RESPONSE OF A NONLINEAR TWO-DEGREE-OF-FREEDOM SYSTEM

SUBJECTED TO AN IMPACT LOADING

by

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II. INTRODUCTION

The solution of the equations of motion of an aircraft fuselage-landing gear configuration during landings is of interest to the designer who must predict the landing loads which an airplane encounters in service. In general such solutions are difficult because of the highly nonlinear characteristics of the oleo-pneumatic shock strut which couples the lower mass of the landing gear to the fuselage.

In the past, attempts to obtain solutions by linearization of these shock strut characteristics have resulted in unrealistic predictions of landing gear motions. Therefore, it has been necessary to carry out most of the theoretical analysis associated with landing gears by means of numerical integration procedures. These numerical methods are tedious, and as a result a large portion of design work has been carried out by means of trial and error drop testing of a system of masses representative of an airplane and landing gear. This in turn has proved to be time consuming and expensive.

This paper presents a method for obtaining an analytical solution of the equations of motion for a basically nonlinear system which closely resembles an actual airplane and landing gear configuration. The nonlinear system considered has two degrees of freedom and is composed of a large mass representative of the fuselage-wing combination connected by an oleo-pneumatic shock strut to a wheel. The shock strut is assumed to have velocity-squared hydraulic damping and coulomb friction forces on
the strut bearings. The nonlinear spring characteristic of the tire is represented by a sectionally linear spring.

In the first part of this paper the equations of motion for this nonlinear system are derived making use of a few simplifications which previous papers have shown to be justified. Also, the degree to which these assumptions limit the results is discussed. Next these equations of motion are solved in analytical form by a method which may be called "equivalent nonlinearization." It is shown that this solution is exact only for a specific combination of impact parameters, but that for a wide range of parameters the solution describes the motion of the system adequately for design purposes. Finally, a few analytical solutions are compared with solutions obtained by numerical integration methods; and the results are compared with experimental data for a typical impact.
III. NOMENCLATURE

$A$ coefficient of quadratic damping term, $\frac{\rho A_b^3}{2(C_d A_o)^2}$

$A_a$ pneumatic area

$A_h$ hydraulic area

$A_o$ area of opening in orifice plate

$B$ coefficient of friction-force term, $\frac{2 u_d}{N}$

$C_d$ orifice discharge coefficient

$E(\varphi, k) = E(\nu)$ Legendre's incomplete elliptic integral of the second kind

$F(\varphi, k) = \nu$ incomplete elliptic integral of the first kind,

\[ \nu = 3 \sqrt{2H u_o'} (1 - e^{-x/3}) \]

$F_a$ air compression force in strut

$F_f$ axial friction force in strut

$F_h$ hydraulic resistance force in the strut

$F_s$ total axial shock-strut force

$F_v$ vertical force, applied at the ground

$F_1$ normal force on upper bearing (attached to inner cylinder)

$F_2$ normal force on lower bearing (attached to outer cylinder)

$H$ constant satisfying equal energy condition, $\sqrt{1 - \frac{2}{u_m' \left( \frac{1}{9} - K_L \right)}}$

$K_L$ nondimensional constant dependent on the wing lift,

\[ \frac{A(L - W)g}{K_2 (1 - B) W_1} \]

$K_2$ slope of the assumed tire deflection curve

$L$ wing-lift force
approximation to the average distance between bearing surfaces during the impact, 
\[ \left( R_1 + \frac{R_2}{2} \right) - \mu(b_1 - b_2) \]
axial distance between upper and lower bearings, for fully extended shock strut

\[ W \]
total weight of system, \( W_1 + W_2 \)

\[ W_1 \]
weight of the upper mass above strut

\[ W_2 \]
weight of the lower mass below strut

\[ b_1 \]
outer radius of upper bearing, attached to inner cylinder

\[ b_2 \]
outer radius of lower bearing, attached to outer cylinder

\[ d \]
distance from the axle to the strut centerline

\[ e \]
base of natural logarithms (\( \approx 2.71828 \ldots \))

\[ g \]
gravitational constant

\[ k \]
modulus of Jacobian elliptic functions and integrals

\[ n \]
polytropic exponent

\[ P_a \]
air pressure in upper strut

\[ P_{a_0} \]
initial air pressure in upper strut

\[ P_h \]
pressure in the lower chamber of the strut

\[ s \]
shock-strut stroke (displacement), \( (s_1 - s_2) \)

\[ s_m \]
maximum possible stroke (fully closed strut)

\[ t \]
time after contact

\[ t_1 \]
time after contact before the equations of motion apply to the system, \( \frac{0.0508}{z_0} \)

\[ u \]
dimensionless lower-mass displacement from position at initial contact
$u_m$  
maximum value of displacement during a given impact

$u_1$  
dimensionless upper-mass displacement from position at initial contact

$u_0'$  
dimensionless initial contact velocity

$v_0$  
initial air volume within strut

$z_1$  
vertical displacement of the upper mass

$z_2$  
vertical displacement of the lower mass

$z_0$  
initial contact velocity

$\tau$  
dimensionless displacement in the transformed equation of motion, $\tau = u^{1/2}$

$\tau_m$  
maximum value of displacement during a given impact

$\Theta$  
dimensionless time from time of initial contact

$\mu$  
coefficient of friction for the bearing surfaces

$p$  
mass density of hydraulic fluid

$\varphi$  
amplitude of $v$

The use of dots over symbols indicates differentiation with respect to time, $t$.

Prime marks indicate differentiation with respect to dimensionless time, $\Theta$, in the text.
IV. PHYSICAL CHARACTERISTICS OF THE SYSTEM

Other investigations have indicated that flexibility of the fuselage and wing structure of an airplane usually has only a small effect on landing gear performance (refs. 1, 2, and 3). For this reason in the dynamical system to be investigated, the fuselage and empennages of the airplane may be represented by a single large mass, \( W_1 \), which is assumed to have freedom only in vertical translation designated by a coordinate \( z_1 \). The lower mass, \( W_2 \), which is considerably smaller lies below the oleo strut and consists principally of the wheel, tire, axle, and strut inner cylinder assembly as shown in figure 1. The strut is assumed to be infinitely rigid in bending. Horizontal forces at the axle, usually called drag or spring back loads, will not be considered in this investigation. Therefore, the lower mass has freedom only in vertical translation designated by the coordinate \( z_2 \). Lift forces are externally applied to the system to represent aerodynamic lift.

Figures 2a and 2b show a schematic representation of a typical oleo-pneumatic shock strut. The lower portion of the cylinder is filled with hydraulic fluid and above this is an air space in an enclosed chamber. The outer cylinder contains a perforated tube which has a small plate at its base containing an orifice. When the strut is compressed the fluid is forced through the orifice at high velocity, and the pressure drop across the orifice resists closure of the strut. The turbulence thus created dissipates a large part of the impact energy. This motion of the strut also increases the air pressure in the upper chamber causing a force which resists closure of the strut.
Figure 1. Two degree of freedom system considered in analysis.
Figure 2a. - Schematic representation of shock strut, wheel and tire with balance of forces on lower mass.
Figure 2b.- Schematic representation of shock strut and balance of forces on upper mass.
The hydraulic resistance force, $F_h$, in the shock strut can be derived from the equation for discharge through an orifice, namely,

$$Q = A_h \dot{s} = C_d A_o \sqrt{\frac{2}{\rho} (P_h - P_a)}$$  \hspace{1cm} (1)

where the dot refers to differentiation with respect to time and

- $Q$  \hspace{1cm} \text{volumetric rate of discharge}
- $A_h$  \hspace{1cm} \text{area subjected to the hydraulic pressure drop where the small difference between $A_h$ and $(A_h - A_o)$ is neglected.}
- $P_h$  \hspace{1cm} \text{pressure in the lower chamber}
- $P_a$  \hspace{1cm} \text{pressure in the air (upper) chamber}
- $\dot{s}$  \hspace{1cm} \text{velocity of closure of the strut, $(\dot{s}_1 - \dot{s}_2)$}
- $C_d$  \hspace{1cm} \text{discharge coefficient}
- $A_o$  \hspace{1cm} \text{orifice area}
- $\rho$  \hspace{1cm} \text{mass density of the hydraulic fluid}

Since

$$F_h = A_h (P_h - P_a)$$  \hspace{1cm} (2)

equations (1) and (2) may be used to derive the expression for the hydraulic resistance force:

$$F_h = \frac{\rho A_h^3}{2 (C_d A_o)^2} \dot{s} |\dot{s}|$$  \hspace{1cm} (3)

where

$$\dot{s} = \dot{s}_1 - \dot{s}_2$$

In this equation the absolute value of one velocity factor is retained
to indicate that this damping force is the usual quadratic damping which retains the sign of the velocity. The orifice coefficient, $C_d$, may be considered as a constant during the compression stroke, according to recent experimental data (ref. 4). Since maximum impact loads are reached prior to maximum strut stroke, the designer is primarily interested in the strut compression process rather than the elongation process. Most struts have a pressure operated check valve which comes into operation when the strut velocity changes sign, thus closing off the main orifice. Fluid is then forced to return to the lower chamber through a number of small passages the total area of which usually varies with the stroke in an indeterminate manner. The orifice coefficient also is unknown for this phase of the impact. Therefore, solutions for the motion are made only for the closure case, although in deriving the equations of motion the sign of the velocity is indicated where appropriate for clarity.

The movement of the inner cylinder compresses the air in the upper part of the strut and this results in an air-pressure force, $F_a$, which may be determined by means of the polytropic law for compression of gases

$$P_a = P_{a_0} \left( \frac{V_0}{V_0 - \Delta s} \right)^n$$

(14)

where

- $P_a$ air pressure in upper strut
- $P_{a_0}$ initial air pressure in upper strut
- $V_0$ initial air volume within strut
strut stroke \( (z_1 - z_2) \)

the pneumatic cross sectional area

polytropic exponent

\( (V_0 - A_s z) \) the air volume for any stroke

Then

\[
F_a = p_a A_a = \rho_0 A_a \left( \frac{V_0}{V_0 - A_s z} \right)^n
\]  

(5)

Other investigations have revealed, however, that the air compression process has only minor effect on the landing gear loads which are developed (refs. 5 and 6). This is true principally because the air pressure force becomes relatively large only for maximum values of strut stroke, which always occur long after the principal part of the impact is over. Therefore the air pressure force will be neglected in this analysis.

Friction forces within the strut result from the relative sliding of bearing surfaces on the inner and outer cylinders. For the type of landing gear being considered, vertical loads applied at the axle are eccentric to the strut center line and produce a bending moment which is resisted by forces \( F_1 \) and \( F_2 \) normal to the upper and lower bearing surfaces respectively as shown in figure 2a. Conditions of relatively high normal pressures, slow sliding velocities of the bearing surfaces, and the poor lubricating properties of the hydraulic fluid lead to the conclusion that the internal friction forces may follow laws similar to those of dry (coulomb) friction; that is, the friction force is
proportional to the normal force. Other landing gear studies have used this same relationship (refs. 5 and 7), although at present no conclusive experimental evidence in support of this assumption is available.

If it is assumed that the friction coefficient is the same on each sliding surface, then the axial friction force, $F_f$, may be determined from the equation

$$F_f = \frac{s}{|s|} \mu \left( |F_1| + |F_2| \right)$$

or, since summation of horizontal forces (fig. 2a) indicates that $F_1 = F_2$ for this strut,

$$F_f = \frac{s}{|s|} 2\mu |F_1|$$

where

- $\mu$ coefficient of friction
- $\frac{s}{|s|}$ factor to indicate sign of friction force
- $F_1$ normal force on the upper bearing
- $F_2$ normal force on the lower bearing

It is desirable to obtain $F_1$ in terms of the resultant axial force within the strut expressed by $F_s = F_f + F_a + F_r$. If $R_1$ is the distance between the upper and lower bearing surfaces for the fully extended strut, then this distance may be expressed as $(R_1 + s)$ for any other strut position. Considering the forces applied to the lower strut piston as a free body, and taking moments about position $M$ in figure 2a, the equation for $F_1$ is
\[(R_1 + s)F_1 = \mu F_1 (d + b_1) + \mu F_2 (d - b_2) + d(F_s - \mu F_1 - \mu F_2) \quad (8)\]

where \(b_1\) and \(b_2\) are the outer radii of the upper and lower bearings, respectively, and \(d\) is the distance from the axle to the strut center line. Since \(F_1 = F_2\), the equation for \(F_1\) may be found as

\[F_1 = \frac{d}{[(R_1 + s) - \mu (b_1 - b_2)]} F_s \quad (9)\]

In order to obtain a solution as presented in this paper, it is necessary to express the quantity \([R_1 + s - \mu (b_1 - b_2)]\) as a constant. A similar assumption was made in reference (7). If \(R_1\) is large with respect to the maximum value of \(s\) which is possible, \(s_m\), then a reasonable constant to use would be \([\left(R_1 + \frac{s_m}{2}\right) - \mu (b_1 - b_2)] = R\) where \(R\) is approximately an average distance between the bearing surfaces. \(s_m\) usually is quite well defined by the strut geometry, and the accuracy of this assumption for \(R\) depends on the particular strut being considered. The friction forces are often quite small for the simple vertical impact case; and the inclusion of friction, as will be shown, does not complicate the final solution obtained. Therefore it was felt that this term should be included for possible future studies on the qualitative effects of friction in the landing gear.

The normal bearing force from equation (9) may now be expressed as

\[F_1 \approx \frac{d}{R} F_s \quad (10)\]

and from equation (7)

\[F_r = \frac{s}{\left|\frac{s}{|s|}(R)\right|} F_s \quad (11)\]
Note that since $F_s$ is the total axial strut force, it can be expressed as the summation of these internal forces:

$$F_s = F_h + F_a + F_r$$  \hspace{1cm} (12)

For this analysis the portions of mass comprising the inner and outer cylinder of the strut are considered to be weightless, and their weights are included with the major portion of the masses to which they are respectively attached. The assumption that the center of gravity of the lower mass lies at the axle center line also is a usual stipulation in such an analysis.

Recent data (refs. 5 and 8) indicates that the dynamic force deflection characteristics of the tire may be sharply nonlinear for small deflections, but can be represented by a linear approximation with only minor loss in accuracy for the major remaining portion of the curve. Thus a linear segment approximation to the true tire spring may be made as in figure 3. For the 27-inch-diameter tire being considered, this assumption gives no force for deflections less than $z_2 = 0.0508$ foot, and a constant slope $K_2$ thereafter. Such a representation seems to be entirely adequate for most practical purposes (ref. 5).

If $F_v$ is defined as the vertical force on the ground resulting from deflection of the spring $K_2$, then the force may be written:

$$F_v = 0 \text{ for } 0 < z < 0.0508 \text{ foot}$$ \hspace{1cm} (13a)

$$F_v = K_2z_2 \text{ for } z_2 > 0.0508 \text{ foot}$$ \hspace{1cm} (13b)
Figure 3. - Comparison of the true force-deflection curve of the tire and a linear approximation to the tire spring.
Finally, the wing lift from aerodynamic forces is represented by the force \( L \) acting through the center of gravity of the upper or fuselage mass. Because of the very short time involved to complete the impact, the wing lift can reasonably be assumed to be a constant. Recent experimental data (ref. 9) indicates that for most landings the wing lift at ground contact is very nearly equal to the weight of the airplane. During the experimental drop tests used for comparisons with theory in this analysis the wing lift force was simulated so that it was equal to the weight of the test model.
V. EQUATIONS OF MOTION

Derivation of the Equations of Motion

With $F_g$ and $F_v$ defined as in the preceding section, the equations of motion may be written in simple form. Considering the summation of vertical forces on the lower mass, (fig. 2a), the motion of the axle is given by the equation

$$F_g - F_v + \frac{W_2}{g} z_2 = 0$$

(14)

Similarly, the vertical motion of the upper mass (fig. 2b) is determined by the equation

$$F_g + L - W_1 = -\frac{W_1}{g} z_1$$

(15)

Where $L$ is the wing-lift force as defined earlier. If equations (14) and (15) are combined, the overall balance of forces can be expressed by

$$F_v + L - W = -\frac{W_1}{g} z_1 - \frac{W_2}{g} z_2$$

(16)

where

$$W = W_1 + W_2$$

Any two of these equations is sufficient to obtain a solution, since there are two degrees of freedom. In this derivation, equations (14) and (16) will be used.

In order to obtain equation (14) in a more desirable form we must express the strut axial force more explicitly. From equation (12)
\[ F_s = F_h + F_a + F_f \]

or, since the air pressure force is neglected for the reasons given in the preceding section,

\[ F_s = \frac{\rho h^3}{2\left(\frac{d^2}{A_o}\right)^2} \dot{s} \left(\frac{s}{s'}\right) + \frac{s}{s'} \left(\frac{2\mu d}{(R)}\right) |F_s| \]  \hspace{1cm} (17)

Some simplifications become obvious at this point. Since the strut compression process only is of interest, the strut velocity and strut axial force will always be positive. Therefore, the absolute value signs may be removed so that

\[ F_s = As^2 + BF_s \]  \hspace{1cm} (18)

where

\[ A = \frac{\rho h^3}{2\left(\frac{d^2}{A_o}\right)^2} \]
\[ B = \frac{2\mu d}{(R)} \]

Equation (18) can be solved for \( F_s \) giving

\[ F_s = \frac{A}{(1-B)} s^2 \]  \hspace{1cm} (19)

Using the definition for \( \dot{s} \), substitute \( F_v \) and \( F_s \) from equations (13b) and (19) into equations (14) and (16) with the result

\[ \frac{w_2}{g} \dot{z}_2 - \frac{A}{(1-B)} \left(\dot{z}_1 - \dot{z}_2\right)^2 + K_2 z_2 - W_2 = 0 \]  \hspace{1cm} (20)
\[
\frac{W_1}{g} \ddot{z}_1 + \frac{W_2}{g} \ddot{z}_2 + K_2 z_2 + L - W = 0
\] (21)

Note that the net effect of including the friction force in the equations of motion is only to change the constant coefficient of the quadratic damping term. In references (5 and 6) it has been determined that since the lower mass is a relatively small fraction (less than 1/16) of the total mass, the system may be simplified even further. Calculations made in these references indicate that the lower mass may be assumed to be equal to zero without greatly modifying the results. Using this assumption, equations (20) and (21) may be written as

\[
\left( \dot{z}_1 - \dot{z}_2 \right)^2 - \left[ \frac{(1 - \beta)}{A} K_2 \right] z_2 = 0
\] (22)

\[
z_1 + \frac{gK_2}{\bar{w}_1} z_2 + (L - \bar{w}_1) \frac{g}{\bar{w}_1} = 0
\] (23)

These are the equations to be solved. The initial conditions for the vertical impact are given by

\[
t = 0 \\
z_1 = z_2 = 0 \\
\dot{z}_1 = \dot{z}_2 = \dot{z}_0
\] (24)

However, the assumption of equation (13a) implies that the system must move a distance equal to 0.0508 foot after initial contact before any finite ground reaction can develop. Since equations (22) and (23) assume that the ground reaction increases linearly with deflection, these
equations do not apply until some time \( t_1 \) after contact. The initial velocity remains essentially constant over this very short interval of time, so \( t_1 \) can be found from

\[
t_1 = \frac{0.0508}{z_o}
\]  

(25)

The initial conditions (24) still apply to the equations of motion in this case with the coordinate system transformed so that the tire force-deflection curve passes through zero. In plotting the solutions, all results must be displaced in time by \( t_1 \) seconds, and the displacements will have 0.0508 foot added, to indicate the actual distance through which the model has moved.

**Introduction of Dimensionless Variables**

The solution of equations (22) and (23) depends on six parameters, namely, \( A, B, \frac{w_1}{g}, K_2, L, \) and \( z_0 \). It is desirable to reduce the number of parameters, and this can be done by introducing generalized variables \( u \) and \( u_1 \) in the following transformations:

\[
u = \frac{Ag}{w_1(1 - B)} z_2
\] 

(26)

\[
u_1 = \frac{Ag}{w_1(1 - B)} z_1
\]

and

\[
\theta = \sqrt{\frac{K_2 g}{w_1}} t
\] 

(27)
so that:

$$u' = \frac{du}{d\theta} = z_2 \left[ \frac{A}{(1 - B)} \sqrt{\frac{g}{K_2 W_1}} \right]$$

$$u_1' = \frac{du}{d\theta} = z_1 \left[ \frac{A}{(1 - B)} \sqrt{\frac{g}{K_2 W_1}} \right]$$

and

$$u_1'' = \frac{d^2u_1}{d\theta^2} = z_1 \left[ \frac{A}{K_2(1 - B)} \right]$$

With these new variables, equations (22) and (23) become

$$(u_1' - u')^2 - u = 0 \quad (28)$$

$$u_1'' + u + K_L = 0 \quad (29)$$

where

$$K_L = \frac{A(L - W_1)g}{K_2(1 - B)W_1}$$

Of special interest in this analysis is the case where the wing lift is equal to the weight of the system, as indicated earlier in this paper. When \( L = W_1 \), then \( K_L = 0 \); and we see that equations (22) and (23) will have a single parameter only, the initial nondimensional contact velocity, which is

$$u_0' = z_0 \left[ \frac{A}{(1 - B)} \sqrt{\frac{g}{K_2 W_1}} \right] \quad (30)$$
Equations (28) and (29) can be solved simultaneously to give a single equation in one variable. Rewrite equation (28) as

$$u_1' - u' = u^{1/2} \tag{31}$$

where the positive root must be used since the equations of motion are applicable only for positive strut-stroke velocity, and this implies that \(u_1' - u'\) must be positive. Differentiate equation (31) with respect to \(\Theta\). Then

$$u_1'' - u'' = \frac{1}{2} u^{-1/2} u' \tag{32}$$

Solve equation (29) for \(u_1''\) and substitute in equation (32). This gives

$$u'' + u + \frac{1}{2} u^{-1/2} u' + K_L = 0 \tag{33}$$

**Transformation of the Equations**

In order to obtain equation (33) in a form more amenable to solution, use the following transformation:

$$u = \tau^2 \tag{34}$$

Then equation (33) may be written

$$\frac{d}{d\Theta} \left[ \frac{d(\tau^2)}{d\Theta} \right] + \frac{1}{2\tau} \frac{d(\tau^2)}{d\Theta} + K_L + \tau^2 = 0 \tag{35}$$

Apply the transformation

$$x = \int_0^\Theta \frac{d\tau}{2\tau(a^2)} \tag{36a}$$
or

\[ dx = \frac{d\theta}{2\tau(\theta)} \]  \quad (36b)

The final equation obtained is

\[ \frac{d^2 \tau}{dx^2} + \frac{d\tau}{dx} + 2k_L \tau + 2\tau^3 = 0 \]  \quad (37)

To determine the initial conditions for this equation, rewrite equation (36b) as

\[ d\theta = \frac{d(\tau^2)}{d\tau} \, dx \]

By substituting equation (36b), it can be seen that

\[ \frac{du}{d\theta} = \frac{dx}{d\tau} \]  \quad (38)

Therefore, the initial conditions are:

for \( x = 0 \)

\[ \tau = u_o^{1/2} = 0 \]  \quad (39)

\[ \frac{d\tau}{dx} = u_o' \]
VI. SOLUTION OF THE TRANSFORMED EQUATION

Considerations Necessary to Obtain a Solution

An exact analytical solution for equation (37) is obtainable only for a specific value of \( K_L \). An equation similar in form to equation (37) for which a solution is known, is

\[
\frac{d^2 \tau}{dx^2} + \frac{d \tau}{dx} + \frac{2}{9} \tau + 2H^2 \tau^3 = 0
\]

where \( H \) is an arbitrary constant. This is a special case of an equation studied in reference (10). It is apparent, then, that the solution of equation (37) is identical to the solution of equation (I40) with the values \( K_L = 1/9 \) and \( H = 1 \).

For any given landing gear configuration, this value of \( K_L \) implies that only one value of wing lift, \( L \neq W_L \), will suffice. As was indicated earlier in this paper, for most landings \( L \neq W_L \); and in general the value of wing lift will not satisfy the condition that \( K_L = 1/9 \). However, it will be shown that with other values of \( K_L \), equation (I40) has solutions reasonably representative of the solutions of equation (37) for certain values of initial conditions and the parameter \( H \).

A device sometimes used in mechanics is to replace a nonlinear spring or damper system by an equivalent system which will have an equal absorption of energy within the range of the variable which is of interest (refs. 11 and 12). Usually this method is used only when the nonlinear equation is a small perturbation from a corresponding linear
differential equation, so that the nonlinear terms may be replaced by equivalent linear expressions which may then be handled analytically. In equation (37) the nonlinearity is so large that equivalent linearization results in solutions that are greatly distorted. Since an analytical solution to equation (40) is available, in this case the energy considerations may be equally useful in replacing the linear and cubic terms in equation (40) by equivalent linear and cubic terms with coefficients adjusted to those in equation (37). Considering these expressions as spring forces, the energy stored within the springs is the integral of the force with respect to the displacement. For this study the primary interest is in the transient response during the first half cycle only; so if the maximum value of displacement in this interval is designated by $\tau_m$, the energy relationship between the actual and equivalent spring is

$$\int_0^{\tau_m} \left[ \frac{2}{9} \tau + 2H^2 \tau^3 \right] d\tau = \int_0^{\tau_m} \left[ 2K_L \tau + 2\tau^3 \right] d\tau$$

(41)

where

$$\tau_m = u_m^{1/2}$$

The determination of the value of $u_m$ is to be discussed later in the section entitled "Qualifications of the Solution." Since the other constants are fixed, $H$ must be adjusted to satisfy the condition of equal energy. By performing the integration, the following expression
is found

\[ H = \left[ 1 - \frac{2}{\tau_m} \left( \frac{1}{2} - K_L \right) \right] = \sqrt{1 - \frac{2}{\tau_m} \left( \frac{1}{2} - K_L \right)} \]  

\( (h2) \)

For a value of \( H \) thus determined, equations \((h0)\) and \((37)\) may be considered as equivalent.

**Presentation of the Solution**

The solution for equation \((h0)\) as derived in appendix A is

\[ \tau = \sqrt{\frac{u_0}{2H}} e^{-x/3} \text{sd} \left[ 3 \sqrt{2Hu_0} \left( 1 - e^{-x/3} \right) \right] \]

\[ (h3) \]

where, for convenience, the modulus \( k^2 = 1/2 \) is understood to apply to all elliptic functions expressed herein. Since \( u = \tau^2 \), it can be seen that

\[ u = \frac{u_0}{2H} e^{-2x/3} \text{sd}^2 \left[ 3 \sqrt{2Hu_0} \left( 1 - e^{-x/3} \right) \right] \]

\( (h4) \)

Keeping in mind that \( \theta \) is defined implicitly as a function of \( x \) by equation \((36c)\), then equation \((36a)\) may be rewritten as

\[ \theta = \int_0^x 2 \tau(\xi) d\xi \]

where the limits on \( x \) are chosen to satisfy the initial conditions for \( \theta \).
Performing the indicated integration, it is seen that

$$\theta = \frac{2}{H} \sin^{-1}\left\{ -\frac{1}{\sqrt{2}} \text{cd}\left[ 3 \sqrt{2Hu_0'} (1 - e^{-x/3}) \right] \right\} - \frac{7\pi}{2H} \tag{45}$$

Thus the solution for $u(\theta)$ is in parametric form with equations (44) and (45). This form is convenient for obtaining the higher derivatives. The solution may be expressed in explicit form, however, as

$$u(\theta) = \frac{u_0'}{2H} \left( 1 - \frac{1}{3 \sqrt{2Hu_0'}} \theta \right)^2 \text{sd}^2 \theta \tag{46}$$

where

$$\theta = \text{cd}^{-1}\left[ -\sqrt{2} \sin\left( \frac{H}{2} \theta + \frac{7\pi}{4} \right), \ \kappa^2 = 1/2 \right] \tag{47}$$

The nondimensional velocity of the lower mass can be obtained by use of equation (38) as

$$u'(\theta) = u_0' \left( 1 - \frac{1}{3 \sqrt{2Hu_0'}} \theta \right)^2 \frac{\text{cd} \theta}{\text{sd} \theta} - \frac{1}{3} \sqrt{\frac{u_0'}{2H}} \left( 1 - \frac{1}{3 \sqrt{2Hu_0'}} \theta \right) \text{sd} \theta \tag{48}$$

For design purposes, the motion of the upper mass is also of interest. The nondimensional acceleration can be obtained from equation (29).

$$u_1''(\theta) = -\frac{u_0'}{2H} \left( 1 - \frac{1}{3 \sqrt{2Hu_0'}} \theta \right)^2 \text{sd}^2 \theta - \kappa_L \tag{49}$$

The nondimensional velocity of the upper mass may be derived by use of equation (28).
\[ u_1'(\theta) = u_0 \left( 1 - \frac{1}{3 \sqrt{2Hu_0}} \right)^2 \frac{\text{d}u_1}{\text{d}n} \theta + \frac{2}{3} \frac{u_0'}{2H} \left( 1 - \frac{1}{3 \sqrt{2Hu_0}} \right) \text{sd} \theta \] (50)

The dimensionless displacement of the upper mass, may be obtained
by graphical integration of equation (50), or more simply, perhaps, by
substituting \( u_1'(\theta) \) from equation (31) in an integral relationship as
follows:

\[ u_1(\theta) = \int_0^{\theta} u_1'(\gamma) \text{d}\gamma = \int_0^{\theta} \left[ u_1^{1/2}(\gamma) + u'(\gamma) \right] \text{d}\gamma = \int_0^{\theta} u_1^{1/2}(\gamma) \text{d}\gamma + u(\theta) \]

(51a)

Then, using the transformation equations (31f) and (36a), and selecting
limits of integration which satisfy the initial conditions, it can be
shown that

\[ u_1(x) = 2 \int_0^x u(\xi) \text{d}\xi + u(x) \] (51b)

Then \( u_1(\theta) \) is available in parametric form with equation (45). How-
ever, using equations (51b) and (45a) the following expression may be
derived:

\[ u_1(x) = u(x) + \frac{2}{3H^2} \left\{ \left[ E(v) - \frac{1}{2} v - \frac{1}{2} \ln \text{sdv} \right] \left( 3 \sqrt{2Hu_0'} - x \right) + \ln(\text{d}v) - \frac{v^2}{4} + \int_0^v E(v) \text{d}v \right\} \] (51c)
\[ v = 3 \sqrt{2Hu_o} \left(1 - e^{-x/3}\right) \]

- \( F(\varphi, k) \) incomplete elliptic integral of the first kind
- \( \varphi \) amplitude of \( v \)
- \( E(v) = E(\varphi, k) \) incomplete elliptic integral of the second kind

The integral term of equation (51c) may be evaluated graphically, and the result is presented for convenience in figure 4.

**Qualifications of the Solution**

Since the solutions contain the parameter \( H \) which is a function of \( u_m \), we must find a value for this maximum displacement before a calculation can be attempted. It may be indicated that the actual value of \( u_m \) which is used as the limit of integration in equation (41) is not critical with respect to the final solution. Any good approximate value for \( u_m \) will suffice since the equivalent spring force-deflection curves will be negligibly different for small variations in the limit of integration. This may be seen from figure 5 where the effects on these curves are very small for large changes in \( u_m \). A method for determining a value of \( u_m \) for use as the above mentioned upper limit of integration is presented in Appendix B, where a very good approximation for \( u_m \) as a function of \( u_o \) is derived using only equations (42) and (44). These values are compared with the true values for \( u_m \) in figure 6.

The degree to which the solution obtained by the method of "equivalent nonlinearity" represents the solution of the original equation (37) depends primarily on the proportion of energy stored by the linear and cubic spring terms. As indicated before, if \( K_L = \frac{1}{9} \) then the
Figure 5.- Comparison of the actual dimensionless spring-force curve and the approximate spring-force curves for various initial conditions.
Figure 6.- Comparison of the true maximum dimensionless lower mass displacement, $u_m$, and the approximation of $u_m$ for the full range of the initial velocity parameter, $u'_o.$
original equation and the equivalent equation (40) are identical and the proportion of energy capable of being stored by each spring must be the same. Usually, however, the case $K_L = 0$ is of the greatest interest so that only a cubic spring term remains in equation (37). In equation (40) the coefficient of the cubic term is a function of $u_m$; so as $u_m$ gets smaller, the coefficient gets smaller, with the result that the linear spring stores a greater proportion of the total spring potential and equation (40) becomes less representative of the original equation.

It may be indicated, however, that this limitation is primarily concerned with the frequency characteristics and shape of the response curves. The energy considerations used in deriving the equivalent non-linear spring imply that the maximum deflection of the approximate spring system will be close to that of the original system, since the work done in deflecting the springs to this maximum position is the same for either spring system.

The energy ratio spoken of indicates, then, to what extent the shape of the response curves may be in error. This ratio of energy, $r$, can be obtained from equation (41) for the case of $K_L = 0$, as

$$r = \frac{\int_0^{\tau_m} \frac{2}{9} \tau^2 \tau d\tau}{\int_0^{\tau_m} \left[ \frac{2}{9} \tau + 2H^2 \tau^3 \right] d\tau} = \frac{1}{1 + \frac{9}{2} \frac{H^2 \tau_m^2}{2}}$$

or, by use of equations (34) and (42)

$$r = \frac{2}{9 \mu_m}$$

(52)
Since $u_m$ is known as a function of $u_o$ (Fig. 6), this ratio of energy may also be presented as a function of $u_o$ as shown in figure 7. Where the ratio of energy stored by the linear spring is a small fraction of the total energy stored in the springs, the solution would be expected to be very good. From figure 7 it can be seen that for $r > \frac{1}{5}$ and for corresponding initial conditions $u_o' \leq 2$, a more rigorous solution would be desirable.
Figure 7.- Plot of the ratio of energy which the linear spring is capable of absorbing to the total energy which both the linear and cubic springs can absorb, as a function of the initial velocity parameter, $u'_0$. 

$$r = \frac{2}{9u_m}$$

Ratio of energy, $r$

Initial dimensionless velocity parameter, $u'_0$
VII. EVALUATION OF THE ANALYTICAL SOLUTION

Comparison with Solutions by the Runge-Kutta Method

Solutions for equations (28) and (29) have been obtained by numerical integration methods in reference (5). The two equations were solved simultaneously and the Runge-Kutta procedure applied as given in reference (13). These results were thoroughly checked for a few specific values of the initial conditions and may be considered as nearly exact solutions for the motion of the system. Therefore, comparisons made between the results presented in reference (5) and the solutions obtained in this paper should be a good indication of the accuracy of the analytical solutions.

The system analyzed in the reference did not have an eccentrically located axle, so that \( d = 0 \) and no normal bearing forces were developed. Thus for these solutions there were no friction forces. Other constants for the physical system are presented in Appendix C. Values for the Jacobian elliptic functions were obtained from reference (14).

The dimensionless displacement and acceleration of the upper mass, and the dimensionless displacement of the lower mass are presented as functions of dimensionless time, \( \theta \), in figures 8 and 9. The ratios of the nondimensional velocities of the upper and lower masses to the initial dimensionless velocity are presented in figures 10 and 11. The solutions are discontinued when \( (u_1' - u') = 0 \), since this corresponds to the time after which \( \dot{\theta} \) becomes negative, and the equations of motion are no longer applicable. A wide range of \( u_0' \) necessary for design purposes is considered, and in most cases the agreement between
Figure 8.- Comparison of solutions for lower-mass displacement and upper-mass acceleration obtained by the Runge-Kutta procedure and from the analytical equation.
Figure 9. Comparison of solutions for upper-mass displacement obtained by the Runge-Kutta procedure and from the analytical equation.
Figure 10.- Comparison of solutions for the lower-mass velocity obtained from the Runge-Kutta procedure and from the analytical equation.
Figure 11.- Comparison of solutions for the upper-mass velocity obtained from the Runge-Kutta procedure and from the analytical equation.
the analytical solution and results obtained by means of the Runge-Kutta procedure is good. Note that the acceleration of the lower mass is not presented, since this quantity is of little importance to the designer, and the corresponding inertia force was neglected in order to obtain equations (28) and (29).

Some differences are apparent at the very lowest initial conditions considered. As was indicated before, for \( u_0' = 1 \) the ratio \( r \) (fig. 7) is rather large, indicating that the linear term in the equivalent spring force is accounting for the major portion of the work done by the springs. Therefore the method of equivalent nonlinearization presented in this paper is approaching the case of equivalent linearization for the very lowest initial conditions. Linearization is known to be impractical in this problem, so the results may be expected to be less good in this range.

**Comparison with Experimental Data**

Reference (5) describes experimental drop tests conducted with a conventional oleo-pneumatic landing gear which closely represents the system in figure 1. Data obtained from these tests, in which a value of wing lift equal to the weight of the test model was simulated, are used for comparisons with the theoretical solutions presented in this paper.

Application of the transformation equations (26) and (27) enables the calculation of the displacements, velocities and acceleration in dimensional terms. These quantities for the upper and lower masses are presented in figures 12 and 13 along with the corresponding experimental data for the impact initial condition of \( z_0' = 8.86 \) feet per second.
Figure 12: A comparison between theoretical results and experimental data for a specific system. The graph illustrates the relationship between displacement and time for upper and lower mass systems.
Figure 13. Comparison between theoretical results and experimental data on the upper-mass acceleration for a typical impact.
This value is equivalent to the dimensionless parameter \( u'_o = 2.39 \).
The ground vertical reaction force \( F_v \) is presented in figure 14 for
comparison with the experimentally obtained forces. In general the
results are in good agreement. The small differences that appear are
attributable to the neglect of the air pressure force, the omission of
the lower mass, to differences between the actual and the assumed tire
characteristics, and to experimental errors.

In view of the observed good agreement between theory and experiment
it seems that the solution presented will enable a designer to determine
with reasonable accuracy the loads a landing gear may experience and thus
aid him in preliminary design calculations.
Figure 14.- Comparison between theoretical results and experimental data on the ground vertical force for a typical impact.
VIII. CONCLUSIONS

An analytical solution has been presented for the equations of motion of a two-degree-of-freedom nonlinear system representing an airplane and landing gear configuration subjected to an impact load. The following conclusions can be made from the evaluation:

1. The friction forces developed on the strut bearing surfaces because of bending forces applied at the axle may be included in the equations of motion without adding to the complexity of the final differential equation. It is necessary to replace the variable distance between bearing surfaces by an approximate mean distance.

2. The equations of motion for the system may be transformed to an equation which closely resembles one having an exact elliptic solution. For a given airplane, a certain value of wing lift during landing will result in the two equations of motion being identical, and the exact solutions will describe the impact.

3. Since in general the wing lift force is approximately equal to the weight of the airplane, a method of "equivalent nonlinearization" must be employed to obtain the equation of motion in a form amenable to solution. The equivalent equation is derived on the basis of equal energy considerations and the major nonlinear character is retained, thereby presenting an improvement over the usual linearization methods which have been unsuccessful in this problem.

4. Comparison with more exact solutions obtained by numerical integration methods reveals that the analytical solutions obtained herein are very accurate within a wide range of initial conditions.
5. Comparison with experimental results indicates that notwithstanding the numerous simplifications made, the solutions obtained describe the motion of the system adequately for design purposes.
IX. BIBLIOGRAPHY


X. VITA

The author was born in St. Cloud, Minnesota on July 1, 1928. He attended Central High School, Minneapolis, Minnesota and graduated in June 1946. The following September he entered the University of Minnesota at Minneapolis and received the Degree of Bachelor of Civil Engineering in March 1951. After graduation, he was called to active military service with the 17th Infantry Division, of the Minnesota National Guard and was honorably discharged in May 1952. Subsequently, he accepted a position with the National Advisory Committee for Aeronautics where he is presently an Aeronautical Research Scientist. After some preliminary graduate studies at the University of Virginia, he entered Virginia Polytechnic Institute in the summer of 1954 to begin work toward the Degree of Master of Science in Applied Mechanics.

J.eorge A. Theisen
XI. APPENDICES

Appendix A. The Derivation of the Solution
to the Transformed Equation

The solution to equation (23) may be determined as follows:

For this appendix only, let differentiation with respect to \( x \) be

\[
\frac{d}{dx} \left( \frac{1}{x'} + \frac{1}{\tau} \right) = \frac{2}{3} \tau + 2H^2 \tau^3 = 0
\]

(A1)

Multiplying through by \( e^{2x/3} \) and arranging terms, equation (A1) may

be written as

\[
\frac{d}{dx} \left[ e^{2x/3} \left( \frac{1}{x'} + \frac{1}{\tau} \right) \right] = -2H^2 \tau^3 e^{2x/3}
\]

(A2)

This equation may be rewritten

\[
\frac{d}{dx} \left[ e^{2x/3} \left( \frac{1}{x'} + \frac{1}{\tau} \right) \right] = -2H^2 \tau^3 e^{2x/3}
\]

(A3)

Multiply through by \( \left[ e^{2x/3} \left( \frac{1}{x'} + \frac{1}{\tau} \right) \right] dx \) to obtain

\[
\left[ e^{2x/3} \left( \frac{1}{x'} + \frac{1}{\tau} \right) \right] \frac{d}{dx} \left[ e^{2x/3} \left( \frac{1}{x'} + \frac{1}{\tau} \right) \right] = -2H^2 \tau^3 \left[ e^{2x/3} \right]^3 \left( \frac{1}{x'} + \frac{1}{\tau} \right) e^{2x/3} dx
\]

(A4)

Equation (A4) may be integrated, and the resulting expression is

\[
e^{2x/3} \left( \frac{1}{x'} + \frac{1}{\tau} \right)^2 = \left[ 1 - \left( \frac{3\tau e^{x/3}}{c_1} \right)^{1/3} \right] \left( \frac{c_1}{3} \right)^{1/3} H^2
\]

(A5)
where $c_1$ is the constant of integration. Take the square root of both sides and solve for $Hc_1^2$

$$Hc_1^2 = \frac{e^{2x/3}}{\frac{1}{9} \sqrt{1 - \frac{(2x e^{x/3})}{c_1}}} \left( \frac{1}{\tau} + \frac{1}{3} \tau \right)$$  \hspace{1cm} (A6)

Multiply through equation (A6) by $\left( \frac{1}{3c_1} e^{-x/3} dx \right)$ and rearrange factors to obtain

$$\frac{1}{3} Hc_1 e^{-x/3} \frac{dx}{\sqrt{1 + \left( \frac{2x e^{x/3}}{c_1} \right)^2 \left[ 1 - \left( \frac{2x e^{x/3}}{c_1} \right)^2 \right]}}$$  \hspace{1cm} (A7)

If a quantity $\psi$ is defined as

$$\psi = \sin^{-1} \left( \frac{3x e^{x/3}}{c_1} \right)$$  \hspace{1cm} (A8)

so that

$$d\psi = \frac{dx}{\sqrt{1 - \left( \frac{3x e^{x/3}}{c_1} \right)^2}} \left[ \left( \frac{1}{\tau} + \frac{1}{3} \tau \right) \frac{3x e^{x/3}}{c_1} \right]$$  \hspace{1cm} (A9)

then substitution of equations (A8) and (A9) into equation (A7) results in the expression

$$\frac{1}{3} Hc_1 e^{-x/3} \frac{dx}{\sqrt{1 + \sin^2 \psi}}$$  \hspace{1cm} (A10)
This may be written in elliptic integral form as follows

$$\int \frac{1}{3} \text{He}e^{-x/3} dx = \int_0^\phi \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

(A11)

where $k^2 = -1$ is the modulus. The result of the integration may be written in standard notation as

$$-\text{He}e^{-x/3} + c_2 = F(c, k) \equiv \nu$$

(A12)

where $c_2$ is a constant of integration and by definition

$$\text{sn} \nu = \sin \phi$$

(A13)

Therefore, in terms of the Jacobian elliptic function equation (A12) may be written as

$$\text{sn} \left( \frac{c_2 - \text{He}e^{-x/3}}{c_1} \right) = \frac{3 \pi e^x/3}{c_1}$$

$k^2 = -1$

(A14)

The solution is, then,

$$\tau = \frac{1}{3} c_1 e^{-x/3} \text{sn} \left( \frac{c_2 - \text{He}e^{-x/3}}{c_1} \right)$$

$k^2 = -1$

(A15)

Using the initial conditions from equation (39), the constants of integration may be determined as

$$c_1 = 3 \sqrt{\frac{u_0}{H}}$$

$$c_2 = 3 \sqrt{Hu_0}$$

(A16)
By substitution of these constants, equation (A15) becomes

\[
\tau = \sqrt{\frac{u_0'}{H}} e^{-x/2} \text{sn} \left[ 3 \sqrt{H u_o'} \left(1 - e^{-x/3}\right) \right]_{k^2=-1} \tag{A17}
\]

This is a valid solution, but the modulus usually is tabulated for positive, real values. By applying a transformation to the modulus (ref. 1h), equation (A17) may be written

\[
\tau = \sqrt{\frac{u_0'}{2H}} e^{-x/3} \text{sd} \left[ 3 \sqrt{2H u_o'} \left(1 - e^{-x/3}\right) \right]_{k^2=1/2} \tag{A18}
\]
Appendix B. The Determination of an Approximate Value for \( u_m \)

To solve for the actual maximum value of \( u \) it would be necessary to evaluate equation (18) for \( u' = 0 \). The corresponding value of \( \theta \) would be very difficult to obtain from this expression. For purposes of this paper any good approximate value of \( u_m \) is sufficient, and equation (18) which is in the parametric form for \( u \) is much simpler to work with as follows:

\[
u = \frac{u_m'}{2H} e^{-2x/3sd} \left[ \frac{3}{\sqrt{2Hu_0}} (1 - e^{-x/3}) \right] \tag{B1} \]

Under the assumption that the exponential term is relatively slowly varying, the value of the argument \( \nu \) corresponding to \( u_m \) can be found by differentiating \( sd(\nu, 1/2) \) with respect to \( \nu \), and equating the result to zero as shown

\[
\frac{d}{d\nu} sd(\nu, 1/2) = \frac{cn(\nu, 1/2)}{dn^2(\nu, 1/2)} = 0 \tag{B2}
\]

or, since the \( dn(\nu, 1/2) \) is always finite

\[
\nu = \text{cn}^{-1}(0, 1/2)
\]

so that

\[
\nu = K \approx 1.85407 \ldots \tag{B3}
\]

where \( K \) is the value of the argument at the quarter period. Substitution of this value in the argument of equation (B1) yields the result

\[
\nu = 3 \sqrt{2Hu_0} (1 - e^{-x/3}) = K \tag{B4}
\]
where the value of \( x \) now corresponds to the value of \( u = u_m \) in the first approximation. From equation (B4) it may be seen that

\[
e^{-x/3} = 1 - \frac{K}{3 \sqrt{2H} u_0'}
\]  

(B5)

Noting that

\[
 sd^2(k, \ell/2) = 2
\]  

(B6)

expressions (B5) and (B6) may be substituted in equation (B1) to obtain

\[
u_m = \frac{u_0'}{H} \left[ 1 - \frac{K}{3 \sqrt{2H} u_0'} \right]^2
\]  

(B7)

where, from equation (B2)

\[
 H = \sqrt{1 - \frac{2}{9 u_m}} \quad \text{(for } K_L = 0) \]

(B8)

From equations (B7) and (B8) the desired approximate solution for \( u_m \) may be obtained. However, by combining the two equations and solving for \( u_0' \), a more explicit expression may be obtained as follows:

\[
u_0' = \frac{H}{9} \left[ 3 \sqrt{u_m} + \frac{K}{H \sqrt{2}} \right]^2
\]  

(B9)

This equation is easier to use for obtaining \( u_m \) as a function of \( u_0' \).
### Appendix C. Constants of the Physical System

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<td>lb</td>
<td>131</td>
</tr>
</tbody>
</table>