sz.-Nagy-Foiaş model theory for row contractions worked out there following ideas of Popescu [110, 111]; an alternative proof via the “lurking isometry argument” can be found in [44, Theorem 3.16]. The implication (3) \( \implies \) (4) is trivial. The content of (4) \( \implies \) (1) amounts to Proposition 4.1.3 in [45]. \( \square \)

We note that formula (4.2.11) has the interpretation that \( S(z) \) is the transfer function of the multidimensional linear system with evolution along \( F_d \) given by the input-state-output equations

\[ \Sigma : \begin{cases} x(1 \cdot \alpha) &= A_1 x(\alpha) + B_1 u(\alpha) \\ \vdots &= \vdots \\ x(d \cdot \alpha) &= A_d x(\alpha) + B_d u(\alpha) \\ y(\alpha) &= C x(\alpha) + D u(\alpha) \end{cases} \quad (4.2.13) \]

initialized with \( x(\emptyset) = 0 \). Here \( u(\alpha) \) takes values in the input space \( U \), \( x(\alpha) \) takes values in the state space \( X \), and \( y(\alpha) \) takes values in the output space \( Y \) for each \( \alpha \in F_d \). If we introduce the noncommutative \( Z \)-transform

\[ \{ x(\alpha) \}_{\alpha \in F_d} \mapsto \hat{x}(z) := \sum_{\alpha \in F_d} x(\alpha) z^\alpha \]

and apply this transform to each of the system equations in (4.2.13), one can solve for \( \hat{y}(z) \) in terms of \( \hat{u}(z) \) to arrive at

\[ \hat{y}(z) = T_\Sigma(z) \cdot \hat{u}(z) \]

where the transfer function \( T_\Sigma(z) \) of the system (4.2.13) is the formal power series with coefficients in \( \mathcal{L}(U, Y) \) given by

\[ T_\Sigma(z) = D + \sum_{j=1}^{d} \sum_{\alpha \in F_d} C A^\alpha B_j z^\alpha z_1 = D + C(I - Z(z)A)^{-1} Z(z)B. \quad (4.2.14) \]

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For complete details, we refer to [33,34,45].

The implication $(4) \implies (2)$ can be seen directly via the explicit identity (4.2.15) given in the next proposition; for the commutative case we refer to [9, Lemma 2.2].

**Proposition 4.2.1** Suppose that $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ is contractive with associated transfer function $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ given by (4.2.11). Then the kernel $K_S(z, w)$ given by (4.2.9) is can also be represented as

$$K_S(z, w) = C(I_X - Z(z)A)^{-1}(I_X - A^*Z(w)^*)^{-1}C^* + D_S(z, w) \quad (4.2.15)$$

where

$$D_S(z, w) = \left[ C(I_X - Z(z)A)^{-1}Z(z) \quad I_Y \right] k_{Sa}(z, w)$$

$$\cdot \left[ \begin{array}{c} Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ I_Y \end{array} \right]. \quad (4.2.16)$$

**Proof:** The result appears in [28]. We include the short proof for completeness.

For a fixed $\alpha \in \mathcal{F}_d$, let us set

$$X_\alpha = z^\alpha w^{\alpha^*} I_Y - S(z)z^\alpha w^{\alpha^*} S(w)^*,$$

$$Y_\alpha = \left[ C(I - Z(z)A)^{-1}Z(z) \quad I_Y \right] z^\alpha w^{\alpha^*} \left[ \begin{array}{c} Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ I_Y \end{array} \right]. \quad (4.2.17)$$

Note that by (4.2.9) and (4.2.5),

$$\sum_{\alpha \in \mathcal{F}_d} X_\alpha = K_S(z, w) \quad \text{and} \quad \sum_{\alpha \in \mathcal{F}_d} Y_\alpha = D_S(z, w).$$

Therefore (4.2.15) is verified once we show that

$$\sum_{\alpha \in \mathcal{F}_d} X_\alpha - \sum_{\alpha \in \mathcal{F}_d} Y_\alpha = C(I - Z(z)A)^{-1}(I - A^*Z(w)^*)^{-1}C^*. \quad (4.2.18)$$

Substituting (4.2.11) into (4.2.17) gives

$$X_\alpha = z^\alpha w^{\alpha^*} I_Y - [D + C(I - Z(z)A)^{-1}Z(z)B] \cdot z^\alpha w^{\alpha^*} \cdot [D^* + B^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^*]$$

$$= z^\alpha w^{\alpha^*} (I_Y - DD^*) - C(I - Z(z)A)Z(z)BD^*z^\alpha w^{\alpha^*}$$

$$- z^\alpha w^{\alpha^*} DB^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^*$$

$$- C(I - Z(z)A)^{-1}Z(z)B \cdot z^\alpha w^{\alpha^*} \cdot B^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^*.$$
On the other hand, careful bookkeeping and use of the identity

\[
I - \mathbf{U}\mathbf{U}^* = \begin{bmatrix}
I - AA^* - BB^* & -AC^* - BD^* \\
-CA^* - DB^* & I - CC^* - DD^*
\end{bmatrix}
\]

gives that

\[
Y_\alpha = C(I - Z(z)A)^{-1}Z(z) \cdot z^\alpha w^{\alpha^T} \cdot (I - AA^* - BB^*)Z(w)^*(I - A^*Z(w)^*)^{-1}C^*
\]

\[
- C(I - Z(z)A)^{-1}Z(z)(AC^* + BD^*)z^\alpha w^{\alpha^T}
\]

\[
- z^\alpha w^{\alpha^T}(CA^* + DB^*)Z(w)^*(I - A^*Z(w)^*)^{-1}C^*
\]

\[
+ z^\alpha w^{\alpha^T}(I - CC^* - DD^*).
\]

Further careful bookkeeping then shows that

\[
X_\alpha - Y_\alpha = z^\alpha w^{\alpha^T}CC^* + C(I - Z(z)A)^{-1}Z(z)AC^*z^\alpha w^{\alpha^T}
\]

\[
+ z^\alpha w^{\alpha^T}CA^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^*
\]

\[
- C(I - Z(z)A)^{-1}Z(z) \cdot z^\alpha w^{\alpha^T} \cdot (I - AA^*)Z(w)^*(I - A^*Z(w)^*)^{-1}C^*
\]

\[
= C(I - Z(z)A)^{-1}(z^\alpha w^{\alpha^T} I_X - Z(z)z^\alpha w^{\alpha^T} Z(w)^*)(I - A^*Z(w)^*)^{-1}C^*. \quad (4.2.19)
\]

Note that

\[
Z(z) \cdot z^\alpha w^{\alpha^T} \cdot Z(w)^* = \sum_{k=1}^{d} z_k z^\alpha w^{\alpha^T} w_k
\]

and hence

\[
\sum_{\alpha \in \mathcal{F}_d: |\alpha| = N} Z(z)z^\alpha w^{\alpha^T} Z(w)^* = \sum_{\alpha \in \mathcal{F}_d: |\alpha| = N+1} z^\alpha w^{\alpha^T} I_X.
\]

Therefore,

\[
\sum_{\alpha \in \mathcal{F}_d} z^\alpha w^{\alpha^T} I_X - \sum_{\alpha \in \mathcal{F}_d} Z(z)z^\alpha w^{\alpha^T} Z(w)^*
\]

\[
= \sum_{N=0}^{\infty} \sum_{\alpha \in \mathcal{F}_d: |\alpha| = N} z^\alpha w^{\alpha^T} I_X - \sum_{N=1}^{\infty} \sum_{\alpha \in \mathcal{F}_d: |\alpha| = N} z^\alpha w^{\alpha^T} I_X = I_X. \quad (4.2.20)
\]

Summing (4.2.19) and combining with (4.2.20) gives the result (4.2.18) as wanted.

\[\square\]

Given a \(d\)-tuple of operators \(A_1, \ldots, A_d\) on the Hilbert space \(\mathcal{X}\), we let \(\mathbf{A} = (A_1, \ldots, A_d)\) denote the operator \(d\)-tuple while \(A\) denotes the associated column
matrix as in (4.2.12) considered as an operator from $\mathcal{X}$ into $\mathcal{X}^d$. If $C$ is an operator from $\mathcal{X}$ into an output space $\mathcal{Y}$, we say that $(C, A)$ is an output pair. The paper [27] studied output pairs and connections with the associated state-output noncommutative linear system (4.2.13). We are particularly interested in the case where in addition $(C, A)$ is contractive, i.e.,

$$A_1^*A_1 + \cdots + A_d^*A_d + C^*C \leq I_\mathcal{X}.$$  \hspace{1cm} (4.2.21)

In this case we have the following result.

**Proposition 4.2.2** Suppose that $(C, A)$ is a contractive output pair. Then:

1. The observability operator

   $$O_{C,A}: x \mapsto \sum_{\alpha \in \mathcal{F}_d} (CA^\alpha x)z^\alpha = C(I - Z(z)A)^{-1}x$$ \hspace{1cm} (4.2.22)

   maps $\mathcal{X}$ contractively into $H_2^2(\mathcal{F}_d)$.

2. The space $\text{Ran } O_{C,A}$ is a NFRKHS with norm given by

   $$\|O_{C,A}x\|_{\mathcal{H}(K_{C,A})} = \|Qx\|_{\mathcal{X}}$$

   where $Q$ is the orthogonal projection onto $(Ker O_{C,A})^\perp$ and with formal reproducing kernel $K_{C,A}$ given by

   $$K_{C,A}(z, w) = C(I - Z(z)A)^{-1}(I - A^*Z(w)^*)^{-1}C^*.$$ \hspace{1cm} (4.2.23)

3. $\mathcal{H}(K_{C,A})$ is invariant under the backward shift operators $S_j^*$ given by (4.2.7) for $j = 1, \ldots, d$ and moreover the difference-quotient inequality

   $$\sum_{j=1}^d \|S_j^*f\|_{\mathcal{H}(K_{C,A})}^2 \leq \|f\|_{\mathcal{H}(K_{C,A})}^2 - \|f_\emptyset\|_{\mathcal{Y}}^2 \text{ for all } f \in \mathcal{H}(K_{C,A})$$ \hspace{1cm} (4.2.24)

   is satisfied.

4. $\mathcal{H}(K_{C,A})$ is isometrically included in $H_2^2(\mathcal{F}_d)$ if and only if in addition $A$ is strongly stable, i.e.,

   $$\lim_{N \to \infty} \sum_{\alpha \in \mathcal{F}_d, |\alpha| = N} \|A^\alpha x\|^2 = 0 \text{ for all } x \in \mathcal{X}.$$ \hspace{1cm} (4.2.25)
**Proof:** We refer to [27, Theorem 2.10] for complete details of the proof. Here we only note that the backward-shift-invariance property in part (3) is a consequence of the intertwining relation

\[ S_j^* \mathcal{O}_{C,A} = \mathcal{O}_{C,A} A_j \quad \text{for} \quad j = 1, \ldots, d \]  

(4.2.26)

and that, in the observable case, (4.2.24) is equivalent to the contractivity property (4.2.21) of \((C,A)\).

Just as in the classical case, the de Branges-Rovnyak space \(\mathcal{H}(K_S)\) has several equivalent characterizations. The following two theorems come from [28].

**Proposition 4.2.3** Let \(S \in \mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})\) and let \(\mathcal{H}\) be a Hilbert space of formal power series in \(\mathcal{Y}\langle\langle \langle z \rangle\rangle\). Then the following are equivalent.

1. \(\mathcal{H}\) is equal to the NFRKHS \(\mathcal{H}(K_S)\) isometrically, where \(K_S(z,w)\) is the non-commutative positive kernel given by (4.2.9).

2. \(\mathcal{H} = \text{Ran} (I - M_S M_S^*)^{1/2}\) with lifted norm

\[
\|(I - M_S M_S^*)^{1/2} g\|_{\mathcal{H}} = \|Qg\|_{H^2_\mathcal{Y}(\mathcal{F}_d)}
\]

(4.2.27)

where \(Q\) is the orthogonal projection of \(H^2_\mathcal{Y}(\mathcal{F}_d)\) onto \((\text{Ker} (I - M_S M_S^*)^{1/2})^\perp\).

3. \(\mathcal{H}\) is the space of all formal power series \(f(z) \in \mathcal{Y}\langle\langle \langle z \rangle\rangle\) with finite \(\mathcal{H}\)-norm, where the \(\mathcal{H}\)-norm is given by

\[
\|f\|_{\mathcal{H}}^2 = \sup_{g \in H^2_\mathcal{Y}(\mathcal{F}_d)} \left\{ \|f + M_S g\|_{H^2_\mathcal{Y}(\mathcal{F}_d)}^2 - \|g\|_{H^2_\mathcal{Y}(\mathcal{F}_d)}^2 \right\}.
\]

(4.2.28)

**Proposition 4.2.4** Suppose that \(S \in \mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})\) and let \(\mathcal{H}(K_S)\) be the associated NFRKHS where \(K_S\) is given by (4.2.9). Then the following conditions hold:

1. The NFRKHS \(\mathcal{H}(K_S)\) is contained contractively in \(H^2_\mathcal{Y}(\mathcal{F}_d)\):

\[
\|f\|_{H^2_\mathcal{Y}(\mathcal{F}_d)}^2 \leq \|f\|_{\mathcal{H}(K_S)}^2 \quad \text{for all} \quad f \in \mathcal{H}(K_S).
\]

2. \(\mathcal{H}(K_S)\) is invariant under each of the backward-shift operators \(S_j^*\) given by (4.2.7) for \(j = 1, \ldots, d\), and moreover, the difference-quotient inequality (4.2.24) holds for \(\mathcal{H}(K_S)\):

\[
\sum_{j=1}^d \|S_j^* f\|_{\mathcal{H}(K_S)}^2 \leq \|f\|_{\mathcal{H}(K_S)}^2 - \|f_0\|^2.
\]

(4.2.29)
3. For each $u \in \mathcal{U}$ and $j = 1, \ldots, d$, the vector $S_j^*(M_Su)$ belongs to $\mathcal{H}(K_S)$ with the estimate
\[
\sum_{j=1}^{d} \|S_j^*(M_Su)\|_{\mathcal{H}(K_S)}^2 \leq \|u\|_{\mathcal{U}}^2 - \|s_\emptyset u\|_{\mathcal{Y}}^2. \tag{4.2.30}
\]
Let us define an operator $E: H_\mathcal{Y}^2(\mathcal{F}_d) \to \mathcal{Y}$ by
\[
E: \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto f_\emptyset. \tag{4.2.31}
\]
As is observed in [27, Proposition 2.9] and can be observed directly,
\[
ES^w f = E \left( \sum_{\alpha \in \mathcal{F}_d} f_{\alpha} z^\alpha \right) = f_{v^\top} \text{ for all } f(z) = \sum_{\alpha \in \mathcal{F}_d} f_{\alpha} z^\alpha \in H_\mathcal{Y}^2(\mathcal{F}_d) \text{ and } v \in \mathcal{F}_d. \tag{4.2.32}
\]
Hence the observability operator $O_{E,S^*}: H_\mathcal{Y}^2(\mathcal{F}_d) \to H_\mathcal{Y}^2(\mathcal{F}_d)$ defined as in (4.2.22) works out to be
\[
O_{E,S^*} = \tau
\]
where $\tau$ is the involution on $H_\mathcal{Y}^2(\mathcal{F}_d)$ given by
\[
\tau: \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto \sum_{v \in \mathcal{F}_d} f_v z^v. \tag{4.2.33}
\]
For this reason we use the “reflected” de Branges-Rovnyak space
\[
\mathcal{H}^r(K_S) = \tau \circ \mathcal{H}(K_S) := \{ \tau(f): f \in \mathcal{H}(K_S) \} \tag{4.2.34}
\]
as the state space for our de Branges-Rovnyak-model realization of $S$ rather than simply $\mathcal{H}(K_S)$ as in the classical case. We define
\[
\|\tau(f)\|_{\mathcal{H}^r(K_S)} = \|f\|_{\mathcal{H}(K_S)}.
\]
Recall that the operator of multiplication on the right by the variable $z_j$ on $H_\mathcal{Y}^2(\mathcal{F}_d)$ was denoted in (4.2.6) by $S_j$ rather than by $S_j^R$ for simplicity. We shall now need its left counterpart, denoted by $S_j^L$ and given by
\[
S_j^L: f(z) = \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto z_j \cdot f(z) = \sum_{v \in \mathcal{F}_d} f_v z^{j^*} \tag{4.2.35}
\]
with adjoint (as an operator on $H_\mathcal{Y}^2(\mathcal{F}_d)$) given by
\[
(S_j^L)^*: \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto \sum_{v \in \mathcal{F}_d} f_{j^*v} z^v. \tag{4.2.36}
\]
For emphasis we now write $S^R_j$ rather than simply $S_j$. We then have the following result (for details, see [28])

**Theorem 4.2.2** Let $S(z) \in S_{nc,d}(U, Y)$ and let $H^r(K_S)$ be the associated de Branges-Rovnyak space given by (4.2.34) Define operators

$$
A_{dBR,j} : H^r(K_S) \rightarrow H^r(K_S), \quad B_{dBR,j} : U \rightarrow H^r(K_S) \quad (j = 1, \ldots, d),
$$

$$
C_{dBR} : H^r(K_S) \rightarrow Y, \quad D_{dBR} : U \rightarrow Y
$$

by

$$
A_{dBR,j} = (S_j^L)^*|_{H^r(K_S)}, \quad B_{dBR,j} = \tau(S_j^R)^*M_S|_U = (S_j^L)^*\tau M_S|_U,
$$

$$
C_{dBR} = E|_{H^r(K_S)}, \quad D_{dBR} = s_0
$$

where $E$ is given by (4.2.31), and set

$$
A_{dBR} = \begin{bmatrix}
A_{dBR,1} \\
\vdots \\
A_{dBR,d}
\end{bmatrix} : H^r(K_S) \rightarrow H^r(K_S)^d, \quad B_{dBR} = \begin{bmatrix}
B_{dBR,1} \\
\vdots \\
B_{dBR,d}
\end{bmatrix} U \rightarrow H^r(K_S)^d.
$$

Then

$$
U_{dBR} = \begin{bmatrix}
A_{dBR} & B_{dBR} \\
C_{dBR} & D_{dBR}
\end{bmatrix} : \begin{bmatrix}
H^r(K_S) \\
U
\end{bmatrix} \rightarrow \begin{bmatrix}
H^r(K_S)^d \\
Y
\end{bmatrix}
$$

is an observable coisometric colligation with transfer function equal to $S(z)$:

$$
S(z) = D_{dBR} + C_{dBR}(I_{H^r(K_S)} - Z(z)A_{dBR})^{-1}Z(z)B_{dBR}.
$$

(4.2.38)

Any other observable, coisometric realization of $S$ is unitarily equivalent to this functional-model realization of $S$.

**Remark 4.2.3** It is possible to make all the ideas and results symmetric with respect to “left versus right”. Then the multiplication operator $M_S$ given here is really the left multiplication operator

$$
M^L_S = \sum_{v \in F_d} s_v(S^L)^v : f(z) \mapsto S(z) \cdot f(z).
$$

It is natural to define the corresponding right multiplication operator $M^R_S$ by

$$
M^R_S = \sum_{v \in F_d} s_v(S^R)^v.
$$
In the scalar case \( \mathcal{U} = \mathcal{Y} = \mathbb{C} \) where \( f(z) \cdot S(z) \) makes sense, we have
\[
M_{S}^{R}: f(z) \mapsto f(z) \cdot (\tau \circ S)(z)
\]
while in general we have
\[
M_{S}^{R}: \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto \sum_{v \in \mathcal{F}_d} \left[ \sum_{\alpha, \beta \in \mathcal{F}_d: \alpha \beta = v} s_{\beta} f_{\alpha} \right] z^v.
\]
The Schur-class \( \mathcal{S}_{nc,d}(U, Y) \) is really the left Schur-class \( \mathcal{S}_{L_{nc,d}}(U, Y) \). The right Schur-class \( \mathcal{S}_{nc,d}(U, Y) \) consists of all formal power series \( S(z) = \sum_{v \in \mathcal{F}_d} s_v z^v \) for which the associated right multiplication operator \( M_{S}^{R} = \sum_{v \in \mathcal{F}_d} s_v (S^R)^v \) has operator norm at most 1. The kernel \( K_{S}(z, w) \) here is really the left kernel \( K_{S}^{L}(z, w) \) given by
\[
K_{S}(z, w) = K_{S}^{L}(z, w) = \{ [I_{Y} - M_{S}^{L}(M_{S}^{L})^*](k_{S_{a}}(\cdot, w))] \}(z).
\]
It is then natural to define the corresponding right kernel
\[
K_{S}^{R}(z, w) = \{ [I_{Y} - M_{S}^{R}(M_{S}^{R})^*](k_{S_{a}}(\cdot, w))] \}(z).
\]
Given an output pair \((C, A)\), the observability operator \( O_{C, A} \) is really the left observability operator \( O_{C, A}^{L} \) with range space invariant under the right backward-shift operators \((S_{j}^{R})^*\); the corresponding right observability operator \( O_{C, A}^{R} \) is given by
\[
O_{C, A}^{R}: x \mapsto \sum_{\alpha \in \mathcal{F}_d} (C A^{\alpha} x) z^\alpha = C(I - Z(S^R)A)^{-1} x
\]
and has range space invariant under the left backward shifts \((S_{j}^{L})^*\). The system (4.2.13) is really a left noncommutative multidimensional linear system with left transfer function
\[
T_{\Sigma^{L}}(z) = D + C(I - Z(S^L)A)^{-1}Z(S^L)B.
\]
For a given colligation \( U = [A \quad B \quad C \quad D] \), there is an associated right transfer function
\[
T_{\Sigma^{R}}(z) = D + C(I - Z(S^R)A)^{-1}Z(S^R)B
\]
associated with the right noncommutative multidimensional linear system
\[
\Sigma^{R}:
\begin{align*}
x(\alpha \cdot 1) &= A_{1}x(\alpha) + B_{1}u(\alpha) \\
& \vdots \\
x(\alpha \cdot d) &= A_{d}x(\alpha) + B_{d}u(\alpha) \\
y(\alpha) &= Cx(\alpha) + Du(\alpha)
\end{align*}
\]
initialized with $x(0) = 0$. With these definitions in place, it is straightforward to formulate and prove mirror-reflected versions of the previous theorems. With all this in hand, it is then possible to identify the state-space $H^\tau(K_S) = \tau \circ H(K^R_S)$ appearing in Theorem 4.2.2 as nothing other than $H(K^L_S)$. Thus, the functional-model realization for a given $S$ as an element of the left Schur-class $S^L_{nc,d}(U, \mathcal{Y})$ uses as state space the functional-model space $H(K^L_S)$ while the realization of $S$ as a member of the right Schur-class $S^R_{nc,d}(U, \mathcal{Y})$ uses as the state space the functional-model $H(K^R_S)$ based on the left kernel $K^L_S$. Presumably it is possible to have an $S$ in the left Schur-class $S^L_{nc,d}(U, \mathcal{Y})$ but not in the right Schur-class $S^R_{nc,d}(U, \mathcal{Y})$ and vice-versa, although we have not worked out an example. With this interpretation, the functional-model realization in Theorem 4.2.2 becomes a more canonical extension of the classical univariate case.

**Remark 4.2.4** We see that a big difference between commutative case in Chapter 3 and noncommutative case here is that: for $S \in S^L_{nc,d}(U, \mathcal{Y})$, $\text{Ker} M_S$ is reducing for the backward shift operators $R_{z_j}^*$, i.e., $Sf \equiv 0 \Rightarrow SR_{z_j}^* f \equiv 0$ for $f \in H^2_\mathcal{Y}(\mathcal{F}_d)$ (see [27] for more detail). Thus if $S$ is a partial isometry, we can always consider a new $S$ which is the restriction on a subspace of $\mathcal{Y}$ and is an isometry. But for $S \in S^L_{d}(U, \mathcal{Y})$, $\text{Ker} M_S$ is not reducing for the backward shift operators $M_{\lambda_j}^*$, i.e., $Sf \equiv 0$ does not imply $SR_{\lambda_j}^* f \equiv 0$ for $f \in H_\mathcal{Y}(k_d)$.

So we say that $S \in S^L_{nc,d}(U, \mathcal{Y})$ is *inner* if the multiplication operator

$$M_S : H^2_\mathcal{U}(\mathcal{F}_d) \to H^2_\mathcal{Y}(\mathcal{F}_d)$$

is isometric; such multipliers are the representers for shift-invariant subspaces in Popescu’s Fock-space analogue of the Beurling-Lax theorem [110] (see also [27]).

We can characterize which functional-model realizations for inner multipliers as the following (see [28])

**Theorem 4.2.5** The Schur-class multiplier $S \in S^L_{nc,d}(U, \mathcal{Y})$ is inner if and only if $S$ has an observable, coisometric realization (4.2.11) such that $A = (A_1, \ldots, A_d)$ is strongly stable (see (4.2.25)).

**Proof:** We refer to [28] for the complete proof.
4.3 Shift-invariant subspaces and Beurling-Lax representation theorems

Suppose that \((Z, X)\) is an isometric input pair, i.e., \(Z = (Z_1, \ldots, Z_d)\) where each \(Z_j : \mathcal{X} \to \mathcal{X}\) and \(X : \mathcal{Y} \to \mathcal{X}\). We say that the input pair \((Z, X)\) is \textit{input-stable} if the associated controllability operator
\[
C_{Z,X} : \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto \sum_{v \in \mathcal{F}_d} Z^v \top X f_v
\]
maps \(H^2_\mathcal{Y}(\mathcal{F}_d)\) into \(\mathcal{X}\). We say that the pair \((Z, X)\) is \textit{exactly controllable} if in addition \(C_{Z,X}\) maps \(H^2_\mathcal{Y}(\mathcal{F}_d)\) onto \(\mathcal{X}\). In this case the associated controllability gramian
\[
G_{Z,X} := (C_{Z,X})^* C_{Z,X}
\]
is strictly positive-definite on \(\mathcal{X}\), and is the unique solution \(H = G_{Z,X}\) of the Stein equation
\[
H - Z_1 H Z_1^* - \cdots - Z_d H Z_d^* = XX^*.
\tag{4.3.1}
\]

By considering the similar pair
\((Z', X')\) with \(Z' = (Z'_1, \ldots, Z'_d)\) where \(Z'_j = H^{-1/2} Z_j H^{1/2}\) and \(X' = H^{-1/2} X\), without loss of generality we may assume that the input pair \((Z, X)\) is \textit{isometric}, i.e., (4.3.1) is satisfied with \(H = I_{\mathcal{X}}\). We are interested in the case when in addition \(Z^*\) is \textit{strongly stable} in the sense of (4.2.25); in this case \(G_{Z,X}\) is the unique solution of the Stein equation (4.3.1). We remark that all these statements are dual to the analogous statements made for observability operators \(O_{C,A}\) since the adjoint \((C, A) := (X^*, Z^*)\) of any input pair \((Z, X)\) is an output pair.

Given any isometric input pair \((Z, X)\) with \(Z^*\) strongly stable, we define a \textit{left functional calculus with operator argument} as follows. Given \(f \in H^2_\mathcal{Y}(\mathcal{F}_d)\) of the form \(f(z) = \sum_{v \in \mathcal{F}_d} f_v z^v\), define
\[
(X f)^\wedge L(Z) = \sum_{v \in \mathcal{F}_d} Z^v \top X f_v =: C_{Z,X} f.
\]
We define a subspace \(M_{Z,X}\) to be the set of all solutions of the associated homogeneous interpolation condition:
\[
M_{Z,X} := \{f \in H^2_\mathcal{Y}(\mathcal{F}_d): (X f)^\wedge L(Z) = 0\}.
\]
That $\mathcal{M}_{Z,X}$ is invariant under the (right) shift operator $S_j$ follows from the intertwining property $C_{Z,X}S_j = Z_jC_{Z,X}$ verified by the following computation:

$$C_{Z,X}S_j f = (XS_j f)^\wedge L(Z) = \sum_{v \in \mathcal{F}_d} Z^{(v)}(v)^T X f_v = Z_j \cdot \sum_{v \in \mathcal{F}_d} Z^{(v)}(v)^T X f_v$$

$$= Z_j \cdot (X f)^\wedge L(Z) = Z_j C_{Z,X} f.$$

It is easily checked that $\mathcal{M}_{Z,X}$ is closed in the metric of $H^2_{\mathcal{F}_d}(Y)$. Hence, by Popescu’s Beurling-lax theorem for the Fock space (see [110]) it is guaranteed that $\mathcal{M}_{Z,X}$ has a representation of the form

$$\mathcal{M}_{Z,X} = \Gamma \cdot H^2_{\mathcal{F}_d}(Y) = \text{Ran } M_{\Gamma}$$

for an inner multiplier $\Gamma \in S_{nc,d}(U,Y)$. Our goal is to understand how to compute a transfer-function realization for $\Gamma$ directly from the homogeneous interpolation data $(Z, X)$. First, however, we show that shift-invariant subspaces $\mathcal{M} \subset H^2_{\mathcal{F}_d}(Y)$ of the form $\mathcal{M} = \mathcal{M}_{Z,X}$ for an admissible input pair $(Z, X)$ as above are not as special as may at first appear.

**Theorem 4.3.1** Suppose that $\mathcal{M}$ is a closed, shift-invariant subspace of $H^2_{\mathcal{F}_d}(U,Y)$. Then there is an isometric input-pair $(Z, X)$ with $Z^*$ strongly stable so that $\mathcal{M} = \mathcal{M}_{Z,X}$.

**Proof:** If $\mathcal{M}$ is invariant for the operators $S_j$, then $\mathcal{M}^\perp$ is invariant for the operators $S_j^*$ for each $j = 1, \ldots, d$. Hence by Theorem 2.8 from [27] there is an observable, contractive output pair $(C, A)$ so that $\mathcal{M}^\perp = \mathcal{H}(K_{C,A}) = \text{Ran } O_{C,A}$ isometrically. As $\mathcal{M}^\perp \subset H^2_{\mathcal{F}_d}(Y)$ isometrically, Proposition 4.2.2 tells us that we may take $(C, A)$ isometric and that $A$ is strongly stable. Let $(Z, X)$ be the input pair $(Z, X) = (A^*, C^*)$. As $\mathcal{M}^\perp = \text{Ran } O_{C,A}$, we may compute $\mathcal{M}$ as

$$\mathcal{M} = (\text{Ran } O_{C,A})^\perp = \text{Ker } (O_{C,A})^* = \text{Ker } C_{A^*,C^*} = \text{Ker } C_{Z,X}$$

and Theorem 4.3.1 follows. \qed

We now suppose that a shift-invariant subspace is given in the form $\mathcal{M} = \mathcal{M}_{Z,X}$ for an admissible homogeneous interpolation data set and we construct a realization for the associated Beurling-Lax representer.
Theorem 4.3.2 Suppose that \((Z, X)\) is an admissible homogeneous interpolation data set and \(M_{Z,X} = \text{Ker } C_{Z,X}\) is the associated shift-invariant subspace. Let \((C, A)\) be the output pair defined by

\[(C, A) = (X^*, Z^*)\]

and choose an input space \(U\) with \(\text{dim } U = \text{rank } (I_{X^d \oplus Y} - [A^* C^*])\) and define an operator \([B \ D] : U \to X^d \oplus Y\) as a solution of the Cholesky factorization problem

\[
\begin{bmatrix}
B \\
D
\end{bmatrix}
\begin{bmatrix}
B^* \\
D^*
\end{bmatrix} = I_{X^d \oplus Y} - \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} A^* \\
C^*
\end{bmatrix}.
\]

Set \(U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\) and let \(\Gamma \in \mathcal{S}_{nc,d}(U, Y)\) be the transfer function of \(U\):

\[
\Gamma(z) = D + C(I - Z(z)A)^{-1}Z(z)B.
\]

Then \(\Gamma\) is inner and \(M_{Z,X} = \Gamma \cdot H^2_U(F_d)\).

**Proof:** If \((Z, X)\) is an admissible homogeneous interpolation data set, then \((Z, X)\) is controllable and \(Z^*\) is strongly stable. Since \((C, A) = (X^*, Z^*)\), we have \((C, A)\) is observable and \(A\) is strongly stable. From the construction of \(U\), we know \(U\) is coisometric. Then by Theorem 4.2.5, \(\Gamma\) is inner and hence \(I - M_{\Gamma}M_{\Gamma}^*\) is the orthogonal projection of \(H^2_Y(F_d)\) onto \((\text{Ran } M_{\Gamma})^\perp\). From part (2) of Proposition (4.2.3) it then follows that

\[
\mathcal{H}(K_{\Gamma}) = H^2_Y \ominus \Gamma \cdot H^2_U(F_d)\text{ isometrically.} \quad (4.3.2)
\]

On the other hand, again since \(U\) is coisometric, we see that \(K_{\Gamma} = K_{C,A}\) and hence \(\mathcal{H}(K_{\Gamma}) = \mathcal{H}(K_{C,A})\). Since \(A\) is strongly stable, Proposition 4.2.2 tells us that \(\mathcal{H}(K_{C,A})\) is isometrically included in \(H^2_Y(F_d)\) and is characterized by

\[
\mathcal{H}(K_{\Gamma}) = \mathcal{H}(K_{C,A}) = \text{Ran } O_{C,A} = \text{Ran } (C_{Z,X})^* \quad (4.3.3)
\]

Comparing (4.3.2) with (4.3.3) and taking orthogonal complements finally leaves us with

\[
\Gamma \cdot H^2_U(F_d) = (\text{Ran } (C_{Z,X})^*)^\perp = \text{Ker } C_{Z,X} = M_{Z,X}
\]

and Theorem 4.3.2 follows. \(\square\)
Now our goal is to do the Krein space analogue of the above results. Similar to Drury-Arveson Krein space, we can form Fock Krein space and define special operators on Fock Krein spaces.

For Kreın spaces $E$ and $E^*$, a formal power series $S$ is said to be a multiplier if the associated multiplication operator $M_S: f(z) \mapsto S(z) \cdot f(z)$ is closed from $H^2_E(F_d) \cap D(S)$ into $H^2_{E^*}(F_d)$; we denote the space of all such formal power series by $C_{nc,d}(E, E^*)$. A multiplier $\Gamma$ in $C_{nc,d}(E, E^*)$ is said to be $J$-phase if the associated multiplication operator $M_\Gamma: H^2_E(F_d) \rightarrow H^2_{E^*}(F_d)$ is a $(J_{E^*}, J_{E^*})$-partial isometry (or $J$-partial isometry). If moreover $(\text{Ran } M_\Gamma)^{[1]}$ is a positive subspace, we say $M_\Gamma$ is $J$-inner. In this case, the space $\mathcal{H}(K_J)$ defined as the reproducing kernel Hilbert space with reproducing kernel

$$K_J(z, w) = \frac{J_{E^*} - \Gamma(z) J_{E} \Gamma(w)}{1 - \langle z, w \rangle},$$

is $J$-isometrically included in $H^2_{E^*}(F_d)$ and

$$\mathcal{H}(K_J) = H^2_{E^*}(F_d) \ominus J\mathcal{H}_{E}(k_d) \quad (4.3.4)$$

With these concepts, we can get the following two theorems.

**Theorem 4.3.3** Let $\mathcal{M}$ be a closed, shift-invariant subspace of $H^2_{E^*}(F_d) := \left[ \begin{array}{c} H^2_E(F_d) \\
H^2_D(F_d) \end{array} \right]$ and suppose $\mathcal{M}$ is $J$-regular ($\mathcal{M} \oplus \mathcal{M}^{[1]} = H^2_{E^*}(F_d)$). Then there is an input space $E$ and $J$-isometric input-pair $(C, A)$ with $A$ strongly stable so that $\mathcal{M} = \mathcal{M}_{A^*, C^*}$. One choice of state space $X$ and operators $A_j: X \rightarrow X$ and $C: X \rightarrow E^*$ is

$$X = \mathcal{M}^{[1]}, A_j = S_j \big|_{\mathcal{M}^{[1]}}, j = 1, \ldots, d, C: f \rightarrow f(0) \quad \text{for } f \in \mathcal{M}^{[1]}$$

where $A$ strongly stable means $\lim_{N \rightarrow \infty} \sum_{x \in F_d: |\alpha| = N} \| A^\alpha x \|^2 = 0$ for $x \in X$.

**Proof:** Similar to the proof of Theorem 4.3.1, it is routine to check the constructed state space and operators work. We omit the details here.

**Theorem 4.3.4** Suppose that $(C, A)$ is a $J$-isometric output-stable pair. Then there is an input space $E$ and a $J$-inner Schur multiplier $S \in S_{nc,d}(E, E^*)$ so that $\mathcal{M}_{A^*, C^*} = \text{clos. Ran } M_\Gamma$. One such $\Gamma$ is given by $\Gamma(z) = D + C(I - Z(z)A)^{-1}Z(z)B$.
where the input space $\mathcal{E}$ and the colligation $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ are chosen such that

$$
\mathbf{X} = \mathcal{M}^{[1]}, A_j = M_j|_{\mathcal{M}[1]}, j = 1, \ldots, d, C : f \to f(0), f \in \mathcal{M}^{[1]}
$$

$$
\mathbf{U} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{M}[1] \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{M}[1] \\ \mathcal{E}_s \end{bmatrix}
$$

is $J$-unitary.

Proof: This is parallel to the proof of Theorem 4.3.2 and Theorem 3.3.2. We omit the details.

Now we know how to construct a $J$-Beurling-Lax representer and from it we can get the existence part of $J$-Beurling-Lax theorem.

Theorem 4.3.5 Suppose that $\mathcal{M}$ is a (right) shift invariant $J\mathcal{E}_s$-regular subspace of $H^2_{\mathcal{E}}(\mathcal{F}_d)$ with $H_{\mathcal{E}(k_d)} \ominus \mathcal{M}$ positive (hence uniformly positive). Then there is an input space $\mathcal{E}$ and an $J$-inner multiplier $\Gamma \in \mathcal{S}_{nc,d}(\mathcal{E}, \mathcal{E}_s)$ so that $\mathcal{M} = \text{clos. Ran } M\Gamma$.

Proof: By Theorem 4.3.3, we can write $\mathcal{M}$ in the form $\mathcal{M} = \mathcal{M}_{A^*, \Gamma S}$. Then by Theorem 4.3.4 we can find our $J$-Beurling-Lax representer $\Gamma$ which meets the requirement.

4.4 Noncommutative left-tangential operator-argument interpolation problem

Similar to the commutative case, here the left-tangential evaluation with operator-argument $f \to (C^*f)^{\wedge L}(A^*)$ amounts to $(C^*f)^{\wedge L}(A^*) = (\mathcal{O}_{C,A})^*f$ for $f \in H^2_{\mathcal{E}}(\mathcal{F}_d)$. This evaluation map extends to multipliers $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ by $(C^*S)^{\wedge L}(A^*) = (\mathcal{O}_{C,A})^*M_S|_{\mathcal{U}}$ and raises the following interpolation problem with operator argument.

Problem 4.4.1 The LTOAnc : Let $\mathcal{U}, \mathcal{Y}$ and $\mathcal{X}$ be Hilbert spaces. Given $(Z, X, Y)$ with $Z = (Z_1, \cdots, Z_d) \in \mathcal{L}(\mathcal{X}, \oplus^d \mathcal{X})$, $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, $Y \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ such that $(Z, X)$ is an input stable pair, i.e., $(X^*, Z^*)$ is an output stable pair. Find $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ such that

$$(XS)^{\wedge L}(Z) := \mathcal{C}_{Z,X}M_S|_{\mathcal{U}} = (\mathcal{O}_{X^*,Z^*})^*M_S|_{\mathcal{U}} = Y.$$
Note that this interpolation problem has been studied by a number of authors (see [25, 61, 113, 116]). Popescu’s commutant lifting theorem implies the existence result for this problem. Moreover the result can be used to get existence result for the commutative case by symmetrizing the Fock space. For parametrization, the Fock-space version of the Beurling-Lax theorem (Hilbert-space version) already appears in the work of Popescu [110] (see also [27]), the proof here through J-inner solution of a homogeneous interpolation problem (Fock-space J-Beurling-Lax Theorem) gives an alternative approach.

As before, we consider the direct sum \( K = \begin{bmatrix} K_+ \\ K_- \end{bmatrix} = \begin{bmatrix} H^2(Y_{Fd}) \\ H^2(U_{Fd}) \end{bmatrix} \) and let \( J \) be the operator with block matrix form

\[
J = \begin{bmatrix} I_{H^2(Y_{Fd})} & 0 \\ 0 & -I_{H^2(U_{Fd})} \end{bmatrix}
\]

Then \( K \) becomes a Krēın space in the inner product \( \langle k, k' \rangle_J \). Suppose \( J \)-contractive operator \( M_S \) from \( H^2(U_{Fd}) \) to \( H^2(Y_{Fd}) \) solves

\[
\begin{bmatrix} X & Y \end{bmatrix} \wedge L(Z) = 0
\]

We can also define the \( J \)-gramian:

\[
\] (4.4.1)

The \( J \)-gramian satisfies the Stein equation \( P - \sum_{j=1}^d Z_j P Z_j^* = [X^*, J [X^*, Z^*]] \). Consider the graph

\[
G = G(S) = \begin{bmatrix} S(z) \\ I \end{bmatrix} : f \in H^2(U_{Fd}) \subset \begin{bmatrix} H^2(Y_{Fd}) \\ H^2(U_{Fd}) \end{bmatrix}
\]

If \( G \) is of this form, we have \( \begin{bmatrix} X & -Y \end{bmatrix} h)^{\wedge L}(Z) = 0 \) for \( h \in G \)

\[
(XSf)^{\wedge L}(Z) = ((XS)^{\wedge L}(Z)f)^{\wedge L}(Z) = (Yf)^{\wedge L}(Z).
\]

Thus \( h(z) = \begin{bmatrix} S(z) f(z) \\ f(z) \end{bmatrix} \) implies

\[
\begin{bmatrix} X & -Y \end{bmatrix} h)^{\wedge L}(Z) = 0
\]
Then it can be checked that $\mathcal{G}$ satisfies:

1. $\mathcal{G}$ is a graph space

2. $\mathcal{G}$ is invariant under $S_j$, $j = 1, 2, \cdots, d$

3. $\mathcal{G}$ is a negative subspace of $\begin{bmatrix} H_2^2(\mathcal{F}_d) \\ H_2^d(\mathcal{F}_d) \end{bmatrix}$

4. $\mathcal{G} \subset \mathcal{M} = \left\{ h \in \begin{bmatrix} H_2^2(\mathcal{F}_d) \\ H_2^d(\mathcal{F}_d) \end{bmatrix} : \begin{bmatrix} X & -Y \end{bmatrix} h \wedge L(Z) = 0 \right\}$

Given a subspace $\mathcal{M}$ one can see that the following conditions are equivalent:

(a) A subspace $\mathcal{G}$ $\mathcal{M}$-maximal negative $\iff$ $\mathcal{G}$ is contained in $\mathcal{M}$ and $\begin{bmatrix} H_2^2(\mathcal{F}_d) \\ H_2^d(\mathcal{F}_d) \end{bmatrix}$-maximal negative.

(b) $\begin{bmatrix} H_2^2(\mathcal{F}_d) \\ H_2^d(\mathcal{F}_d) \end{bmatrix} \ominus \mathcal{M}$ is positive

(c) If $\mathcal{M} = \mathcal{M}_{A^*, C^*}$ with output stable pair $(C, A)$, the $J$-gramian is positive.

It follows that $\mathcal{G}$ satisfying the conditions (1), (3) and (4) is equivalent to $\mathcal{G}$ $\mathcal{M}$-maximal negative given positivity of $J$-gramian. Conversely, if $\mathcal{G}$ as a subspace of $\begin{bmatrix} H_2^2(\mathcal{F}_d) \\ H_2^d(\mathcal{F}_d) \end{bmatrix}$ satisfying the above conditions, then $\mathcal{G} = \begin{bmatrix} M_S \\ I \end{bmatrix} H_2^d(\mathcal{F}_d)$ for a solution $S$ of the interpolation problem.

Hence again solving this LTOAnc interpolation problem for a solution $S$ translates to finding subspaces of $\begin{bmatrix} H_2^2(\mathcal{F}_d) \\ H_2^d(\mathcal{F}_d) \end{bmatrix}$ which satisfy the above conditions. We are interested in finding corresponding $J$-Beurling-Lax representer and here we will get it via

Step 1: reducing the problem to a homogeneous interpolation problem and

Step 2: finding a $J$-inner solution of the homogeneous interpolation problem via state-space realization technique.

Note that the two steps have been completed by Theorem 4.3.3 and Theorem 4.3.4 in last section. Now, we are ready to solve the LTOAnc for the nondegenerate case.
Theorem 4.4.2 Suppose we are given the admissible data in the LTOAnc. Then the solutions of LTOAnc exist if and only if \( J \)-gramian \( G_{[[Y_\ast],[Z_\ast]]}^J \geq 0 \). Moreover, if we assume that \( J \)-gramian \( G_{[[Y_\ast],[Z_\ast]]}^J \) is strictly positive, then the class of all solutions \( S \) of the problem coincides with the class of all \( S \) with the representation

\[
S = (\Gamma_{11} T + \Gamma_{12})(\Gamma_{21} T + \Gamma_{22})^{-1}
\]

for some free \( T \) in noncommutative Schur-class \( S_{nc,d}(U, \tilde{Y}) \), where \( \Gamma \) is the \( J \)-Beurling-Lax representer.

**Proof:** Recall that we set \( M = \left\{ h \in \left[ \begin{array}{c} H_2^2(F_d) \\ H_2^0(F_d) \end{array} \right] : \left( \begin{array}{cc} X & -Y \\ \end{array} \right) h \right\} \). By Theorem 4.3.5, suppose we can write \( M \) as \( M = \text{clos.} \text{Ran} \Gamma \) and \( M \Gamma \) is \( J \)-isometry. Then if \( \Gamma \) is bounded, it preserves the Kreǐn space geometry, and a subspace \( G \) is \( M \)-maximal negative subspace if and only if \( G = M \Gamma G' \), where \( G' \) is \( H_2^2(F_d) \)-maximal negative subspace. If \( H_{\varepsilon}(F_d) \cap M \) positive, then \( M \)-maximal negative implies \( H_{\varepsilon} \)-maximal negative. While \( M \)-maximal negative subspace is easily characterized as subspaces of the form \( G_T = M \Gamma [T(z)] H_2(F_d) \) where \( T(z) \in S_{nc,d}(U, \tilde{Y}) \). Then we may get the solution \( S \) with

\[
\begin{bmatrix}
S(z) \\
I
\end{bmatrix} H_2^2(F_d) = M \Gamma \begin{bmatrix}
T(z) \\
I
\end{bmatrix} H_2(F_d)
\]

which follows that \( S = (\Gamma_{11} T + \Gamma_{12})(\Gamma_{21} T + \Gamma_{22})^{-1} \) for some \( T(z) \in S_{nc,d}(U, \tilde{Y}) \). If \( M \Gamma \) is unbounded, the situation is more delicate. Parallel to Theorem 3.2.5, we get the same solution. \( \square \)
Chapter 5

Conclusion and Future Work

In this chapter we give conclusions of the dissertation and a view of future research.

5.1 Conclusion

In this dissertation, we solve the left tangential interpolation problems on the commutative or noncommutative unit ball. For the commutative setting, we consider left tangential operator argument interpolation problems for Schur class multipliers on the Drury-Arveson space and for the noncommutative setting, we consider interpolation problems for Schur class multipliers on Fock space. In both settings, we apply the Krein-space geometry approach (also known as the Grassmannian Approach). To implement this approach $J$-versions of Beurling-Lax representers for shift-invariant subspaces are required. Here we obtain these $J$-Beurling-Lax theorems by the state-space method. We see that the Krein-space geometry method in solving the interpolation problems is particularly simple when the Beurling-Lax representer is bounded. The Potapov approach applies equally well whether the representer is bounded or not.

5.2 Future research

In addition to the current research, we intend to study other aspects of multivariable operator theory and its applications. Since multivariable operator theory is an
interplay of Algebra, Analysis, Geometry and Topology, it would be interesting to
explore the understanding from different points of view and enhance interdisciplinary
research. Specifically, the following directions seem to be natural:

5.2.1 Unification of interpolation problems

Various multivariable generalizations of the operator-valued Schur class have ap-
peared recently. One of the most encompassing is the sophisticated generalized
Schur class which has been introduced by Muhly and Solel (see [101, 103]). In this
setting the unit disk is replaced by the strict unit ball of the elements of a dual
correspondence $E^*$ associated with a $W^*$-correspondence $E$ over a $W^*$-algebra $A$,
i.e., a Hilbert $W^*$-module $E$ over $A$ endowed with a structure of a left module over
$A$ via a $*$-representation $\sigma$ of $A$. In [23], we introduced the notion of a reproducing
kernel Hilbert correspondence and an analogue of the Fourier (or $Z$-) transform for
the Muhly-Solel setting. We identified the Muhly-Solel Fock space as a reproducing
kernel Hilbert correspondence associated with a particular completely positive
kernel, which can be viewed as an analogue of the classical Szeg"o kernel. In this
way we were able to make the analogy between the Muhly-Solel Schur class and the
classical Schur class more complete. We also illustrated the theory by specializing
it to some well-studied special cases; in some instances there result new kinds of re-
alization theorems. Particularly, we discussed one of the main examples motivating
the work in [100, 101, 103], namely the setting of analytic crossed-product algebras.
It is interesting to conclude that the realization theorem for a particular instance of
this example, after some translation, amounts to the realization theorem for input-
output maps of conservative time-varying linear systems obtained in [11], which
clarified the relation between the two point evaluations appearing in [11] and [102].

The Muhly-Solel setting incorporates time-varying and multivariable interpo-
lation problems for different Schur classes, as well as examples involving quiver
algebras, semi-crossed products, and directed graphs. One can refer to a more com-
prehensive survey including this setting written by Ball and ter Horst (see [40]).
There are still other types of generalized Schur classes which are not subsumed un-
der the Muhly-Solel Hardy space or correspondence setup mentioned above, e.g., the
Schur-Agler class for the polydisk (see [1,3,42] and for more general domains [12,24]),
the noncommutative Schur-Agler class (see [34,38]), and higher-rank graph algebras
(see [90]). Recently, Dritschel, Marcantognini, and McCullough introduced another
abstract framework in [68], which incorporates all the settings in [1, 3, 34, 42, 90].
However the theory in [68] does not appear to include the analytic crossed-product
algebras included in the Muhly-Solel scheme since it does not allow for the action of
a $W^*$-algebra $\mathcal{A}$ acting on the ambient Hilbert space. Hence it would be somewhat
challenging to see whether we can have some sort of synthesis of these two settings.
The recent work on product decompositions over general semigroups (see [124]) ap-
pears to be a start in this direction. It would be nice to compare these two general
settings with the abstract interpolation problems discussed in [26,85] also.

5.2.2 Multivariable operator theory and algebraic geometry

Because of the lack of a sufficiently developed function theory in several variables,
the development of multivariable operator theory has been hampered. For example,
we need much more function theory on the unit ball to consider the operator the-
ory of the Hardy space over the unit ball. On the one hand, I plan to work more
on multivariable function theory and on the other hand I would like to study how
to use tools from other areas (e.g., algebra, algebraic geometry) to solve problems
in multivariable operator theory. Since algebraic geometry combines techniques of
many different fields, I am interested in the interplay of algebraic geometry and
multivariable operator theory. With Ball and Vinnikov I have been working on
the bidisk analogue of the homogenous interpolation problems done for the case of
the ball (see [32]). Specifically, we know that a transfer-function realization for the
function (of Givone-Roesser or Fornasini-Marchesini type), if minimal in the Popov-
Belevitch-Hautus sense, determines a linearization (i.e., a determinantal representa-
tion) for the pole variety and for the zero variety. We discuss the converse question
of constructing a realization (of either Givone-Roesser or Fornasini-Marchesini type)
for a function having prescribed zero and pole varieties. The basic idea follows the
solution for the one-variable case due to Ball-Gohberg-Rodman (see [35]), but with
additional ingredients from the theory of determinantal representations for algebraic
curves due to Vinnikov.

5.2.3 Further applications

It is well known for the single-variable setting that the theory of interpolation problems has close connection with 1-D systems theory and robust control theory. This suggests the exploration of similar application in the multivariable setting. Certain types of multidimensional input/state/output linear systems provide a convenient formalism for the study of linear-fractional-transformation (LFT) models for plants with structured uncertainty and for linear parameter-varying (LPV) plants (see [47, 67, 107]). During the past decade a lot of work has been done on robust feedback-control for such linear systems by making use of this approach. In a recent paper [31], we clarified the precise connection between robust stability and robust performance. As a project for future work, I would like to make broader exploration on this topic.

It would be also very interesting to consider applications of multivariable operator theory to other fields. For example, if we can develop a control theory for the structured noncommutative multidimensional linear systems studied in [33, 34], we might be able to see some applications in mathematical finance topics, say, option pricing and contingent claim hedging. There is a recent literature on the control-theoretic aspects of interest rate theory (see [54, 76]). The ideas of realization can be applied to modeling of interest rates, particularly, term structure. We may want to see how the broadly developed theories on control contribute to modeling of the ever-changing financial market.
Bibliography


