A micromechanics-based method for off-axis strength prediction of unidirectional laminae – Approach for a nonlinear rubber based lamina

Jérémy Duthoit

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Kenneth L. Reifsnider (Chair)
Scott W. Case
Stephen L. Kampe
Ronald G. Kander

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(ABSTRACT)

In this study, a micromechanics-based method is developed to predict the off-axis strength of unidirectional linear elastic laminae. These composites fail by matrix cracking along a plane parallel to the fiber direction. The stresses in the matrix are calculated using a local stress analysis based on a concentric cylinder model. This model consists of a unique fiber embedded in matrix; both constituents are represented by cylinders. A finite element model is also constructed and the results of the two models compared. The stresses and strains from the concentric cylinder model are averaged over the volume of the matrix and used in a local failure function. This failure function has the form of a reduced and normalized strain energy density function where only transverse and shear terms are considered. The off-axis strength prediction method is validated using data from the literature.

This failure function will be used in the near future for composites with a matrix having nonlinear properties. Experimental tensile tests on steel-cord/rubber laminae and laminates as well as on the nonlinear rubber matrix were performed. Stress-strain behavior and off-axis strength data were obtained. An approach for off-axis strength prediction for these laminae is defined based on a finite element stress analysis. The finite element analysis approach is motivated by the one used for linear composites.
DEDICATION

This work is dedicated to my parents Liliane and Bruno Duthoit.
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CHAPTER 1 INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

Because of their high strength-to-weight ratio as well as corrosion and fatigue resistance, composite materials are a more and more common alternative to traditional materials such as metals and polymers. They are used in a wide range of applications: from the automotive and aerospace industry to the sporting goods industry. With this steady rise in the use of composite materials, failure theories and life prediction tools have been developed to help the designer and the engineer. Most of these theories are macromechanical; the composite is modeled as a homogeneous anisotropic material. They were derived from the ones used for “classical” homogeneous isotropic materials such as metals. However, to take into account the interaction of the fiber and the matrix (the fundamental constituents of the composite) and to directly relate the structural performance to the fundamental make-up of the composite, a micromechanical study is necessary. The present work focuses on developing a micromechanics-based method to predict off-axis strength of unidirectional laminae. The understanding of the failure mechanisms within a lamina is of prime importance since these laminae are building blocks of the widely used laminated composites. A special local failure function coupled with a local stress analysis is used for composites with linear elastic constituents. An approach for off-axis strength prediction of composites with nonlinear matrix is also discussed.

In the following section, the failure modes of off-axis unidirectional composites are presented. Static strength failure criteria and existing micromechanical stress analyses are then briefly reviewed. The last part of the literature review focuses on nonlinear elastic constitutive laws for rubber.
1.2 Literature review

1.2.1 Failure modes of off-axis unidirectional composites

Numerous previous experiments have been performed on off-axis composite laminae. Aboudi\(^1\) worked on an E-Glass/Epoxy lamina under static loading. The same composite system was used by Hashin and Rotem\(^2\) but under cyclic loading. Interestingly enough, the failure modes observed were similar for both static and oscillatory loadings. Described in the early work of Rosen and Dow\(^3\) and also of Hashin and Rotem\(^2\), failure takes two basic different configurations:

- For small off-axis angles (\(< \theta_{\text{thr}}\)), failure occurs due to cumulative fiber failure. An explanation for this phenomenon is that for small angles, most of the load is carried by the fibers. As shown by Rosen\(^3\), the failure load is then a function of fiber strength and of matrix and fiber elastic properties. \(\theta_{\text{thr}}\) is usually of the order of 1-2 \(^\circ\); the value of the threshold angle \(\theta_{\text{thr}}\) depends on the type of composites tested.

- For larger angles, the failure mode is matrix cracking along the fiber direction. When the off-axis angle \(\theta\) increases, matrix shear and transverse stresses increase whereas fiber stresses decrease. Subramanian\(^4\) observed that for an Glass/Epoxy system, when \(5^\circ<\theta<30^\circ\), the matrix failed in shear (Figure 1 (a)) and when \(\theta>30^\circ\), normal stresses in the matrix became dominant and caused final failure (Figure 1 (b)). These observations are dependent on the nature of the constituents of the composite. Generally speaking, we can say that rupture is due to combined transverse tensile and shear stresses.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{matrix_cracking.png}
\caption{Matrix cracking in (a) transverse tension and (b) shear}
\end{figure}

\textit{From Rahman and Pecknold}\(^5\)
Hashin and Rotem\textsuperscript{2} suggest that the fundamental difference between the two failure modes described (fiber breaking and matrix cracking) makes it reasonable to assume that the two phenomena are independent of each other.

Several authors\textsuperscript{6,7,8,9} investigated failure of unidirectional fiber-reinforced laminates which consist of unidirectional fiber-reinforced laminae stacked together. Ultimate failure of these laminates usually results from a sequence of events. The first of these events is the matrix cracking phenomenon at the lamina level discussed before. Huang and Yeoh\textsuperscript{10} and Lee and Liu\textsuperscript{11} describe the sequence of events leading to failure as:
- fiber-matrix debonding developing into matrix cracks within a lamina
- coalescence of matrix cracks to form a line crack that turns into an interply crack
- propagation of the interply crack due to interlaminar shear strain, delamination and final failure

This process is illustrated in Figure 2. It is common to a wide range of composites, from Graphite/Epoxy to Steel/Rubber. When the fiber/matrix interface is strong, the debonding phenomenon is skipped and failure occurs within the matrix.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Debonding leading to crack coalescence and delamination for a [+38/-38/-38/+38] nylon/elastomer composite. From Lee and Liu\textsuperscript{11}.}
\end{figure}
A typical delamination pattern for elastomer based laminates is shown in Figure 3 (a).

Fiber-matrix debonding, referred to as “socketing”, was first described by Breidenbach and Lake\textsuperscript{13,14}. Large interlaminar shear strains are developed in angle-ply elastomeric composites loaded in tension. According to Lee and Liu\textsuperscript{11}, ±19° steel/rubber laminates exhibit up to 130% interply shear strain when subjected to a global static tensile strain of 10%. For fiber orientations smaller than 30°, angle-ply laminates with non-elastomeric matrices may exhibit a “mixed” failure mode also called “Partial FB (fiber breakage)” by Sun and Zhou\textsuperscript{12}. Failure starts as matrix cracking and delamination and then converts to fiber breakage (Figure 3 (b)). Independence of the two failure mechanisms may not be assumed for laminates.

The importance of traction free edges has been noted by Lee and Liu\textsuperscript{11} and Huang and Yeoh\textsuperscript{10} for rubber based composites and Sun and Zhou\textsuperscript{10} for Graphite/Epoxy systems. Matrix cracking is thought to initiate close to the cord ends because of higher stress and strain concentrations.
1.2.2 Static Strength Theories

Tools to predict failure of laminated composite systems are of prime importance to the designer or engineer. A constant effort has been made to develop failure criteria applicable to composite materials. These criteria were, as expected, derived from those used for “classical” isotropic materials. Special efforts were made to modify the existing failure functions or to apply them appropriately to take into account the anisotropy in strength of composites. The failure criterion determines a surface in the stress space surface called the failure envelope. As for isotropic materials, this envelope defines an acceptable state of stress for the composite.

In the case of laminates, the global strength of the system will be dependent on the strength of the individual plies within the laminate. A straightforward and general procedure for laminate strength analysis exists and is well established. Using Classical Lamination Theory, stresses in each ply are calculated. The global applied load is increased till a lamina fails (usually referred to as First Ply Failure: FPF) according to a predefined criterion. The properties of the failed lamina are then discounted and a new stress and strain distribution in each lamina is calculated. If we have a stress-controlled loading condition, the stresses in the undiscounted plies increase to satisfy equilibrium conditions. The two failure modes normally observed for FPF are delamination and matrix cracking. The discount operation involves setting the transverse and shear modulus to a reduced value. When delamination occurs, a single laminate turns into two or more uncoupled laminates treated in parallel. Other plies may fail at the increased stress level resulting from FPF. If all of them fail, the laminate is said to have suffered “gross failure”. If no more laminae fail, the load is increased and the process repeated. As pointed out by Jones\textsuperscript{15}, “the entire procedure for strength analysis is independent of the failure criterion, but the results of the procedure, the maximum loads and deformations, do depend on the specific failure criterion”.

In the next sections, we will survey commonly employed failure criteria. Most of these criteria are macroscopic, that is, applied at the lamina level. But the use of microscopic (at the fiber/matrix level) failure criteria, coupled with micromechanics-
based analysis is being used increasingly. For an extensive review of existing failure criteria, the reader is referred to Nahas\textsuperscript{16}.

1.2.2.1 Macroscopic failure criteria

We will use the classification first introduced by Tsai\textsuperscript{17} who differentiates non-interactive and interactive criteria. A criterion is said to be interactive when it takes into account interaction of the different mechanisms and modes of failure.

1.2.2.1.1 Non-Interactive Criteria

1.2.2.1.1.1 Maximum Stress Criterion

\textbf{Equation Set 1}

\begin{align*}
\sigma_x &< X_t & \sigma_x &> X_c \\
\sigma_y &< Y_t & \text{for tensile (+) stresses} & \sigma_y &> Y_c & \text{for compressive (-) stresses} \\
\tau_{xy} &< S & \tau_{xy} &< S
\end{align*}

where \(x\) is the fiber direction, \(y\) the transverse direction.

\(X_{t,c}\) = Tensile and Compressive strength in \(x\) direction

\(Y_{t,c}\) = Tensile and Compressive strength in \(y\) direction

\(S\) = In-Plane Shear Strength

Failure occurs when one of the inequality is met. To use this criterion, the five previously defined strengths must be experimentally measured.

1.2.2.1.1.2 Maximum Strain Criterion

This criterion is very similar to maximum stress.
Equation set 2

\[
\varepsilon_x < \frac{X_t}{E_{tx}} \quad \text{or} \quad \varepsilon_x > \frac{X_c}{E_{cx}}
\]

\[
\varepsilon_y < \frac{Y_t}{E_{ty}} \quad \text{for tensile strains} \quad \text{or} \quad \varepsilon_y > \frac{Y_c}{E_{cy}} \quad \text{for compressive strains}
\]

\[
\gamma_{xy} < \frac{S}{G} \quad \gamma_{xy} < \frac{S}{G}
\]

\[E_{ij} = \text{Elastic modulus in tension (i = t) or compression (i = c) in j direction}\]

\[G = \text{In-Plane (xy) Shear modulus}\]

1.2.2.1.1.3 Hashin Criterion

Hashin\textsuperscript{18} also models the various failure modes of a composite separately. A general failure criterion should be piecewise smooth, each piece modeling a distinct failure mode. In plane stress:

Equation Set 3

Fiber modes

Tensile failure

\[
\left( \frac{\sigma_1}{X_t} \right)^2 + \left( \frac{\sigma_{12}}{S} \right)^2 = 1
\]

Compressive failure

\[
\left( \frac{\sigma_1}{X_c} \right)^2 = 1
\]

Matrix modes

Tensile failure

\[
\left( \frac{\sigma_2}{Y_t} \right)^2 + \left( \frac{\sigma_{12}}{S} \right)^2 = 1
\]

Compressive failure

\[
\left( \frac{\sigma_2}{2X_{23}} \right)^2 + \left[ \left( \frac{Y_c}{2X_{23}} \right)^2 - 1 \right] \cdot \frac{\sigma_2}{Y_c} + \left( \frac{\sigma_{12}}{S} \right)^2 = 1
\]

\[X_{23} = \text{transverse shear strength}\]

where 1 is the fiber direction, 2 and 3 are the transverse directions.
1.2.2.1.2 Interactive Quadratic Criteria

The quadratic criteria developed for orthotropic plies of unidirectional composites were first introduced by workers in the USSR about 20 years ago. Goldenblat and Kopnov\textsuperscript{19} proposed a very general failure criterion in terms of stress components:

\textbf{Equation 4}

\[ [F_i F_j] \sigma_i [F_k F_L] F_j + [F_{jk} \sigma_j \sigma_k] Y + \ldots = 1 \]

The reader should keep in mind that the choice of relationships is based on curve fitting considerations. There is no theoretical or physical basis for them.

1.2.2.1.2.1 Tsai-Wu criterion

Tsai and Wu\textsuperscript{20} reduced Equation 4 to:

\textbf{Equation 5}

\[ [F_i F_j] + [F_{ij}] = 1 \]

also written:

\textbf{Equation 6}

\[ [G_i \varepsilon_i] + [G_{ij} \varepsilon_i \varepsilon_j] = 1 \quad i, j = 1, \ldots, 6 \quad \text{in strain space} \]

\( F_i \) and \( F_{ij} \) are strength tensors of the second and fourth ranks respectively. If we consider plane stress conditions, Equation 5 can be reduced to:

\textbf{Equation 7}

\[ [F_i F_j] + F_2 \sigma_2 + F_6 \sigma_6 + F_{11} \sigma_1^2 + F_{22} \sigma_2^2 + F_{66} \sigma_6^2 + 2 F_{12} \sigma_1 \sigma_2 = 1 \]

Since the shear strength is independent of shear stress sign, we have \( F_6 = 0 \). Except for the interaction term \( F_{12} \), all the components of the strength tensors appearing in Equation 7 can be determined using simple uniaxial tests:
Equation Set 8

\[ F_i X_i + F_{11} X_i^2 = 1 \]  \( \text{under tensile load in the } 1 \text{- direction} \)

\[ F_i X_c + F_{11} X_c^2 = 1 \]  \( \text{under compressive load in the } 1 \text{- direction} \)

Equation Set 8 can be solved for \( F_1 \) and \( F_{11} \). Similar reasoning for 2-direction and for shear leads to \( F_2, F_{22} \) and \( F_{66} \). Finally,

Equation Set 9

\[
\begin{align*}
F_1 &= \frac{1}{X_t} + \frac{1}{X_c} \\
F_{11} &= -\frac{1}{X_t X_c} \\
F_2 &= \frac{1}{Y_t} + \frac{1}{Y_c} \\
F_{22} &= -\frac{1}{Y_t Y_c} \\
F_6 &= 0 \\
F_{66} &= \frac{1}{S^2}
\end{align*}
\]

If we consider a state of biaxial tension described by \( \sigma = \sigma_1 = \sigma_2 \), we can derive an expression for \( F_{12} \) using Equation 7.

Equation 10

\[
F_{12} = \frac{1}{2\sigma_c^2} \left[ \left( \frac{1}{X_t} + \frac{1}{X_c} + \frac{1}{Y_t} + \frac{1}{Y_c} \right) \sigma_c + \left( \frac{1}{X_t X_c} + \frac{1}{Y_t Y_c} \right) \sigma_c^2 \right]
\]

\( \sigma_c \) : biaxial tensile failure stress

Because biaxial experiments are difficult to conduct, approximate expressions of \( F_{12} \) have been empirically derived. Tsai and Hahn\(^{21}\) suggested that

Equation 11

\[
F_{12} = -\frac{1}{2} \frac{1}{\sqrt{X_t X_c Y_t Y_c}}
\]

Hoffman\(^{22}\) proposed

Equation 12

\[
F_{12} = -\frac{1}{2} \frac{1}{X_t X_c}
\]
1.2.2.1.2.2  Tsai-Hill

Tsai\textsuperscript{23} postulated that the failure criterion of a unidirectional fiber composite has the same mathematical form as the yield criterion of an orthotropic plastic material. This yield criterion proposed by Hill\textsuperscript{24} is based on the Von-Mises criterion for isotropic materials. The resulting “Tsai-Hill” criterion is:

\textbf{Equation 13}

\[ F_{ij} \sigma_i \sigma_j = 1 \]

It is basically Equation 5 without the linear terms. In plane stress, it is written as:

\textbf{Equation 14}

\[
\frac{\sigma_1^2}{X^2} + \frac{\sigma_2^2}{Y^2} + \frac{\tau_{12}^2}{S^2} = 1
\]

1.2.2.2  Micro-level Failure criteria

Most of the failure criteria are phenomenological (macromechanical). There are very few “local” criteria using mechanistic (micromechanics) analysis. Even though micromechanics-based analyses have received more and more attention in the recent years, their main purpose remains as the determination of effective composite properties from the constituents’ properties. Because interaction of failure mechanisms at the local (micro) level is thought to be complex to model, local stresses and strains are rarely used in failure predictions.

However, attempts have been made to use common macroscopic failure theories at the micro level. Huang\textsuperscript{25} applied a maximum stress criterion to the constituent materials of Glass/Epoxy and graphite/Epoxy systems. This criterion was also used by Rahman and Pecknold\textsuperscript{8} and by Aboudi\textsuperscript{1}. The results were in good agreement with experimental data.

Subramanian\textsuperscript{26} proposed a specific local failure criterion to predict the static strength of off-axis unidirectional laminates. It is written as follows:
\[
\sigma_{zz}^f = X^f \quad \text{(fiber failure)}
\]
\[
\frac{\sigma_{xx}^m}{X^m} + \frac{\sigma_{yy}^m}{X^m} + \frac{\sigma_{zz}^m}{X^m} + \frac{\sigma_{yz}^m}{S^m} = 1 \quad \text{(matrix failure)}
\]

where

- \(X^m\) = In-situ tensile strength of the matrix
- \(S^m\) = In-situ shear strength of the matrix
- \(X^f\) = In-situ Tensile Strength of the fiber
- \(\sigma_{ij}\) = Average stress

The stresses were obtained from a Concentric Cylinder Model (See 2.3) and averaged over the entire volume of the matrix. This approach was extended by Subramanian\textsuperscript{4} to predict the fatigue response of unidirectional laminates.

The results and predictions obtained with a failure criterion applied locally are dependent on the micromechanics model used. This is an important feature of this type of analysis.

### 1.2.3 Micromechanical Stress Analysis

Many micromechanics stress analyses based on both analytical and numerical solutions have been presented in the literature in the past. The Concentric Cylinder Model is the simplest and the most commonly used. It is a closed-form analysis, but it neglects the effect of neighboring fibers on the local stress state; therefore, it is expected to work well for composites with small fiber volume fractions. However, this model was successfully used to determine effective properties of composites with large volume fractions. Fiber interactions have been accounted for in models using a regular, periodic arrangement of the fibers. Common arrangements such as square and hexagonal arrays were considered. Aboudi\textsuperscript{1,27} presented theories based on the analysis of a rectilinear repeating cell representing a unidirectional composite. These theories are referred to as the Method of Cells and the Generalized Method of Cells.
Adams and Doner\textsuperscript{28} used a finite difference method. Later on, finite element analyses and boundary element techniques were proposed, respectively, by Adams and Crane\textsuperscript{29} and by Achenbach and Zu\textsuperscript{30}. The main disadvantages of these numerical solutions are that they are complex and require large amounts of computing time. To overcome these problems, analytical solutions have been proposed.

Series-type elasticity solutions are one of the big families of analytical methods. An extensive review of these elasticity methods is given by Chamis and Sendeckyj\textsuperscript{31}. Carman and Averill\textsuperscript{32} developed a refined series-type elasticity solution: it takes into account all six components of mechanical loading, cylindrical orthotropy of the constituents and interphase regions. Airy’s stress function approaches were also used, first by Kobayashi and Ishikawa\textsuperscript{33,34}, and later extended by Naik and Crews\textsuperscript{35} among others.

We now present in a more detailed form some of the micromechanical analysis previously introduced.

1.2.3.1 Method of Cells

In this method, first presented by Aboudi\textsuperscript{37}, a continuously reinforced unidirectional fibrous composite is modeled as a rectangular, double-periodic array of square fibers embedded in a matrix phase (Figure 4 (a)). Because of the symmetry of this arrangement, a representative cell consisting of four subcells can be identified (Figure 4 (b)). One subcell is occupied by the fiber, the other three by the matrix.
The analysis is performed on the representative cell. The overall behavior of the composite is determined from the interactions between the subcells. The theory consists of equilibrium and of continuity of displacements and tractions at the interfaces between the subcells and between neighboring cells on an average basis. Linear displacement fields in each subcell are also assumed.

1.2.3.2 Generalized Method of Cells

The Generalized Method of Cell (GMC) is an extension by Paley and Aboudi\textsuperscript{36} of the Method of Cells. The representative unit consists here of an arbitrary number of subcells (Figure 5). It is capable of modeling multiphase periodic composites.
Compared to the original Method of Cells, the GMC has extended modeling capabilities. It includes:
- elastic-plastic response of multiphased composites
- modeling of various fiber geometry (both shape and packing arrangements)
- modeling of porosities and damage
- modeling of interfacial regions and inclusions

If accurate, both the Method of Cells and the Generalized Method of Cells require a considerable amount of calculation. A computer Micromechanics Analysis Code (MAC) based on the GMC was developed at NASA. It is available at http://www.lerc.nasa.gov/WWW/LPB/mac.descriptions/index.html.

1.2.3.3 The Concentric Cylinder Model (CCM)

This model was used in the present work. It is discussed in detail in section 2.3 p.24.
1.2.3.4 Elasticity Solution

We present here the Averill and Carman\textsuperscript{32} elasticity solution. They consider a diamond packed fiber-reinforced composite (Figure 6).

![Figure 6](image)

**Figure 6** (a) Cross-section of a fiber-reinforced composite material with fibers packed in a diamond array (b) Representative Volume Element

*From Averill and Carman\textsuperscript{32}*

The fibers have circular cross-section and are of infinite length. Fiber coatings or interphases may be represented by concentric circular cylinders around the fiber. All materials are homogeneous and linearly elastic. The matrix is isotropic and the other phases cylindrically orthotropic.
The governing equations of the model are basically the same as those of the Concentric Cylinder Model (See 2.3 p.24). The solutions of the two models differ because of the boundary conditions. In the Averill and Carman model, they are:

- No singularity at \( r = 0 \).
- Continuity of displacement and normal stress at the fiber matrix interface and between adjacent cells.
- On the diagonal boundary (See Figure 6 (b)), a collocation technique is applied.

For more details, the reader is referred to the work of Averill and Carman$^{32}$.

1.2.4 Nonlinear elastic constitutive laws for rubber-like materials – Strain energy density based theories

Large elastic extensibility or hyperelasticity often characterizes rubber-like materials more generally called elastomers. Some elastomers can indeed be stretched up to 1000 % of their original length. A typical stress/strain curve for rubber is shown in Figure 7.

![Figure 7: Typical stress/strain curve for a pure vulcanized gum](image)

The issue of rubber elasticity can be addressed through two different approaches. The first one is based on statistical models using molecular and structural considerations.
Rubber is described as a network of flexible molecular chains that can deform and change conformation when subjected to a stress. The second approach makes no reference to molecular structure and assumes that the material is characterized by a purely mechanical constitutive relation. It is referred to as the continuum theory of rubber elasticity. Because the present work is mainly mechanics based, this theory is more suitable to our purposes and will be discussed next.

The continuum theory includes several models; all are based on the strain energy density function $W$. Two categories of models can be distinguished. In the first category, $W$ is written as a polynomial function of the principal strain invariants $I_1$, $I_2$ and $I_3$ defined by:

$$
I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 \\
I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2
$$

where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the three principal extension ratios.

In the second category, $W$ is assumed to be a separable function of the extension ratios $\lambda_1$, $\lambda_2$ and $\lambda_3$. For both categories, the strain tensor is the Lagrangian strain tensor $\varepsilon$ and the stress tensor thermodynamically conjugate to this strain is the second Piola-Kirchhoff stress tensor $\sigma$. It relates to $W$ as follows:

$$
\sigma = \frac{\partial W}{\partial \varepsilon}
$$

The constitutive relation tensor $D$ is written as:

$$
D = \frac{\partial \sigma}{\partial \varepsilon} = \frac{\partial^2 W}{\partial \varepsilon^2}
$$
1.2.4.1 \( W = f (I_1, I_2, I_3) \)

1.2.4.1.1 Mooney-Rivlin and Generalized Mooney-Rivlin Models

For isotropic materials, the strain energy density function is a symmetrical function of \( I_1, I_2 \) and \( I_3 \). Rivlin\(^{37}\) showed that all possible forms of \( W \) could be represented in terms of the three invariants. If the material is incompressible, \( I_3 = 1 \) and \( W \) is a function of only \( I_1 \) and \( I_2 \). Oden\(^{38}\) proposed the following form for \( W \):

Equation 18

\[
W = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{ij} (I_1 - 3)^i (I_2 - 3)^j \\
\text{where } C_{ij} \text{ are material constants}
\]

The particular form of Equation 18 with only linear terms in \( I_1 \) and \( I_2 \) was originally proposed by Mooney\(^{39}\). It is often called the Mooney-Rivlin equation and written as:

Equation 19

\[
W = C_{10} (I_1 - 3) + C_{01} (I_2 - 3)
\]

Equation 18 is sometimes called the generalized Mooney-Rivlin equation. Equation 18 and Equation 19 are the most widely used constitutive relationship in the stress analysis of elastomers. The constants appearing in both equations are obtained by curve fitting experimental data.

1.2.4.1.2 Modified Generalized Mooney-Rivlin model

The modified generalized Mooney-Rivlin model was proposed by Gadala\(^{40}\):

Equation Set 20

\[
W = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{ij} (\hat{I}_1 - 3)^i (\hat{I}_2 - 3)^j
\]

where \( \hat{I}_1 \) and \( \hat{I}_2 \) are modified invariants defined by

\[
\hat{I}_1 = I_1 - (I_3 - 1)
\]

\[
\hat{I}_2 = I_2 - 2(I_3 - 1)
\]
1.2.4.1.3 Swanson\textsuperscript{41} Model

Swanson wrote the strain energy density function as:

\begin{equation}
W = \sum_{i=1}^{\infty} \frac{3A_i}{2(1+a_i)} (I_i/3)^{1+a_i} + \sum_{j=1}^{\infty} \frac{3B_j}{2(1+b_j)} (I_j/3)^{1+b_j} + \int_{0}^{t_1} \frac{g(t)}{t} dt - \left( \sum_{i=1}^{\infty} A_i + 2 \sum_{j=1}^{\infty} B_j \right) \frac{\ln(I_3)}{2}
\end{equation}

where \( A_i, B_j, a_i \) and \( b_j \) are material constants and the function \( g \) is defined by:

\begin{equation}
g(I_3) = C(I_3 - 1) \quad \text{where} \quad C = \frac{\chi}{2} - \frac{1}{3} \left[ \sum_{i=1}^{\infty} A_i (1 + a_i) + 4 \sum_{j=1}^{\infty} B_j (1 + b_j) \right]
\end{equation}

and \( \chi \) is the bulk modulus.

1.2.4.1.4 Blatz-Ko\textsuperscript{42} model

Blatz-Ko proposed:

\begin{equation}
W = \frac{1}{2} \mu \left[ (I_1 - 3) + \frac{2}{a} (I_3 - a^2/2 - 1) \right] \quad \text{where} \quad a = \frac{2v}{(1-2v)}
\end{equation}

and \( \mu \) is a material constant.

1.2.4.1.5 Hart-Smith model\textsuperscript{43}

Hart-Smith used:

\begin{equation}
W = C \int e^{k_1(I_1-3)^2} dI_1 + k_2 \cdot \ln \left( \frac{I_2}{3} \right)
\end{equation}

where \( C, k_1 \) and \( k_2 \) are material constants.
The second Piola-Kirchhoff stress tensor \( \sigma \) and the constitutive relation tensor \( D \) were derived by Gadala\(^{44} \) for all the previous models.

1.2.4.2 \( W = f(\lambda_1, \lambda_2, \lambda_3) \)

1.2.4.2.1 Ogden material model

Ogden\(^{45} \) expressed the strain energy density function as:

\[
W = \sum_{i=1}^{3} \sum_{j=1}^{m} \frac{c_j}{b_j} (\lambda_i^{b_j})^m - 1
\]

where \( c_j \) and \( b_j \) are materials coefficients and \( m \) is fixed by the user to achieve the desired accuracy.

1.2.4.2.2 Peng\(^{46} \) material model

Peng expressed \( W \) as:

\[
W = u(\lambda_1) + u(\lambda_2) + u(\lambda_3)
\]

A detailed expression for \( u \) is given by Valanis and Landel\(^{47} \).

1.2.4.2.3 Peng-Landel\(^{48} \) material model

This is the simplest model of hyperelasticity. Only one material constant has to be specified using experimental data. For incompressible materials, \( W \) is expressed as:
\[ W = \sum_{i=1}^{3} c \left[ \lambda_i - 1 - \ln(\lambda_i) - \frac{1}{6} (\ln(\lambda_i))^2 + \frac{1}{18} (\ln(\lambda_i))^3 - \frac{1}{216} (\ln(\lambda_i))^4 \right] \]

where \( c \) is the initial tensile modulus.

Methods of developing the materials constants in the previous models from experimental data are demonstrated by Finney and Kumar\( ^{49} \).
CHAPTER 2  OFF-AXIS STRENGTH PREDICTIONS FOR
LINEAR ELASTIC UNIDIRECTIONAL SINGLE PLY
COMPOSITES

2.1 Introduction

Figure 8 illustrates a typical single-ply off-axis composite. This type of composite is common and has been extensively tested over the years. A fair amount of strength versus off-axis angle data can be found in the literature for different composite assemblages. This chapter presents a modeling procedure to predict the strength of linear off-axis single-ply composites. The term linear indicates that both the matrix and the fiber exhibit a linear stress/strain relation. Strength predictions for different linear composite systems are performed using a specific local failure criterion (2.2 pp. 23) which is coupled with a concentric cylinder based local stress analysis (2.3 pp. 24). The accuracy and potential of the modeling procedure is evaluated by comparing the strength predictions to data in the literature (2.4 pp. 31).

Figure 8 Single ply off-axis specimen
A model developed with a commercial finite element code (ANSYS) is also presented in this chapter (2.5 pp. 43). The results from the Concentric Cylinder Model based code were verified using the code NDSANDS written by Pagano. The CCM solution was used to check the finite elements results. The objective was to obtain a Finite Element Model (FEM) leading to stresses and strains comparable to the ones obtained with CCM (at least in an average sense). The resulting FEM was then used as a core to develop a nonlinear model (CHAPTER 3). This nonlinear version is later adopted in the strength prediction procedure of nonlinear single-ply composites. Indeed, for nonlinear materials, the Concentric Cylinder Model is not valid anymore and finite element methods are commonly employed in the absence of any analytical models.

2.2 Failure criterion

The type of composite considered and more importantly the failure mode(s) encountered serve as guidelines to choose an adequate failure criterion. In off-axis composites, for angles greater than the threshold angle $\theta_{\text{thr}}$ under which fiber failure takes place, failure is defined by matrix failure due to combined transverse and shear stresses (See 1.2.1 pp. 2). Based on our experimental observations and results presented in the literature (1.2.1 pp. 2), a few assumptions were made. First we will assume that interfaces between the fibers and the matrix are strong. In this case, failure occurs due to cracking in the matrix close to the interface (the exact location is not known) along a plane parallel to the fiber direction. Since the largest matrix stresses and strains occur at the interface and since failure is not interfacial, we can assume that failure is not controlled by point stress and/or strain values. As a result, we will employ a local failure criterion using matrix-averaged values of stresses and strains as opposed to point values. To keep the criterion as simple as possible, we only use contributions in shear and transverse directions in the plane of the lamina since cracking results from combined stresses in these two directions. We will also assume that the life of the composite is controlled by the initiation of the matrix cracks. In other words, the failure strength corresponds to the initiation stress; propagation of the cracks is assumed to be quasi-immediate. The proposed failure criterion is then a crack initiation criterion.
One of the objectives of this work was to construct a criterion that would work for both linear and nonlinear materials. Energy-based failure functions are known to be more effective for nonlinear materials. Inspired by a paper by Plumtree\textsuperscript{50}, the idea was to \textbf{multiply the stress terms by the corresponding strain} value to obtain energy terms. To get a value for the failure criterion between 0 and 1, both the transverse and shear terms have to be \textbf{normalized}. The proposed failure criterion has the form of a reduced and normalized strain energy density function:

\begin{equation}
F_a = \frac{\bar{\sigma}_{xx} \cdot \bar{e}_{xx}}{A} + \frac{\bar{\tau}_{xz} \cdot \bar{\gamma}_{xz}}{B}
\end{equation}

where bars indicate matrix-averaged quantities, \(z\) is the fiber direction, \(x\) is the transverse direction and \(A\) and \(B\) are the normalizing constants.

Since stresses and strains in Equation 28 are local, \(A\) and \(B\) are the products of in-situ strength and in-situ strain-to-failure of the matrix in the direction of interest. The method used to obtain \(A\) and \(B\) is discussed in 2.4.3 p. 35.

\section{2.3 Local stress analysis: Concentric Cylinder Model (CCM)}

Several papers address the problem of computation of local stresses in unidirectional continuous fiber composites subjected to thermo-mechanical loadings and prediction of its effective stiffness. Several of these papers use models where the composite constituents are represented by cylinders. The papers by Pagano\textsuperscript{51,52} present the advantage of giving a clear and general formulation of a Concentric Cylinder based model. This formulation will be used in the present work in a simplified version.

\subsection{2.3.1 Theory}

\subsubsection{2.3.1.1 Geometry}

A continuous fiber reinforced composite is modeled by a representative volume element composed of two concentric cylinder elements in which the inner cylinder is the
fiber and the outer ring is the matrix (Figure 9). The ratio of the two radii is calculated from the fiber volume fraction. The fiber-to-fiber interaction is neglected in the Concentric Cylinder Model.

\[ \text{Volume of fiber} \div \text{Total volume} = \text{fiber volume fraction} = \left( \frac{r_f}{r_m} \right)^2 \]

where \( r_f \) = radius of the fiber
\( r_m \) = outer radius of the matrix

**Figure 9 Concentric Cylinder**

2.3.1.2 Driving equations

Both the fiber and the matrix are assumed to be linear elastic and homogeneous. The fiber is assumed to be perfectly bonded to the matrix. The constituent materials have transversely isotropic properties.

2.3.1.2.1 Equilibrium

The equilibrium equations in cylindrical coordinate system are:
Equation set 29

\[ \sigma_{rr} + \frac{1}{r} \cdot \tau_{r\alpha,\alpha} + \frac{1}{r} \cdot (\sigma_r - \sigma_{\alpha}) = 0 \]

\[ \tau_{r\alpha,\alpha} + \frac{1}{r} \cdot \sigma_{\alpha,\alpha} + \frac{2}{r} \cdot \tau_{r\alpha} = 0 \]

\[ \tau_{rz,\alpha} + \frac{1}{r} \cdot \tau_{z\alpha,\alpha} + \frac{1}{r} \cdot \tau_{zz} = 0 \]

Stresses are *only* functions of \( r \) and \( \alpha \) and differentiation is indicated by a comma.

2.3.1.2.2 Constitutive equations

For each constituent material:

Equation set 30

\[ \sigma_z = C_{11} \cdot \varepsilon_z + C_{12} \cdot \varepsilon_r + C_{12} \cdot \varepsilon_\alpha \]

\[ \sigma_r = C_{12} \cdot \varepsilon_z + C_{22} \cdot \varepsilon_r + C_{23} \cdot \varepsilon_\alpha \]

\[ \sigma_z = C_{12} \cdot \varepsilon_z + C_{23} \cdot \varepsilon_r + C_{22} \cdot \varepsilon_\alpha \]

\[ \tau_{r\alpha} = C_{44} \cdot \gamma_{r\alpha} \]

\[ \tau_{z\alpha} = C_{55} \cdot \gamma_{z\alpha} \]

\[ \tau_{rz} = C_{55} \cdot \gamma_{rz} \]

where \( C_{mn} (m, n = 1, 2, \ldots, 6) \) are the elastic stiffness constants of the individual material.

For transversely isotropic material, \( C_{44} = \frac{C_{22} - C_{23}}{2} \).

2.3.1.2.3 Kinematics equations
2.3.1.2.4 Governing equations

Substituting Equation set 30 and Equation set 31 into Equation set 29, we obtain the governing field equation in terms of displacements ($u_r, u_{\alpha}, u_z$):

\[
\begin{align*}
\varepsilon_r &= u_{z,r} \\
\gamma_{r\alpha} &= \frac{1}{r} \cdot u_{r,\alpha} + u_{\alpha,r} - \frac{1}{r} \cdot u_{\alpha} \\
\varepsilon_\alpha &= \frac{1}{r} \cdot u_{\alpha,\alpha} + \frac{1}{r} \cdot u_r \\
\gamma_{r\alpha} &= u_{r,\alpha} + u_{\alpha,r} \\
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
\varepsilon_r &= u_{z,r} \\
\gamma_{r\alpha} &= \frac{1}{r} \cdot u_{r,\alpha} + u_{\alpha,r} - \frac{1}{r} \cdot u_{\alpha} \\
\varepsilon_\alpha &= \frac{1}{r} \cdot u_{\alpha,\alpha} + \frac{1}{r} \cdot u_r \\
\gamma_{r\alpha} &= u_{r,\alpha} + u_{\alpha,r} \\
\end{align*}
\]

Equation set 32

\[
\begin{align*}
C_{22} \cdot \left( u_{r,rr} + \frac{1}{r} u_{r,r} - \frac{1}{r^2} u_r \right) + C_{44} \cdot \frac{1}{r^2} u_{r,\alpha\alpha} + (C_{23} + C_{44}) \cdot \frac{1}{r} u_{\alpha,\alpha} - (C_{22} + C_{44}) \cdot \frac{1}{r^2} u_{\alpha} + C_{12} u_{z,zz} &= 0 \\
C_{44} \cdot \left( u_{\alpha,rr} + \frac{1}{r} u_{\alpha,r} - \frac{1}{r^2} u_\alpha \right) + C_{22} \cdot \frac{1}{r^2} u_{\alpha,\alpha\alpha} + (C_{23} + C_{44}) \cdot \frac{1}{r} u_{r,\alpha\alpha} + (C_{22} + C_{44}) \cdot \frac{1}{r^2} u_{r,\alpha} + C_{12} u_{z,\alpha\alpha} &= 0 \\
C_{55} \cdot \left( u_{z,zz} + \frac{1}{r} u_{z,z} + \frac{1}{r} u_{z,\alpha\alpha} + u_{z,rr} + \frac{1}{r} u_{z,r} + \frac{1}{r^2} u_{z,\alpha\alpha} \right) &= 0
\end{align*}
\]

2.3.1.3 General solution

The general solution of Equation set 32 is a series solution. A solution containing the terms necessary to satisfy the class of boundary conditions that are imposed (See 2.3.1.4) is as follows:
2.3.1.4 Boundary and interface conditions

To solve for the constants \( A_1^{(k)}, \ldots, Y_3^{(k)} \) used to define \( \alpha_1^{(k)}(r), \ldots, \delta_5^{(k)}(r) \) (See Appendix 1), we have to prescribe interface and boundary conditions. Two types of boundary conditions can be prescribed on the outer surfaces of our model: displacement based\(^{51}\) or traction based\(^{52}\) boundary conditions. We chose to apply traction based conditions to be consistent with the Finite Element model. The surface tractions employed in the FE model were also derived from the global applied stress and the off-axis angle.

Finally, the conditions to prescribe are:

(i) Continuity of displacements and stresses across the fiber/matrix interface
(ii) Non-singular displacements and stresses at the origin

(iii) Elimination of rigid body motion

(iv) Boundary conditions derived from the global applied stress (Figure 10)

(v) Equilibrium of the entire body

\[ \sigma = \sigma_{\text{glob}} \cdot \cos^2 \theta \]
\[ \sigma_x = \sigma_{\text{glob}} \cdot \sin^2 \theta \]
\[ \tau_{xz} = -\sigma_{\text{glob}} \cdot \cos^2 \theta \cdot \sin^2 \theta \]

Figure 10 Traction boundary conditions for the CCM from the global applied stress and the off-axis angle

2.3.2 Code

A computer code called NDSANDS based on the Concentric Cylinder Model was developed by Pagano and Tandon. It can be used either to analyze a composite or to conduct parametric study on the effects of the material properties, the geometry of the composite, etc. The code used in the present work was written by Dr. S. Case at Virginia
Tech based on the papers by Pagano. This program was modified to calculate the average stress in the matrix in the x, y and z directions:

Equation 35

\[ \overline{\sigma_{ij}} = \frac{1}{V} \iiint_V \sigma_{ij}(r, \alpha) \, dV \]

where \( V \) is the volume of the matrix

\[ \sigma_{ij}(r, \alpha) \]

are stresses in the (x,y,z) coordinate system transformed from Equation Set 34.

\( \sigma_{ij} \) are not z dependent and the cylinders have two axes of symmetry. Equation 35 can then be reduced to:

Equation 36

\[ \overline{\sigma_{ij}} = \frac{1}{A} \int_{r_1}^{r_2} \int_{\alpha=0}^{\alpha=\frac{\pi}{2}} \sigma_{ij}(r, \alpha) \cdot r \cdot dr \cdot d\alpha \]

2.3.2.1 Input

The code is written in FORTRAN and uses two input files:

- “Prop” is a database of properties of fiber and matrix materials. This file can be modified by the user and new materials can be added. The properties to be entered are the elastic constants (longitudinal and transverse Young’s modulus, Poisson’s ratios and shear modulus), the coefficients of thermal expansion and the coefficients of moisture absorption.

- “Composit” is a database of composite systems. For each of these systems, the fiber and matrix materials have to be chosen from the “Prop” database.

The composite to be analyzed, the temperature, the moisture and the applied stresses or strains are entered interactively when the executable file is run.
2.3.2.2 Output

The calculated volume averaged stresses and strains in the matrix are given in an interactive window.

2.4 Validation of the approach

2.4.1 Composite systems

Experimental data from the literature for five different unidirectional composites were used to validate the micromechanics-based method. The experimental strength vs. off-axis angle data were plotted along with the predicted curves. The constituent properties and fiber volume fraction of the five systems are reported in Table 1.

<table>
<thead>
<tr>
<th>Table 1 Properties of the five unidirectional composites used to validate the analysis</th>
<th>E₁ (GPa)</th>
<th>E₂ (GPa)</th>
<th>G₁₂ (GPa)</th>
<th>ν₁</th>
<th>ν₂₃</th>
</tr>
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<tbody>
<tr>
<td>Tsai-Hahn&lt;sup&gt;21&lt;/sup&gt; V&lt;sub&gt;f&lt;/sub&gt; = 0.66</td>
<td>Graphite AS</td>
<td>213.7</td>
<td>13.8</td>
<td>13.8</td>
<td>0.2</td>
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<tr>
<td></td>
<td>Epoxy 3501</td>
<td>3.45</td>
<td>3.45</td>
<td>1.3</td>
<td>0.35</td>
</tr>
<tr>
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<td>29.5</td>
<td>24.1</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>Polyimide</td>
<td>3.1</td>
<td>3.1</td>
<td>1.1</td>
<td>0.39</td>
</tr>
<tr>
<td>Pindera et al.&lt;sup&gt;54&lt;/sup&gt; V&lt;sub&gt;f&lt;/sub&gt; = 0.55</td>
<td>Kevlar</td>
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<td>4.1</td>
<td>2.9</td>
<td>0.35</td>
</tr>
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<td>3.45</td>
<td>1.3</td>
<td>0.35</td>
</tr>
<tr>
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<td>0.35</td>
</tr>
</tbody>
</table>
2.4.2 Generation of the theoretical Strength vs. Off-Axis angle curves

We recall that the failure function has the following form (for convenience, reduced index notation has been adopted here):

\[ F_a = \frac{\sigma_2 \cdot \bar{\varepsilon}_2}{A} + \frac{\tau_6 \cdot \bar{\gamma}_6}{B} \]

Reduced index notation: 1 → 11 → zz, 2 → 22 → xx, 3 → 33 → yy, 4 → 23 → xy,
5 → 13 → yz and 6 → 12 → xz

This failure function is applied in the matrix. Because the matrix is considered as linear and isotropic (\( E_1 = E_2 = E_3 \)), the strains appearing in Equation 37 can be expressed as:

\[ \bar{\varepsilon}_2 = \frac{1}{E_2^m} \left( \bar{\sigma}_2 - \nu^m \cdot \bar{\sigma}_1 - \nu^m \cdot \bar{\sigma}_3 \right) \]
\[ \bar{\gamma}_6 = \frac{1}{G_{12}^m} \cdot \bar{\tau}_6 \quad \text{and} \quad G_{12}^m = \frac{E_2^m}{2 \cdot (1 + \nu^m)} \]

where \( \nu^m \) is the Poisson ratios of the matrix, \( E_2^m \) the elastic modulus of the matrix and \( G_{12}^m \) the shear modulus of the matrix.

Substituting Equation Set 38 into Equation 37 we have:

\[ F_a = \frac{\bar{\sigma}_2^2 - \nu^m \cdot \bar{\sigma}_1 \cdot \bar{\sigma}_2 - \nu^m \cdot \bar{\sigma}_2 \cdot \bar{\sigma}_3}{E_2^m \cdot A} + \frac{2 \cdot (1 + \nu^m) \cdot \bar{\tau}_6^2}{E_2^m \cdot B} \]

Because the stress analysis is linear, we can write the volume averaged local stresses as a linear combination of the applied stresses:
\[ \bar{\sigma}_i = K_{ij} \cdot \sigma_j \quad i, j = 1, \ldots, 6 \]

where \( K_{ij} \) are constants.

The loading direction (See Figure 8) results in a plane stress state. Therefore Equation 40 can be reduced as follow for the stresses appearing in Equation 39:

**Equation Set 41**

\[
\begin{align*}
\bar{\sigma}_1 &= K_{11} \sigma_1 + K_{12} \sigma_2 + K_{16} \sigma_6 \\
\bar{\sigma}_2 &= K_{21} \sigma_1 + K_{22} \sigma_2 + K_{26} \sigma_6 \\
\bar{\sigma}_3 &= K_{31} \sigma_1 + K_{32} \sigma_2 + K_{36} \sigma_6 \\
\bar{\tau}_6 &= \sigma_6 = K_{61} \sigma_1 + K_{62} \sigma_2 + K_{66} \sigma_6 
\end{align*}
\]

There is no coupling between volume averaged normal stresses and applied shear stresses. As a consequence, \( K_{16} = K_{26} = K_{36} = 0 \). Similarly, there is no coupling between volume averaged shear stresses and applied normal stresses and \( K_{61} = K_{62} = 0 \). Equation Set 4 can then be rewritten as:

**Equation Set 42**

\[
\begin{align*}
\bar{\sigma}_1 &= K_{11} \sigma_1 + K_{12} \sigma_2 \\
\bar{\sigma}_2 &= K_{21} \sigma_1 + K_{22} \sigma_2 \\
\bar{\sigma}_3 &= K_{31} \sigma_1 + K_{32} \sigma_2 \\
\bar{\tau}_6 &= \sigma_6 = K_{66} \sigma_6 
\end{align*}
\]

The remaining \( K_{ij} \) constants are determined by applying unit stresses successively in the 1, 2 and 6 direction and solving for the volume averaged stresses with the Concentric Cylinder Model code:
\[ \begin{align*}
(1) & \quad \sigma_1 = 1, \sigma_2 = \sigma_6 = 0 \\
\Rightarrow & \quad \sigma_1 = K_{11} \\
& \quad \sigma_2 = K_{21} \\
& \quad \sigma_3 = K_{31}
\end{align*} \]

\[ \begin{align*}
(2) & \quad \sigma_2 = 1, \sigma_1 = \sigma_6 = 0 \\
\Rightarrow & \quad \sigma_1 = K_{12} \\
& \quad \sigma_2 = K_{22} \\
& \quad \sigma_3 = K_{32}
\end{align*} \]

\[ \begin{align*}
(3) & \quad \sigma_6 = 1, \sigma_1 = \sigma_2 = 0 \\
\Rightarrow & \quad \tau_6 = K_{66}
\end{align*} \]

The constants $K_{ij}$ obtained for the five unidirectional composites considered are given in Table 2.

**Table 2 $K_{ij}$ constants for the five composite systems**

<table>
<thead>
<tr>
<th></th>
<th>Tsai-Hahn</th>
<th>Pindera/Herakovich</th>
<th>Pindera et al.</th>
<th>Pipes and Cole</th>
<th>Hashin and Rotem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{11}$</td>
<td>2.522×10^{-2}</td>
<td>2.310×10^{-2}</td>
<td>4.953×10^{-2}</td>
<td>2.638×10^{-2}</td>
<td>7.882×10^{-2}</td>
</tr>
<tr>
<td>$K_{21}$</td>
<td>1.429×10^{-3}</td>
<td>5.234×10^{-4}</td>
<td>0</td>
<td>1.235×10^{-3}</td>
<td>3.707×10^{-4}</td>
</tr>
<tr>
<td>$K_{31}$</td>
<td>1.429×10^{-3}</td>
<td>5.234×10^{-4}</td>
<td>0</td>
<td>1.235×10^{-3}</td>
<td>3.707×10^{-4}</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>3.137×10^{-1}</td>
<td>3.398×10^{-1}</td>
<td>3.470×10^{-1}</td>
<td>2.983×10^{-1}</td>
<td>2.803×10^{-1}</td>
</tr>
<tr>
<td>$K_{22}$</td>
<td>9.112×10^{-1}</td>
<td>8.965×10^{-1}</td>
<td>1.016</td>
<td>8.975×10^{-1}</td>
<td>8.764×10^{-1}</td>
</tr>
<tr>
<td>$K_{32}$</td>
<td>2.033×10^{-3}</td>
<td>-4.679×10^{-3}</td>
<td>2.472×10^{-2}</td>
<td>-2.585×10^{-2}</td>
<td>-1.781×10^{-2}</td>
</tr>
<tr>
<td>$K_{66}$</td>
<td>6.463×10^{-1}</td>
<td>6.424×10^{-1}</td>
<td>8.271×10^{-1}</td>
<td>6.705×10^{-1}</td>
<td>6.448×10^{-1}</td>
</tr>
</tbody>
</table>

The applied stresses $\sigma_1$, $\sigma_2$ and $\sigma_6$ can be derived from the global applied stress $\sigma_{\text{glob}}$ and the off-axis angle $\theta$ using a simple tensor transformation (See Figure 10). It leads to:
Equation Set 44

\[ \sigma_1 = \sigma_{\text{glob}} \cdot \cos^2(\theta) \]
\[ \sigma_2 = \sigma_{\text{glob}} \cdot \sin^2(\theta) \]
\[ \tau_6 = -\sigma_{\text{glob}} \cdot \cos(\theta)\sin(\theta) \]

Substituting Equation Set 42 then Equation Set 44 into Equation 39, we obtain:

Equation 45

\[
F_a = \sigma_{\text{glob}}^2 \left[ \frac{1}{E_2^m} \cdot A \right] \left[ \left( K_{21} \cos^2(\theta) + K_{22} \sin^2(\theta) \right)^2 \right. \\
\left. - v^m \cdot \left( K_{11} \cos^2(\theta) + K_{12} \sin^2(\theta) \right) \cdot \left( K_{21} \cos^2(\theta) + K_{22} \sin^2(\theta) \right) \right] \\
+ \left[ \frac{2 \cdot (1 + v^m)}{E_2^m} \cdot B \cdot K_{66} \cdot \cos^2(\theta) \sin^2(\theta) \right]
\]

At failure \( F_a = 1 \) and Equation 45 can be solved for the global failure stress, \( \sigma^f_{\text{glob}} \) as a function of \( \theta \). \( \sigma^f_{\text{glob}} \) is the off-axis strength.

The concentric cylinder model is very stable. It is not very sensitive to changes in the constituents properties: matrix averaged stresses obtained from Equation Set 41 vary within less than 0.5% when matrix and fiber properties vary by 5%. The resulting error on the \( \sigma^f_{\text{glob}} \) curve is less than 4%.

2.4.3 Results and discussion

A and B constants still need to be defined to plot \( \sigma^f_{\text{glob}}(\theta) \). Ideally A and B would be directly related to matrix strengths and strains-to-failure. An attempt was made to express A and B as follows:
Equation Set 46

\begin{align*}
A &= Y_t^m \cdot \frac{Y_t^m}{E_t^m} \\
B &= S^m \cdot \frac{S^m}{G_{12}^m}
\end{align*}

where \( Y_t^m \) = the transverse strength of the matrix
\( S^m \) = the shear strength of the matrix

The obtained off-axis strength curves are far from the experimental data for the five composites considered. The deviation on the experimental data is less than 0.5 \%. The graphite/Polyimide curve is given as illustration in Figure 11.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graphite_polyimide.png}
\caption{Off-Axis Strength prediction of graphite/polyimide lamina along with the experimental results of Pindera and Herakovich\textsuperscript{53}. A and B from Equation Set 46}
\end{figure}

Interestingly enough all the predicted curves overestimate the off-axis strength and very good predictions are obtained if A and B from Equation Set 46 are multiplied by a scaling
factor. This scaling factor seems to be independent of the type of composite and is close to 0.3. This factor could eventually be related to the fiber volume fraction and/or the moduli of the fiber and the matrix. It indeed seems reasonable to think that the intrinsic strength of the matrix is modified by the presence of the fiber. The in-situ strength is so slightly different from the matrix strength. Future research in this direction would be of particular interest.

The effect on the predicted off-axis strength curve of each matrix strength term is illustrated in Figure 12 and Figure 13. In Figure 12, the transverse strength of the matrix is maintained constant and the shear strength varies between $S^m$ and $S^m/2.5$. In Figure 13, the transverse strength varies and the shear strength is constant.

![Graphite/Polyimide](image)

**Figure 12** Effect of the matrix shear strength value on the predicted off-axis strength curve
Another possibility would be to use A and B from Equation Set 46 but to redefine the critical element volume i.e. average the stresses and strains appearing in the numerator of the failure function over a reduced or more localized volume of matrix. Future work will also be conducted in this direction.

The method to determine A and B in the present work is the following:

- For a pure transverse loading, the shear term of Equation 37 is equal to zero and the composite theoretically fails, i.e. \( F_a = 1 \), when the transverse composite strength \( Y_t \) is reached. As a consequence, if we apply \( \sigma_1 = 0 \), \( \sigma_2 = Y_t \) and \( \sigma_{12} = \sigma_6 = 0 \) as boundary conditions for the CCM, and using Equation Set 42 and Equation Set 38, Equation 37 can be written:
Equation 47 can be solved for $A$:

$$\frac{K_{22} \cdot Y_t \cdot K_{22} \cdot Y_t \cdot \frac{1}{E_m^m}}{A}$$

Equation 48

$$A = K_{22} \cdot Y_t \cdot \frac{1}{E_m^m}$$

The $Y_t$ used in Equation 48 is the experimental strength at 90º off-axis angle.

- Similarly, for a pure shear loading the transverse term of Equation 37 is equal to zero and the composite theoretically fails, i.e. $F_a = 1$, when the shear composite strength $S$ is reached. However, we can’t obtain a pure shear strength value from tensile tests data. We will assume that the matrix fail in shear for the experimental strength - off-axis angle configuration that maximizes the product of averaged shear stress and strain. The 15º experimental point, if available, fulfills this condition. We apply $\sigma_1 = \sigma_{15}^{e\exp} \cos^2(15)$, $\sigma_2 = \sigma_{15}^{e\exp} \sin^2(15)$ and $\sigma_{12} = \sigma_6 = \sigma_{15}^{e\exp} \cos^2(15) \sin^2(15)$ as boundary conditions for the CCM. Equation 37 can be then be solved for $B$.

The off-axis strength predictions obtained in this fashion are presented in Figure 14 to Figure 18. We recall that the deviation is less than 0.5% for the experimental data and less than 4% for the predicted curves.
Figure 14  Off-Axis Strength prediction of graphite/epoxy lamina along with the experimental results of Tsai and Hahn$^{21}$. 

Figure 15  Off-Axis Strength prediction of graphite/polyimide lamina along with the experimental results of Pindera and Herakovich$^{53}$. 

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Figure 16 Off-Axis Strength prediction of kevlar/epoxy lamina along with the experimental results of Pindera et al.\textsuperscript{54}.

Figure 17 Off-Axis Strength prediction of Boron/epoxy lamina along with the experimental results of Pipes and Cole\textsuperscript{55}.
The predictions of off-axis strength with the method discussed in this chapter are in very good agreement with the experimental data for the entire set of unidirectional single-ply composites considered. The local failure function was built to predict matrix crack initiation assumed to correspond to gross failure (See 2.2). This failure mode occurs for off-axis angles greater than a threshold angle $\theta_{\text{thr}}$ (See 1.2.1). At $0^\circ$ off-axis angle, the failure mode is different: fiber breakage is responsible for gross failure. This explains why the predicted curve is far from the $0^\circ$ tensile data point. We can notice on Figure 15, Figure 16 and Figure 18 that the predicted curves are also further from the $5^\circ$ point than from higher angle data points. A possible explanation is that since this angle is very close to the threshold angle, fiber breakage or a mix fiber breakage/matrix cracking failure mode might occur.

The predicted curves pass exactly through the $15^\circ$ and $90^\circ$ points since these 2 points were used to determine the A and B constants (See 2.2). We recall here that since the transverse and shear terms in Equation 37 are normalized respectively by A and B and since these two constants are not related to physical properties but instead determined
by setting $F_a = 1$ at $15^\circ$ and $90^\circ$ off-axis angle, the strength prediction capability of the method is \textbf{not} a way of validating the stress analysis used.

2.5 Finite Element Model

A finite element based local stress analysis was performed using the commercial code ANSYS. The reason a finite element approach was implemented for the linear problem is that it serves as a foundation for the construction of a finite element based local stress analysis valid for nonlinear materials too. Indeed, finite element based micromechanical studies are the most commonly employed to investigate the nonlinear behavior of fiber-reinforced composites. Commercial finite element codes also give the user the possibility to easily change the constitutive nonlinear law of the material. The linear FE model discussed next was validated using the concentric cylinder model. For the comparison to be meaningful, the geometry and boundary conditions of the FEM were chosen to be identical or at least comparable to those of the CCM.

From the set of five composite systems used to validate the micromechanical model, two were modeled with finite elements. The first one is the Graphite/Polyimide tested by Pindera and Herakovich\textsuperscript{53}. Orthotropic properties for the fiber and isotropic properties for the matrix were used according to Table 1. The second composite considered is the E-Glass/Epoxy tested by Hashin and Rotem\textsuperscript{2}. In this case both the fiber and the matrix are isotropic (Table 1).

2.5.1 Geometry

2.5.1.1 Full geometry

The continuous fiber reinforced composite is modeled by a representative volume element composed of an inner cylinder representing the fiber and of an extruded square representing the matrix (See Figure 19). This geometry, slightly different from the one used in the Concentric Cylinder Model where both the fiber and the matrix are represented by cylinders, is more representative of the actual tested material.
The fiber volume fraction of the composite is defined by the ratio cross-sectional area of the fiber/cross-sectional area of the square. The dimensions of the square were fixed to $4 \times 4$ (no units). The fiber volume fraction was then enforced by fixing the radius of the fiber. An appropriate choice of the longitudinal dimension $L$ (See Figure 19) will be discussed in 2.5.2.3 p. 47.

2.5.1.2 Reduced geometry

The configuration of the stresses applied on the boundaries of the model can help reduce its size saving precious computation time. The applied boundary conditions are discussed in detail in 2.5.3; the stresses applied on the boundaries are derived from the global applied stress and off-axis angle in exactly the same way as for the CCM (See Figure 30 p. 53).

As a consequence, there are only global stresses in the $xz$ plane (See Figure 30). For the finite element model the $xz$ plane may be treated as a plane of symmetry and only half the entire volume need to be modeled as shown in Figure 20.
2.5.2 Mesh

2.5.2.1 Element types

The fiber and the matrix were meshed using 8 noded 3D solid elements. This element is referred to as SOLID45 in ANSYS and is presented in Figure 21 (a). Special surface elements called SURF22 had to be used to apply the shear stresses. They were superposed to the existing mesh on all surfaces where shear stresses are applied. SURF22 element is presented in Figure 21 (b).

Figure 21 Pictorial description of elements SOLID45 and SURF22. From ANSYS Elements Reference\textsuperscript{56}. 

Figure 20 Reduced Finite Element Model geometry
2.5.2.2 Mesh size

Several mesh sizes were investigated. The configuration shown in Figure 22 and Figure 23 was found to lead to a good compromise between accuracy and computation time. The two models were defined to have the same number of nodes and elements. Decreasing the longitudinal element size to 0.25 changes the results by less than 1% but considerably increases the computation time.

Figure 22 Meshed model. Isometric view.
2.5.2.3 **Longitudinal dimension of the model**

An important issue in building the FE model was to fix the dimension L. (See Figure 20). According to the St Venant’s principle, if an actual distribution of forces is replaced by a statically equivalent system, the distribution of stress and strain is altered only near the regions of load application. Ideally, the stresses and strains should reach a plateau far enough from where the loads are applied on the model. In the Concentric Cylinder Model, the stresses are not z dependent. In order for the CCM/FEM comparison to be meaningful, the FEM results had to be retrieved in a location where they are not (or very slightly) z dependent. Several L dimensions were investigated for the Graphite/Polyimide. The shortest satisfactory dimension, i.e. were a plateau is noticeable, was found to be $L = 10 \ (-5 \leq z \leq 5 \)$. Picking an optimum L length reduces the total number of elements and as a consequence the computation time. The $z$-variations of $\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_z, \bar{\tau}_{xz}$ and $\bar{\varepsilon}_x$ (matrix averaged values) for the Graphite/Polyimide under $\sigma_{glob} = 60.6 \text{ MPa}$ at $\theta = 60^\circ$ (experimental off-axis strength value) are shown in Figure
24 to Figure 28. Other global conditions could have been used. The “characteristic” distance over which the stresses reach a plateau was found to be independent of the magnitude of the load applied. The way the values are averaged is discussed in details in 2.5.4 p. 53. The abscissa represents the elements’ layer number for an element size along z of 0.5. The elements’ layer 1 is between z = 0 and z = 0.5 and the elements’ layer 10 is between z = 4.5 and z = 5. More generally,

Equation 49

\[ z = 0.5 \times (i - 1) \leq \text{Elements’ layer } i \leq z = 0.5 \times i. \]

Figure 24 Variation of \( \sigma_x \) along the z direction (represented by the elements’ layer number) for the Graphite/Polyimide composite.
Figure 25 Variation of $\sigma_y$ along the z direction (represented by the elements’ layer number) for the Graphite/Polyimide composite.

Figure 26 Variation of $\sigma_z$ along the z direction (represented by the elements’ layer number) for the Graphite/Polyimide composite.
Figure 27 Variation of $\tau_{xz}$ along the $z$ direction (represented by the elements’ layer number) for the Graphite/Polyimide composite.

Figure 28 Variation of $\varepsilon_x$ along the $z$ direction (represented by the elements’ layer number) for the Graphite/Polyimide composite.
The same longitudinal length was adopted for the modeling of the E-Glass/Epoxy composite. The number of elements, the number of nodes and the average run time for the composites investigated are given in Table 3 for information.

**Table 3 Finite Element Models information**

<table>
<thead>
<tr>
<th></th>
<th>Graphite/Epoxy and E-Glass/Epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of SOLID45 elements</td>
<td>5760</td>
</tr>
<tr>
<td>Number of SURF22 elements</td>
<td>896</td>
</tr>
<tr>
<td>Total number of elements</td>
<td>6656</td>
</tr>
<tr>
<td>Total number of nodes</td>
<td>6636</td>
</tr>
<tr>
<td>Average run time</td>
<td>5 min. 20s</td>
</tr>
</tbody>
</table>

2.5.3 **Boundary conditions**

Displacement symmetry condition is applied on the xz bottom surface of the reduced model and five nodes are pinned to prevent rigid body motions (See Figure 29). The node referenced A in Figure 29 is pinned in all directions to prevent translations in the 3 directions. Nodes B and C are pinned in the x direction and nodes D and E in the z direction to prevent rotation of the entire structure around the y-axis. Rotation around the x and z-axes are prevented through the symmetry condition on the xz bottom surface.
Figure 29 Pinned nodes on the y = 0 xz plane.

Traction based boundary conditions are applied to the model (See Figure 30). As previously pointed out, they are very similar to the stress boundary conditions applied on the Concentric Cylinder Model and are also derived from the global applied stress $\sigma_{\text{glob}}$ and the off-axis angle $\theta$. 
\[ \sigma_z = \sigma_{\text{glob}} \cdot \cos^2 \theta \]
\[ \sigma_x = \sigma_{\text{glob}} \cdot \sin^2 \theta \]
\[ \tau_{xz} = -\sigma_{\text{glob}} \cdot \cos^2 \theta \sin^2 \theta \]

Figure 30 Traction boundary conditions for the FE Model from the global applied stress and the off-axis angle

2.5.4 Results and discussion

The results are retrieved in the plateau region at the elements’ layer number 1 (0 < z < 0.5). The layer of elements is represented in Figure 31. Stresses are averaged as follow:

\[ \sigma_{ij} = \frac{\sum_k \sigma_{ij}^k \times \text{volume}^k}{\sum_k \text{volume}^k} \]

where k is the element number in the selected layer
The averaged strains are obtained by multiplying the averaged stresses by the compliance matrix of the matrix material.

The stresses and strains obtained for $\sigma_{\text{glob}} = 100$ MPa as a function of off-axis angle $\theta$ are plotted along with the corresponding CCM stresses and strains in Figure 32 to Figure 39. The finite element model is also very stable. If the properties of the fiber and the matrix are change by 5%, the averaged stresses and strains do not change by more than 0.5%.
Figure 32 $\sigma_x$ vs. $\theta$ from the Concentric Cylinder Model and the Finite Element Models for $\sigma_{\text{glob}} = 100$ MPa. EGlass/Epoxy (properties in Table 1 p. 31).

Figure 33 $\sigma_z$ vs. $\theta$ from the Concentric Cylinder Model and the Finite Element Models for $\sigma_{\text{glob}} = 100$ MPa. EGlass/Epoxy (properties in Table 1 p. 31).
Figure 34 $\tau_{xz}$ vs. $\theta$ from the Concentric Cylinder Model and the Finite Element Models for $\sigma_{\text{glob}} = 100$ MPa. EGlass/Epoxy (properties in Table 1 p. 31).

Figure 35 $\varepsilon_x$ vs. $\theta$ from the Concentric Cylinder Model and the Finite Element Models for $\sigma_{\text{glob}} = 100$ MPa. EGlass/Epoxy (properties in Table 1 p. 31).
Figure 36 $\overline{\sigma_x}$ vs. $\theta$ from the Concentric Cylinder Model and the Finite Element Models for $\sigma_{\text{glob}} = 100$ MPa. Graphite/Polyimide (properties in Table 1 p. 31).

Figure 37 $\overline{\sigma_z}$ vs. $\theta$ from the Concentric Cylinder Model and the Finite Element Models for $\sigma_{\text{glob}} = 100$ MPa. Graphite/Polyimide (properties in Table 1 p. 31).
Figure 38 $\tau_{xz}$ vs. $\theta$ from the Concentric Cylinder Model and the Finite Element Models for $\sigma_{\text{glob}} = 100$ MPa. Graphite/Polyimide (properties in Table 1 p. 31).

Figure 39 $\bar{\varepsilon}_x$ vs. $\theta$ from the Concentric Cylinder Model and the Finite Element Models for $\sigma_{\text{glob}} = 100$ MPa. Graphite/Polyimide (properties in Table 1 p. 31).
The obtained values of stresses and strains are in good agreement with the CCM results. The difference between the two models is slightly larger in the case of the Graphite/Polyimide. For both composites, the maximum CCM/FEM difference is found for $\tau_{xz}$ at 45°. It doesn’t exceed 8%. All the other stress and strain values match within less than 3%. 
CHAPTER 3 AN OFF-AXIS STRENGTH PREDICTION APPROACH FOR RUBBER BASED UNIDIRECTIONAL COMPOSITES

3.1 Introduction

The failure function used in CHAPTER 2 was constructed to eventually capture the nonlinear behavior of the material where it is applied. To check this feature, an attempt was made to predict off-axis strength for a steel-cord/rubber unidirectional composite using the failure function. The rubbery matrix is indeed expected to be nonlinear. The challenge here lies in constructing a local stress analysis with nonlinear capabilities. A finite element stress analysis based on the FE model constructed in CHAPTER 2 was considered. This nonlinear FE analysis is still under development. Divergence problems were encountered in the solution process. These problems along with possible solutions are discussed in detail in 3.3 p. 77. The general methodology that we planned to use is also discussed in this section.

Rubber based materials often follow an hyperelastic law (See 1.2.4 p. 16). The stress/strain behavior of the rubber used in the steel-cord/rubber composite considered was determined from experimental static tensile tests on dogbone specimens. To generate experimental off-axis strength data, the composite laminae and two plies laminates were tested in static tensile tests for different cord orientations. To obtain the instantaneous elastic response of the material, the rate of loading was high (high enough for relaxation and viscoelastic behaviors not to occur).

3.2 Experimental work

3.2.1 Materials

In order to protect proprietary information, the source and composition of the materials tested in this work will not be described. The geometry and dimensions
(including gage length) of the different specimens are presented in Figure 40 to Figure 42.

3.2.1.1 Rubber dogbone specimens

Figure 40 Dogbone Specimen

3.2.1.2 Off-axis unidirectional steel-cord/rubber laminae

Figure 41 Unidirectional steel-cord/rubber lamina

Specimens with 15°, 18°, 30°, 45°, 60° and 90° off-axis angle (angle between the steel cord direction and the loading direction) were tested. The steel cords are brass coated to ensure a good bonding with the rubber.
3.2.1.3  +30°/-30° and +18°/-18° steel-cord/rubber laminates

The total length and gage length of the laminates are the same as those of the laminae. However, the thickness is twice that of the laminae, i.e. 4 mm. A side view is shown Figure 42.

![Figure 42 Side view of +30°/-30° steel-cord/rubber specimen](image)

3.2.2  Description of the experiments

Quasi-static tensile tests were performed on an “Instron” machine (See Figure 43) at room temperature. The load cells used for the laminae and laminates were respectively 5 kN and 10kN. The dogbone were tested using a 1000 N load cell and pneumatic grips specially adapted to small samples. The tests were performed in displacement control at a rate of 50 mm per minute. The data acquisition was 250 msec for all the specimens. Because crosshead displacement could not be used to obtain accurate strain data due to stiffening effects in the grips, an extensometer capable of recording strains up to 1000% was used for the laminae and laminates. It was placed on the center of the tested specimen. This extensometer could not be used for the dogbones because of their small size. A 14% engineering strain amplification factor (also from a confidential source) was used to account for the grip effects.
3.2.3 Experimental results and discussion

3.2.3.1 Tensile tests on the rubber dogbones

3.2.3.1.1 Engineering stress-strain curve

The engineering stress ($\sigma_{\text{eng}} = \frac{\text{Force}}{\text{Initial Area}}$) - engineering strain ($\varepsilon_{\text{eng}} = \frac{\Delta\text{Length}}{\text{Initial Length}}$) curve of the rubber is given in Figure 44.
After a small kink between 0 and 15% strain, the curve is very close to a straight line. The kink is usually more pronounced for rubber materials. The strain at failure is close to 450% at a stress level of 17 MPa. A method to determine if a material follows a Mooney-Rivlin model (See p.18) was presented by Finney and Kumar. This method was used here. For an incompressible Mooney-Rivlin material under uniaxial tension, the stress-strain equation can be expressed as:

\[ S = 2 \cdot \left( a - a^{-2} \right) \left( C_{10} + C_{01} \cdot a \right) \]

where \( S \) is the engineering stress, \( a \) is the stretch ratio \( \left( 1 + \frac{\Delta \text{Length}}{\text{Original Length}} \right) \) and \( C_{10} \) and \( C_{01} \) come from Equation 19 p. 18. From Equation 51, a plot of \( \frac{S}{2} \cdot \left( a - a^{-2} \right) \) against \( a^{-1} \) will give a straight line if the material follows a Mooney-Rivlin model. The slope will give \( C_{01} \) and the intercept \( C_{10} \). The plot obtained for the tested rubber is shown in Figure 44.
45. It clearly indicates that the rubber specimen does not follow a Mooney-Rivlin law. An alternative constitutive relation will have to be used in the Finite Element stress analysis.

![Rubber dogbone specimen graph](image)

**Figure 45** Mooney-Rivlin based plot for the rubber

3.2.3.1.2 True stress-strain curve

The nonlinearity of the material is more obvious in Figure 46 where true stress and true strain are plotted. We recall that:

**Equation Set 52**

\[
\sigma_{true} = \frac{\text{Force}}{\text{Current Area}} = \sigma_{eng} \cdot (1 + \varepsilon_{eng}) \quad \text{and} \quad \varepsilon_{true} = \ln\left(1 + \frac{\Delta \text{Length}}{\text{Original Length}}\right) = \ln\left(1 + \varepsilon_{eng}\right)
\]
Figure 46 True stress/strain curve

The nonlinear structural behavior of the rubber is expected to be mainly due to material non-linear behavior as opposed to geometric nonlinearities in testing the dogbone specimen. A good way to show the presence of material nonlinearity is to plot the Second Piola-Kirchoff stress vs. the Green-Lagrange strain in the direction of loading.

3.2.3.1.3 Second Piola-Kirchoff stress - Green-Lagrange strain curve

Second Piola-Kirchoff stress and Green-Lagrange strain tensors are used in large deformation theory of continuum mechanics. The Green-Lagrange strain tensor is formulated to disregard the geometric nonlinearity effects while accounting for actual material stretching. In the direction of loading, it is calculated as follows:

\[
\varepsilon_{GL} = \frac{\text{Length}^2 - \text{Initial Length}^2}{2 \cdot \text{Initial Length}^2}
\]

\text{Equation 53}
The thermodynamically conjugate stress tensor $\overline{\mathbf{S}}$, Second Piola-Kirchoff, does not have any physical meaning. It was derived from the following relation:

**Equation 54**

$$
[\overline{\mathbf{S}}] = J \cdot \left[ F^{-1} \right] \cdot \left[ T \right] \cdot \left[ F^{-1} \right]^T = \left[ F^{-1} \right] \cdot [\sigma]
$$

- $F_{ij} = \frac{\partial x_i}{\partial X_j}$ is the deformation gradient where $x$ is the deformed coordinate and $X$ is the undeformed coordinate.
- $J = \text{Det}[F]$.
- $[T]$ is the true stress tensor.
- $[\sigma]$ is the Cauchy or engineering stress tensor.

$[\overline{\mathbf{S}}]$ is calculated for the dogbone specimen. We consider the gage length of the dogbone specimen and assume a uniform strain field. We also assume that the originally rectangular shape of the gage region of the dogbone will remain rectangular as the specimen deforms (See Figure 47).

![Figure 47 Deformation of the gage part of the rubber dogbone](image-url)
Because of symmetry, the problem can be reduced to 1/8\textsuperscript{th} of the original rectangle as shown in Figure 48.

Figure 48 Reduced geometry

We have

Equation Set 55

\[ x_1 = X_1 \cdot \frac{1_1 / 2}{L_1 / 2} = X_1 \cdot \frac{1_1}{L_1} \quad 0 \leq X_1 \leq \frac{L_1}{2} \]

\[ x_2 = X_2 \cdot \frac{1_2 / 2}{L_2 / 2} = X_2 \cdot \frac{1_2}{L_2} \quad 0 \leq X_2 \leq \frac{L_2}{2} \]

\[ x_3 = X_3 \cdot \frac{1_3 / 2}{L_3 / 2} = X_3 \cdot \frac{1_3}{L_3} \quad 0 \leq X_3 \leq \frac{L_3}{2} \]

For an incompressible material, the volume is constant, i.e.

Equation 56

\[ l_1 \cdot l_2 \cdot l_3 = L_1 \cdot L_2 \cdot L_3 \]

We assume that

Equation Set 57

\[ L_1 = k \cdot L_2 \quad \text{and} \quad l_1 = k \cdot l_2 \]
Substituting Equation Set 57 into Equation 56 and solving for $l_2$ we obtain:

Equation 58

\[ l_2 = \sqrt{\frac{L_3}{I_3}} \cdot L_2 \]

From Equation Set 57 and Equation 58, we get:

Equation 59

\[ l_1 = \sqrt{\frac{L_2}{I_3}} L_1 \]

Using Equation Set 55 and Equation 59 we have:

Equation 60

\[ x_1 = X_1 \sqrt{\frac{L_3}{I_3}} \quad 0 \leq X_1 \leq \frac{L_1}{2} \]

Similarly,

Equation 61

\[ x_2 = X_2 \sqrt{\frac{L_3}{I_3}} \quad 0 \leq X_2 \leq \frac{L_1}{2} \]

We can now form the [F] matrix:

Equation 62

\[
[F] = \begin{bmatrix}
\sqrt{\frac{L_3}{I_3}} & 0 & 0 \\
0 & \sqrt{\frac{L_3}{I_3}} & 0 \\
0 & 0 & \frac{1}{L_3} \\
\end{bmatrix}
\]
The only non-zero stress component is in the direction of the applied normal load. From Equation 54 and Equation 62 we finally obtain:

\[
\hat{S}_{33} = \frac{L_3}{l_3} \cdot \sigma_{33}
\]

Equation 63

Figure 49 is a plot of the calculated Second Piola-Kirchoff stress vs. the corresponding Green-Lagrange strain. It shows the presence of material nonlinearity since the obtained curve is not a straight line.

![Figure 49 Second Piola-Kirchoff stress vs. Green-Lagrange strain](image)

3.2.3.2 Tensile tests on the off-axis unidirectional steel-cord/rubber laminae

As expected, failure occurred within the rubber close to the steel/rubber interface and along a direction parallel to the direction of the cords. The broken specimens are shown in Figure 50.
Three specimens for each off-axis angle were tested. The results were very consistent and a representative curve for each angle is presented next.
Figure 51 Stress/strain curve for the 15° off-axis lamina

Figure 52 Stress/strain curve for the 18° off-axis lamina
Figure 53 Stress/strain curve for the 30° off-axis lamina

Figure 54 Stress/strain curve for the 45° off-axis lamina
Figure 55 Stress/strain curve for the 60° off-axis lamina

Figure 56 Stress/strain curve for the 90° off-axis lamina
As the off-axis angle increases the contribution of the rubber phase on the overall response of the composite increases and the nonlinear tendency of the stress-strain curve is more pronounced. The larger slope of the curves obtained from crosshead displacement compared to the ones obtained with the extensometer clearly show the stiffening effect of the grips. The early part of the 18° curve and the kink at 120% strain of the 90° curve are due to testing problems and do not represent any actual physical trend or phenomenon.

3.2.3.3 Tensile tests on the off-axis unidirectional steel-cord/rubber laminates

Coalescence of matrix cracks along the fiber in each lamina leads to the V-shape type of failure shown in Figure 57.

![Figure 57 Broken +30°/-30° unidirectional laminate](image)

The stress-strain curves of the +18°/-18° and +30°/-30° laminates are presented in Figure 58 and Figure 59.
Figure 58 Stress/strain curve for the +18°/-18° off-axis laminate

Figure 59 Stress/strain curve for the +30°/-30° off-axis laminate
3.2.3.4 Strength vs. off-axis angle curve for the laminae and the laminates

The off-axis strength for the three series of laminae tested are presented in Figure 60. The results are very consistent. The off-axis strength of the $+18^\circ/-18^\circ$ and $+30^\circ/-30^\circ$ laminates are also given.

![Experimental Off-Axis Strength](image)

Figure 60 Off-axis strength for the steel-cord/rubber laminae and laminates

3.3 The Finite Element Model

As previously pointed out the model in based on the one described in 2.5 p. 43 and is still under development. The fiber here is steel and the matrix rubber. There are two main differences between the nonlinear Finite Element local stress analysis and its linear version:

- the matrix constitutive law. The rubber matrix is the nonlinear material in the system.
- the boundary conditions applied on the model.
3.3.1 Nonlinear structural analysis in ANSYS - Newton-Raphson procedure

3.3.1.1 Basics

To solve nonlinear problems, a series of successive linear approximations with corrections are needed. The load is broken into a series of load increments. These load increments can be applied either over several load steps or over several substeps within a load step. To identify load steps and substeps, ANSYS uses time. In rate-independent analyses, the time parameter is only a counter or a tracker.

For a nonlinear structural analysis, the typical finite element discretization process yields:

\[
[K(u)] \cdot \{u\} = \{F_a\}
\]

where \(\{u\}\) = vector of unknown displacement values
\n\[K(u)\] = stiffness matrix function of the displacements (in a linear analysis the stiffness matrix is a constant)
\n\{F_a\} = vector of applied loads

ANSYS uses the Newton-Raphson method to solve nonlinear problems. This method is the most commonly used iterative process of solving static nonlinear equations. It is briefly presented in the next section.

3.3.1.2 The Newton-Raphson procedure (from ANSYS Theory Manual)

The Newton-Raphson method can be written as:

\[
[K(u_i)^T] \cdot \{\Delta u_i\} = \{F^a_i\} - \{F^{nr}_i\} = R_i
\]

\[u_{i+1} = u_i + \{\Delta u_i\}\]

where \([K(u_i)^T]\) = tangent stiffness matrix
\ni = equilibrium iteration number
\{F_{i}^{nr}\} = \text{restoring force vector calculated from the elements stresses}

and \(R_{i} = \text{residual}\)

For a given loadstep or substeps, the general algorithm is the following (See also Figure 61):

1. Assume \(\{u_{0}\}\). \(\{u_{0}\}\) is usually the converged solution from the previous time step. On the first time step, \(\{u_{0}\} = \{0\}\).
2. Compute the updated tangent matrix \([K(u)^{T}]\) and the restoring force \(\{F_{i}^{nr}\}\) form configuration \(\{u_{i}\}\).
3. Calculate \(\{\Delta u_{i}\}\) from Equation Set 65.
4. Calculate \(\{\Delta u_{i+1}\}\)
5. Repeat steps 2 to 4 until convergence is achieved, i.e. the residual \(R_{i}\) is zero ± tolerance.

Figure 61 Newton-Raphson method
Different methods exist to help the Newton-Raphson algorithm to converge. For more details the reader will refer to ANSYS Theory Manual\textsuperscript{57}.

### 3.3.2 Rubber constitutive relation

We first planned to use 3D hyperelastic elements following a Mooney-Rivlin law. Because of the experimental results showing the inadequacy of this type of constitutive law, another approach was considered. The behavior of the rubber was entered in ANSYS by discretizing the experimental stress-strain curve into small linear pieces. Because of their large deformations and large strains capabilities the 3D structural elements Solid45 (2.5.2.1 p. 45) were used, coupled with the user entered stress-strain relation.

### 3.3.3 Stress or displacement boundary conditions ?

Stresses are easy to apply as boundary conditions because they can be directly derived from the global applied stress (See Figure 30 p. 53). The derivation is valid for linear materials as well as for non-linear materials. However, the ANSYS nonlinear analysis does not converge when relatively high stresses are applied on the model. For nonlinear analysis, displacement boundary conditions are known to give a more stable model and to converge more easily. The problem is now to derive the displacements to apply on the model from the global displacement in the direction of loading. In the case of nonlinear materials unlike stresses, displacements can not be easily transformed from one coordinate system to the other. A technique called Submodeling was used. It is discussed in the next section.

### 3.3.4 Submodeling

The submodeling technique is used in finite element analysis to obtain more accurate results in a region of the model. For our problem, ideally, the entire lamina including all the steel-cords should be modeled and the results retrieved far from the free edges and far from the region where the loads are applied. However, this type of approach would be too computationally intense. An alternative is to consider the lamina
as an homogeneous material having the effective composite properties. This approximation, employed by Pagano\textsuperscript{51,52} in his Concentric Cylinder Model, might however be too coarse in the case of a steel-cord/rubber composite where matrix and fiber properties are very different. Further investigations should allow us to confirm or invalidate this issue. The effective composite behavior is given by the experimental stress-strain curves at the off-axis angle considered (Figure 51 to Figure 56). These stress-strain curves can be entered using the same multilinear approximation technique as that used for the rubber dogbones.

The submodeling procedure is summarized in Figure 62. A small rectangle oriented at the off-axis angle considered is drawn in the center of the “coarse” model of the equivalent homogeneous lamina. Calculated displacements are retrieved on this rectangle and applied as boundary conditions to the “fine” submodel after interpolation. A shell-to-solid submodeling technique can be used to reduce the computation time. The coarse model is meshed with shell elements and the submodel with 3D solid elements.

Figure 62 Submodeling procedure for the steel-cord/rubber lamina
CHAPTER 4 SUMMARY AND CONCLUSIONS

4.1 Summary

A new micromechanics-based model for the off-axis strength prediction of unidirectional linear elastic laminae has been established. It uses:

- An existing concentric cylinder model to obtain the stresses and strains at the matrix and fiber level.
- A new local failure function written as a reduced and normalized strain energy density function.

The feasibility of this approach has been demonstrated by applying the model to different composite laminae and comparing the results with data from the literature.

The same type of approach will be used for nonlinear laminae and the method has been outlined in CHAPTER 3. The challenge here lies in constructing a local stress analysis. A finite element analysis is currently under development for this nonlinear problem. It is similar to the linear finite element analysis approach described in the present work and which was validated with the concentric cylinder model. The comparison of the CCM and the FEM is possible because:

- Comparable geometry and boundary conditions were used
- Stresses and strains obtained from the CCM do not vary along the longitudinal z direction and FEM results were retrieved in a region where they don’t vary with z either.

Tensile tests were performed on steel-cord/rubber laminae and laminates and the constitutive behavior of the nonlinear rubber matrix was investigated.

4.2 Future work

4.2.1 Linear analysis

The following represent suggestions for future work:
• When A and B are expressed with matrix strength, we have seen that the predicted off-axis strength are not correct. If we multiply A and B by 0.3, we consistently obtain very good predictions. Investigations on how to relate this factor of 0.3 to fiber volume fraction and/or stiffness of the matrix and the fiber would be a challenging future direction of research.

• Different critical element volumes could be investigated. If matrix strength and strain-to-failure properties are used for A and B, the currently calculated averaged stresses and strains are too low (the off-axis strength is overestimated). If we consider a smaller volume where the high stresses are concentrated, we should get better predictions. Ultimately, we could use the maximum stress and strain values in the numerator of the failure function.

### 4.2.2 Nonlinear analysis

The nonlinear finite element model is currently under development. Possible methods to deal with the convergence problems are:

• modify the time step and the nonlinear analysis options
• find a way to enter more information about how the rubber behaves in order to better describe the complex state of stress and strain that the matrix undergoes.
• ultimately, try to run the model on finite element code with explicit formulation (ABAQUS explicit for example).
REFERENCES


APPENDIX 1
(from Pagano$^5$)

\[ U_1(r) = A_1 r^3 + \frac{A_2}{r^3} + A_3 r + \frac{A_4}{r} \]
\[ V_1(r) = -\frac{(3C_{22} + C_{23})}{2C_{23}} A_1 r^3 + \frac{A_2}{r^3} - A_3 r - \frac{(C_{22} - C_{23}) A_4}{2C_{22}} \frac{1}{r} \]
\[ W_1(r) = S_1 r^2 + S_2 \frac{1}{r^2} \]
\[ U_2(r) = B_1 r^3 + \frac{B_2}{r^3} + B_3 r + \frac{B_4}{r} \]
\[ V_2(r) = \frac{(3C_{22} + C_{23})}{2C_{23}} B_1 r^3 - \frac{B_2}{r^3} + B_3 r + \frac{(C_{22} - C_{23}) B_4}{2C_{22}} \frac{1}{r} \]
\[ W_2(r) = T_1 r^2 + T_2 \frac{1}{r^2} \]
\[ U_3(r) = D_1 r + \frac{D_2}{r} \]
\[ V_3(r) = F_1 r + \frac{F_2}{r} \]
\[ W_3(r) = H_1 + H_2 \ln r \]
\[ U_4(r) = \left[ \frac{2C_{12}}{5C_{22} + C_{23}} P_3 + P_1 \right] r^2 + \frac{P_2}{r^2} + P_4 \ln r + P_5 \]
\[ V_4(r) = \frac{(5C_{22} + C_{23})}{C_{22} - 3C_{23}} P_1 r^2 + \frac{P_2}{r^2} - P_4 \left[ \ln r + \frac{(C_{22} + C_{23})}{3C_{22} - C_{23}} \right] - P_5 \]

\[ W_4(r) = (X_1 + X_3)r + \frac{X_2}{r} \]

\[ U_5(r) = \left[ -\frac{2C_{12}}{5C_{22} + C_{23}} Q_3 + Q_4 \right] r^2 + \frac{Q_2}{r^2} + Q_4 \ln r + Q_5 \]

\[ V_5(r) = \frac{(5C_{22} + C_{23})}{C_{22} - 3C_{23}} Q_3 r^2 - \frac{Q_2}{r^2} + Q_4 \left[ \ln r + \frac{(C_{22} + C_{23})}{3C_{22} - C_{23}} \right] + Q_5 \]

\[ W_5(r) = (Y_1 - Y_3)r + \frac{Y_2}{r} \]

\[ U_6(r) = -V_6(r) = Y_3 \]
\[ W_6(r) = Q_3 r \]
\[ U_7(r) = V_7(r) = -X_3 \]
\[ W_7(r) = -P_3 r \]
\[ U_8(r) = -V_8(r) = 0.5P_3 \]
\[ U_9(r) = V_9(r) = -0.5Q_3 \]
\[ V_{10}(r) = F_2 r \]
\[ W_{10}(r) = D_3 \]

\[ x_1(r) = C_{12}(C_{23} - C_{22}) \left\{ \frac{3A_1 r^2}{C_{23}} + \frac{A_4}{C_{22} r^2} \right\} \]

\[ \zeta_1(r) = (C_{22} - C_{23}) \left\{ -\frac{3A_2}{r^4} + A_3 - \frac{(C_{22} + C_{23}) A_4}{C_{22} r^2} \right\} \]

\[ \beta_1(r) = (C_{22} - C_{23}) \left\{ -\frac{3(C_{22} + C_{23}) A_1 r^2}{C_{23}} + \frac{3A_2}{r^4} - A_3 \right\} \]

\[ \gamma_1(r) = 0.5(C_{22} - C_{23}) \left\{ -\frac{3(C_{22} + C_{23}) A_1 r^2}{C_{23}} - \frac{6A_2}{r^4} - 2A_3 - \frac{(C_{22} + C_{23}) A_4}{C_{22} r^2} \right\} \]

\[ \zeta_1(r) = 2C_{55} \left\{ S_1 r + \frac{S_2}{r^3} \right\} \]

\[ \delta_1(r) = 2C_{55} \left\{ S_1 r - \frac{S_2}{r^3} \right\} \]
\[ \mathbf{z}_2(r) = C_{12}(C_{23} - C_{22}) \left\{ \frac{3B_1 r^2}{C_{23}} + \frac{B_4}{C_{22} r^2} \right\} \]

\[ \mathbf{\zeta}_2(r) = (C_{22} - C_{23}) \left\{ -\frac{3B_2}{r^2} + B_3 - \left( \frac{C_{22} + C_{23}}{C_{22}} \right) \frac{B_3}{r^2} \right\} \]

\[ \mathbf{\beta}_2(r) = (C_{22} - C_{23}) \left\{ -\frac{3(C_{22} + C_{23})}{C_{23}} B_1 r^2 + \frac{3B_2}{r^2} - B_3 \right\} \]

\[ \mathbf{\gamma}_2(r) = 0.5(C_{22} - C_{23}) \left\{ \frac{3(C_{22} + C_{23})}{C_{23}} B_1 r^2 + \frac{6B_2}{r^2} \right\} \]

\[ + 2B_3 + \left( \frac{C_{22} + C_{23}}{C_{22}} \right) \frac{B_4}{r^2} \]

\[ \mathbf{\xi}_2(r) = -2C_{55} \left\{ T_1 r + \frac{T_2}{r^3} \right\} \]

\[ \mathbf{\delta}_2(r) = 2C_{55} \left\{ T_1 r - \frac{T_2}{r^3} \right\} \]

\[ \mathbf{x}_3(r) = C_{14}(D_3 - e_2) + 2C_{12}(D_1 - e_r) \]

\[ \mathbf{\zeta}_3(r) = C_{14}(D_3 - e_2) + (C_{22} + C_{23}) k(D_1 - e_r) - (C_{22} - C_{23}) \frac{D_2}{r^2} \]

\[ \mathbf{\beta}_3(r) = C_{14}(D_3 - e_2) + (C_{22} + C_{23}) k(D_1 - e_r) + (C_{22} - C_{23}) \frac{D_2}{r^2} \]

\[ \mathbf{\gamma}_3(r) = -2C_{44} \frac{F_2}{r^2} \]

\[ \mathbf{\xi}_3(r) = C_{55} F_3 r \]

\[ \mathbf{\delta}_3(r) = C_{55} \frac{H_2}{r} \]

\[ \mathbf{x}_4(r) = \left\{ \frac{8C_{12}(C_{22} - C_{23})}{C_{22} - 3C_{23}} P_1 + \left[ -C_{11} + \frac{6C_{12}^2}{5C_{22} + C_{23}} \right] P_3 \right\} r \]

\[ + \frac{2C_{12}(C_{22} - C_{23})}{3C_{22} - C_{23}} \frac{P_4}{r} \]

\[ \mathbf{\zeta}_4(r) = (C_{22} - C_{23}) \left\{ \left[ \frac{2(C_{22} + C_{23})}{C_{22} - 3C_{23}} P_1 - \frac{C_{12}}{5C_{22} + C_{23}} P_3 \right] r \right\} \]

\[ - \frac{2P_2}{r^3} + \left( \frac{3C_{22} + C_{23}}{3C_{22} - C_{23}} \right) \frac{P_4}{r} \]
\[ \beta_4(r) = (C_{22} - C_{23}) \left\{ \frac{6(C_{22} + C_{23})}{C_{22} - 3C_{23}} P_1 - \frac{3C_{12}}{5C_{22} + C_{23}} P_3 \right\} r \\
\quad + \frac{2P_2}{r^3} - \frac{C_{22} - C_{23}}{3C_{22} - C_{23}} P_4 \right\} r \]

\[ \gamma_4(r) = 0.5(C_{22} - C_{23}) \left\{ \frac{4(C_{22} + C_{23})}{C_{22} - 3C_{23}} P_1 - \frac{2C_{12}}{5C_{22} + C_{23}} P_3 \right\} r \\
\quad - \frac{4P_2}{r^3} - \frac{2(C_{22} - C_{23})}{3C_{22} - C_{23}} P_4 \right\} r \]

\[ \xi_4(r) = C_{55} \left\{ X_1 + \frac{X_2}{r^2} \right\} \]

\[ \delta_4(r) = C_{55} \left\{ X_1 - \frac{X_2}{r^2} \right\} \]

\[ \chi_4(r) = \left\{ \frac{8C_{12}(C_{22} - C_{23})}{C_{22} - 3C_{23}} Q_1 + \frac{6C_{12}}{5C_{22} + C_{23}} Q_3 \right\} r \\
\quad + \frac{2C_{12}Q_4}{3C_{22} - C_{23}} \right\} r \]

\[ \zeta_5(r) = (C_{22} - C_{23}) \left\{ \frac{2(C_{22} + C_{23})}{C_{22} - 3C_{23}} Q_1 + \frac{C_{12}}{5C_{22} + C_{23}} Q_3 \right\} r \\
\quad - \frac{2Q_2}{r^3} + \frac{3C_{22} + C_{23}}{3C_{22} - C_{23}} Q_4 \right\} r \]

\[ \beta_5(r) = (C_{22} - C_{23}) \left\{ \frac{6(C_{22} + C_{23})}{C_{22} - 3C_{23}} Q_1 + \frac{3C_{12}}{5C_{22} + C_{23}} Q_3 \right\} r \\
\quad + \frac{2Q_2}{r^3} - \frac{C_{22} - C_{23}}{3C_{22} - C_{23}} Q_4 \right\} r \]

\[ \gamma_5(r) = 0.5(C_{22} - C_{23}) \left\{ \frac{-4(C_{22} + C_{23})}{C_{22} - 3C_{23}} Q_1 - \frac{2C_{12}}{5C_{22} + C_{23}} Q_3 \right\} r \\
\quad + \frac{4Q_2}{r^3} + \frac{2(C_{22} - C_{23})}{3C_{22} - C_{23}} Q_4 \right\} r \]

\[ \xi_5(r) = -C_{55} \left\{ Y_1 + \frac{Y_2}{r^2} \right\} \]

\[ \delta_5(r) = C_{55} \left\{ Y_1 - \frac{Y_2}{r^2} \right\} \]
VITA

Jérémy Duthoit, son of Liliane Oter and Bruno Duthoit, was born on February 29, 1976 in Béthune, France. He grew up in Beuvry in the north of France. After obtaining his Baccalauréat in Mathematics and Physics in 1993 at the Lycée Louis Blaringhem in Béthune, France, Mr. Duthoit was admitted in the Université de Technologie de Compiègne (UTC, France). After two years of general engineering, he entered the department of Mechanical Engineering and specialized in Materials and Technologic Innovation. He was then selected for an exchange program between UTC and Virginia Tech. In August 1997, he moved to Blacksburg, Virginia, USA to enroll in the Master’s Program of the Department of Materials Science and Engineering. He joined the Materials Response Group where he conducted his research on mechanics of composites. Jeremy intends to continue his work in the field of composite materials in academia or in industries.

Jérémy Duthoit