TORSION OF ELLIPTICAL COMPOSITE CYLINDRICAL SHELLS

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Dissertation submitted to the Faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY in ENGINEERING MECHANICS

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August 2, 2007
Blacksburg, Virginia

Keywords: Noncircular Cylinders, Composite Materials, Buckling, Postbuckling, Progressive Failure Analysis, Configuration Changes

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Engineering Mechanics

(ABSTRACT)

The response of elliptical composite cylindrical shells under torsion is studied. The torsional condition is developed by rotating one end of the cylinder relative to the other. Prebuckling, buckling, and postbuckling responses are examined, and material failure is considered. Four elliptical cross sections, defined by their aspect ratio, the ratio of minor to major radii, are considered: 1.00 (circular), 0.85, 0.70, and 0.55. Two overall cylinder sizes are studied; a small size with a radius and length for the circular cylinder of 4.28 in. and 12.85 in., respectively, and a large size with radii and lengths five times larger, and thicknesses two times larger than the small cylinders. The radii of the elliptical cylinders are determined so the circumference is the same for all cylinders of a given size. For each elliptical cylinder, two lengths are considered. One length is equal to the length of the circular cylinder, and the other length has a sensitivity of the buckling twist to changes in the length-to-radius ratio the same as the circular cylinder. A quasi-isotropic lamination sequence of a medium-modulus graphite-epoxy composite material is assumed. The STAGS finite element code is used to obtain numerical results. The geometrically-nonlinear static and transient, eigenvalue, and progressive failure analysis options in the code are employed. Generally, the buckling twist and resulting torque decrease with decreasing aspect ratio. Due to material anisotropy, the buckling values are generally smaller for a negative twist than a positive twist. Relative to the buckling torque, cylinders with aspect ratios of 1.00 and 0.85 show little or no increase in capacity in the postbuckling range, while cylinders with aspect ratios of 0.70 and 0.55 show an increase. Postbuckling shapes are characterized by wave-like deformations, with ridges and valleys forming a helical pattern due to the nature of loading. The amplitudes of the deformations are dependent on cross-sectional geometry. Some elliptical cylinders develop wave-like deformations prior to buckling. Instabilities in the postbuckling range result in shape changes and loss of torque capacity. Material failure occurs on ridges and in valleys. Cylinder size and cross-sectional geometry influence the initiation and progression of failure.
Acknowledgments

This research was supported by a grant to Virginia Tech from NASA Langley Research Center through the National Institute of Aerospace. The financial support is gratefully acknowledged. I would like to thank my advisor Professor M. W. Hyer for his technical and professional guidance during the course of this research. I would also like to thank Professors R. D. Kriz, M. J. Patil, S. A. Ragab, and S. Thangjitham for serving on my Ph.D. committee and for their valuable comments. I am grateful to M. Rouse for serving as my contact at NASA Langley Research Center. I would like to thank the Branch Heads, T. S. Gates and H. K. Rivers, for the computational facilities made available to me, and for providing me with office space. I would also like to thank Drs. M. W. Hilburger, N. F. Knight, Jr., J. C. Riddick, and R. P. Thomburgh for their technical advice. Finally, I would like to thank my family for their love and support.
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CHAPTER 1
INTRODUCTION

Cylindrical shell structures with noncircular cross sections are not as widely used as ones with circular cross sections. In many applications a circular cross section may be the most efficient structural shape; however, certain applications may benefit from a non-circular cross section. In aircraft design, for example, a fuselage with a noncircular cross section may be better suited for blended wing-body construction. In other aerospace applications, constraints on the dimensions or the nature of the payload may require a cylindrical structure to have a noncircular cross section.

As cylindrical shell structures with circular cross sections, especially thin-walled structures, are particularly efficient in supporting a range of loads, including internal pressure, axial compression, bending, torsion, and combinations of these, the question naturally arises as to what the loss of load capacity will be when a noncircular cross section is employed. For a circular cylinder, a loss of load capacity occurs either when there is material failure or a loss of stability. Either of these may occur at a lower applied load level when a noncircular geometry is employed. The reason is that with a noncircular geometry, even the simplest loadings, loadings which result in axisymmetric responses in circular geometries, produce a stress state that varies with circumferential location. This leads to what could be considered a stress concentration effect at certain circumferential locations, which in turn creates likely sites for instabilities or initial material failure and subsequent progression of failure. When considering noncircular cylindrical structures in the context of aircraft and other aerospace applications, as most contemporary aerospace structures are manufactured using fiber-reinforced composite materials, it is important to understand the stability and material failure characteristics of noncircular composite cylinders.

The present study focuses on the numerically-predicted response of noncircular cylindrical shells constructed of a fiber-reinforced composite material and subjected to a torsional loading. An elliptical cross section is examined, as opposed to an oval cross section, for example, primarily because past research at Virginia Tech considered this class of a non-
circular geometry, and because mandrels exist for constructing small-scale cylinders with
elliptical cross sections for any experimental activities that may follow the present work.
The specific loading considered is a prescribed angle of twist of one end of the cylinder
relative to the other. A torsional loading is considered because a noncircular cylinder could
be at a distinct disadvantage in reacting torsion. In fact, a noncircular geometry most likely
would not be used to resist torsion, but it could be that some component of a combined
loading condition is torsional. Therefore it is believed to be prudent to understand the tor-
sional response of noncircular cylinders. The range of loading investigated is well into
the post-buckling range, far enough to produce material failure. A progressive material
failure analysis is used and loading is continued to the first fiber failure. The numerical
results are computed using a finite-element analysis, in particular, using the finite-element
code STAGS (STructural Analysis of General Shells) [1]. To evaluate cross-sectional ge-
ometric effects, a range of cross-sectional aspect ratios, defined here as the ratio of the
minor diameter to major diameter, are considered. A circular cylinder with an aspect ratio
of unity is included in this range. Cylinders with what could be considered small overall
dimensions, dimensions suitable for use in commonly-available loading fixtures, and cylin-
ders with larger overall dimensions are studied. A single lamination sequence, namely a
quasi-isotropic laminate, is considered because the material properties of a quasi-isotropic
laminate in the axial and circumferential directions are the same. The effects of cylinder ge-
ometry, an important aspect of the study here, will not be influenced by directional material
property effects. The considerations of laminates with more directional material property
characteristics could certainly constitute future studies.

As an example of the characteristics of structural response that must be addressed when
investigating noncircular cylinders, a qualitative comparison of the post-buckled deforma-
tion of a circular cylinder and an elliptical cylinder, each subjected to a torsional load, is
illustrated in Figure 1.1. As seen, the amplitude of the spiral wave-like, or wrinkle-like, de-
formation pattern for the circular cylinder, Figure 1.1a, is much the same around the entire
circumference. On the other hand, for the elliptical cylinder, Figure 1.1b, the amplitude of
the deformation pattern varies with circumferential position. The amplitude is greatest in
the flatter portions of the cross section, herein referred to as the crown and keel regions,
and least in the more highly curved side regions. Such behavior clearly leads to specific
circumferential locations where buckling instabilities and material failures initiate.

Another example of the characteristics of structural response that must be addressed
when investigating noncircular cylinders, and one which has dictated much of the direc-
tion of the work described in this dissertation, is the issue of length-to-radius ratio effects,
illustrated in Figure 1.2. For a circular cylinder the relation between the critical twist angle per unit cylinder length, hereafter referred to as simply the critical twist and defined as the twist angle per unit length that causes a torsional instability in the cylinder, and the cylinder length-to-radius ratio is shown qualitatively in Figure 1.2a. As seen in the figure, the smaller the ratio of length to radius, the larger the critical twist. This means that stiffening of the cylinder due to boundary effects increases with smaller length-to-radius ratios. This clearly makes sense, but when the relation is computed for specific cylinders, the relationship quantifies the stiffening effect. Importantly, the slope of the relation tends to be larger for smaller values of the length-to-radius ratio. This means that the smaller the length-to-radius ratio, the more sensitive the critical twist is to changes in the length-to-radius ratio. So, 'short' cylinders are stiffened considerably by boundary effects and small changes in the length-to-radius ratio result in non-trivial changes in the critical twist.
With non-circular cylinders, since the radius of curvature changes with circumferential position, there is the issue of what radius of curvature to use if relations such as shown in Figure 1.2a are to be developed. Using the maximum radius of curvature for the cross section, which occurs in the flatter crown and keel regions, the relations between critical twist and length-to-radius ratio for elliptical cylinders with four cross-sectional aspect ratios, the same circumference, and the same wall thickness are illustrated schematically in Figure 1.2b. For any given length, since the four cylinders considered in Figure 1.2b all have the same circumference wall thickness, they all have the same weight. As seen, aspect ratio has an impact on the critical twist, the cylinders with the smaller aspect ratio becoming unstable at lower twists. More importantly, however, the results in Figure 1.2b raise the question as to how to compare the performance of cylinders with different aspect ratios. Is it sufficient to simply require that the length-to-radius ratios of the four cylinders be the same when making comparisons? Or should the lengths of all four cylinders be the same? If the lengths are the same, then since the maximum radius of curvature increases with decreasing aspect ratio, the length-to-radius ratios would decrease with decreasing aspect ratio and the stiffening effects of the boundary would become increasingly important even though the lengths of the cylinders were all the same. Also, the slopes of the relations would increase with decreasing aspect ratio. With this constant-length approach, boundary effects could mask the results when evaluating the influence of aspect ratio on stability. The changing slopes would also be an issue with the constant length-to-radius ratio approach. Another approach would be to require that the length of each cylinder be chosen so that the slope of each of the four relations is the same for the aspect ratios of interest. The lengths of the cylinders would not be the same, but the sensitivity of the critical twist angle to changes in the length-to-radius ratio would be the same. This could be interpreted to mean that the sensitivities of the four cylinders to boundary effects would be the same. These three options for making comparisons are illustrated schematically in Figure 1.3. The loci of the length-to-radius ratios for the three different approaches for comparing the performance of elliptical cylinders with various aspect ratios vary with respect to the critical twist angle relations for the four aspect ratios. This discussion of comparing the performance of cylinders with different aspect ratios has focused on cylinder length because of the practicality of manufacturing cylinders, testing cylinders, and their usage. Long cylinders are difficult, and expensive, to manufacture and test, and typically stiffeners, bulkheads, or other elements are used in structures, so the length of a monocoque section is limited in practice.
As the above discussion illustrates, there are many interesting, and important, aspects to consider when investigating the structural response of noncircular cylinders. As will be seen, some of the characteristics of noncircular cylinders can be a disadvantage, but others could be interpreted as an advantage.

Before discussing the details of the present investigation, a brief review of previous studies on the stability of cylinders subjected to torsion is presented. This chapter then concludes with a statement of the specific objectives of this study and a roadmap to the chapters to follow.
1.1 Literature Review

Stability of cylindrical structures has been studied extensively throughout the years. However, when considering cylinders under a torsional load, stability has not been studied as extensively as with other types of loading. The breadth of coverage is even narrower for noncircular cylinders, and almost non-existent for noncircular composite cylinders. Many of the studies dealing with the stability of cylindrical structures under a torsional load are discussed below.

1.1.1 Stability of Circular Isotropic Cylinders under Torsion

In one of the earliest studies of the stability of circular isotropic cylinders under torsion, Donnell [2] compiled results from prior experiments [3–6] and performed additional experiments on the stability of circular cylinders. The experimental results were compared to results from a thin-shell theory presented with the work, and both of these were also compared with other theoretical solutions developed by Schwerin [7] and Greenhill [8]. A comparison of the initial postbuckling shapes and critical loads is presented for many different cylinder geometries, as well as the different materials used in the compiled results. The difference between the experimental buckling stresses and the predicted buckling stresses was significant, and this discrepancy was believed to be a result of not including imperfections as part of the theoretical analysis.

Batdorf [9] modified the buckling equation of Donnell [2] to make the solution easier to develop for cylinders with clamped boundary conditions. Using the modified equation,
Batdorf et al. [10] calculated the critical stresses under torsion for a wide range of cylinder geometries, and the results were compared to experimental results and results using the buckling equation from Donnell [2]. There was a slight improvement in the predicted buckling stresses when using the modified buckling equation, but the difference between the predicted and experimental buckling stresses was still significant.

Following the methodology Donnell [11] and Donnell and Wan [12] used for including large deflections and initial imperfections when considering the buckling response of cylinders under axial compression, Loo [13] extended the theory to include cylinders under a torsional load, and defined the imperfection so it could take the form of the buckling mode shape for axial compression or torsion. Using the relationship between shear stress and shear strain, postbuckling response was examined for different cylinder geometries and imperfection amplitudes. Immediately after buckling, with increasing shear strain, there was a decrease in the shear stress. This was true for all cylinder geometries that were examined; however, for certain geometries the shear stress would then begin to increase with increasing shear stress. To simplify the solution, Loo assumed two of the unknown parameters of the solution could be solved for using the theory from Donnell [2], and then those values were used to obtain the complete solution. Nash [14] used the same theory as Loo, but made no assumptions regarding the accuracy of the small-deflection theory of Donnell [2] and solved for all of the unknown parameters. For the cylinder geometries that were considered in both studies, Nash observed that the consequence of the assumption by Loo resulted in a smaller decrease in the shear stress in the initial postbuckling response.

Nash [15] compared the theoretical results from refs. [2, 13, 14] to experimental results for a wide range of cylinder geometries. The initial imperfections in each of the cylinders were measured prior to loading, and the theories from refs. [13, 14] were capable of including initial imperfections of the approximate shape of the buckling mode shapes. When imperfection amplitudes similar to the measured amplitudes were included in the theoretical solutions, much better agreement was seen between predicted buckling stress and the experimental buckling stress than was observed for a comparison with the predicted buckling stress using the theory from ref. [2]. Despite this, there was still a significant discrepancy between the predicted and observed values for the angle of the ridges and valleys of the buckling wave with respect to the cylindrical axis (see Figure 1.1a).

Yamaki and Matsuda [16] and Yamaki [17] also studied the postbuckling response, including comparisons between theoretical solutions and experimental results. For the theoretical response, a geometrically perfect cylinder was used, so in the initial postbuckling
region there was a significant difference between the theoretical and experimental results, when considering the torque-twist angle and axial end displacement-twist angle relations. Further into the postbuckling response, the difference between the solutions became negligible. Yamaki et al. [18] extended the theory from refs. [16, 17] to include geometric imperfections in the form of the buckling mode shapes, and examined the effect of increasing the imperfection amplitude on the postbuckling response. Much better agreement was seen between the theoretical and experimental results when an imperfection was included in the analysis.

In the preceding paragraphs the development of the study of the elastic buckling and postbuckling of circular cylinders under torsion was discussed. The focus remained on the stability of circular isotropic cylinders, and while the review did not cover all of the work on the stability of circular isotropic cylinders under torsion, it did cover the gamut as it progressed from a study concerned with the critical stresses and initial postbuckling shapes to one that studied the response deep into the postbuckling range, and was capable of including geometric imperfections in the analysis. It is possible that the theories presented for the isotropic cylinders could be refined because there is still a noticeable difference between the theoretical and experimental responses; however, even before some of the more recent studies for isotropic cylinders had been conducted [16, 17], a shift toward the study of composite cylinders had begun.

1.1.2 Stability of Circular Composite Cylinders under Torsion

While the majority of the recent work dealing with the stability of composite cylinders has focused on fiber-reinforced laminated materials, some of the earlier studies considered orthotropic (single layer) [19] and laminated plywood cylinders [20–22]. For studies of fiber-reinforced laminated composites, a number of studies have considered torsion alone [23–36], but also several have considered combined loading with one of the loads being torsion [37–44]. In most cases where combined loads were considered, the results under torsion alone were also given. Most of the work was concerned with predicting the critical loads, and the initial postbuckling response was only of interest when it could be used to generate some form of an imperfection.

Tabiei and Simitses [25] compared the buckling loads using shear deformation theories and a classical theory to determine the range of cylinder thickness for which each was applicable. A higher-order shear deformation theory and a first-order shear deformation theory with and without a shear correction factor were considered as alternatives to the classical theory. Several laminates were considered as well as four different wall thicknesses. For
cylinders with radius-to-thickness ratios greater than 30, a classical solution appeared to be sufficient. A shear deformation theory was better suited for thicker cylinders (radius-to-thickness ratio less than 30). Tabiei and Simitses [26] expanded this analysis to compare the effect of using Sanders-type kinematic relations instead of Donnell-type relations. A greater difference between the two approaches was seen in shorter and thicker cylinders, with the critical loads using the Sanders-type relations always less than when the Donnell-type relations were used. Dong and Etitum [27] developed a three-dimensional elasticity solution based on Biot’s incremental theory to study stability, and used this to examine the effect of cylinder length and thickness, and the effect of the angle in two-layer angle-ply laminates. Then Etitum and Dong [28] used the same theory to compare results with a first-order shear deformation theory and a classical theory. Similar to the results seen in ref. [25], a classical theory was sufficient for thin cylinders that were neither extremely short nor long. Kim et al. [29] developed an elasticity solution, and made comparisons to the results from ref. [25]. For long and thin cylinders, there was good agreement between the elasticity solution and the shell theories, with the buckling loads from the shell theories slightly lower than the elasticity solution in most cases.

Within several of the studies of combined loads, comparisons were made between analytical and experimental results. Wilkins and Love [37] developed a combined loading test procedure for uniaxial compression and torsion of cylinders constructed of symmetric laminates. To determine the locations of the first buckle, Wilkins and Love used a lateral stiffness survey. Comparisons with an analytical solution were limited to single loading cases, and for pure torsion the difference between the predicted and measured critical values was significant. Imperfections were not taken into account for the analytical solution. Herakovich and Johnson [38] conducted a similar study by creating buckling interaction diagrams for symmetric and unsymmetric laminates. The predictions of the buckling loads were generated from a theory based on Flügge shell theory [45]. Reasonable agreement was seen between the experimental and theoretical results. Again initial imperfections were not included in the theoretical analysis.

In an effort to improve the agreement between the analytical results and experimental results, Booton and Tennyson [39] included an axisymmetric imperfection based on the buckling mode shape under axial compression. Three load conditions were considered in this study: (1) internal pressure and axial compression, (2) torsion and axial compression, and (3) internal pressure, torsion, and axial compression. When comparing the theoretical solution to the experimental results, the imperfection of the unloaded cylinder was measured, and the root mean square of the amplitude was used as the amplitude for the imper-
fection in the theoretical analysis. Shaw et al. [40] in a study of combined torsion and axial compression used two different imperfections, one an axisymmetric imperfection similar to the buckling mode shape in axial compression, and another based on the buckling mode shapes for torsion. The imperfection shape was not measured, but the amplitude was varied within the theoretical analysis in an effort to approximate the imperfection amplitude in the cylinders that were tested. As might be expected, agreement between theoretical and experimental results was better on one end of the interaction diagram than the other, depending on the shape of the theoretical imperfection. For instance, when a torsional buckling mode imperfection was used, the shape of the experimental interaction diagram was closer to the shape of the theoretical interaction diagram near the pure torsion end of the interaction diagram, and less agreement was seen near the pure axial compression end of the interaction diagram. A similar argument could be made for the axisymmetric imperfection and the better agreement with the experimental results near the end of the interaction diagram where there was pure axial compression.

Hilburger et al. [41], using a finite element analysis, developed interaction curves for combined torsion and axial compression. The imperfections of the cylinders were measured using a coordinate measurement machine, and the measured imperfection was used as one of the imperfections in the model. Because of the similarity of the measured imperfections for different cylinder specimens, a mean imperfection was also calculated. Interaction diagrams were generated for geometrically perfect cylinders, cylinders with the measured imperfection, cylinders with the mean imperfection plus or minus one standard deviation, and cylinders with an imperfection based on the buckling mode shape from a linear bifurcation analysis. While experimental data was only available for pure axial compression, the eigenmode imperfection produced an ultraconservative prediction of the critical load. Better agreement was seen when the experimental value was compared to both the mean imperfection plus one standard deviation and the measured imperfection.

Bisagni and Cordisco [42] examined the postbuckling response of circular composite cylinders under combined loads. An experimental study of the response of cylinders under combined torsion and axial compression was carried out to determine the postbuckling strength capacity of the cylinders. Before loading the cylinders, the imperfection was measured. This study was the only one to consider the postbuckling response of circular composite cylinders. For the single cylinder geometry and the depth into the postbuckling range that was considered, there was no increase in the torque within the postbuckling range as was seen for certain geometries for the isotropic cylinders.
1.1.3 Noncircular Cylinders under Torsion

There has been very little work that considers the elastic buckling response of oval or elliptical cylinders, composite or otherwise, under a torsional load, as can be seen by looking at the references from a review such as the one by Soldatos [46]. One of the earliest studies was a strength test by Lundquist and Burke [47] in 1935. This was an experimental analysis of the failure loads under torsion, bending, and combined torsion and bending of duralumin cylinders with an elliptical cross section. The cylinder wall was thin and imperfect, so the cylinders buckled before material failure occurred, and failure was defined to occur when the cylinder lost its load carrying capacity. As the elliptical cylinders were being loaded, it was observed that wrinkles first appeared near the end of the minor axis, i.e., the flatter regions of the cross section (see Figure 1.1b), and as the applied load was increased these wrinkles spread around the circumference of the cylinder. It was also observed, when comparing the results with the results from ref. [2], that the shear stress at the failure load was similar to the shearing stress for a circumscribed circular cylinder with the same length and wall thickness. However, the shear stress in the elliptical cylinders was determined from a membrane analogy and was considered the average stress.

The remaining studies on the response of elliptical cylinders under a torsional load were analytical studies. Kozarov and Mladenov [48] developed a theoretical approach for determining the critical torque for isotropic elliptical cylinder. Parbery and Karihaloo [49] showed that the minimum-weight cylinder that is optimized for torsional and bending stiffness has a cross-sectional shape that is very close to an ellipse. Though not capable of predicting critical conditions, Kardomateas [50, 51] developed a linear elasticity solution for thick filament-wound anisotropic elliptical cylinders. A generalized plane-strain assumption was made, as any variation in the stresses along the cylinder axis was neglected. In ref. [50] solutions were obtained for orthotropic cylinders, while in ref. [51] laminates with different ply angles were considered. The solution at a given cross section was used to analyze the variation in transverse shear stresses and axial warping around the circumference and through the thickness.

There appear to be no definitive references on the post-buckling behavior of noncircular cylinders under torsion. As stated earlier, non-circular geometries are not particularly suited for torsional loads, which may account for the lack of studies. However, as also stated previously and as reported in some of the references cited above which addressed circular cylinders, a torsional load could be one component of a combined load condition and could represent an extreme condition on an interaction diagram. There is therefore
merit in considering torsional loading of non-circular cylinders. Furthermore, within the context of today’s high-performance materials, there is merit in considering non-circular cylinders constructed of laminated fiber-reinforced composite materials. Haynie et al. [52] studied the responses of a circular cylinder and an elliptical cylinder with an aspect ratio of 0.70 under a torsional load using a finite-element analysis based on STAGS. The circular and elliptical cylinders had the same circumference and length, and were constructed from a graphite-epoxy material with a quasi-isotropic lamination sequence. The torsional load was created by applying a known twist angle to one end of the cylinder, so one end rotated about the cylinder axis relative to the other. Geometric imperfections in the form of the first few buckling mode shapes were included and the focus was on the predicted failure initiation and its subsequent progression as the twist angle was increased past the value to cause buckling. For increasing twist, wrinkling developed in the cylinder wall as observed by Lundquist and Burke [47]. In the elliptical cylinder these wrinkles first appeared in the flatter regions of the cross section, at midlength, near the ends of the minor diameter, and for higher levels of twist wrinkling progressed to the more curved regions of the cross section near the ends of the major diameter. Consequently, initial material failure in the elliptical cylinder occurred at a midlength location in the wrinkles in the flatter region of the cross section, and the damage progressed to the more highly curved regions with increasing twist. By comparison, the deformed cylinder wall of the circular cylinder had wrinkles that were distributed more uniformly around the entire circumference. The initiation and progression of material failure in the circular cylinder also occurred uniformly around the circumference, with any lack of uniformity attributed to the initial geometric imperfection.

In order to have a clearer picture of the role of cylinder geometry on cylinder behavior, Haynie and Hyer [53] expanded on their earlier finite element analyses [52] by investigating quasi-isotropic fiber reinforced composite cylinders with no geometric imperfections. Also, so-called small and large cylinders were considered to determine the effect of overall cylinder size on behavior. Circular cylinders and elliptical cylinders with an aspect ratio of 0.70 were again considered. The torsional loading was again applied by twisting one end of the cylinder relative to the other. The character of the torque vs. twist relation as the postbuckling response began depended on cross sectional aspect ratio and cylinder size. Also since they were not influenced by any \textit{a priori} imperfection shape, the deformation patterns in the postbuckling range of response of some cylinders experienced sudden changes, accompanied by drops in torque. These changes in deformation pattern were studied by considering a dynamic analysis. Interestingly, when material failure and progressive damage was included for those cylinders, the sudden changes in the deformation
Chapter 1

Introduction

pattern did not occur. For the smaller elliptical cylinders initial failure began at specific circumferential locations in the flatter regions of the cross section in the wrinkles of the postbuckled deformations, near the midlength of the cylinder. For the larger cylinders, initial failure occurred near the ends of the cylinders, not at the midlength, in the wrinkles of the postbuckled deformations, but not necessarily in the flatter region of the cross section.

1.2 Statement of Objectives

The present study continues the investigation into the response of elliptical composite cylinders under a torsional load. The study expands on the earlier work of Haynie et al. [52] and Haynie and Hyer [53], and as with the latter study, in order to have a clearer picture of the role of cylinder geometry on the response, geometric imperfections are not included, and only a quasi-isotropic lamination sequence is considered. As a loss in load capacity occurs with a noncircular cross section when compared to a circular one, so too is there a loss in load capacity when imperfect cylinders are considered. How much of a loss in load capacity can depend on the type and form of the imperfection. The type imperfection included and the particular form assumed can strongly influence predicted results, so to serve as a baseline it is important to know the response of perfect cylinders. In addition to the circular cylinder and an elliptical cylinder with an aspect ratio of 0.70, cylinders with aspect ratios of 0.85 and 0.55 will also be considered. The cross-sectional geometry of each cylinder is determined so the circumference of the cylinders is the same for each aspect ratio. As with the more recent study [53], two cylinder sizes, referred to here simply as small cylinders and large cylinders, will be considered.

In summary, the objectives of this study are:

– To examine the effect of the aspect ratio of the noncircular geometry on the buckling and postbuckling response of geometrically perfect quasi-isotropic composite cylinders under torsion. Specifically, aspect ratios of 0.55, 0.70, 0.85, and 1.00 are of interest, the last case being an equivalent circular cylinder. Included here is the analysis of the prebuckling deformations, and how they affect the level at which buckling occurs.

– To determine how the response changes when failure initiation and progression are considered. Failure initiation and progression will be based on the maximum stress failure criterion and a scheme for degrading material properties due to failure.
– To consider the influence of a larger scale on cylinder response. Here the circumference and length of the cylinders will be increased by a factor of five, and the cylinder wall by a factor of two.

Due to the anisotropy of the cylinders, and since it has been observed in the previous studies [52, 53], the effect of the direction of the applied twist will be considered as part of the above objectives. Also, regarding the discussion of the choice of cylinder length-to-radius ratio, two different lengths will be considered for the elliptical cylinders, and the difference in the response due to cylinder length will also be considered.

1.3 Overview of Remaining Chapters

In the chapter to follow, a description of the specifics of the problem investigated here is presented. The geometry of the cylinders that are studied is introduced. Then the specific geometries of the cylinders are defined, and the material properties of the lamina from which the cylinders are constructed are given. The chapter concludes with a description of the finite element model.

The important issues involved with obtaining credible numerical results are discussed in Chapter 3. As one end of the cylinder is rotated relative to the other end, a component of displacement of the cylinder wall is a rigid-body component. So Chapter 3 begins with a description of the approach that was used to subtract rigid-body rotation from the total displacement, so the elastic deformation of the cylinder wall can be more clearly seen. The remainder of the chapter is used to describe the numerical issues involved with determining the response of the cylinders, and to validate the results. A mesh refinement study is presented, and the effect on the predicted results of the twist rate during a transient analysis is examined. Validation includes comparing the initial slope of the torque-twist relation and the buckling conditions to values determined from theoretical and alternative numerical approaches. The chapter concludes by explaining how the lengths of the cylinders to be studied in depth are determined.

The prebuckling, buckling, and postbuckling responses of the small and large cylinders are presented in Chapters 4 and 5, respectively. The postbuckling responses are discussed when material failure is not considered, and then when material failure is. Finally, conclusions that can be drawn from the presented results are discussed in Chapter 6.
CHAPTER 2

DESCRIPTION OF PROBLEM

A description of the problem is presented in this chapter. In describing the problem, the geometry of the cylinders and the boundary conditions are discussed. Then the specific geometries of the cylinders that are studied are defined. Next, the material properties are described as well as the failure criterion used to predict failure. The chapter concludes with a description of the finite element model used to obtain numerical results.

2.1 Description of Geometry

The geometry of the cylinders being studied is described in Figure 2.1. The global coordinate system (X, Y, Z), with its origin at the geometric center of the cylinder, is identified. The X-axis will be referred to as the cylinder axis, and hence \(x\) is the axial coordinate. The arclength coordinate \(s\) is measured from the crown of the cylinder, as shown. As mentioned, the crown and keel are the flatter portions of the cylinder cross section, and through those portions the \(+Z\)-axis and \(−Z\)-axis extend, respectively. The \(±Y\)-axes extend through the sides of the cylinder, the more highly curved regions of the cross section. The cross-sectional geometry of the cylinder is described by the major diameter \(2a\) and the minor diameter \(2b\). The circumference of the cylinder is denoted as \(C\). The cylinder has a length \(L\) and wall thickness \(H\). Not shown is the limiting case of a circular cylinder when \(a = b = R\), where \(R\) is the radius of the circular cylinder. The angle between the direction of the fibers in any layer and the \(+X\)-axis is defined as \(θ\), and the positive direction for \(θ\) is shown.

The cylinder end at \(x = −L/2\) is clamped, while the end at \(x = +L/2\) is subjected to the rotation, or twist angle, \(φ\) about the cylindrical axis, for which the positive sense is identified in the figure. The positive sense of the torque \(T\) required to produce the applied rotation \(φ\) is also shown.
Figure 2.1. Cylinder geometry and coordinate system

The major and minor radii, $a$ and $b$, for each of the cylindrical cross sections considered in this study are determined under the constraint that the circumference $C$ be the same for cylinders with different aspect ratios. The equation for the circumference of an ellipse (See Appendix A) is

$$C = \int_{0}^{2\pi} \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} \, dt,$$

where $t$ is a parameter with no physical meaning. Note that for a circular cylinder when $a = b = R = \text{constant}$, Equation (2.1) reduces to $C = 2\pi R$.

For an elliptical cross section, which has a radius of curvature that varies with circumferential position, the maximum and minimum radii of curvature occur at the ends of the minor and major diameters, respectively, and are given by the relationships

$$R_{\text{max}} = \frac{a^2}{b} \quad \text{and} \quad R_{\text{min}} = \frac{b^2}{a}. \quad (2.2)$$

The expression for the radius of curvature for any $t$ is given in Appendix A when presenting the parameters used to describe a surface for an elliptical cylinder. Also in Appendix A are
the governing equations for the Donnell-Mushtari-Vlasov (DMV) shell theory as presented by Brush and Almroth [54] and Sanders [55], and adapted for an elliptical cylinder. The governing equations are included only as a reference, because a finite element analysis is the only approach used to analyze the response of the cylinders in this study. The governing equations as written in Appendix A will not be solved in any form.

2.2 Specific Cylinder Geometries Studied

As mentioned in the Introduction, two overall sizes, so-called small and large cylinders, are considered in this study. The small circular cylinder, with a radius of 4.28 in. and length of 12.85 in. is considered the base geometry. This radius is chosen because the circumference of this circular cross section is the same as the circumference of the elliptical cylinder with an aspect ratio of 0.70 that was used in experimental studies for different load conditions [56, 57]. Using Equation (2.1), and the circumference of the circular cylinder, \( C = 26.9 \) in., the cross-sectional geometries of the elliptical cylinders are determined so all cylinders have the same circumference. Also as mentioned earlier, three different elliptical cross sections, with aspect ratios of 0.85, 0.70, and 0.55, are investigated. Two different lengths are considered for each of the elliptical cylinders. The length of the circular cylinder is used for one length, and these cylinders are referred to as constant-length cylinders. The other length is determined from the sensitivity of the critical twist to changes in \( L/R \), and are referred to as constant-twist-sensitivity cylinders. How these lengths are found is discussed section 3.7.1, and will not be addressed here. The major and minor radii and lengths for all of the small cylinders investigated are shown in Table 2.1. Also tabulated for reference purposes are the minimum and maximum radii of curvature, \( R_{\text{min}} \) and \( R_{\text{max}} \), the minimum and maximum radius-to-thickness ratios, \( R_{\text{min}}/H \) and \( R_{\text{max}}/H \), and the length-to-radius ratios, \( L/R_{\text{min}} \) and \( L/R_{\text{max}} \). Note for the circular cylinder, \( R = R_{\text{min}} = R_{\text{max}} \) so the values are only listed once in Table 2.1. The seven geometries are depicted in Figure 2.2. The circular cylinder is the first cylinder shown (Figure 2.2a), the constant-length cylinders are on the left (Figures 2.2b, 2.2d, and 2.2f), and the constant-twist-sensitivity cylinders on the right (Figures 2.2c, 2.2e, and 2.2g). The wall of the small cylinder is assumed to be constructed of eight layers, each of thickness \( h = 0.0055 \) in., for a total thickness \( H = 0.044 \) in.

In addition to the constant-length and constant-twist-sensitivity cylinders, a third alternative for the lengths of the elliptical cylinders was discussed in the Introduction. It was to use the same value of the length-to-radius ratio to determine the length of the cylinder for
Table 2.1. Geometries of small cylinders

<table>
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<tr>
<th>$\frac{b}{a}$</th>
<th>$a$ [in]</th>
<th>$b$ [in]</th>
<th>$L$ [in]</th>
<th>$R_{\text{max}}$ [in]</th>
<th>$R_{\text{min}}$ [in]</th>
<th>$\frac{R_{\text{max}}}{H}$</th>
<th>$\frac{R_{\text{min}}}{H}$</th>
<th>$\frac{L}{R_{\text{max}}}$</th>
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<td>2.65</td>
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</tr>
<tr>
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<td>55.7</td>
<td>1.800</td>
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<td></td>
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each elliptical cross section. Since the maximum radius of curvature is used to calculate the length-to-radius ratio of the elliptical cylinders, the length of the elliptical cylinders with aspect ratios of 0.85, 0.70 and 0.55 would be approximately 1.3, 1.7 and 2.3 times, respectively, the length of the circular cylinder. Mainly because of the large increase in length with respect to the constant-length cylinders, and specifically for the cylinder with an aspect ratio of 0.55, cylinders with the same length-to-radius ratio are not considered in this study.

The geometry of the large cylinders is determined by increasing the cross-sectional dimensions and length of the small cylinders by a factor of five, and the wall thickness by a factor of two. The radii and lengths for the large cylinders are given in Table 2.2. The wall thickness for the large cylinders is achieved by doubling the number of plies in the laminate, and the total wall thickness for the large cylinders is $H = 0.088$ in. To illustrate the difference in size, the small and large circular cylinders are shown to scale in Figure 2.3.

Table 2.2. Geometries of large cylinders

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<thead>
<tr>
<th>$\frac{b}{a}$</th>
<th>$a$ [in]</th>
<th>$b$ [in]</th>
<th>$L$ [in]</th>
<th>$R_{\text{max}}$ [in]</th>
<th>$R_{\text{min}}$ [in]</th>
<th>$\frac{R_{\text{max}}}{H}$</th>
<th>$\frac{R_{\text{min}}}{H}$</th>
<th>$\frac{L}{R_{\text{max}}}$</th>
<th>$\frac{L}{R_{\text{min}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>21.4</td>
<td>21.4</td>
<td>64.2</td>
<td>21.4</td>
<td>243</td>
<td>3.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.85</td>
<td>23.1</td>
<td>19.64</td>
<td>64.2</td>
<td>27.2</td>
<td>16.70</td>
<td>309</td>
<td>190</td>
<td>2.36</td>
<td>3.85</td>
</tr>
<tr>
<td>0.70</td>
<td>25.0</td>
<td>17.50</td>
<td>64.2</td>
<td>35.7</td>
<td>12.25</td>
<td>406</td>
<td>139</td>
<td>1.798</td>
<td>5.24</td>
</tr>
<tr>
<td>0.55</td>
<td>27.1</td>
<td>14.89</td>
<td>64.2</td>
<td>49.2</td>
<td>8.19</td>
<td>559</td>
<td>93.0</td>
<td>1.306</td>
<td>7.84</td>
</tr>
</tbody>
</table>
(a) $b/a = 1.00$, Length = 12.85 in

(b) $b/a = 0.85$, Length = 12.85 in

(c) $b/a = 0.85$, Length = 14.42 in

(d) $b/a = 0.70$, Length = 12.85 in

(e) $b/a = 0.70$, Length = 16.29 in

(f) $b/a = 0.55$, Length = 12.85 in

(g) $b/a = 0.55$, Length = 18.86 in

Figure 2.2. Geometries of small cylinders
Figure 2.3. Comparison of small and large circular cylinders
2.3 Material Properties

All cylinders are assumed to be fabricated from a medium-modulus graphite-epoxy composite material, and, as mentioned in the Introduction, only a quasi-isotropic lamination sequence is considered. For the small cylinders the specific lamination sequence is \([\pm 45/0/90]_S\) and for the large cylinders the sequence is \([\pm 45/0/90]_{2S}\). The material properties of a layer of the graphite-epoxy material are given in Table 2.3, where the standard subscript notation is used for the engineering properties in the principal material directions, i.e., the 1-direction is the fiber direction and the 2-direction perpendicular to the fiber direction. It can be seen that the wall thicknesses of the cylinders given in the previous section were determined by multiplying the thickness of a single layer, \(h\), by the number of layers in the laminate, e.g., the wall thickness for the 8-layer laminate for the small cylinders is 0.044 in.

Table 2.3. Properties of a layer of graphite-epoxy

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_1)</td>
<td>18.85 Msi</td>
</tr>
<tr>
<td>(E_2)</td>
<td>1.407 Msi</td>
</tr>
<tr>
<td>(G_{12})</td>
<td>0.725 Msi</td>
</tr>
<tr>
<td>(\nu_{12})</td>
<td>0.3</td>
</tr>
<tr>
<td>(h)</td>
<td>0.0055 in.</td>
</tr>
<tr>
<td>(X_t)</td>
<td>(215 \times 10^3) psi</td>
</tr>
<tr>
<td>(X_c)</td>
<td>(180 \times 10^3) psi</td>
</tr>
<tr>
<td>(Y_t)</td>
<td>(7.25 \times 10^3) psi</td>
</tr>
<tr>
<td>(Y_c)</td>
<td>(29 \times 10^3) psi</td>
</tr>
<tr>
<td>(S_{12})</td>
<td>(14.5 \times 10^3) psi</td>
</tr>
</tbody>
</table>

The maximum stress failure criterion is used to predict material failure. Only intralaminar failures are considered. There are five assumed failure modes: tensile or compressive matrix failure, tensile or compressive fiber failure, and in-plane shear failure. The terms matrix, fiber, and shear failure are generalized terms used in STAGS, and adopted in this study, to denote the direction or component of stress that causes material failure. A matrix failure is a failure in the direction perpendicular to the fiber direction due to tensile or compressive failure of the matrix or the bond at the fiber-matrix interface or the fiber. A fiber failure is when the failure is in the direction parallel to the fiber direction due to tensile or compressive failure of the fiber. Shear failure is due to matrix or fiber-matrix
interface shear failure, or fiber shear failure in the plane of the layer. The stresses for the five intralaminar modes of failure in the principal material coordinate system are denoted as:

\[ X_t = \text{tensile failure stress in the fiber direction} \]
\[ X_c = \text{compressive failure stress in the fiber direction (a positive number)} \]
\[ Y_t = \text{tensile failure stress perpendicular to the fiber direction} \]
\[ Y_c = \text{compressive failure stress perpendicular to the fiber direction (a positive number)} \]
\[ S_{12} = \text{intralaminar shear failure stress} \]

The numerical values for these failure stress levels were included in Table 2.3. The compressive failure stresses are positive values because of the manner in which STAGS applies the criterion and distinguishes between tensile and compressive failures. Within STAGS the failure criterion is applied at points within each layer of every element. How the maximum stress failure criterion is implemented as part of a progressive failure analysis within STAGS is discussed in section 3.3.

### 2.4 Description of Finite Element Model

A representation of the finite element model for all cylinder geometries is shown in Figure 2.4. The model is comprised of two types of elements. A four-noded quadrilateral 410 shell element is used for the surface of the cylinder. This element enforces the Kirchhoff hypothesis and employs the nonlinear Lagrangian strain tensor. Each node has six degrees of freedom, three translational degrees of freedom and three rotational degrees of freedom, one of which is the drilling degree of freedom, which is the rotation about the normal to the element surface. Except for different mesh densities used in the the mesh refinement study, the mesh for the circular cylinder has 200 elements in the circumferential direction and 96 elements in the axial direction. Details of the meshes used in the mesh refinement study are given in section 3.4. For all meshes used the element length in the axial direction is approximately the same as the element length in the circumferential direction. The finite element meshes for the elliptical cylinders have the same number of circumferential elements as the circular cylinder, and when different lengths of elliptical cylinders are considered, the number of elements in the axial direction is adjusted so that the element length does not change. The element length in the circumferential direction of the elliptical cylinders is
uniform around the entire circumference. The second element type in the model is a rigid link element, or the E120 element in the STAGS nomenclature. This element, through a rigid-link constraint enforced using Lagrange multipliers, makes the displacements of one end of the link dependent on the displacements and rotations of the other end.

For the end of the cylinder at $x = -L/2$ all degrees of freedom are restrained from translating or rotating. The rigid links are used to enforce the boundary conditions on the other end of the cylinder, where the twist angle is applied. At $x = +L/2$ each node is attached to a rigid link and the other end of the rigid link is attached to a common node that lies on the X-axis and in the plane of the end of the cylinder. Therefore all rigid links also lie in the plane of the end of the cylinder. The common node is free to move along the X-axis and rotate about the X-axis, but is constrained from translating or rotating in any other direction. The rigid links are constrained to rotate with the common node, and the nodes of the shell elements are prevented from rotating with respect to the rigid links. Therefore, when an angle of rotation about the cylinder axis, or twist angle, is applied at the common node, the rigid links rotate and the end cross section of the cylinder rotates as a rigid ellipse about the cylinder axis. The rigid links also enforce a clamped boundary condition.
condition with respect to bending in the direction of the cylinder axis. The torque required to produce the applied twist angle is computed by STAGS at the common node.

The cylinder geometries and the material properties were defined in this chapter. Also, the finite element model was described. Now, the approach used to analyze the cylinders is described in the following chapter, and the preliminary results used to validate the finite element analysis are presented.
CHAPTER 3

DETAILS OF FINITE ELEMENT PROCEDURE AND VALIDATION OF SOLUTION

The foundation for the solution of the problem is laid out in this chapter. First, the approach used to remove the displacements due to the rigid-body rotation of the cross section is described. This procedure helps evaluate the elastic component of displacement of the cylinder, the component of interest. Second, the approach used in the finite element analysis is described. This is followed by a discussion of the progressive failure analysis. Next, results are presented for a mesh refinement study used to validate that the mesh density used to compute results presented in Chapters 4 and 5 is sufficient. Results are then presented to show that the twist rates employed in the transient analysis, which are used to predict the postbuckling response discussed in Chapters 4 and 5, do not have a significant effect on the response. Then to validate the results of the finite element analyses, two comparisons are made to results from other formulations of the problem. In one, the slope of the initial torque-twist response is compared to the slope calculated using a strength-of-materials approach; and in the other the critical twist angles for circular cylinders are compared to values predicted using a formulation of Batdorf [9]. Finally, results from a study on the effect of cylinder length on the critical twist are presented for both the small and the large cylinders to show how the lengths of the cylinders presented in Tables 2.1 and 2.2 were chosen. Note that some of the results presented in this chapter include the postbuckling responses of cylinders. However, since the postbuckling responses of the cylinders will be discussed in detail in Chapters 4 and 5, the postbuckling responses presented in this chapter will not be described except when needed to describe differences in responses regarding different mesh densities or twist rates.
3.1 Displacement Adjustment due to Rotation of the Elliptical Cross Section

The displacements in STAGS are reported relative to the coordinate systems of the undeformed cylinder. For cylinders under an applied twist as are considered in this study, this means that for a given axial location along the length of the cylinder, the rigid-body rotation of the cross section is included in the displacements. As will be shown, a better understanding of the elastic deformation of the cylinder wall under a torsional load is possible when the displacements due to the rigid-body rotation of the cross section are removed from the total displacements. Before showing an example to illustrate the influence of rigid-body displacements on the total displacements, the approach that is used to subtract the displacements due to the rigid-body rotation of the cross section from the total displacements will be described.

A cross section of the mid-surface of the elliptical cylinder shown in Figure 2.1, at an arbitrary axial location \( x_c \) along the length of the cylinder, is shown in Figure 3.1. The global coordinate system \((Y, Z)\) and the circumferential arclength coordinate \( s \) are the same as previously shown in Figure 2.1. Also shown for an arbitrary point A, on the mid-surface of the cylinder, is a local coordinate system \((y_n, z_n)\). This local system is a Cartesian coordinate system defined such that \( y_n \) is tangent to the mid-surface and \( z_n \) is normal to the mid-surface at point A. A Cartesian system is used for the local coordinate system because the local nodal coordinate system in STAGS is a Cartesian coordinate system. Not shown is a local global coordinate system at point A, \((y_g, z_g)\), which is obtained by translating the origin of the global system to point A. The vector \( r_0 \) is the position vector of point A for cross section in its original position, before any twist is applied. The \( Y \)- and \( Z \)-components of the position vector \( r \) can be defined using parametric equations, so \( r_0 \) is a function of the parameter \( t \), and can be written as (see Equation (A.2) in Appendix A)

\[
r_0(t) = a \sin t \hat{e}_2 + b \cos t \hat{e}_3, \quad 0 \leq t \leq 2\pi.
\]

When a twist angle \( \phi \) is applied at \( x/L = +0.5 \), the angle of rotation, \( \phi_c \), of the cross section at \( x_c \), is

\[
\phi_c = \phi \left( \frac{x_c}{L} + \frac{1}{2} \right), \quad -\frac{L}{2} \leq x_c \leq +\frac{L}{2}.
\]

The rotation of the cross section in Figure 3.1 is shown in Figure 3.2 for a rotation of \( -\phi_c \). Point A moves to position A’. For the sake of the discussion here, a rotated global coordinate system \((Y', Z')\) is defined for the rotated cross section, and the displacement vector \( v_{rot} \) between points A and A’ is shown. Only the \( v \) and \( w \) components of the displacement
vector are present in $\mathbf{v}_{\text{rot}}$, because the $u$ component is assumed to not influence the rotation of the cross section. The position vector for the rotated, but not elastically deformed, cross section in $(Y',Z')$ coordinates, $\mathbf{r}'_1$, can be defined using the same parametric equations as in Equation (3.1). When expressed in the unrotated coordinate system, $\mathbf{r}'_1$ becomes $\mathbf{r}_1$. Subtracting the original position vector $\mathbf{r}_0$ from the transformed position vector of the rotated cross section $\mathbf{r}_1$ yields the displacement vector for the rotation of the cross section in local global coordinates $\mathbf{v}_g$ which can be expressed as

$$\mathbf{v}_g = \mathbf{r}_1 - \mathbf{r}_0.$$  \hspace{1cm} (3.3)

Using the angle $\psi$, $\mathbf{v}_g$ is transformed to the local coordinate system $(y_n, z_n)$. The angle
ψ is the angle between the local coordinate system and the local global coordinate system, and is depicted in Figure 3.3. In the local coordinate system, the displacement vector due to the rotation of the cross section is denoted by $v_{\text{rot}}$. The deformation vector $v$ is found by subtracting $v_{\text{rot}}$ from the total displacement of the cross section, and can be written as

$$v = v_{\text{total}} - v_{\text{rot}}.$$  \hspace{1cm} (3.4)

The deformation vector $v$ is written without a subscript because most of the discussion of displacements will refer to the elastic deformation of the cylinder wall, specifically the normal deformation $w$, so mainly for simplicity in later discussions no descriptive subscript is used for the elastic deformations.

![Figure 3.3. Angle between local and local global coordinate systems](image)

The normal displacements before and after subtracting the displacements due to the rotation of the cross section are shown in Figure 3.4 in the prebuckling range for a small constant-length elliptical cylinder with an aspect ratio of 0.70 and a positive twist. The displacements have been normalized by the cylinder wall thickness $H$, the axial location has been normalized by the cylinder length $L$, and the arc length location has been normalized by the cylinder circumference $C$. The surface of the cylinder has been unrolled, a format that will be used throughout this dissertation. The applied twist angle for which the displacements are shown is about 90% of the critical twist angle for this cylinder and twist direction. In Figure 3.4a, the total normal displacements are shown. These are the displacements before the displacements due to the rigid-body rotation of the cross section have been subtracted. The displacements due to the rigid-body rotation of the cross-section are shown in Figure 3.4b. The elastic deformations of the cross section are shown in Fig-
Note that the scale for the color contours in Figure 3.4c is significantly smaller than the scales in Figures 3.4a and 3.4b. By noting the similarities of Figures 3.4a and 3.4b, it is seen that in the prebuckling range, the displacements due to rotation are such a large part of the total displacements that the normal elastic deformations are almost completely masked by them. For example, near the cylinder midlength, i.e., $x/L = 0$, and near the crown ($s/C = 0$) and keel ($s/C = 0.5$) circumferential wrinkle-like patterns are visible in Figure 3.4c. These deformations are not fully formed wrinkles, but the displacements due to the rotation of the cross section are so large compared to the amplitude of these deformations that there is no indication that wrinkling is starting to occur when looking at the total displacements. Also, as observed in Figure 3.4c, there are slight inward and outward bulges in the deformation pattern that occur near the quarter spans (i.e., $x/L = \pm 0.25$) and closer to the sides ($s/C = 0.25, 0.75$) then the crown or keel. The bulges are elastic deformations, and the magnitudes of these bulges are also so small compared to the displacements due to the rotation of the cross section that the bulges cannot be seen in the plot of total normal displacements. As the normal deformations continue to increase and the number and amplitude of the wrinkles in the cylinder wall increase, the elastic deformations will be a larger part of the total displacements; however, even in the postbuckling range the displacements due to the rotation of the cross section are large enough, especially near $x/L = +L/2$, to obscure some of the details in the deformation pattern from being seen in the total displacements. So most of the discussion of the deformation of the cylinder wall will be of the deformation after the rotation of the cross section has been subtracted from the total displacement.
Figure 3.4. Displacement adjustment due to rigid body rotation of elliptical cross section
3.2 Procedure to Predict the Postbuckling Response

STAGS is a nonlinear finite element code developed for the analysis of shell structures. The capabilities of STAGS that are utilized for this study include a linear buckling analysis, geometrically nonlinear static analyses, and geometrically nonlinear transient stress analyses. The transient analysis capability in STAGS was used extensively for computing postbuckling response. The approach used to predict the postbuckling response of the cylinders is illustrated schematically using a portion of a hypothetical torque-twist angle relation, i.e., a $T$ vs. $\phi$ relation, in Figure 3.5. Initially, a geometrically nonlinear static analysis is used to determine the prebuckling response of the cylinder. The twist angle is increased until the cylinder becomes unstable. An unstable configuration in STAGS is recognized by negative roots appearing in the tangent stiffness matrix, which indicates there are multiple equilibrium solutions, i.e., equilibrium configurations, for this applied twist angle. Once this critical twist angle has been reached, the analysis is continued to a twist angle value slightly greater than the critical value. The increase in twist angle past the critical value is generally no more than 0.5% of the critical value. At this increased twist angle the static analysis is stopped, and the cylinder is in an unstable equilibrium configuration.

![Figure 3.5. Schematic of approach to move from unstable primary branch to stable postbuckling branch](image-url)

*Figure 3.5. Schematic of approach to move from unstable primary branch to stable postbuckling branch*
From this increased twist angle, with the applied twist angle held constant, a geometrically nonlinear transient restart with a variable time step is used to move the cylinder from the unstable equilibrium configuration to a stable equilibrium configuration. At the start, when the time step is very small, the cylinder will remain in the unstable configuration. Because of the ease in which convergence is initially obtained, the time step will increase. When the time step becomes large enough, the cylinder will begin to change configuration. As the cylinder changes configuration, recognized by the reported kinetic energy increasing several orders of magnitude, the time step will no longer increase and will sometimes decrease due to convergence difficulties. The change in configuration is accompanied by a slight drop in torque. Proportional damping is used in the transient analyses so the cylinder will settle into the new stable configuration. The cylinder is considered settled when the kinetic energy has decreased to a level near the level before the change in configuration. Another indication that the cylinder has settled is that the time step will again increase, because in this settled state the difference in the configuration between successive solution steps is small and convergence is easily obtained.

Once the cylinder has settled into the new configuration, this transient analysis is stopped, and another nonlinear transient analysis is started. In this restart, the time step is held constant and the twist angle is increased linearly over time. The rate at which the twist angle is increased will be discussed in more detail in section 3.5. With the twist angle increasing, a postbuckling equilibrium path can then be followed. In most of the cylinders in this study, instabilities also occur in the postbuckling region. When an instability is encountered along the postbuckling path, the transient analysis is more suitable for determining the change from an unstable configuration to a stable configuration. The sudden drop in torque and postbuckling deformation pattern are easily accommodated by the transient analysis. The twist rate is kept the same for the entire range of the postbuckling response that is considered.

It should be mentioned that early in this study, static analyses were used in an attempt to move the solution from the primary prebuckling path to the postbuckling path. In many cases the static analysis required considerable fine-tuning as regards step sizes, restarts of the analysis, etc., and even then using a static analysis was not always successful. The dynamic approach was far less complicated and was successful in every case. In addition, use of a transient analysis simulates exactly the conditions when testing cylinders in the laboratory, i.e., time is always moving forward, even with a so-called static test.
3.3 Progressive Failure Analysis

The progressive failure analysis within STAGS is executed at the element level and within each layer of an element. The failure criterion is evaluated at different layer, or ply, points within each element. The locations of these points are determined to some extent by the location of the surface integration points, or Gauss points, in the plane of the element. The locations of these various points are shown in Figure 3.6. For the 410 shell element used in this model there are four surface integration points, as shown in Figure 3.6a. For a laminate, additional points are considered through the thickness of each layer, points where the stress will be calculated, and these points are called layer integration points, or ply points. The locations of the ply points within the plane of a layer are the same as the locations of the surface Gauss points. There are two ply points through the thickness of each layer, one near the top of the layer and one near the bottom. Specifically, the top and bottom ply points are a distance of \( \pm 1/\sqrt{3}(h_k/2) \) from the middle of the layer, here \( h_k \) refers to the thickness of layer \( k \). A representation of the shell element in Figure 3.6a is shown as a four-layer laminate in Figure 3.6b, and ply points through the thickness can be seen. However, because all of the ply points cannot be clearly seen through the thickness of the laminate, a single layer from the laminate is shown in Figure 3.6c, were the top and bottom ply points can be seen. Considering the small cylinders in this study, which have eight plies, there will be 16 points that are checked for each mode of failure at each of the four surface integration points. There are therefore a total of 64 ply points at which the stress will be calculated per element in the small cylinders. For the 16-ply large cylinders, there will be a total of 128 ply points per element.

An important point needs to be made regarding how STAGS counts the points that register failure within an element. As described above, there are two ply points through the thickness of a layer that are checked for failure. When counting the number of failures, only the initial failure through the thickness of a layer at each Gauss point location is counted. So the maximum possible failure count within an element is half of the total number of ply points. The maximum failure count in the eight-layer walls of the small cylinders is 32 per element, and for the 16-layer walls of the large cylinders is 64 per element.

Before calculating the stress state at the ply points, equilibrium is first established in the STAGS model. From the equilibrium state the stresses are calculated at each ply point in the model and the failure criterion is applied. Three stress components are needed to determine if failure has occurred at this point: the normal stress perpendicular to the fiber direction, the normal stress in the fiber direction, and the intralayer shear stress. As described in
section 2.3, these stresses are compared to the maximum failure stresses to determine if a tensile or compressive matrix failure, tensile or compressive fiber failure, or in-plane shear failure has occurred. If the maximum stress is exceeded, the material is degraded.

To degrade the material, the appropriate stiffnesses are multiplied by a degradation factor of 0.2 [58]. When failure due to excess stress perpendicular to the fiber direction is predicted, the modulus $E_2$ and shear modulus $G_{12}$ are multiplied by the degradation factor. At a point where excess shear stress is predicted, the shear modulus $G_{12}$ is multiplied by the degradation factor. Each of these two types of failures will be identified as matrix failures, and will be qualified as either tensile, compressive, or shear failures when discussed. The modulus $E_1$ and the shear modulus $G_{12}$ will be multiplied by the degradation factor when excess stress in the fiber direction, identified as a fiber failure, is predicted. When the moduli $E_1$ or $E_2$ are degraded, the appropriate Poisson’s ratio is also degraded so the reciprocity
relationship remains valid. After the material has been degraded, equilibrium is reestablished in the model, and STAGS proceeds to the next solution step to find the equilibrium conditions for the next level of twist angle $\phi$. In this study a recursive degradation scheme is used for all types of failures. Recursive degradation means that at any ply point where failure is detected for a given equilibrium solution step, the degradation factor is applied to the appropriate material properties at that point for the equilibrium solution step when failure is detected and applied again for each subsequent equilibrium solution step. With this scheme, the affected properties can become small fractions of their original values a few solution steps after failure is first detected.

To summarize the progressive failure analysis within STAGS, the first step is to determine equilibrium for the current applied twist angle and degraded material property state. Then, using this solution, the stresses are calculated at each ply point. The stress components are then checked against the failure criterion. If failure has occurred at more ply points, the appropriate stiffnesses are degraded at those points. Using the degraded material properties at the failed ply points, the constitutive matrix is recalculated for the surface integration point to which the failed ply points belong. Then, equilibrium is reestablished before proceeding to the next level of twist angle $\phi$.

### 3.4 Mesh Convergence Study

A mesh convergence study is performed using three different finite element mesh densities and two cylinder geometries. The finite element meshes have 100, 200 or 400 elements in the circumferential direction, and 48, 96, and 192 elements, respectively, in the axial direction. The meshes will be referred to by the number of elements in the circumferential direction, i.e., the mesh with 200 circumferential elements will be referred to as mesh-200. One of the cylinders is a large circular cylinder with a positive twist, and the other cylinder is a large constant-length elliptical cylinder with an aspect ratio of 0.55 and a negative twist. Material failure is considered for both geometries. The buckling and postbuckling responses are generated for both cylinders, and a postbuckling path is followed until the first fiber failures are detected. The circular cylinder is chosen because it is one of the cylinders that has the largest number of circumferential waves in the initial postbuckling configuration. The elliptical cylinder represents the most extreme noncircular cross section that is considered. And for the different sizes, lengths, and twist directions that are considered, this elliptical cylinder has the largest number of circumferential waves. The results that will be presented and discussed in this section are results that have not been
previously discussed in this study, and because this section focuses on the convergence of
the finite element mesh, the details and differences in the responses of the circular and el-
liptical cylinders will not be discussed here except when relevant to the discussion of mesh
convergence. More detailed discussions on the effect of an elliptical cross section on the
response of the cylinder are in Chapters 4 and 5.

The torque-twist relations for the two cylinder geometries are shown in Figures 3.7
and 3.8. Both the torque and the twist for a given mesh are normalized by the critical
values for the large circular cylinder with a positive twist for that mesh. As a matter of
fact, it is the normalized twist, twist angle per unit length of cylinder, that is plotted in
normalized form. Thus the normalization factor for the horizontal axis is $\phi^{cir+}_{cr} / L^{cir}$, where
‘cir’ refers to circle, the ‘+’ to positive twist, and, of course, ‘cr’ to critical. With the
normalization scheme used, the large circular cylinder buckles at normalized torque and
twist values of (1,1) as indicated in Figure 3.7. The critical twists and torques for the
circular cylinder are tabulated in Table 3.1 and for the elliptical cylinder in Table 3.2. Twist
and torque values when the first failures occurred, which are matrix failures, and when the
first fiber failures occurred are also shown. There is very good agreement between mesh-
200 and mesh-400 for all of the twists and torques considered and for both geometries.
Considering the responses shown in Figures 3.7 and 3.8, the responses for mesh-200 and
mesh-400 are almost indistinguishable over the entire range of twist. The critical values
for mesh-100 are within an acceptable range of the mesh-400 values; however, as can be
seen in Figures 3.7 and 3.8 the difference in the torque between mesh-100 and mesh-400
increases with increasing twist in the postbuckling range. For the large elliptical cylinder,
Figure 3.8, there is an instability in the postbuckling range near $\phi = 0.844$, resulting in a
sudden decrease in the torque. Instabilities such as this will be discussed in more detail in
Chapters 4 and 5.

A second point to make regarding the agreement of the torque and twist values is that
the tabulated values are also a measure of convergence with respect to material failure. In
this regard there is still very good agreement between mesh-200 and mesh-400. In terms of
the overall accumulation of damage in the circular cylinder, mesh-200 and mesh-400 each
have failures in 2.1% of the total number of ply points. Mesh-100 has failures in 2.4% of
the total number of ply points. The percentage of ply points that have failed in the elliptical
cylinder is 0.12% for all of the meshes.

Also important for the mesh convergence study are the predicted deformations of the
cylinder walls, in particular the waviness as shown in Figure 1.1, and in particular the
magnitudes of the outward (ridges) and inward (valleys) normal deformations within the wrinkles. The normal deformations around the circumference at the midlength for the circular cylinder are shown in Figure 3.9, and for the elliptical cylinder in Figure 3.10. As seen, at the twist where the first failures occur, there is very good agreement between mesh-200 and mesh-400 for both cylinders, and for each of the circumferential locations and magnitudes of the peak deformations. There is still good agreement between the two meshes in the deformations in the valleys at the twists where first fiber failures occur. As can be seen from the values in Table 3.1, there is a large percent difference between the deformations of the ridge in the circular cylinder at the twist where the first fiber failures occur; however, in terms of the magnitude of the difference between the meshes, it is less than the difference in $w/H$ for the adjacent valley (not shown), and because the deformation of the ridge is so close to zero, the percent difference is very large. The total normal displacements of a ridge and valley are shown over the entire range of twist in Figure 3.11 for the circular cylinder, and are shown for ridges and valleys near the ends of the major and minor axes of the elliptical cylinder in Figure 3.12. From these figures it can also be concluded that the results from mesh-200 and mesh-400 are in good agreement.
The computational times for each mesh and geometry are shown in Table 3.3. Note that when convergence difficulties occur, STAGS will cut the time step until a converged solution can be found. After the convergence difficulties are resolved, STAGS does not increase the time step, because a constant time step analysis is being used. The only way to continue at the original time step is to stop and then restart the analysis. Because of the length of time for these analyses, the restart is not always from the last saved step, but from a step in the middle of the analysis (and for a step after the convergence problems have been resolved). When this happens, steps are discarded. The computational time for these discarded steps is included in the times reported in Table 3.3. Because the computational time for a given step is less for a coarser mesh, typically the number of discarded steps for an analysis increases as the number of elements in the mesh decreases.

Based on the comparisons of the twist, torque, and deformations discussed above, the agreement between mesh-200 and mesh-400 is good. The computational savings when using mesh-200 rather than mesh-400 are significant. The computational time of mesh-200 for the circular cylinder is 35% of the time of mesh-400. For the elliptical cylinder, the time of mesh-200 is only 20% of the mesh-400 time. While the savings in computational time is even greater for mesh-100, the differences that are observed in the response relative
Table 3.1. Twist, torque, and deformations for different mesh densities, large circular cylinder

<table>
<thead>
<tr>
<th></th>
<th>mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
</tr>
<tr>
<td>Critical Twist ([\text{rad/in}] \times 10^5) ((% \Delta))(^a)</td>
<td>5.78 (2.7)</td>
</tr>
<tr>
<td>Critical Torque ([\text{in-lb}] \times 10^{-5})</td>
<td>8.79 (2.4)</td>
</tr>
<tr>
<td>First Failures</td>
<td></td>
</tr>
<tr>
<td>Twist ([\text{rad/in}] \times 10^5)</td>
<td>29.0 (−5.2)</td>
</tr>
<tr>
<td>Torque ([\text{in-lb}] \times 10^{-5})</td>
<td>6.52 (8.9)</td>
</tr>
<tr>
<td>(\frac{w}{H}) (ridge)</td>
<td>2.33 (4.5)</td>
</tr>
<tr>
<td>(\frac{w}{H}) (valley)</td>
<td>−14.60 (−6.4)</td>
</tr>
<tr>
<td>First Fiber Failures</td>
<td></td>
</tr>
<tr>
<td>Twist ([\text{rad/in}] \times 10^5)</td>
<td>69.6 (17.4)</td>
</tr>
<tr>
<td>Torque ([\text{in-lb}] \times 10^{-5})</td>
<td>5.80 (11.3)</td>
</tr>
<tr>
<td>(\frac{w}{H}) (ridge)</td>
<td>0.0023 (−99)</td>
</tr>
<tr>
<td>(\frac{w}{H}) (valley)</td>
<td>−24.9 (−8.7)</td>
</tr>
</tbody>
</table>

\(^a\) \(\% \Delta\) is the percent difference between the current value and the value for mesh-400.

\(^b\) At \(x/L = 0\) and considering the first ridge or valley that occurs for \(s/C \geq 0\).

to mesh-200 and mesh-400 are not acceptable. The difference in the computational time between mesh-200 and mesh-400 is large enough that the differences that are seen are considered acceptable, and the mesh with 200 circumferential elements is used for the remaining analyses in this study.
Table 3.2. Twist, torque, and deformations for different mesh densities, large $b/a = 0.55$ cylinder

<table>
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<tr>
<td>Critical Twist [rad/in]×$10^5$ (%$\Delta$)$^a$</td>
<td>5.53 (2.9)</td>
</tr>
<tr>
<td>Critical Torque [in-lb]×$10^{-5}$</td>
<td>5.53 (2.2)</td>
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</table>

<table>
<thead>
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<tbody>
<tr>
<td>Twist [rad/in]×$10^5$</td>
<td>16.55 (9.9)</td>
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<tr>
<td>Torque [in-lb]×$10^{-5}$</td>
<td>6.33 (5.7)</td>
</tr>
<tr>
<td>$\frac{w}{H}$ (ridge, $s/C \approx 0.25$)</td>
<td>1.986 (25)</td>
</tr>
<tr>
<td>$\frac{w}{H}$ (valley, $s/C \approx 0.31$)</td>
<td>-6.08 (5.7)</td>
</tr>
<tr>
<td>$\frac{w}{H}$ (ridge, $s/C \approx 0.5$)</td>
<td>4.17 (-4.1)</td>
</tr>
<tr>
<td>$\frac{w}{H}$ (valley, $s/C \approx 0.58$)</td>
<td>-12.14 (6.8)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>First Fiber Failures</th>
<th></th>
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</thead>
<tbody>
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<td>Twist [rad/in]×$10^5$</td>
<td>22.7 (4.6)</td>
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<tr>
<td>Torque [in-lb]×$10^{-5}$</td>
<td>5.54 (5.3)</td>
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<tr>
<td>$\frac{w}{H}$ (valley, $s/C \approx 0.25$)</td>
<td>-2.28 (8.6)</td>
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<tr>
<td>$\frac{w}{H}$ (ridge, $s/C \approx 0.29$)</td>
<td>1.169 (23)</td>
</tr>
<tr>
<td>$\frac{w}{H}$ (valley, $s/C \approx 0.5$)</td>
<td>-13.33 (1.8)</td>
</tr>
<tr>
<td>$\frac{w}{H}$ (ridge, $s/C \approx 0.58$)</td>
<td>4.04 (-6.7)</td>
</tr>
</tbody>
</table>

$^a$ $\%\Delta$ is the percent difference between the current value and the value for mesh-400.

Table 3.3. Computational time for different meshes and geometries

<table>
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<tr>
<th>mesh</th>
<th>circular cylinder</th>
<th>elliptical cylinder</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>$8.1 \times 10^4$</td>
<td>$1.28 \times 10^4$</td>
</tr>
<tr>
<td>200</td>
<td>$3.6 \times 10^5$</td>
<td>$4.5 \times 10^4$</td>
</tr>
<tr>
<td>400</td>
<td>$1.02 \times 10^6$</td>
<td>$2.5 \times 10^5$</td>
</tr>
</tbody>
</table>
Figure 3.9. Normal deformations around the circumference at $x/L = 0$ for large circular cylinder, positive twist, mesh convergence study
Figure 3.10. Normal deformations around the circumference at $x/L = 0$ for large constant-length $b/a = 0.55$ cylinder, negative twist, mesh convergence study
\[ \phi = \left| \phi \right| \frac{L}{L^*} \]

Figure 3.11. Normal displacements for a valley and ridge at \( x/L = 0 \) for large circular cylinder, positive twist, mesh convergence study
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Figure 3.12. Normal displacement of valleys and ridges at $x/L = 0$ for large constant-length $b/a = 0.55$ cylinder, negative twist, mesh convergence study.

(a) normal displacement near $y/C = 0.25$

(b) normal displacement near $y/C = 0.29$

(c) normal displacement near $y/C = 0.50$

(d) normal displacement near $y/C = 0.58$
3.5 Twist Rate Comparisons

The twist rate with the transient analysis used by Haynie and Hyer [53] was approximately one degree per minute (≈ 3 × 10^{-4} radians/second). This rate was chosen by them so the twist rate would be comparable to a rate that might be used in physical testing under quasi-static conditions. However, for a given twist rate, there is a maximum time step with which the STAGS finite element analysis can proceed and with which convergence can be obtained easily enough so that the time step will not be cut. For the twist rates used in the present study, the maximum time step is determined through trial and error. Despite thoughts to the contrary, the cross-sectional geometry does not have a significant effect on the maximum value. Due to both the length of time of the analyses and the amount of disk space required for the analysis of a single cylinder, it is desirable to use as fast a twist rate as possible to reduce computational time and disk space. To that end, for the small cylinders in this study, a twist rate of approximately four degrees per minute (≈ 1.2 × 10^{-3} radians/second) is used. For the large cylinders, the twist rate is chosen so that at \( x/L = +L/2 \) the velocity of a point on the surface of a large circular cylinder is the same as the velocity of a point on the surface of a small circular cylinder. Using the relation \( \dot{v} = R\dot{\phi} \), where \( \dot{v} \) is the tangential velocity of a point on the surface of the circular cross section of radius \( R \) and \( \dot{\phi} \) is the twist rate, the twist rate of the large cylinder should be five times less than the twist rate of the small cylinders. Again, to save computational time and disk space, a twist rate only 2.5 times less than the twist rate for the small cylinders is used. A twist rate of 1.6 degrees per minute (≈ 4.8 × 10^{-4} radians/second) is used for the large cylinders.

To assess the effect of changing the twist rate, the responses of four cylinders are examined at slow and fast twist rates. For the small cylinders the slow and fast rates are one and four degrees per minute (≈ 3 × 10^{-4} and ≈ 1.2 × 10^{-3} radians/second), respectively; and for the large cylinders the slow and fast rates are 0.8 and 1.6 degrees per minute (≈ 4.8 × 10^{-4} and ≈ 2.4 × 10^{-4} radians/second), respectively. The four cylinders that are considered are a small circular cylinder with a negative twist, a small constant-length elliptical cylinder with an aspect ratio of 0.70 and a positive twist, a large circular cylinder with a positive twist and including material failure, and a large constant-twist-sensitivity elliptical cylinder with an aspect ratio of 0.85 and a positive twist.

The torque-twist relations for the small circular cylinder are shown in Figure 3.13. The normal displacement around the circumference at the mid-length of the cylinder is shown in Figure 3.14, and the total normal displacements of the ridge and valley marked
by the arrows in Figure 3.14 are plotted in Figure 3.15 for the entire range of twist that is considered. The only discernible difference that can be observed in these results is that the ridges and valleys illustrated in Figure 3.14 do not occur at the same circumferential location; however, this is not a concern because this is a circular cylinder. The difference in the torque level and the amplitudes of the ridges and valleys is negligible for the two twist rates considered. The benefit from using the fast twist rate is that the computational time is $4 \times 10^4$ seconds, while it is $6 \times 10^4$ seconds for the slow twist rate. Also the size of the file that stores the data for each equilibrium solution step are 3.5 GB and 7.5 GB for the fast and slow twist rates, respectively.

![Figure 3.13. Comparison of torque-twist relations of small circular cylinder, negative twist, for slow and fast twist rates](image)

Material failure is included in the models when comparing the slow and fast twist rates for the large circular cylinder. The torque-twist relation, the normal deformation around the circumference at the cylinder midlength, and the total normal displacements of a ridge and valley are shown in Figures 3.16, 3.17, and 3.18, respectively. The analysis is stopped at the twist at which first fiber failures are detected. Additionally, the number of ply points that have failed are plotted as a percentage of the total number of ply points over the same range of twist, and this is shown in Figure 3.19. There is not a significant difference in the torque response, the normal deformations, total normal displacements, or the progression...
of failure for the slow and fast twist rates. The twists for both the first failures and first fiber failures in the cylinder are predicted for the fast rate at levels that are approximately 0.2% less than the respective levels for the slow rate. The computational time and amount of disk space for the slow twist rate are $1.2 \times 10^6$ seconds and 13.4 GB, and for the fast twist rate are $4 \times 10^5$ seconds and 5.7 GB.

The results for the circular cylinders show the computational savings that can be realized by using a faster twist rate, but neither of the circular cylinders has any instabilities in the postbuckling range. Most of the elliptical cylinders in this study have instabilities in the postbuckling range, and the two cases mentioned above are used to examine the effect of different twist rates on the postbuckling responses of elliptical cylinders. Again, the characteristics of the responses may not be fully described here because the intent in this section is to examine the effect of the twist rate on the response. The characteristics of the responses of the elliptical cylinders are described in greater detail in Chapters 4 and 5.

The torque-twist relations of the small constant-length elliptical cylinder with an aspect ratio of 0.70 are shown in Figure 3.20 for the slow and fast twist rates. The normal deformations around the circumference at the cylinder mid-length are shown for a twist of $\phi \approx 1.25$ in Figure 3.21, and the normal displacements of the ridge at $s/C = 0.5$ and valley at $s/C = 0.25$ that are marked by the arrows are plotted for the entire range of twist in Figure 3.22. As can be seen in Figures 3.20 and 3.22, an instability occurs near a twist of $\phi \approx 1.2$, and there is a sharp drop in torque and a change in the deformation pattern as the cylinder moves to a stable configuration. The twist at which the instability occurs
for the fast rate is approximately 1% greater than the level for the slow rate. The reason for this difference in the twist levels is not fully understood, but is believed to be at least partially due to the artificial damping that is included in the finite element transient analysis. Additional analyses that would be needed to determine the cause are not considered necessary, since the difference in the twist levels of the instability is so small. Due to the details of executing the particular finite element analyses, the analysis for the fast twist rate includes a significant range of twist greater than what is reported and greater than the range of twist for the slow rate, so comparisons of the computational expense are not valid for this cylinder.

For the large constant-twist-sensitivity elliptical cylinder with an aspect ratio of 0.85, the torque-twist responses are shown in Figure 3.23 for the slow and fast twist rates. The normal deformation around the circumference at the mid-length is shown in Figure 3.24 for a twist of \( \bar{\phi} \approx 5.6 \), and the total normal displacements, plotted over the range of twist that is considered, are shown in Figure 3.25 for the circumferential locations that are marked in Figure 3.24. Instabilities where there is a sharp, but small, drop in torque and a change in the deformation pattern occur at twists of \( \bar{\phi} \approx 4.6 \) and \( \approx 6.4 \). Both instabilities occur at twists for the fast twist rate that are less than 1% greater than for the slow rate. After the
second instability at $\bar{\phi} \approx 6.4$, which results in a change in the deformation pattern that is a local change near the ends of the major axes, there is another change in the deformation pattern during which the wrinkles change circumferential location. There is no sharp drop in torque that accompanies the latter configuration change, and this change occurs for the fast rate at a twist that is $\approx 5\%$ greater than for the slow rate. Again this difference in the responses for the slow and fast twist rates is at least partially due to the artificial damping included in the transient analysis, but the difference is not considered to be large enough to warrant additional analyses to try to get better agreement between the two twist rates.

To be noted in Figure 3.25 is the fact that for this case the displacement responses for the slow and fast twist rates deviate significantly over the range of twist $2.4 < \bar{\phi} < 3$. For the slow twist rate the circumferential location of the wrinkles is changing. At a twist of $\bar{\phi} \approx 2.4$, the wrinkles start to shift in one direction, but near a twist of $\bar{\phi} \approx 2.7$, the direction of the shift reverses, and by a twist of $\bar{\phi} \approx 3$, the wrinkles have returned to the same circumferential locations as at $\bar{\phi} \approx 2.4$. No change in the circumferential location of the wrinkles is seen for the fast twist rate over the same range of twist. Despite the difference in the displacement response, the largest difference in the torque over this range is less than $0.3\%$. The strain energy over this range of twist is plotted in Figure 3.26, and
the largest difference between the strain energy for the slow and fast twist rates is less than 0.01%. Considering the value of the strain energy in this range is \( \approx 6000 \) lb-in, the small difference in the strain energy implies that despite the difference in the deformation pattern, the cylinders are in equivalent energy states and both are valid solutions. So, the combination of twist rate, time step, and possibly other factors, including the artificial damping, that are part of the numerical solution scheme, result in different displacement responses for the slow and fast twist rates over this range of twist. Because the change in the deformation pattern does not have a significant effect on either the torque or strain energy response, and the deviation in the displacement response only occurs in an isolated range of twist, additional analyses are not considered to try to determine why the displacement responses are different in this range of twist for the two twist rates.

In summary, the responses of the cylinders under two twist rates have been examined for both small and large cylinders. For the small and large circular cylinders, there is very little difference in the torque and displacement response for the slow and fast twist rates, and in the large cylinder initial failure and the progression of failure until the first fiber failures are detected are almost the same for the two twist rates. The computational expense of the fast rate is much less than the slow rate for both the small and large circular cylinders. In the large and small elliptical cylinders, there are small differences in the twist value at which instabilities occur. Otherwise, except for the range of twist just discussed for the large elliptical cylinder, the torque and displacement responses are almost the same for both twist rates. Because of the agreement between the responses for the two twist rates
Figure 3.18. Comparison of normal displacement-twist relations of large circular cylinder, positive twist, for slow and fast twist rates

and the computational savings over the slow rate, the fast rate is used in most subsequent analyses.
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Figure 3.19. Comparison of progression of failure of large circular cylinder, positive twist, for slow and fast twist rates

Figure 3.20. Comparison of torque-twist relations of small constant-length \( \frac{b}{a} = 0.70 \) cylinder, positive twist, for slow and fast twist rates
Figure 3.21. Comparison of normal deformations around the circumference at $x/L = 0$ of a small constant-length $b/a = 0.70$ cylinder, negative twist, for slow and fast twist rates, at a twist of $\bar{\phi} \approx 1.25$

Figure 3.22. Comparison of normal displacement-twist relations of small constant-length $b/a = 0.70$ cylinder, negative twist, for slow and fast twist rates
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Figure 3.23. Comparison of torque-twist relations of large constant-twist-sensitivity $b/a = 0.85$ cylinder, positive twist, for slow and fast twist rates

Figure 3.24. Comparison of normal deformations around the circumference at $x/L = 0$ of large constant-twist-sensitivity $b/a = 0.85$ cylinder, positive twist, for slow and fast twist rates
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Figure 3.25. Comparison of normal displacement-twist relations of large constant-twist-sensitivity $b/a = 0.85$ cylinder, positive twist, for slow and fast twist rates

\[
\text{Twist, } \bar{\phi} = \frac{\left| \phi \right|}{L} \frac{L_{\text{cir}}}{\phi_{\text{cir}}^{\text{cr}}}
\]

Figure 3.26. Comparison of strain energy of large constant-twist-sensitivity $b/a = 0.85$ cylinder, positive twist, for slow and fast twist rates

\[
\text{Twist, } \bar{\phi} = \frac{\left| \phi \right|}{L} \frac{L_{\text{cir}}}{\phi_{\text{cir}}^{\text{cr}}}
\]
3.6 Validation of Model

The finite element model used in this study will be validated using two approaches. First, the initial slope of the linear part of the torque-twist response will be compared to a theoretically-predicted slope. The cylinder geometries defined in Tables 2.1 and 2.2 will be used when comparing the theoretical and numerical results. Second, the values of the critical twist angles from a linear buckling analysis will be compared to values predicted from a numerical solution developed by Batdorf [9] for a circular cylinder. After establishing confidence in the prediction of the critical value for circular cylinders, additional finite element analyses will be used to do the same for the elliptical cylinders.

3.6.1 Validation of Initial Slope of Torque-Twist Relation

Using a strength of materials approach, Shames [59] derived an equation relating the twist, $\phi/L$, to the torque for thin-walled cylinders with an arbitrary cross-sectional shape. To use the equation with the cylinders in this study, the equation is modified so it can be used with cylinders made from anisotropic materials. With the modified equation, there are still limitations with respect to the degree of material anisotropy for which the equation is valid, but for the quasi-isotropic laminate that is considered here, with $A_{16} = A_{26} = 0$, the equation is valid. The modified equation is given by

$$\frac{d\phi}{dx} = \frac{TC}{4A_c^2A_{66}},$$

(3.5)

where $\phi$ is the twist angle at an arbitrary point along the length of the cylinder, $T$ is the torque on the cylinder, $C$ the cylinder circumference, $A_c$ the area of the cross section enclosed by the mid-surface of the cylinder, and $A_{66}$ the shear stiffness parameter. Integrating over the length of the cylinder, Equation (3.5) becomes

$$\frac{\phi}{L} = \frac{TC}{4A_c^2A_{66}}.$$  

(3.6)

One of the measures of the response of the cylinders in this study is the torque-twist relation, i.e., a $T$ vs. $\phi/L$ relation. Initially, in the prebuckling range, the torque-twist response is linear. Equation (3.6) is only valid when the torque-twist response is linear. Equation (3.6) can be rewritten to define the slope of torque-twist response in terms of geometric and material parameters. Rewriting the equation, the slope of the torque-twist relation, $m$, is
using Shames’ approach is

\[
m_{\text{Shames}} = \frac{T}{(\int_L \phi)} = \frac{4A_c^2A_{66}}{C}.
\] (3.7)

The cross-sectional area of an elliptical cylinder is

\[
A_c = \pi ab.
\] (3.8)

The slope calculated using Equation (3.7) is compared to the slope of the torque-twist relations from the finite element analysis in Table 3.4 for the small cylinders and Table 3.5 for the large cylinders. Some large and small cylinders had a nonlinear torque-twist relation that developed after a moderate amount of twist had been applied. In those cases only the initial linear portion of the relation is used in calculating the slope.

By comparing the numbers in Tables 3.4 and 3.5, it can be concluded that there is very good agreement between \(m_{\text{Shames}}\) and \(m_{\text{STAGS}}\). The percent difference between the slopes increases for smaller aspect ratios, but even for the most extreme elliptical cross section considered, with an aspect ratio of 0.55, the difference between \(m_{\text{Shames}}\) and \(m_{\text{STAGS}}\) is less than 2%. When considering the difference between \(m_{\text{Shames}}\) and \(m_{\text{STAGS}}\) for the elliptical cylinders, \(m_{\text{Shames}}\) should not be considered exact and part of the difference can be attributed to approximations that are made in deriving Equation (3.5). For example, end conditions are not taken into account in Shames’ formulation, and both the constant-length and constant-twist-sensitivity cylinders are short enough that end conditions can affect the slope.

<table>
<thead>
<tr>
<th>aspect ratio, (\frac{b}{a})</th>
<th>(L) [in]</th>
<th>(m_{\text{Shames}}) [lb-in/rad/in] ((10^{-7}))</th>
<th>(m_{\text{STAGS}}) [lb-in/rad/in] ((10^{-7}))</th>
<th>% difference</th>
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<tr>
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Table 3.5. Comparison of slope of torque-twist relations for large cylinders

<table>
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<th>aspect ratio, $b/a$</th>
<th>$L$ [in]</th>
<th>$m_{\text{Shames}}$ [lb-in/ rad/in] $(10^{-10})$</th>
<th>$m_{\text{STAGS}}$ [lb-in/ rad/in] $(10^{-10})$</th>
<th>% difference</th>
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<td>1.523</td>
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<td>1.188</td>
<td>1.11</td>
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</table>

3.6.2 Validation of Buckling Conditions

The second approach used to validate the finite-element calculations is a comparison of the buckling conditions predicted from the finite element model to those predicted from another approach. Batdorf [9] developed a buckling analysis for circular isotropic cylinders under various load conditions, and the solution for torsion was discussed in Batdorf et al. [10]. The solution uses the Galerkin approach to determine the critical stress for circular cylinders with clamped boundary conditions. Details of the solution procedure are given in [10], and will not be repeated here. The solution is recreated, and results for cylinders of interest in this study are calculated. Since Batdorf’s solution was derived for isotropic cylinders, comparisons of the buckling values from Batdorf’s analysis and STAGS are first made using aluminum material properties.

The solution for Batdorf’s analysis yields a critical shear stress coefficient, $k_s$, which is defined in terms of the shear stress $\tau_{xs}$. Because shell elements are used in the STAGS analysis, and an average shear stress through the thickness is assumed in Batdorf’s analysis, the shear stress resultant, $N_{xs}$, will be used instead of the shear stress in the definition of $k_s$. Therefore, using the relation $N_{xs} = \tau_{xs}H$, $k_s$ is defined as

$$k_s = \frac{N_{xs}^c L^2}{\pi^2 D},$$  \hspace{1cm} (3.9)$$

The bending stiffness term, $D$, in Equation (3.9) is defined as

$$D = \frac{E H^3}{12(1 - \nu^2)},$$  \hspace{1cm} (3.10)$$

where $E$ is Young’s Modulus of the material and $\nu$ is Poisson’s ratio. For a circular cylinder,
assuming the shear stress, and consequently the shear stress resultant, is constant along the length of the cylinder, the critical twist angle, $\phi_{cr}$, can be determined from the shear stress coefficient and is defined by

$$\phi_{cr} = \frac{k_s \pi^2 H^2}{6(1 - \nu)LR}.$$ (3.11)

Using Batdorf’s analysis along with Equation (3.11), the critical twist angle is calculated for circular aluminum cylinders. The geometry for the large and small cylinders was given in Tables 2.1 and 2.2. The wall thicknesses are 0.044 in. and 0.088 in. for the small and large cylinders, respectively. The material properties are assumed to be $E = 10^7$ psi and $\nu = 0.3$. The critical twist angles are given in Table 3.6. As can be observed, the predicted values for the critical twist angle for the two approaches are within 3% of each other, confirming that the finite-element formulation is correct when applied to circular isotropic cylinders.

<table>
<thead>
<tr>
<th>size</th>
<th>$\phi_{cr}$ [rad] (Batdorf)</th>
<th>$\phi_{cr}$ [rad] (STAGS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>0.01196</td>
<td>0.01166</td>
</tr>
<tr>
<td>large</td>
<td>0.00381</td>
<td>0.00374</td>
</tr>
</tbody>
</table>

To use Batdorf’s analysis with composite materials, effective engineering properties, as defined by Hyer [60], are used for the modulus $E$ and Poisson’s ratio $\nu$. Batdorf’s analysis cannot account for the fact that the buckling conditions for composite cylinders are different for positive and negative twists. This difference is due to the presence of the bending stiffness terms $D_{16}$ and $D_{26}$, which are not present in isotropic materials. To be able to predict this difference, a formulation such as the one developed by Nemeth [61], in which the Batdorf analysis is extended for symmetrically laminated composite shells, should be used. For a more extensive study, Nemeth’s approach would be preferred; however, since the purpose here is only to validate the results for the finite-element formulation in this study, Batdorf’s analysis is considered sufficient.

The predicted values of $\phi_{cr}$ for small and large circular composite cylinders are given in Table 3.7. The cylinder geometries are the same as for the aluminum cylinders, and were given in Tables 2.1 and 2.2. Values for a positive twist from STAGS are close to the values predicted by Batdorf’s analysis. For a negative twist, the critical twist angles for the two approaches do not compare as well. For the large cylinder, the effect of $D_{16}$ and $D_{26}$ is less
than in the small cylinder, and the difference between the magnitudes of the positive and negative critical twist angles is less than 10%. Using the average of the positive and negative twist angles, there is about 10% difference with the prediction using Batdorf’s analysis for the large cylinders. Based on these comparisons, the circular composite cylinders are considered to be properly modeled with the finite-element analysis.

Table 3.7. Comparison of predicted critical twist angles for composite circular cylinders as computed using Batdorf’s analysis and STAGS

<table>
<thead>
<tr>
<th>size</th>
<th>$\phi_{cr}$ [rad] (Batdorf)</th>
<th>$+\phi_{cr}$ [rad] (STAGS)</th>
<th>$-\phi_{cr}$ [rad] (STAGS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>0.01210</td>
<td>0.01196</td>
<td>-0.00940</td>
</tr>
<tr>
<td>large</td>
<td>0.00386</td>
<td>0.00364</td>
<td>-0.00336</td>
</tr>
</tbody>
</table>

Having accepted the validity of the finite-element formulations for circular cylinders, results from STAGS analyses for circular cylinders will be used to validate the results for elliptical cylinders. Lundquist [47], in an experimental study of elliptical cylinders, postulated that for geometrically perfect cylinders the shear stress resultant in the elliptical cylinder when the wall begins to wrinkle will be equal to or greater than the critical shear stress resultant in a circular cylinder of the same length and wall thickness and with a radius equal to the maximum radius of curvature of the elliptical cylinder. Accepting Lundquist’s postulate as a valid statement, it follows that the circular cylinder with a radius equal to the maximum radius of curvature provides a lower bound for the shear stress resultant for buckling in an elliptical cylinder.

For an elliptical cylinder with an aspect ratio close to unity, it would be expected that the buckling values of shear stress resultant and twist angle would be close, but less, than those for a circular cylinder with the same circumference. As the aspect ratio is decreased, the difference between the buckling values for the elliptical and circular cylinders would increase. Similarly, the buckling values for an elliptical cylinder with an aspect ratio close to unity would have buckling values greater than those of an elliptical cylinder with an aspect ratio further from unity. So the upper bound to the buckling conditions of an elliptical cylinder are given by the buckling conditions of the circular cylinder or another elliptical cylinder with an aspect ratio closer to unity.

Without a specific formula to validate the results for elliptical cylinders, the upper and lower bounds as described above will be used to establish a range of buckling conditions for which the buckling conditions of the elliptical cylinders in this study are acceptable. The buckling conditions for small and large constant-length quasi-isotropic elliptical cylinders

60
are given in Table 3.8 for both positive and negative twists. The critical twist angle, $\phi_{cr}$, is calculated using a linear buckling analysis, the shear stress resultant, $N_{xs}$, at the end of the minor radius and near the midlength is determined using a linear static analysis at the predicted critical twist angle, and the average shear stress resultant, $N_{xs}^{avg}$, is calculated from the variation of $N_{xs}$ around the circumference near the mid-length of the cylinder from the same static analysis.

For each of the three noncircular cylinders, the cylinder with the next largest aspect ratio provides the upper bound buckling conditions, i.e., the buckling conditions of the cylinder with an aspect ratio 0.85 serves as the upper bound for the buckling conditions of a cylinder with an aspect ratio of 0.70. The shear stress resultant for a circular cylinder with a radius equal to the maximum radius of curvature of the elliptical cylinder provides the lower bound for the shear stress resultant in the elliptical cylinder. In this manner, at least for the upper bound, it is necessary to validate the buckling conditions for the cylinder with an aspect ratio of 0.85 before the buckling conditions for an aspect ratio 0.70 can be validated, which in turn is needed to validate the buckling conditions for the cylinder with an aspect ratio of 0.55. These bounding criteria are satisfied for each of the elliptical cylinders for both positive and negative twists, as can be seen in Table 3.8. These comparisons provide confidence that the finite-element models of the elliptical cylinders are formulated properly.
### Table 3.8: Comparisons of buckling conditions predicted by STAGS for quasi-isotropic elliptical and circular cylinders

<table>
<thead>
<tr>
<th>Aspect Ratio, $b/a$</th>
<th>elliptical cylinder</th>
<th>circular cylinder</th>
<th>$\phi_{cr}$</th>
<th>$N_{cr}(s/C = 0)$</th>
<th>$N_{crx}(s/C = 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>size</td>
<td>positive twist</td>
<td>negative twist</td>
<td></td>
<td>[rad]</td>
<td>[lb · in]</td>
</tr>
<tr>
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<td>0.01196</td>
<td>-492</td>
<td>0.00940</td>
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<tr>
<td></td>
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<td>0.01171</td>
<td>-477</td>
<td>0.00910</td>
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<td>-434</td>
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<td>0.55</td>
<td>-299</td>
<td>0.00364</td>
<td>-299</td>
<td>0.00795</td>
</tr>
<tr>
<td>large</td>
<td>1.00</td>
<td>-492</td>
<td>0.01196</td>
<td>-492</td>
<td>0.00940</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>-284</td>
<td>0.01042</td>
<td>-284</td>
<td>0.00910</td>
</tr>
<tr>
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<td>0.70</td>
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<td>0.00364</td>
<td>-233</td>
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</tr>
<tr>
<td></td>
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<td>-188</td>
<td>0.00295</td>
<td>-188</td>
<td>0.00336</td>
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</table>

R = $R_{max}$ (See Tables 2.1 and 2.2)
3.7 Influence of Length-to-Radius Ratio on Buckling

Before proceeding to discuss specific results for the small and large cylinders, the influence of the cylinder length-to-radius ratio on the critical twist, as illustrated in Figures 1.2 and 1.3, should be discussed in more detail than was mentioned in the Introduction. For the elliptical cylinders, the maximum radius of curvature $R_{\text{max}}$ is used in the definition of the length-to-radius ratio. In the present study, different values of the length-to-radius ratio are obtained by varying the length of the cylinder and keeping the cross-sectional geometry constant. The cross-sectional geometry for the cylinders was defined in Tables 2.1 and 2.2. In what follows, a linear eigenvalue analysis is used to predict the critical twist values as a function of length-to-radius ratio and for the different aspect ratios. In the finite element mesh, the number of elements in the axial direction is adjusted so the elements are approximately the same size for all lengths considered.

3.7.1 Small Cylinders

The relationships between the length-to-radius ratio and the critical twist are shown in Figure 3.27 for the small cylinders for positive and negative twists. Each symbol represents a single finite element analysis, so a number of analyses are represented in the figure. Shown in Figure 3.27, and to be discussed shortly, are loci representing the values of $L/R$ and critical twists for the seven constant-length and constant-twist-sensitivity geometries defined in Table 2.1, which are also identified with solid symbols in Figure 3.27. As described in the Introduction, the critical twist increases for decreasing values of the length-to-radius ratio. Additionally, for any value of the length-to-radius ratio, the critical twist of a cylinder with a smaller aspect ratio is less than the critical twist of a cylinder with a larger aspect ratio, with the circular cylinder having the largest critical twist. Consider, for example, cylinders with $L/R = 3$. The critical twists of cylinders with aspect ratios of 0.85, 0.70, and 0.55 are roughly 85%, 69%, and 54%, respectively, of the critical twist of the circular cylinder. This decrease in the critical twist occurs for both positive and negative twists.

For all cylinders, applying a negative twist causes the cylinder to buckle at a lower twist than applying a positive twist. This has been observed in other studies on composite cylinders [38] and is due to the presence of the bending stiffness terms $D_{16}$ and $D_{26}$. This point is illustrated in Figures 3.28 and 3.29. The data from Figure 3.27 is repeated in Figure 3.28 for cylinders with aspect ratios of 1.00 and 0.55. The percent difference between the positive and negative critical twist values for all aspect ratios is shown in Figure 3.29.
With decreasing length-to-radius ratio the difference between positive and negative critical twists becomes larger. This is an indication that boundary effects influence this difference, and from Figure 3.29, it is seen that this difference is almost independent of aspect ratio for length-to-radius ratios less than three. There is about a 25\% difference between positive and negative critical twists for the constant-length and constant-twist-sensitivity small cylinders that are defined in Table 2.1.

As discussed in the Introduction, the slope of the relation between the length-to-radius ratio and the critical twist can, at least in a comparative sense, be used as a measure of the influence of the boundaries on the critical twist value. So, when the slope of the relation of a circular cylinder is the same as the slope of the relation for a cylinder with an aspect ratio of 0.85, then in a general sense the sensitivities of the two cylinders to boundary effects are the same. For sure, with the relations having the same slope, the sensitivity of the critical twist to changes in the length-to-radius ratio is the same, and thus these cylinders are identified herein as constant-twist-sensitivity cylinders. To determine the lengths of the constant-twist-sensitivity cylinders, the twist sensitivity $S_\phi$ of a cylinder for a given $L/R$ is defined as the slope of the $L/R$ vs. critical twist relation at that $L/R$. A power law equation of the form

$$
\left( \frac{|\phi|}{L} \right)_{cr} = P \left( \frac{L}{R_{\text{max}}} \right)^Q,
$$

is used to find an approximate functional relation between the critical twist and $L/R$ for the various cases shown in Figure 3.27. The constants $P$ and $Q$ are determined using the curve fit tool within Tecplot\textsuperscript{\textregistered} [62]. Data within the range $1 \leq L/R \leq 4$ are used to find equations for each of the relations in Figures 3.27a and 3.27b. After the constants $P$ and $Q$ are calculated, the slope of the relations in Figure 3.27 is determined by taking the derivative of Equation (3.12) with respect to $L/R$. The mathematical equation for the twist sensitivity is thus

$$
S_\phi = \frac{d \left( \frac{|\phi|}{L} \right)_{cr}}{d \left( \frac{L}{R_{\text{max}}} \right)} = PQ \left( \frac{L}{R_{\text{max}}} \right)^{Q-1}.
$$

The twist sensitivity of the circular cylinder for a positive twist is calculated, and a value of the length-to-radius ratio is found for the elliptical cylinders with a positive twist so that the twist sensitivity is equivalent to the twist sensitivity of that circular cylinder with a positive twist. These length-to-radius ratios, for the constant-twist-sensitivity cylinders, are given in Table 3.9, along with the twist sensitivities of the constant-length cylinders. Twist sensitivities for a negative twist are calculated, but are shown only as reference since
additional cylinder lengths were not calculated for a negative twist. For the constant-length cylinders, the twist sensitivity increases as the aspect ratio decreases. The largest increase in twist sensitivity relative to the circular case is about 80% for the elliptical cylinder with an aspect ratio of 0.55. The increases in length from the constant-length cylinders to the constant-twist-sensitivity cylinders are 12%, 27%, and 47% for the cylinders with aspect ratios of 0.85, 0.70, and 0.55, respectively. For the geometries chosen for the constant-twist-sensitivity cylinders, the slopes of the relations for the four aspect ratios are close enough in value that the sensitivities can be considered the same.
Figure 3.27. Effect of cylinder length on the critical twist, small cylinders
Figure 3.28. Effect of cylinder length on the critical twist, small circular and $b/a = 0.55$ cylinders, positive and negative twist.

Figure 3.29. Percent difference between critical twists for positive and negative twists, small cylinders.
Table 3.9. Twist sensitivities for small cylinders

<table>
<thead>
<tr>
<th>( \frac{b}{a} )</th>
<th>twist</th>
<th>( R_{\text{max}} )</th>
<th>( L ) [in]</th>
<th>( \frac{L}{R_{\text{max}}} )</th>
<th>( S_{\phi} ) [rad/in]</th>
<th>% increase( ^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>(+)</td>
<td>4.28</td>
<td>12.85</td>
<td>3.00</td>
<td>-0.0001975</td>
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</tr>
<tr>
<td>1.00</td>
<td>(-)</td>
<td>4.28</td>
<td>12.85</td>
<td>3.00</td>
<td>-0.0001975</td>
<td>-0.00001364</td>
</tr>
</tbody>
</table>

constant-length cylinders

<table>
<thead>
<tr>
<th>( \frac{b}{a} )</th>
<th>twist</th>
<th>( R_{\text{max}} )</th>
<th>( L ) [in]</th>
<th>( \frac{L}{R_{\text{max}}} )</th>
<th>( S_{\phi} ) [rad/in]</th>
<th>% increase( ^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85</td>
<td>(+)</td>
<td>5.44</td>
<td>12.85</td>
<td>2.36</td>
<td>-0.0002374</td>
<td>20.2</td>
</tr>
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<td>(-)</td>
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<td>12.85</td>
<td>2.36</td>
<td>-0.0001679</td>
<td>20.4</td>
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<td>0.70</td>
<td>(+)</td>
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<td>12.85</td>
<td>1.31</td>
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<td>77.1</td>
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</tbody>
</table>

constant-twist-sensitivity cylinders

<table>
<thead>
<tr>
<th>( \frac{b}{a} )</th>
<th>twist</th>
<th>( R_{\text{max}} )</th>
<th>( L ) [in]</th>
<th>( \frac{L}{R_{\text{max}}} )</th>
<th>( S_{\phi} ) [rad/in]</th>
<th>% increase( ^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85</td>
<td>(+)</td>
<td>5.44</td>
<td>14.42</td>
<td>2.65</td>
<td>-0.0001969</td>
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<td>14.42</td>
<td>2.65</td>
<td>-0.0001402</td>
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<td>(+)</td>
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<td>16.29</td>
<td>2.28</td>
<td>-0.0001973</td>
<td>-0.12</td>
</tr>
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<td>18.86</td>
<td>1.92</td>
<td>-0.0001375</td>
<td>-1.3</td>
</tr>
</tbody>
</table>

\( ^a \) % increase is the percent difference between the current twist sensitivity and the twist sensitivity of the cylinder with \( b/a = 1.00 \) and the same direction of twist.

3.7.2 Large Cylinders

The influence of the length-to-radius ratio on the critical twist of the large cylinders is shown in Figure 3.30 for positive and negative twists. Similar to what was seen for the small cylinders, smaller values of the length-to-radius ratio result in larger critical twist values. And again, the critical twist of a cylinder with a smaller aspect ratio is less than the critical twist of a cylinder with a larger aspect ratio. When \( L/R = 3 \) and for cylinders with either a positive or negative twist, the critical twists for cylinders with aspect ratios of 0.85, 0.70, and 0.55 are roughly 84\%, 67\%, and 52\%, respectively, of the critical twist of the circular cylinder for the same direction of twist. As illustrated in Figures 3.31 and 3.32, the difference between the critical twist values for positive and negative twists is less for the large cylinders than it was for the small cylinders. The difference is practically independent of aspect ratio for the range of \( L/R \) considered. For the cylinder geometries defined in
Table 2.2, there is approximately a 9% difference between the magnitudes of the positive and negative critical twists.

The geometries of the large cylinders are determined by multiplying the radii and lengths of the small cylinders by a scale factor of five, so the lengths of the constant-twist-sensitivity cylinders are predetermined. Using the same approach as for the small cylinders, the twist sensitivities of the large cylinders have been calculated and are shown in Table 3.10. The same trend is seen for the constant-length cylinders, with cylinders with smaller aspect ratios being more sensitive to changes in the length-to-radius ratio than cylinders with a larger aspect ratio. Even though the lengths of the constant-twist-sensitivity cylinders are not determined from the twist sensitivity of the larger circular cylinder, the difference between the twist sensitivities of cylinders with different aspect ratios is small enough that the cylinders are considered to have the same twist sensitivity.

Attention can now turn to specific results for the cylinders. As has been discussed in this chapter, there are many issues to be addressed to develop credible results. Mesh convergence, twist rates, and the procedure to move from the prebuckling to the postbuckling path more or less independent of cylinder geometry are all important issues that had to be resolved before numerical results and physical interpretation of these results are possible. Results for the small cylinders are addressed in the next chapter, and results for the large cylinder in the subsequent chapter.
Figure 3.30. Effect of cylinder length on the critical twist, large cylinders
Chapter 3  
Details of Finite Element Procedure and Validation of Solution

Figure 3.31. Effect of cylinder length on the critical twist, large circular and $b/a = 0.55$ cylinders, positive and negative twist

Figure 3.32. Percent difference between critical twists for positive and negative twists, large cylinders
Table 3.10. Twist sensitivities for large cylinders

<table>
<thead>
<tr>
<th>$\frac{b}{a}$</th>
<th>twist</th>
<th>$R_{max}$</th>
<th>$L$ [in]</th>
<th>$\frac{L}{R_{max}}$</th>
<th>$S_\phi (10^6)$ [rad/in]</th>
<th>% increase$^a$</th>
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</thead>
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<tr>
<td>1.00</td>
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<td></td>
</tr>
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<td>1.917</td>
<td>-10.34</td>
<td>-0.610</td>
</tr>
<tr>
<td></td>
<td>(-)</td>
<td></td>
<td></td>
<td></td>
<td>-8.97</td>
<td>-1.699</td>
</tr>
</tbody>
</table>

$^a$ % increase is the percent difference between the current twist sensitivity and the twist sensitivity of the cylinder with $b/a = 1.00$ and the same direction of twist.
CHAPTER 4

RESULTS FOR SMALL CYLINDERS

Important details of the numerically-predicted response of the small circular and elliptical cylinders are presented, discussed, interpreted, and compared in this chapter. The previous chapter established the credibility of the results to be presented by investigating convergence and other important issues, so the results in this chapter can be viewed with some level of confidence.

The geometries of the small cylinders discussed in this chapter were given in Table 2.1. To provide a quick overview, and introduce the different aspects of cylinder responses to be discussed, the torque-twist relation, illustrations of cylinders with color contours used to describe the normal deformation, and illustrations of deformed cylinder shapes are shown for a circular cylinder and a constant-length elliptical cylinder with an aspect ratio of 0.70 both with a positive twist in Figures 4.1 and 4.2, respectively. The torque and twist, in Figures 4.1a and 4.2a, are normalized by the critical torque and critical twist of the circular cylinder twisted in the positive direction. The critical values will be reported in the following section. The critical values of the circular and elliptical cylinders are marked by symbols in Figures 4.1a and 4.2a, respectively. Arrows in Figures 4.1a and 4.2a point to certain twist levels and the label on each arrow denotes the subfigure in that figure in which the normal deformation or deformed shape for that twist level is shown.

The response for twist levels less than the critical twist is referred to herein as the prebuckling response of the cylinder. An example of the normal deformation in the prebuckling range for the circular cylinder is shown in Figure 4.1b. This deformation is for a twist level slightly below the critical twist. Color contours are used to show the normal deformation because the deformations are so small, with an inward normal deformation of approximately 0.1% of the wall thickness for most of the cylinder wall away from the ends of the cylinder. For the circular cylinder, the prebuckling response is strictly axisymmetric, and no wrinkling is exhibited. The largest deformations are confined to the regions near the
boundaries, so-called boundary layers. In contrast, the normal deformation for the elliptical cylinder for a twist level slightly below the critical twist is shown in Figure 4.2b, and there is wrinkling of the cylinder wall at all axial and circumferential locations. These wrinkles form outward deformations, ridges, and inward deformations, valleys, at different circumferential locations, and these ridges and valleys are arranged in a helical manner along the length of the cylinder. For this cylinder the amplitudes of the wrinkles are as great as 40% of the wall thickness in the flatter part of the cross section, with smaller amplitudes in the more curved part of the cross section. When the wrinkles develop to the extent shown in Figure 4.2b, the torque-twist response exhibits nonlinear behavior. Not all of the elliptical cylinders develop wrinkling in the prebuckling range of twist. The prebuckling response of all small cylinders will be discussed in the following section.

Following the approach described in Figure 3.5, after determining the critical value, the cylinder is twisted to a level slightly greater than the critical value so the cylinder is in an unstable state. Then, using a transient analysis, the cylinder moves, dynamically, to a stable postbuckling configuration. The initial postbuckling configuration is shown for the circular cylinder and elliptical cylinder in Figures 4.1c and 4.2c, respectively. In the initial postbuckling configuration, the circular cylinder exhibits helically-oriented wrinkles. The magnitudes of the amplitudes of the ridges and valleys are uniform around the entire circumference, with a magnitude of roughly 25% of the wall thickness. For the elliptical cylinder, the wrinkles of the initial postbuckling configuration are very similar to those of the prebuckling configuration, except the circumferential locations of the ridges and valleys have changed. The initial postbuckling configurations of all of the small cylinders will be discussed in Section 4.2.

After the initial postbuckling configuration has been found, as has been described a postbuckling path is determined for each cylinder using a transient analysis during which the twist angle is increased linearly over time. The deformed shape of the circular cylinder is shown at two twist levels along the postbuckling path in Figures 4.1d and 4.1e. The circumferential locations of the ridges and valleys are the same at both twist levels, and are the same as in the initial postbuckling shape. The difference in the shapes is an increase in magnitude of the deformations. Similar growth in the amplitudes of the wrinkles in the elliptical cylinder occurs in the range of twist between the initial postbuckling shape, Figure 4.2c, and the shape before the sharp drop in torque near a twist of $\phi = 1.2$, Figure 4.2d. The sharp drop in torque indicates an instability in the postbuckling path, and the drop in torque occurs because the cylinder is changing shape. The deformed shape of the cylinder after the drop in torque is shown in Figure 4.2e. In the new shape, the number
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of circumferential waves, or wrinkles, is the same as before the instability; however, the circumferential locations of the ridges and valleys have changed. Instabilities occur in the postbuckling paths of almost all of the elliptical cylinders that are considered; however, the change in shape that occurs is not the same in each case. The character of the different types of changes in shape and the postbuckling behavior will be discussed in Section 4.3.

Lastly, the response of the cylinders when material failure is considered is discussed in Section 4.4. As described in Chapter 2, a progressive failure analysis using the maximum stress failure criterion is used to determine the response of the cylinder until the first fiber failures are detected. The effect of aspect ratio, cylinder length, and twist direction on the response will be discussed in Section 4.4.

In summary, the response of all small composite cylinders as the torque-twist relations proceed along paths like those shown in Figures 4.1a and 4.2a will be discussed in this chapter. First, the prebuckling response is examined. Then the initial postbuckling configuration is presented. Next the postbuckling path of the cylinders are examined, including descriptions of changes in configurations seen when instabilities are encountered. Finally, the effect of material failure on the cylinder response is shown.
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Figure 4.1. Behavior of small circular cylinder, positive twist

(a) torque-twist relation

(b) normal deformation for prebuckling state

(c) normal deformation for initial postbuckling state

(d) postbuckling shape at $\phi = 1.5$

(e) postbuckling shape at $\phi = 2.0$
Figure 4.2. Behavior of small constant-length \( b/a = 0.70 \) cylinder, positive twist
Chapter 4 Results for Small Cylinders

4.1 Prebuckling Response

As described in Section 3.2, the prebuckling response of each cylinder is determined using a geometrically nonlinear static analysis within STAGS. The prebuckling range of twist ends at the critical twist, which corresponds to the first unstable equilibrium configuration reported in the finite element analysis. The prebuckling torque-twist relations for all cylinders are shown in Figure 4.3. The relations for both positive and negative twist are shown. The torque and twist are normalized by the critical values of the circular cylinder twisted in the positive direction. Symbols in Figure 4.3 identify the critical values for each cylinder and twist direction, with the solid symbols identifying the critical values for a negative twist. These critical values are reported in Table 4.1. It should be noted that the initial slope for a given geometry is the same for either direction of twist. Recall, these slopes were given in Table 3.4.

<table>
<thead>
<tr>
<th>aspect ratio, b/a</th>
<th>length, L [in]</th>
<th>positive twist</th>
<th>negative twist</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>twist, φ/L <a href="10%5E4">rad/in</a></td>
<td>Torque, T [lb · in]</td>
<td>twist, φ/L <a href="10%5E4">rad/in</a></td>
</tr>
<tr>
<td>1.00</td>
<td>9.28^a</td>
<td>56500^a</td>
<td>-7.32</td>
</tr>
<tr>
<td>0.85</td>
<td>9.08</td>
<td>54300</td>
<td>-7.08</td>
</tr>
<tr>
<td></td>
<td>14.42</td>
<td>8.49</td>
<td>-6.65</td>
</tr>
<tr>
<td>0.70</td>
<td>8.65</td>
<td>48000</td>
<td>-6.61</td>
</tr>
<tr>
<td></td>
<td>16.29</td>
<td>7.50</td>
<td>-5.88</td>
</tr>
<tr>
<td>0.55</td>
<td>8.22</td>
<td>38800</td>
<td>-6.24</td>
</tr>
<tr>
<td></td>
<td>18.86</td>
<td>6.40</td>
<td>-5.13</td>
</tr>
</tbody>
</table>

^a values used as normalization factors

The torsional stiffness, defined here as the slope of the torque-twist relation, is largest for the circular cylinder. The elliptical cylinders with aspect ratios of 0.85, 0.70, and 0.55 have torsional stiffnesses that are approximately 98%, 92%, and 78%, respectively, of the stiffness of the circular cylinder. The critical torque and critical twist are also smaller for smaller aspect ratios, and for a given aspect ratio the critical torque and critical twist are less for the constant-twist-sensitivity cylinders than for the constant-length cylinders, which are
shorter. For a given geometry, the critical torque and critical twist for a negative twist are roughly 20% less than for a positive twist.

The torque-twist relations for the circular cylinders are linear for the prebuckling range of twist. For some of the elliptical cylinders there is a softening of the torsional stiffness over a small range of twist near the critical value. Softening is seen in eight cases for the elliptical cylinders: constant-length cylinder with an aspect ratio of 0.70 and a positive twist, constant-length cylinder with an aspect ratio of 0.55 and positive and negative twists, constant-twist-sensitivity cylinder with an aspect ratio of 0.85 and a positive twist, and constant-twist-sensitivity cylinders with aspect ratios of 0.70 and 0.55 and positive and negative twists. For the other small elliptical cylinders, the prebuckling torque-twist response is linear.

Softening of the torsional stiffness is a result of the deformation of the cylinder wall. Even though the change in the torsional stiffness occurs over such a small range of twist, and is barely noticeable in some cases, the change in the behavior of the cylinder is significant. Evidence of this can be seen by examining the axial end displacement for the prebuckling range of twist, which is shown in Figure 4.4 for the constant-length cylinders and Figure 4.5 for the constant-twist-sensitivity cylinders. The axial end displacement is normalized by the cylinder wall thickness. In these figures, the plots of the axial end displacement end at the critical twist value. For each of the eight cylinders listed above as having a nonlinear torque-twist relation, there is a small range of twist, near the critical value, over which the axial end displacement increases significantly, and the slope appears almost infinite. The cylinders that have a linear torque-twist response do not have the same sharp increase in the axial end displacement near the critical twist value. Which cylinder has which behavior is clear from Figures 4.4 and 4.5.

To relate softening of the torsional stiffness and the sharp increase in the axial end displacement to the deformation of the cylinder wall, a detailed view of the torque and axial end displacement responses over the range of twist near the critical twist are shown along with color contour plots of the normal deformation of the cylinder wall at specific twist levels in Figure 4.6 for the constant-length elliptical cylinder with an aspect ratio of 0.70 and a positive twist. In Figure 4.6a, the scale for torque is on the left vertical axis, and the scale for the axial end displacement is on the right vertical axis. The scale for the twist, on the horizontal axis, is the same for the torque and axial end displacement responses. At a twist of $\Phi = 0.781$, the normal deformation, shown in Figure 4.6b, is representative of the deformation in the cylinder for all twist levels less than this value.
The largest normal deformations occur near the sides of the cylinder. The torque-twist relation is linear at this twist level, and there is no significant increase in the axial end displacement. Although the normal deformations are still greatest near the sides of the cylinder at a twist of $\phi = 0.882$, deformations near the crown and the keel are beginning to have the character of a ridge of a wrinkle, as shown in Figure 4.6c. The torque-twist relation is still linear at a twist of $\phi = 0.882$, and there is no significant change in the rate of increase in the axial end displacement. With a twist of $\phi = 0.914$, the wrinkle-like deformations, shown in Figure 4.6d, are spreading toward the more curved region of the cross section, and the normal deformations are greater in the wrinkle-like deformations in the flatter crown and keel regions than near the more highly curved sides of the cylinder. The torque-twist relation still appears to be linear, and there is no significant change in the rate of increase in the axial end displacement. At a twist of $\phi = 0.924$, the normal deformation near the sides of the cylinder has the character of a valley of a wrinkle, as shown in Figure 4.6e. The torque-twist relation still appears to have the same slope as for smaller twist values, but there does appear to be a slight increase in the rate at which the axial end displacement is increasing. Between a twist of $\phi = 0.924$ and $\phi = 0.932$, slightly less than the critical level, the normal deformations increase by an order of magnitude, with maximum inward and outward deformations greater than 40% of the wall thickness, as shown in Figure 4.6f. Because the normal deformations are a significant percentage of the wall thickness, the deformation pattern can be described as wrinkles or circumferential waves in the cylinder wall. Also, the axial end displacement more than doubles and there is significant softening of the torsional stiffness between $\phi = 0.924$ and $\phi = 0.932$.

In the eight cylinders that have a nonlinear torque-twist relation and a sharp increase in the axial end displacement over a small range of twist below the critical value, wrinkles develop in each of the cylinders. To illustrate this, and to contrast the deformations with those in which wrinkles do not develop, color contours of the normal deformations at a level of twist slightly less than the critical twist value are plotted on rolled-out views of the cylinder for the constant-length cylinders in Figures 4.7 and 4.8, and for the constant-twist-sensitivity cylinders in Figures 4.9 and 4.10. A rolled out view of the cylinder is used, so the entire surface of the cylinder can be seen in one view. The normalized cylinder length, $x/L$, is plotted on the vertical axis, and the normalized circumferential arc length, $s/C$, on the horizontal axis. Color contours depict the normal deformation, with the legend to the right of the plot defining the scale. Note the case of a circular cylinder with a positive twist is shown in both Figures 4.7a and 4.9a, and similarly for a negative twist in both Figures 4.8a and 4.10a.
In a circular cylinder for both twist directions, as is shown in Figures 4.7a and 4.8a, the normal deformation is axisymmetric at a level of twist slightly less than the critical value. Actually the normal deformation is axisymmetric for the entire prebuckling range. For a positive twist, the normal deformation is inward and roughly 0.1% of the total wall thickness. The normal deformation is also inward for a negative twist, and is roughly 0.07% of the total wall thickness.

The normal deformation in the elliptical cylinders is not axisymmetric in the prebuckling range, regardless of whether or not wrinkles develop. The deformations in the cylinders in which wrinkles develop just prior to the critical level of twist are at least an order of magnitude greater than in the cylinders in which wrinkles do not develop. Considering the the elliptical cylinders with an aspect ratio of 0.85, wrinkles only develop in the constant-twist-sensitivity cylinder with a positive twist, Figure 4.9b. For this geometry and twist direction, the magnitudes of the deformations are almost two orders of magnitude greater than in the other three cases, cases for which wrinkles do not develop, Figures 4.7b, 4.8b, and 4.10b. In the constant-twist sensitivity elliptical cylinder with an aspect ratio of 0.85 with a positive twist, there are six circumferential waves in the deformation pattern. Continuing with other aspect ratios, wrinkles do not develop in the constant-length elliptical cylinder with an aspect ratio of 0.70 with a negative twist, Figure 4.8c, but do in the other three cases of elliptical cylinders with this aspect ratio, Figures 4.7c, 4.9c, and 4.10c. In the cylinders that do develop wrinkles, there are six circumferential waves in the deformation pattern. The difference in the normal deformation between the constant-length elliptical cylinder with an aspect ratio of 0.70 with a negative twist and the other three cases is approximately one order of magnitude. All of the cases of elliptical cylinders with an aspect ratio of 0.55 develop wrinkles in the prebuckling range of twist, Figures 4.7d, 4.8d, 4.9d, and 4.10d. The constant-length cylinder with positive and negative twists has six circumferential waves in the deformation pattern, as does the constant-twist sensitivity cylinder with a negative twist. Wrinkles only develop in the flatter part of the cross section in the constant-twist-sensitivity cylinder with a positive twist, Figure 4.9d, and do not develop in the sides of the cylinder. An explanation of why, for a given geometry, wrinkles can develop for one twist direction but not the other will be given in the following section when discussing the initial postbuckling response of the cylinders.

The responses of the cylinders for the prebuckling range of twist has been described in this section. An elliptical cross section is shown to reduce the torsional stiffness of the cylinder when compared to a circular cylinder. The critical torque and critical twist occur at smaller twist levels for cylinders with smaller aspect ratios, and for a given aspect ratio, the
critical torque and critical twist are smaller for longer cylinders. Also, for any geometry the critical torque and critical twist are less for a negative twist than for a positive twist due to the anisotropy of the laminate. Finally, wrinkles develop in the prebuckling range for certain geometries and twist directions, and in those cylinders the development of the wrinkles results in softening of the torsional stiffness and a sharp increase in the axial end displacement of the cylinder.
Figure 4.3. Torque-twist relations of small cylinders
Figure 4.4. Axial end displacement-twist relations of small constant-length cylinders
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Figure 4.5. Axial end displacement-twist relations of small constant-twist-sensitivity cylinders
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Torque, \( T = \frac{|T|}{T_{cr}^{cir+}} \)

Axial End Displacement, \( \frac{u(x=L/2)}{H} \)

Twist, \( \bar{\phi} = \frac{|\phi| L_{cr}^{cir}}{L \phi_{cr}^{cir+}} \)

(a) torque-twist relation and axial end displacement-twist relation

(b) \( \bar{\phi} = 0.781 \)

(c) \( \bar{\phi} = 0.882 \)

(d) \( \bar{\phi} = 0.914 \)

(e) \( \bar{\phi} = 0.924 \)

(f) \( \bar{\phi} = 0.932 \)

Figure 4.6. Prebuckling response near critical value of small constant-length \( b/a = 0.70 \) cylinder, positive twist

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Figure 4.7. Normal deformation at a twist level slightly less than the critical value for small constant-length cylinders, positive twist
Figure 4.8. Normal deformation at a twist level slightly less than the critical value for small constant-length cylinders, negative twist
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Figure 4.9. Normal deformation at a twist level slightly less than the critical value for small constant-twist-sensitivity cylinders, positive twist
Figure 4.10. Normal deformation at a twist level slightly less than the critical value for small constant-twist-sensitivity cylinders, negative twist
4.2 Initial Postbuckling Response

Herein, the initial postbuckling configuration will refer to the state at the end of the initial transient analysis, as discussed in relation to Figure 3.5 in Section 3.2. To review, during the initial transient analysis, the applied twist angle is held constant. At the beginning of the analysis, the cylinder is in an unstable configuration, and because a variable time step is used, the numerical instabilities introduced as the time step increases cause the cylinder to change configuration. The analysis is stopped when the cylinder has settled into a stable configuration, which is the initial postbuckling state. In theory, since the cylinder is in an unstable state at the critical twist, any disturbance will cause motion to begin. Here, however, the ‘disturbance’ is numerical and is large enough to overcome the artificial damping used to aid convergence to the stable equilibrium state after the change in configuration.

Plots of the deformed shape and color contours of the normal deformation on rolled-out views of the cylinder in the initial postbuckling configuration are shown in Figures 4.11 and 4.12 for the constant-length cylinders, and Figures 4.13 and 4.14 for the constant-twist-sensitivity cylinders. Note that Figures 4.11a and 4.13a are the same initial postbuckling configuration for a circular cylinder with a positive twist, and similarly Figures 4.12a and 4.14a are the same for a negative twist.

In all cylinders the deformation pattern has a number of circumferential waves, or wrinkles, that are comprised of ridges and valleys which are arranged in a helical manner around the circumference of the cylinder. For the circular cylinder the amplitudes of the ridges and valleys are uniform around the circumference of the cylinder. For the elliptical cylinders, the amplitudes of the ridges and valleys are greater in the flatter part of the cross section than in the more curved regions of the cross section. The difference in the magnitudes of the amplitudes from the flatter to more curved regions of the cross section is greater in cylinders with smaller aspect ratios.

The magnitudes of the largest amplitudes in all the cylinders are a significant percentage of the wall thickness. When compared to the magnitudes of the deformations of prebuckling configuration just prior to the critical value, Figures 4.7a and 4.8a, the magnitudes of the wrinkles in the initial postbuckling state of the circular cylinder, Figures 4.11a and 4.12a, are more than two orders of magnitude greater. Similarly for the three cases of elliptical cylinders with an aspect ratio of 0.85 in which wrinkles did not develop in the prebuckling range, Figures 4.7b, 4.8b, and 4.10b, the magnitudes of the wrinkles in the initial postbuckling state, Figures 4.11b, 4.12b, and 4.14b, are almost two orders of mag-
nitude greater. In the constant-length elliptical cylinder with an aspect ratio of 0.70 and a negative twist, the only case for an aspect ratio of 0.70 without clearly defined wrinkles in the prebuckling range, Figure 4.8c, the increase in the magnitudes of the deformations is approximately one order of magnitude, as can be observed in Figure 4.12c. For the cases when wrinkles did develop in the prebuckling range, the magnitudes of the wrinkles in the initial postbuckling state are generally the same order as the magnitudes of the wrinkles in the prebuckling range. In general, the magnitudes of the largest normal deformations increase for smaller aspect ratios. For example, the maximum inward normalized normal deformations, \( w/H \), in the constant-length cylinders with a positive twist, Figure 4.11, are approximately -0.29, -0.41, -0.86, and -1.37 for cylinders with aspect ratios of 1.00, 0.85, 0.70, and 0.55, respectively.

The number of circumferential waves in the initial postbuckling configuration is due to both the cylinder geometry and the material properties. Here the number is counted using the normal deformations at the cylinder midlength. In the circular cylinder there are seven circumferential waves for either twist direction. Three of the cases of the elliptical cylinder with an aspect ratio of 0.85 also have seven circumferential waves, Figures 4.11b, 4.12b, and 4.14b, but the constant-twist-sensitivity cylinder with a positive twist, Figures 4.13b, has six circumferential waves. This is one example where for a given cylinder geometry the number of circumferential waves is different for a positive and negative twist directions, and shows that the material anisotropy that causes differences in the magnitudes of the critical twist values also affects the initial postbuckling configuration. A specific example of the effect of cylinder length on the number of circumferential waves can be illustrated using the cylinders with an aspect ratio of 0.85 and a positive twist. The constant-length cylinder, with a length of 12.85 in., Figure 4.11b, has seven circumferential waves, and the constant-twist-sensitivity cylinder, with a length of 14.42 in., Figure 4.13b, has six circumferential waves. For any cylinder length, the initial postbuckling configuration will have an integer number of circumferential waves, and for this example, there is a length somewhere between 12.85 in. and 14.42 in. for which any cylinder longer than this value the cylinder will buckle into a configuration with six circumferential waves and for any cylinder shorter than this value will buckle into a configuration with seven circumferential waves. Since both cylinder lengths for the elliptical cylinders with an aspect ratio of 0.85 and a negative twist, Figure 4.12b and 4.14b, have seven circumferential waves, the threshold length value for the initial postbuckling configuration to have less than seven circumferential waves must be greater than 14.42 in.
Continuing to describe the number of circumferential waves for each cylinder geometry in the initial postbuckling state, there are seven circumferential waves in the constant-length elliptical cylinder with an aspect ratio of 0.70 and a negative twist, Figure 4.12c, but for the other three cases of elliptical cylinders with an aspect ratio of 0.70 there are six circumferential waves in the initial postbuckling state, Figures 4.11c, 4.13c, and 4.14c. Three cases of the elliptical cylinder with an aspect ratio of 0.55 have six circumferential waves, Figures 4.11d, 4.12d, and 4.14d, but the constant-twist-sensitivity cylinder under a positive twist has five circumferential waves, Figure 4.13d.

For the elliptical cylinders, there are seven cases that have an even number of circumferential waves in the initial postbuckling configuration. All seven of these cases developed wrinkles in the prebuckling range, as can be seen in Figures 4.7-4.10. The eighth case in which wrinkles developed in the prebuckling range was the constant-twist-sensitivity elliptical cylinder with an aspect ratio of 0.55 and a positive twist, Figure 4.13d. The normal deformation in the prebuckling range for this case is shown in Figure 4.9d. As previously stated, it appears wrinkles have only developed in the flatter regions of the cross section near the crown and keel, and not in the more curved regions near the sides. This is different from the other cases that develop wrinkles in the prebuckling range, because in each of those cases wrinkles develop around the entire circumference.

An interesting observation can be made by looking at the path the torque-twist relation follows when unloading, or untwisting, the cylinders from the initial postbuckling configuration. One case that is considered is the constant-length elliptical cylinder with an aspect ratio of 0.70 and a positive twist, and it is considered because it is one of the cases when wrinkles develop in the prebuckling range, Figure 4.7c. Another case is the constant-length elliptical cylinder with an aspect ratio of 0.85 and a positive twist, which is a case in which wrinkles do not develop in the prebuckling range, Figure 4.7b. Detailed views of the torque-twist relations including twisting and untwisting for these two cases are shown in Figure 4.15. The torque-twist relations for both cases are generated using the same approach. At the end of the initial transient analysis when the initial postbuckling state is established, a second transient analysis is started during which the twist is initially increased linearly over a short time, then the twist is stopped for a short time, and finally the twist is decreased linearly. For these cases the analysis is stopped when the torque-twist relation returns to the prebuckling response. In addition to showing the torque-twist relations for twisting and untwisting, the critical value from the nonlinear static analysis is denoted by the symbol.
For the constant-length cylinder with an aspect ratio of 0.70 and a positive twist, Figure 4.15a, there is a hysteresis-like behavior as the torque-twist relation returns to the prebuckling path at a twist of approximately $\phi = 0.925$. The critical value is at a twist of $\phi = 0.933$. The hysteresis-like effect is expected because of the character of the difference between the deformation patterns at a twist level slightly less than the critical value, Figure 4.7c, and the initial postbuckling state, Figure 4.11c, i.e., a reversal of the ridges and valleys. By way of contrast, when untwisting the constant-length elliptical cylinder with an aspect ratio of 0.85 and a positive twist, Figure 4.15b, there is a very little hysteresis-like behavior as the untwisting torque-twist relation smoothly joins the prebuckling torque-twist relation at a twist level slightly less than the critical twist value. It should also be noted that because the approach used to make the transition from the prebuckling state at the critical value of twist to the initial postbuckling state is to twist the cylinders past the critical value before starting the initial transient analysis, the portion of the torque-twist relation beyond the critical value is artificial and can not physically exist. So making the statement that there is a slight hysteresis-like behavior in this case is based on the assumption that for a geometrically perfect cylinder the initial postbuckling path would bifurcate from the prebuckling path at the critical value. These two cases are expected to be representative of the behavior, in terms of the torque-twist relation, of all cases when unloading the cylinders from initial postbuckling twist levels. It should be noted that the term 'hysteresis-like' behavior is used, despite the lack of any dissipation mechanism in this pattern, because of the lack of a better term.
Figure 4.11. Deformed shape and normal deformation of initial postbuckling state for small constant-length cylinders, positive twist
Figure 4.12. Deformed shape and normal deformation of initial postbuckling state for small constant-length cylinders, negative twist
Figure 4.13. Deformed shape and normal deformation of initial postbuckling state for small constant-twist-sensitivity cylinders, positive twist
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Figure 4.14. Deformed shape and normal deformation of initial postbuckling state for small constant-twist-sensitivity cylinders, negative twist
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Torque, \( \bar{T} = \frac{|T|}{T_{cr}^{cir}} \)

Twist, \( \bar{\phi} = \frac{\phi}{L} \frac{L^{cir}}{\phi_{cr}^{cir}} \)

(a) constant-length cylinder with \( b/a = 0.70 \) and positive twist

(b) constant-length cylinder with \( b/a = 0.85 \) and positive twist

Figure 4.15. Untwisting of small cylinders from twist at initial postbuckling state
4.3 Postbuckling Response

After the initial postbuckling state has been established, a postbuckling path is determined using a second transient analysis during which the twist angle is increased linearly over time. Again, refer to Figure 3.5 and Section 3.2. The torque-twist relations of the small cylinders including the postbuckling path are shown in Figure 4.16 for the constant-length cylinders and Figure 4.17 for the constant-twist-sensitivity cylinders. The range of twist that is considered ends at a twist of $\varphi = 2.5$ because when material failure is included in the model, first fiber failures occur in all cases before a twist of $\varphi = 2.5$ is reached, and the analysis is stopped when the first fiber failures are predicted.

For the circular cylinder and the elliptical cylinders with an aspect ratio of 0.85, the torque-twist relation of the initial postbuckling response has a negative slope. The slopes remain negative over the entire range of twist considered. The initial slope of the postbuckling torque-twist relations are also negative for most of the cases of the elliptical cylinders with aspect ratios of 0.70 and 0.55, with the exceptions being the constant-length cylinders with a negative twist, Figure 4.16b. In these two cases, the initial slope of the postbuckling path is positive. Even for the cases of the other elliptical cylinders with aspect ratios of 0.70 and 0.55, when the initial slope of the postbuckling response is negative, the slope becomes positive deeper in the postbuckling range. It should be noted that in the literature review in Chapter 1 it was noted that Loo’s [13] extended theory for circular cylinders predicted a decrease in shear stress for increasing shear stress immediately after buckling, and for some geometries, the shear stress would begin to increase again with increasing shear strain. All of the cases for the small elliptical cylinders with aspect ratios of 0.70 and 0.55 have maximum torques that are greater than the critical torques. At the end of the twist range, most of the cases for elliptical cylinders with aspect ratios of 0.70 and 0.55 have a negative slope. Two cases that are exceptions are the constant-length cylinder with an aspect ratio of 0.55 under positive and negative twists, Figures 4.16a and 4.16b. In both cases the slope is approximately zero at a twist of $\varphi = 2.5$.

For both constant-length and constant-twist-sensitivity cylinders and positive and negative twists, the difference in the torque level between cylinders with different aspect ratios decreases for increasing twist values, i.e., the torque-twist relations tend to converge for increasing twist. This is most evident in the constant-length cylinders with a negative twist, Figure 4.16b, where the torque levels of cylinders of all aspect ratios are converging at a torque level of $\overline{T} = 0.6$ as the applied twist approaches a value of $\overline{\varphi} = 2.5$. The difference in the torque levels at the end of the twist range are greater in the constant-twist-sensitivity
cylinders than the constant-length cylinders, and this is expected since the effect of the boundaries are less in the longer constant-twist-sensitivity cylinders. For the constant-length cylinders, the boundary effects are greater in cylinders with smaller aspect ratios, so the boundary effects are greatest in the constant-length elliptical cylinder with an aspect ratio of 0.55.

In the postbuckling response of most of the elliptical cylinders there is at least one, and in some cases more, locations on the postbuckling torque-twist relation where there is a sharp drop in torque. These drops identify locations where there are instabilities in the postbuckling paths of these cylinders. When an instability is encountered, there is a change in the deformation pattern that occurs when the cylinder moves from the unstable configuration before the drop in torque to the stable configuration at the end of the drop in torque. Because the postbuckling response when material failure is not included in the finite element analysis is computed only to use as a reference when examining the effect of material failure on the response, only the instabilities that occur prior to the twist at which the first fiber failures are predicted are of interest. The twist at which first fiber failure occurs in each case will be discussed in following section. Also when material failure is included in the analysis, the twist at which the instability occurs may be effected, but the character of the change in configuration will be the same as in the cylinder without material failure. The types of changes in configuration will be described below.

Using the twist at which first fiber failure occurs as the end of the range, there are only two types of changes in configuration that occur as a result of instabilities in the postbuckling range of small cylinders. One is when there is a change in the circumferential location of the circumferential waves, or wrinkles, but the number of wrinkles remains the same, and the other is when there is a change in the number of circumferential waves. For each of these situations, the torque level drops. After discussing the changes in configuration that are seen for each case, an example of each type of change in configuration will be shown. Of the instabilities that occur at twist levels outside the range of twist that is of interest in this study, only one additional type of change in configuration is seen. It is a local change in configuration where the changes occur only in the ridges and valleys near the sides of the cylinder. An example of this type of change in configuration will not be shown for the small cylinders, but does occur in the range of twist before first fiber failures occur in the large cylinders, and it will be discussed in more detail in Chapter 5.

No instabilities occur in the circular cylinder for either direction of twist, but it is believed that if the postbuckling analysis were continued over a larger range of twist, instabil-
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ities would occur. Also, no instabilities occur in three cases of elliptical cylinders with an aspect ratio of 0.85. These cases are the constant-length cylinder with a positive or negative twist, Figures 4.16a and 4.16b, and the constant-twist-sensitivity cylinder with a negative twist, Figure 4.17b. Instabilities that result in changes of the circumferential location of the wrinkles occur in eight cases: the constant-twist-sensitivity cylinder with an aspect ratio of 0.85 and a positive twist, all four cases of elliptical cylinders with an aspect ratio of 0.70, the constant-length cylinder with an aspect ratio of 0.55 and a positive and negative twist, and the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a positive twist. An instability that results in a change in the number of circumferential waves occurs only in the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a negative twist.

For some of the cases mentioned above, there is only one instability in the postbuckling range, and the twist at which the instability occurs can be identified by where the drop in torque occurs on the postbuckling path. In four cases there are multiple drops in torque in the postbuckling response. In two of these cases, only the first instability is of interest because the second occurs at a twist that is larger than the twist at which the first fiber failures occur. These two cases are the constant-twist-sensitivity cylinder with an aspect ratio of 0.70 and a negative twist, Figure 4.17b, and the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a negative twist, Figure 4.17b. The other two cases have three instabilities in the postbuckling response, and only the first two instabilities are within the range of interest. One of these cases is the constant-length cylinder with an aspect ratio of 0.70 and a negative twist, Figure 4.16b, and these three instabilities result in very small drops in torque. The other case is the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a negative twist, Figure 4.17b. In three of these four cases the type of change in configuration is the same for all of the instabilities in the postbuckling response. The one case with different types of changes in shape is the constant-twist-sensitivity cylinder with an aspect ratio of 0.70 and a positive twist, Figure 4.17a, where the second instability results in a localized change in configuration, and as mentioned above, this type will not be described in this chapter.

Considering the four aspect ratios, constant-length and constant-twist-sensitivity cylinders, positive and negative twist, and multiple load-drops in some cases, the most common change in configuration seen in the small cylinders is when the wrinkles change circumferential locations, but there is no change in the number of circumferential waves. This change in configuration is illustrated in Figure 4.18 for the constant-length cylinder with an aspect ratio of 0.70 and a positive twist, Figure 4.16a. The arrows in the plot of the torque-twist response in Figure 4.18a identify the twist values for which the deformed shape and normal
deformations are shown in Figures 4.18b and 4.18c. At a twist of approximately $\bar{\phi} = 1.2$ before the drop in torque, a valley of a wrinkle crosses the crown and keel at the cylinder midlength, as shown in Figure 4.18b. After the drop in torque, the wrinkles have shifted so that a ridge crosses the crown and keel at the midlength, as shown in Figure 4.18c. For the cylinders that develop wrinkles in the prebuckling range of twist, this type of change in configuration is the same as is seen in the change in configuration from the prebuckling configuration to the initial postbuckling configuration.

The second type of change in configuration seen in the small cylinders is a change in the number of circumferential waves. This type of change is seen in only one of the small cylinders, the constant-twist-sensitivity elliptical cylinder with an aspect ratio of 0.55 and a negative twist, Figure 4.17b, and is illustrated in Figure 4.19. At a twist of approximately $\bar{\phi} = 1.1$ and before the drop in torque, there are six circumferential waves and the distribution of these waves is symmetric about either the major or minor axes, as shown in Figure 4.19b. Approaching a twist of $\bar{\phi} = 1.1$, the amplitudes of the valleys near the $s/C = 0.25$ side of the cylinder are decreasing. Then when the sharp drop in torque occurs, the ridge at the side near $s/C = 0.25$ disappears and the adjacent valleys merge into a single valley, resulting in a deformation pattern with five circumferential waves, as shown in Figure 4.19c. The second instability in the postbuckling range occurs at a twist greater than the twist at which first fiber failures occur, but is being illustrated in the torque-twist relation for the sake of general interest. The second instability for this case occurs near a twist of $\bar{\phi} = 1.8$ and is a change from a configuration with five circumferential waves back to a configuration with six circumferential waves.

Having discussed the response when increasing the twist level, and continuing to consider how the cylinder will respond when decreasing the twist level, as was done after determining the initial postbuckling state, as an example the twist level is decreased for one case after a change in configuration has occurred. Similar to the hysteresis-like behavior seen in the torque-twist response when untwisting a cylinder which has a nonlinear torque-twist response in the prebuckling range, Figure 4.15, untwisting after a change in configuration also results in a hysteresis-like behavior in the torque-twist response. An example of this is shown in Figure 4.20 for the constant-length cylinder with an aspect ratio of 0.70 and a positive twist. The applied twist is decreased after the change in configuration illustrated in Figure 4.18. Using the same approach as used when unloading the cylinders from the initial postbuckling state, the twist is increased slightly after the drop in torque, then the direction of twist is changed. As can be seen in Figure 4.20, there is a significant range of untwisting before the torque level returns to the postbuckling path.
predicted during twisting of the cylinder. The change in configuration is the opposite of the change predicted with increasing twist, and the deformation pattern after the sharp increase in torque returns to a configuration similar to the one predicted while the cylinder was being twisted.

As has been seen, interesting behavior occurs in the elliptical cylinders when the twist level is increased beyond the critical value. Changes in the configuration, drops in the applied torque, and hysteresis-like behavior upon unloading all can occur. However, since the deformations in the postbuckling range are quite large, the question arises as to the effect material failure will have, if any, on the behavior just discussed. This is the subject of the remaining section in this chapter.
Figure 4.16. Torque-twist relations of small constant-length cylinders, including postbuckling paths
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Torque, $T = \left| \frac{T}{T_{cir}^{+}} \right|$  
Twist, $\bar{\phi} = \left| \frac{\phi}{\phi_{cir}^{+}} \right| \frac{L_{cir}}{L}$

(a) positive twist

(b) negative twist

Figure 4.17. Torque-twist relations of small constant-twist-sensitivity cylinders, including postbuckling paths
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(a) torque-twist response

(b) shape and normal deformation before change in configuration

(c) shape and normal deformation after change in configuration

Figure 4.18. Postbuckling change in deformation pattern, change in circumferential location of ridges and valleys, small constant-length $b/a = 0.70$ cylinder, positive twist

\[ T = \left| \frac{T}{T_{cr}^\text{circ}} \right| \]

\[ \Phi = \frac{\left| \phi \right| L_{cr}^\text{circ}}{\Phi_{cr}^\text{circ}} \]
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Figure 4.19. Postbuckling change in deformation pattern, change in number of circumferential waves, small constant-twist-sensitivity $b/a = 0.55$ cylinder, negative twist
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Figure 4.20. Untwisting after instability in postbuckling range for small constant-length $b/a = 0.70$ cylinder, positive twist
4.4 Influence of Material Failure

The response of the cylinders when material failure is included in the finite element analysis is determined using the same approach as when material failure is not included, except at every solution step the stress state at every ply point is compared to the failure criterion (see Figure 3.6 and associated discussion in Section 3.3). As will be shown, material failure does not occur until the postbuckling range, so the prebuckling and initial postbuckling responses do not change from when material failure is not considered. Using the maximum stress failure criterion, the progression of failure from when the initial failures are predicted to when the first fiber failures are predicted is determined for each of the cylinder geometries defined in Table 2.1 and for both a positive and negative twist. Recall from the discussion in Section 3.3 that failure at integration points predicted to be in the direction perpendicular to the fiber direction are referred to as matrix failures, and failures in the direction parallel to the fiber direction are referred to as fiber failures. Intralaminar shear failures are also possible, but for the cylinders in this study no shear failures are observed. The results for each case are presented in this section and compared to the results from when material failure is not included in the finite element analysis.

Torque-twist relations and illustrations of the deformed shapes of the cylinders with color contours showing the percent failure in each element are shown for the small circular cylinder with a positive twist in Figure 4.21. The torque-twist relation, Figure 4.21a, is shown for the cases that include and do not include material failure, and the relation when material failure is included terminates at the twist level when first fiber failures are predicted to occur. As seen, there is only a slight decrease in the torsional stiffness as a result of including material failure. Also, the torque level at the twist level at which the first fiber failures are detected is very close to the torque level of that same twist level but not considering material failure. The initial failures are predicted to occur near a twist of $\phi = 1.242$, and the locations of the initial failures are shown in Figure 4.21b. In Figure 4.21b and all subsequent figures depicting the failure state in the cylinder, any element that contains failed ply points is assigned a color that depends on the failure count within the element. The failure count for an element is expressed as a percentage of the maximum possible failure count for the element, and the legend to the right of the subfigures defines the relation between the percentage of the element that has failed and the color shown. In the caption of Figure 4.21b, the failure count for the entire cylinder is reported, along with the percentage of the maximum possible failure count for the cylinder. In the caption for Figure 4.21, the maximum possible failure count for the entire cylinder is indicated, i.e.,
614,400. The initial failures in the small circular cylinder with a positive twist all occur in the outermost $+45$ degree layer, ply number 8, near the midlength of the cylinder and on each of the ridges of the circumferential waves. All of the initial failures are matrix tensile failures. Note that since initial failure occurs on each of the ridges, and since the waviness is of uniform amplitude around the circumference of this circular cylinder, initial failure occurs uniformly around the cylinder. With increasing positive twist, failures spread along the ridges of the wrinkles, as shown in Figures 4.21c and 4.21d. Again, the spread of failure occurs uniformly along each ridge. At a twist of $\phi = 1.726$, the initial fiber failures are detected, and the accumulation of failure at this twist value is shown in Figure 4.21e. The white circle in Figure 4.21e highlights the region where the first fiber failures occur. For this twist value, there are only two failed points due to fiber failure, and both are compressive failures that occur in the innermost $+45$ degree ply, ply number 1. With a slight increase in twist, additional fiber failures occur on each of the ridges, returning to the uniform behavior observed with matrix failures, e.g. Figure 4.21a. Failure does not occur near the ends of the cylinder, but is limited to the more developed regions of the wrinkles, or what will be referred to as the wrinkled area. Until the first fiber failures are detected, all of the failures are matrix tensile failures. The largest percentage of failure in any element is roughly 20%, and based on the maximum possible failure count for the cylinder only 1.38% contains failure. Details of the twist level, layers with failure, and number of failed points are summarized in Table 4.2 for the twist level at which the initial failures are detected and the twist level at which the first fiber failures are detected.

Failure characteristics for the small circular cylinder with a negative twist are shown in Figure 4.22. As for a positive twist, the change in the slope of the torque-twist response due to material failure is small, but in terms of the difference in the magnitudes of the torque level at the twist level at which first fiber failures are detected, the difference is larger due to the failure count being over four times greater than for a positive twist. The locations of initial failures are shown in Figure 4.22b, and occur in the innermost $+45$ degree layer, ply number 1, near the midlength in each of the valleys of the circumferential waves. At a twist of $\phi = 1.235$ failures have occurred on the ridges of each of the circumferential waves, as shown in Figure 4.22c. With increasing negative twist, the regions with failure spread along the ridges and valleys, and the accumulation of damage in the cylinder when the first fiber failures are detected is shown in Figure 4.22e. Again, the white circle highlights the region where the initial fiber failures occur. Also, the initial fiber failures, of which there are only six, all occur on one ridge. The fiber failures are compressive and occur in the innermost $-45$ degree ply, ply number 2. With just a slight increase in twist, fiber failures
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... occur on the other ridges around the circumference. Even though the initial failures occur in the valleys of the wrinkles, failure progresses at a faster rate along the ridges. Over 45% of the ply points have failed in some of the elements when a negative twist is applied, and approximately 6.8% of the cylinder contains failure. Note that prior to the first fiber failures, the uniformity of failure progression around the circumference is also present for a circular cylinder with a negative twist. Again refer to Table 4.2 for a summary of the details.

For the elliptical cylinders, like in the circular cylinder, there is not a significant impact on the slope of the torque-twist relation when material failure is included in the analysis compared to when it is not. This can be observed in all of the torque-twist relations for elliptical cylinders, Figures 4.23a-4.34a. Also like the circular cylinder, a larger difference in the torque levels is seen in the cases in which more failure accumulates as compared to the cases with less accumulated failure, e.g., Figure 4.24a at point (e) compared to Figure 4.23a at point (e), and Figure 4.25a at point (e) compared to Figure 4.26a at point (e). Failure can also change the twist value at which an instability occurs. In the constant-twist-sensitivity cylinder with an aspect ratio of 0.70 and a negative twist, when material failure is not included, there is an instability that occurs at a twist near $\varphi = 1.0$. This was seen in Figure 4.17b and the torque-twist relation for this case is repeated in Figure 4.32a. When material failure is included in the analysis, and initial failures occur at a twist of $\varphi = 0.957$, the twist at which the instability occurs is roughly 5% greater than when failure is not included. The character of the change in deformed shape is the same, namely, a change in the circumferential locations of the wrinkles. Similar behavior is seen in the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a negative twist, as shown in Figure 4.34a, and that is the one case in which there is a change in the number of circumferential waves. In the constant-length cylinder with an aspect ratio of 0.70 and a negative twist, Figure 4.26, and the constant-twist-sensitivity cylinder with an aspect ratio of 0.85 and a positive twist, Figure 4.29, there are instabilities that occur in the case when material failure is not included, and when material failure is included, and the initial failures occur before the instability, a change in configuration still occurs, but there is not a sharp drop in torque. There is a drop in torque, but it is a more gradual decrease with increasing twist. In terms of loss of torsional stiffness, the effect of matrix failures is barely measurable. However, the effect of a small amount of matrix failures on the twist angle at which instabilities occur can be noticeable.

There are general trends for the progression of failure in the small elliptical cylinders, and for this reason only the constant-length cylinder with an aspect ratio of 0.70, Fig-
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Figures 4.25 and 4.26, will be discussed in detail. For this cylinder with a positive twist, the initial failures occur on the ridge of wrinkle near the crown and the cylinder midlength, Figure 4.25b. Unlike the circular cylinder when initial failures occurred at a number of locations around the circumference, Figure 4.21b, for the elliptical cylinder initial failure is very localized at just two locations around the circumference (one location is hidden in Figure 4.25b). The initial failures are matrix tensile failures, and these failures occur in the outermost +45 degree ply, ply number 8. A change in shape, characterized by the wrinkles changing circumferential location, is taking place when the initial failures occur. With an increase in twist, Figure 4.25c, failures have spread to ridges at the sides of the cylinder.

As the twist increases, failure occurs in the the valleys of wrinkles in the flatter part of the cross section, Figure 4.25d. When the first fiber failures occur, Figure 4.25e, 1.30% of the cylinder contains failure. Recall, the location of the first fiber failures is highlighted by the white circle, and, incidentally, the size of the white circle denotes, roughly, the size of the region where the first fiber failures occur. The first fiber failures are compressive failures, and occur in the outermost +45 degree ply, ply number 8. The largest amount of failure in any element is approximately 28%. Also, at the twist when first fiber failures occur, there are also matrix failures at the ends of the cylinder. This is one of two cases in which failures at the ends of a cylinder were seen. (The other is the constant-twist-sensitivity cylinder with an aspect ratio of 0.85 and a positive twist, Figure 4.29.)

The failure characteristics of the small constant-length cylinder with an aspect ratio of 0.70 and a negative twist are illustrated in Figure 4.26. The initial failures occurs in a single valley of a wrinkle in the flatter part of the cross section, Figure 4.26b. Note this is one of the cases with an odd number of circumferential waves in the deformed shape, and the failure pattern on the part of the cylinder that can be seen is not the same as on the underside of the cylinder that cannot be seen in the figures. As the twist increases, failure spreads to other ridges and valleys of wrinkles around the circumference, Figures 4.26c and 4.26d, and also spread along the length of the ridges and valleys. When the first fiber failures occur, Figure 4.26e, failures have occurred in all valleys and on all ridges of the wrinkles. The first fiber failures are compressive failures, and occur in the outermost −45 degree ply, ply number 7. At the twist for first fiber failures 3.96% of the cylinder contains failure. The largest amount of failure in any element is approximately 38%. As in the circular cylinder, even though the initial failures occur in the valleys of the wrinkles, failure occurs at a faster rate on the ridges. The progression of failure for these two cases is representative of the progression of failure in the small elliptical cylinders.
The plots of the failure states for all the small elliptical cylinders are shown in the b, c, d, and e subfigures of Figures 4.23 - 4.34. With one exception, the initial failures in the elliptical cylinders occur on ridges and in valleys in the flatter part of the cross section, and with increasing twist failures spread to the ridges and valleys of the more curved regions of the cross section. The exception is the constant-length cylinder with an aspect ratio of 0.55 and a positive twist, and in this case the initial failures occur on ridges near the more curved region of the cross section slightly away from the cylinder midlength, as shown in Figure 4.27b. In all cases initial failures occur on ridges of wrinkles for cylinders with a positive twist, and in valleys of wrinkles for cylinders with a negative twist, Figures 4.23b - 4.34b, as with the circular cylinder, Figures 4.21b and 4.22b. Also, as in the circular cylinder under a negative twist, damage accumulates at a faster rate along the ridges than in the valleys for the elliptical cylinders under a negative twist. All failures that occur prior to the first fiber failures are matrix tensile failures, and the first fiber failures are always compressive failures for the cases in this study. Details of the state of damage in the elliptical cylinders at the twist at which the initial failures occur and the twist at which the first fiber failures occur are also included in Table 4.2.

As stated above, the direction of the applied twist has a significant effect on the initial failure state and the progression of failure up to the first fiber failures. In all cases with a positive twist, the initial failures occur in the outermost $+45$ degree ply, ply number 8, and for a negative twist the initial failures occur in the innermost $+45$ degree ply, ply number 1. Also, for a given geometry the twist value at which initial failures occur for a negative twist is less than the twist value at which initial failures occur with a positive twist; however, with the exception of the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a negative twist, the twist value at which the first fiber failures occur for a negative twist is larger than the twist value with a positive twist. First fiber failures occur in a $+45$ degree ply for a positive twist, and a $-45$ degree ply for a negative twist. Also, the percent failure in a cylinder for a given geometry with a negative twist is at least twice the percent failure for the cylinder with a positive twist. More damage also accumulates through the thickness of the cylinders with a negative twist, with failures occurring in at least two more plies with a negative twist than with a positive twist. For most of the cylinders with a negative twist, failures occur in all but the two middle 90 degree plies, ply numbers 4 and 5, but for most of the cylinders with a positive twist, failures occur only in the inner $+45$ and $-45$ and the outer $+45$ degree plies, ply numbers 1, 2, and 8. Failures do not occur in the two middle 90 degree plies, ply numbers 4 and 5, for any case.
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Cylinder geometry also affects the failure characteristics of the small cylinders. For both twist directions, the amount of failure that accumulates before the first fiber failures is smaller for cylinders with smaller aspect ratios. This is due, to some extent, to the localized nature of damage initiation and spreading for the elliptical cylinders, as opposed to every ridge and valley experiencing failure for the circular cylinder. For example, the percent failure in the circular cylinder with a negative twist is 6.75%, and the percent failure in the constant-length elliptical cylinders with aspect ratios of 0.85, 0.70, and 0.55 and a negative twist is 5.43%, 3.96%, and 1.45%, respectively. Also for smaller aspect ratios, the locations of failures in the more curved region of the cross section are skewed away from the midlength of the cylinder, and occur more towards the ends of the wrinkled area. A good example of this is the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a positive twist, shown in Figure 4.33e. This trend is also true for the locations of the first fiber failures. In the circular cylinder, Figures 4.21e and 4.22e, and the cylinders with an aspect ratio of 0.85, Figures 4.23e, 4.24e, 4.29e, and 4.30e, the locations of the first fiber failures are close to the midlength. For the elliptical cylinders with aspect ratios of 0.70 and 0.55, the first fiber failures occur in valleys or on ridges near the end of the wrinkled area, Figures 4.25e, 4.26e, 4.31e, 4.32e, 4.27e, 4.28e, 4.33e, and 4.34e. Although the responses are different for the small constant-length and constant-twist-sensitivity cylinders of each aspect ratio, there do not appear to be general trends across all aspect ratios that would help explain the influence of cylinder length on the response of the cylinder when material failure is included in the analysis.

Finally, to observe the response when untwisting the cylinder after material failure and an instability have occurred, the case of the constant-twist-sensitivity cylinder with an aspect ratio of 0.70 and a negative twist, Figure 4.32, is untwisted after a postbuckling instability. The change in shape caused by the instability is a change in the circumferential location of the wrinkles. Using the same approach as in the previous sections, details of the loading-unloading torque-twist relation for this case are shown in Figure 4.35. The failure state in the cylinder when untwisting starts from a twist of $\phi = 1.063$ is similar to the state shown in Figure 4.32c, but the failure count is only 9474 as opposed to 9800 at a twist of $\phi = 1.106$. All of the failures are matrix tensile failures, and no additional failures occur when untwisting the cylinder. Near a twist of $\phi = 0.77$ there is a sharp increase in torque, like the response is returning to the twisting path, then the torque suddenly drops, and eventually settles in a state at a lower torque level. The torque-twist relation then follows a path back to the prebuckling path that is similar in character to the twisting path from the initial postbuckling state, but is at a lower torque level. At the end of the untwisting
torque-twist relation, the slope is nearly identical to the slope of the prebuckling torque-twist relation. While there is a loss in energy when ply points fail, it is clear that when the cylinder is unloaded to the prebuckling range of twist the small amount of matrix failures for this case have only a small effect on the torsional stiffness of the cylinder.

Having examined the effect of material failure on the postbuckling response of the small cylinders, all of the results for the small cylinders have been presented for the different aspects of the response that are considered: prebuckling, initial postbuckling, postbuckling, and postbuckling with material failure. Now the effect of increasing the size of the cylinder will be examined. The results of large cylinders, for which the geometry is obtained by increasing the radii and length of the small cylinders by a factor of five and by doubling the thickness, will be discussed in the following chapter.
Table 4.2. Summary of failure characteristics for small cylinders

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<th>first failures</th>
<th>first fiber failures</th>
<th>% of maximum possible failure count</th>
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<td>layer nos. with fiber failures</td>
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<td>1.906</td>
</tr>
<tr>
<td>16.29</td>
<td></td>
<td>(+)</td>
<td>1.241</td>
<td>8</td>
<td>1.705</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(−)</td>
<td>0.957</td>
<td>1</td>
<td>1.805</td>
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<tr>
<td>0.55</td>
<td>12.85</td>
<td>(+)</td>
<td>1.446</td>
<td>8</td>
<td>1.826</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(−)</td>
<td>1.047</td>
<td>1</td>
<td>1.845</td>
</tr>
<tr>
<td>18.86</td>
<td></td>
<td>(+)</td>
<td>1.266</td>
<td>8</td>
<td>1.686</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(−)</td>
<td>1.084</td>
<td>1</td>
<td>1.600</td>
</tr>
</tbody>
</table>

$^a$ outermost layer
Figure 4.21. Failure progression in small circular cylinder, positive twist (maximum possible failure count: 614,400)
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![Graph](image)

**Torque,**

\[ T = \frac{|T|}{T_{cir}^{circ}} \]

**Twist,**

\[ \phi = \frac{|\phi| L_{cir}}{L_{cir}^{circ}} \]

(a) torque-twist relation

(b) \( \bar{\phi} = 1.186, \) failure count: 236
   (0.04% of max. possible failure count)

(c) \( \bar{\phi} = 1.235, \) failure count: 1981
   (0.32%)

(d) \( \bar{\phi} = 1.765, \) failure count: 24488
   (3.99%)

(e) \( \bar{\phi} = 2.349, \) failure count: 41477
   (6.75%) (fiber failure region)

**Figure 4.22.** Failure progression in small circular cylinder, negative twist (maximum possible failure count: 614,400)
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Figure 4.23. Failure progression in small constant-length $b/a = 0.85$ cylinder, positive twist  
(maximum possible failure count: 614,400)
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Figure 4.24. Failure progression in small constant-length $b/a = 0.85$ cylinder, negative twist (maximum possible failure count: 614,400)
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Figure 4.25. Failure progression in small constant-length $b/a = 0.70$ cylinder, positive twist (maximum possible failure count: 614,400)
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Figure 4.26. Failure progression in small constant-length $b/a = 0.70$ cylinder, negative twist
(maximum possible failure count: 614,400)

(a) torque-twist relation

(b) $\bar{\phi} = 1.033$, failure count: 38
(0.01% of max. possible failure count)

(c) $\bar{\phi} = 1.188$, failure count: 2813
(0.46%)

(d) $\bar{\phi} = 1.576$, failure count: 11111
(1.81%)

(e) $\bar{\phi} = 1.906$, failure count: 24327
(3.96%) (fiber failure region)
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Figure 4.27. Failure progression in small constant-length $b/a = 0.55$ cylinder, positive twist (maximum possible failure count: 614,400)
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Torque, \( T = \frac{|T|}{T_{circ}^{cr}} \)

Twist, \( \phi = \frac{\phi}{L_{circ}^{cr}} \)

(a) torque-twist relation

(b) \( \phi = 1.047 \), failure count: 52
   (0.01% of max. possible failure count)

(c) \( \phi = 1.283 \), failure count: 3062
   (0.50%)

(d) \( \phi = 1.666 \), failure count: 6948
   (1.13%)

(e) \( \phi = 1.845 \), failure count: 8910
   (1.45%) (fiber failure region)

Figure 4.28. Failure progression in small constant-length \( b/a = 0.55 \) cylinder, negative twist
(maximum possible failure count: 614,400)
Figure 4.29. Failure progression in small constant-twist-sensitivity $b/a = 0.85$ cylinder, positive twist (maximum possible failure count: 691,200)
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Figure 4.30. Failure progression in small constant-twist-sensitivity $b/a = 0.85$ cylinder, negative twist  
(maximum possible failure count: 691,200)
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Figure 4.31. Failure progression in small constant-twist-sensitivity $b/a = 0.70$ cylinder, positive twist
(maximum possible failure count: 780,800)
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Torque,
\[ T = \frac{|T|}{T_{cr}^{\text{cir}+\text{cr}}} \]

Twist,
\[ \phi = \frac{\phi_L^{\text{cir}+\text{cr}}}{L} \]

(a) torque-twist relation

Figure 4.32. Failure progression in small constant-twist-sensitivity \( b/a = 0.70 \) cylinder, negative twist (maximum possible failure count: 780,800)

(b) \( \phi = 0.957 \), failure count: 324
(0.04\% of max. possible failure count)

(c) \( \phi = 1.106 \), failure count: 9800
(1.26\%)

(d) \( \phi = 1.505 \), failure count: 16830
(2.16\%)

(e) \( \phi = 1.805 \), failure count: 24782
(3.17\%) (fiber failure region)
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Figure 4.33. Failure progression in small constant-twist-sensitivity $b/a = 0.55$ cylinder, positive twist (maximum possible failure count: 902,400)

(a) Torque-twist relation

(b) $\bar{\phi} = 1.266$, failure count: 52  
(0.01% of max. possible failure count)

(c) $\bar{\phi} = 1.381$, failure count: 756  
(0.08%)

(d) $\bar{\phi} = 1.487$, failure count: 1465  
(0.16%)

(e) $\bar{\phi} = 1.686$, failure count: 3603  
(0.40%)( fiber failure region)
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Figure 4.34. Failure progression in small constant-twist-sensitivity $b/a = 0.55$ cylinder, negative twist (maximum possible failure count: 902,400)
Figure 4.35. Untwisting after instability in postbuckling range for small constant-twist-sensitivity $b/a = 0.70$ cylinder with material failure, negative twist.
CHAPTER 5

RESULTS FOR LARGE CYLINDERS

With the response of the small cylinders discussed in the previous chapter, details of
the numerically-predicted response of the large circular and elliptical cylinders are pre-
presented and discussed in this chapter. Comparisons of responses are made among the large
cylinders and with small cylinders, so the effect of cylinder size can be examined.

The geometries of the large cylinders that are studied were given in Table 2.2. Recall
the geometries for the large cylinders were determined by increasing the length and radii
of the small cylinders by a factor of five. The wall thickness was increased by a factor of
two by doubling the number of plies in the laminate. The same aspects of the response as
discussed for the small cylinders are again discussed for the large cylinders: prebuckling,
initial postbuckling, postbuckling, and response with material failure. However, in most
cases the hysteresis-like behavior that was discussed for the small cylinder is expected
to be similar in the large cylinders. For that reason the behavior of unloading the large
cylinders will not be discussed unless the behavior is significantly different from what was
observed in the small cylinders.

The same approach used to determine the response of the small cylinders, which is
described in Figure 3.5, is used to predict the response of the large cylinders. The prebuck-
ling response of the large cylinders is discussed in Section 5.1, followed by a discussion
of the initial postbuckling state in Section 5.2. The postbuckling response of the cylinders
without including material failure in the analysis is presented in Section 5.3, and finally,
the postbuckling response when material failure is included in the analysis is presented in
Section 5.4.

5.1 Prebuckling Response

As for the small cylinders, the prebuckling response of the large cylinders is determined
using a geometrically nonlinear static analysis, and the prebuckling range of twist ends, by
Chapter 5 Results for Large Cylinders

definition, at the critical twist. The prebuckling torque-twist relations for the large cylinders are shown in Figure 5.1. For the large cylinders, the torque and twist are normalized by the critical torque and critical twist of the large circular cylinder twisted in the positive direction. As in Figure 4.3 for the small cylinders, the symbols identify the critical values for each cylinder and twist direction, and the critical values are reported in Table 5.1. The critical twist values for the large cylinders range from 10 to 17 times less than the critical twist values of their respective small cylinders, and the critical torque values of the large cylinders range from 15 to 18 times greater than the critical torque values of their respective small cylinders. The decreased critical twist values for the large cylinders is attributed to the proportionally thinner wall of these cylinders, in the sense of the ratio of the radius to the wall thickness, \( R/H \), as compared to the small cylinders. Because of the scale factor used to determine the geometry of the large cylinders, \( R/H \) is 2.5 times greater for the large cylinders. The increase in the critical torque is attributed to the increase in overall cross-section dimensions.

Table 5.1. Critical torque and critical twist for large cylinders

<table>
<thead>
<tr>
<th>aspect ratio, ( b/a )</th>
<th>length, ( L ) [in]</th>
<th>positive twist</th>
<th>negative twist</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>twist, ( \phi/L ) <a href="10%5E5">rad/in</a></td>
<td>Torque, ( T ) [lb · in]</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>64.2</td>
<td>5.66&lt;sup&gt;a&lt;/sup&gt;</td>
<td>862000&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>0.85</td>
<td>64.2</td>
<td>5.42</td>
<td>810000</td>
</tr>
<tr>
<td></td>
<td>72.1</td>
<td>6.22</td>
<td>771000</td>
</tr>
<tr>
<td>0.70</td>
<td>64.2</td>
<td>6.21</td>
<td>732000</td>
</tr>
<tr>
<td></td>
<td>81.4</td>
<td>4.38</td>
<td>610000</td>
</tr>
<tr>
<td>0.55</td>
<td>64.2</td>
<td>5.96</td>
<td>595000</td>
</tr>
<tr>
<td></td>
<td>94.3</td>
<td>3.70</td>
<td>439000</td>
</tr>
</tbody>
</table>

<sup>a</sup> values used as normalization factors

Considering the initial ranges of twist over which the torque-twist relations are linear, the circular cylinder has the greatest torsional stiffness. The reduction in the torsional stiffness, with changes in the aspect ratio of the cylinders, is the same as in the small cylinders with the torsional stiffness of the elliptical cylinders with aspect ratios of 0.85, 0.70, and 0.55 being approximately 98%, 92%, and 78% of the torsional stiffness of the circular
cylinder. Recall the torsional stiffnesses for the large cylinders were listed in Table 3.5. Also as with the small cylinders, there is nonlinear torque-twist behavior in several of the elliptical cylinders. However, when compared with the torque-twist relations for the small cylinders, Figure 4.3, the nonlinearities are more obvious for the large cylinders. In five cases, the range of twist over which the nonlinear behavior occurs is more than 20% of the prebuckling range of twist. The five cases are the constant-length cylinder with an aspect ratio of 0.70 and a positive twist, the constant-length cylinder with an aspect ratio of 0.55 and positive and negative twists, the constant-twist-sensitivity cylinder with an aspect ratio of 0.85 and a positive twist, and the constant-twist-sensitivity cylinder with an aspect ratio of 0.70 and a negative twist. These cases are identified in Figure 5.1 by having an initially constant torsional stiffness that is followed by a range of twist with a torsional stiffness that is significantly reduced from the initial stiffness. Because of their appearance, the shape of the torque-twist relation for these cases will be referred to as a dogleg-shaped relation. In three other cases of large elliptical cylinders, the behavior is similar to the behavior in the small cylinder, where softening of the torsional stiffness occurs over a very small range of twist near the critical level. The degree of softening in these three cases is not as severe as in the five previously mentioned cases with the dogleg-shaped torque-twist relation. Softening of the torsional stiffness in these three cases is difficult to see even in detailed views of the torque-twist relation. The three cases are the constant-twist-sensitivity cylinder with an aspect ratio of 0.70 and a positive twist, and the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and positive and negative twists. Interestingly, the twist at which the change in torsional stiffness occurs in the cases with the dogleg-shaped relation is slightly less than the critical value from a linear eigenvalue analysis, and for the other cases the difference between the critical twist levels from a linear eigenvalue analysis and the nonlinear static analysis is less than 0.5%.

The five cases with the dogleg-shaped torque-twist relation make it difficult to make statements regarding general trends due to changes in aspect ratio, length, and twist direction like the statements that were able to be made for the small cylinders. Recall that in the small cylinders, Figure 4.3, the critical twist and torque were smaller for smaller aspect ratios. This is still true in the large cylinders for the torque, but in four of the five cases with the dogleg-shaped torque-twist relation, the critical twist is greater than the critical twist of the circular cylinder, as seen in Figure 5.1. In the fifth case, the constant-length cylinder with an aspect ratio of 0.55 and a negative twist, the critical twist is larger than the critical twists of the constant-length cylinders with aspect ratios of 0.85 and 0.70 and negative twists. Considering only the cases for which there is no dogleg-shaped relation
for either a positive or negative twist, the critical torque and critical twist for a negative twist are roughly 8% less than for a positive twist. This comparison is made for the circular cylinder, the constant-length elliptical cylinder with an aspect ratio of 0.85, and the constant-twist-sensitivity cylinder with an aspect ratio of 0.55.

As in the small cylinders, the onset of nonlinear behavior in the torque-twist relation coincides with a sharp increase in the axial end displacement. The axial end displacement for the prebuckling range of twist is shown for the constant-length and constant-twist-sensitivity cylinders in Figures 5.2 and 5.3, respectively. In the cases with the dogleg-shaped torque-twist relation, the axial end displacement is approximately two orders of magnitude greater than the other cases. When the scale is set to show the full range of axial end displacement for these cases, details of the relations of the remaining cases cannot be seen. To show the details of the axial end displacement response for the cases that do not have a dogleg-shaped torque-twist relation, Figures 5.2 and 5.3 are repeated in Figures 5.4 and 5.5, but with the scale of the vertical axis reduced by two orders of magnitude. In Figure 5.5a the sharp increases in the axial end displacement that could not be seen in Figure 5.3a are seen for the constant-twist-sensitivity cylinders with aspect ratios of 0.70 and 0.55 and positive twists. Similarly the sharp increase in the axial end displacement of the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a negative twist can be seen in Figure 5.5b. These last three cases are the cases that have nonlinear torque-twist behavior for a small range of twist near the critical value. Wrinkling occurs in the cylinders that have softening of the torsional stiffness and a sharp increase in the axial end displacement.

A detailed view of the torque and axial end displacement responses over the range of twist near the change in slope of the torque and axial end displacement responses is shown along with color contour plots of the normal deformation of the cylinder wall at certain twist levels in Figure 5.6 for the constant-length elliptical cylinder with an aspect ratio of 0.70 and a positive twist. The results presented in Figure 5.6 are used to describe the development of wrinkles in the prebuckling range and relate this to the softening of the torsional stiffness and sharp increase in the axial end displacement. In Figure 5.6a, the scale for the torque is on the left vertical axis and the scale for the axial end displacement is on the right vertical axis. The scale for the twist is on the horizontal axis. Notice the range of twist does not extend to the critical value because to show the axial end displacement response for the range of twist up to the critical value would require the scale for the right vertical axis to be increased by two orders of magnitude, and would make it difficult to see the details of the response where the change in slope occurs. The normal deformation
at a twist of $\phi = 0.844$ is shown in Figure 5.6b, and is representative of the deformation for all twist levels below $\phi = 0.844$, with the largest normal deformations occurring near the sides of the cylinder. The torque-twist relation is linear at this twist value, and there is not a significant change in the slope of the axial end displacement response. At a twist of $\phi = 0.876$, the largest normal deformations still occur near the sides of the cylinder, but near the crown and keel the deformations are starting to have the character of a valley of a wrinkle, as shown in Figure 5.6c. There is still not a change in the slope of the torque-twist relation, nor is there a significant change in the slope of the axial end displacement response. With a small increase in twist to $\phi = 0.879$, wrinkle-like deformations have developed near the crown and keel, and the amplitudes of these deformations are larger than the deformations at the sides of the cylinder, as can be seen in Figure 5.6d. The slope of the torque-twist relation is not changed, and there is no significant change in the slope of the axial end displacement response. For twist values greater than $\phi = 0.879$, there is a significant softening of the torsional stiffness and a significant increase in the slope of the axial end displacement response. Wrinkles have developed around the entire circumference at a twist of $\phi = 0.882$, as is shown in Figure 5.6e, and the magnitudes of the largest normal deformations have increased by an order of magnitude from a twist of $\phi = 0.879$. Between a twist of $\phi = 0.882$, Figure 5.6e, and $\phi = 1.095$, Figure 5.6f, the normal deformations increase again by an order of magnitude, with the wrinkles deepening around the entire circumference. Although the entire range is not shown in the detailed view in Figure 5.6a, the slopes of the torque and axial end displacement responses are almost linear between twists of $\phi = 0.879$ and $\phi = 1.095$. When compared to the somewhat gradual softening of the torsional stiffness in the small cylinder, Figure 4.6a, softening of the torsional stiffness in the large cylinder is an almost discrete change in slope of the torque-twist response. An analogous statement can also be made about the change in slope of the axial end displacement.

As a result of wrinkles developing in the prebuckling responses for some of the cases, the character of the normal deformations can be quite different depending on whether no wrinkles develop, wrinkles develop in only the flatter part of the cross section, or wrinkles develop around the entire circumference of the cylinder. Color contour plots of the normal deformations are shown on rolled-out views of the cylinder for the constant-length cylinders in Figures 5.7 and 5.8 and for the constant-twist-sensitivity cylinders in Figures 5.9 and 5.10. The normal deformation is shown for a twist value that is slightly below the critical twist value. The case for the circular cylinder with a positive twist is shown in both Figures 5.7a and 5.9a, and for a negative twist in Figures 5.8a and 5.10a.
As in the small circular cylinders, the deformation in the large circular cylinders is axi-symmetric and is an inward deformation that is a small fraction of the total wall thickness. This is true for both positive and negative twists. The four cases of elliptical cylinders with linear torque-twist responses have normal deformations that are less than 1.5% of the wall thickness, as shown in Figures 5.7b, 5.8b, 5.8c, and 5.10b, and the largest deformations occur nearer the sides than the crown or keel of the cylinder. For the three cases in which there is nonlinear torque-twist behavior over a small range of twist near the critical value, the normal deformation patterns have wrinkles deformations that have developed in the flattest part of the cross section near the crown and keel, and the normal deformation patterns for these three cases are shown in Figures 5.9c, 5.9d, and 5.10d. The largest deformations occur on the ridges and valleys of the wrinkles, with the largest amplitudes ranging from approximately 4% of the wall thickness in the constant-twist-sensitivity cylinder with an aspect ratio of 0.70 and a positive twist (Fig. 5.9c) to approximately 20% of the wall thickness in the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a positive twist (Fig. 5.9d). In the five cases with dogleg-shaped torque-twist relations, the largest normal deformations at a twist slightly less than the critical value, shown in Figures 5.7c, 5.7d, 5.8d, 5.9b, and 5.10c, are at least one wall thickness for the ridges of the wrinkles and at least four wall thicknesses for the valleys of the wrinkles. As with the small cylinders, the reason why wrinkles do not develop in the prebuckling range for all cylinders will be discussed in the following section when discussing the initial postbuckling response, and as in the small cylinders it is related to the number of circumferential waves in the initial postbuckling configuration.
Figure 5.1. Torque-twist relations of large cylinders
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Figure 5.2. Axial end displacement-twist relations of large constant-length cylinders
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Figure 5.3. Axial end displacement-twist relations of large constant-twist-sensitivity cylinders
Figure 5.4. Axial end displacement-twist relations of large constant-length cylinders, detailed view
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Figure 5.5. Axial end displacement-twist relations of large constant-twist-sensitivity cylinders, detailed view

(a) positive twist

(b) negative twist
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Figure 5.6. Prebuckling response in large constant-length $b/a = 0.70$ cylinder, positive twist
Figure 5.7. Normal deformation at a twist level slightly less than the critical value for large constant-length cylinders, positive twist
Figure 5.8. Normal deformation at a twist level slightly less than the critical value for large constant-length cylinders, negative twist
Figure 5.9. Normal deformation at a twist level slightly less than the critical value for large constant-twist-sensitivity cylinders, positive twist
Figure 5.10. Normal deformation at a twist level slightly less than the critical value for large constant-twist-sensitivity cylinders, negative twist
5.2 Initial Postbuckling Response

The initial postbuckling configuration for the large constant-length cylinders is shown in Figures 5.11 and 5.12 and for the constant-twist-sensitivity cylinders in Figures 5.13 and 5.14. As with the figures for the small cylinders, within each subfigure the deformed shape is shown on the left and a color contour plot of the normal deformations on a rolled-out view of the cylinder is shown on the right. The initial postbuckling configuration of the circular cylinder under a positive twist is shown in both Figures 5.11a and 5.13a and under a negative twist in both Figures 5.12a and 5.14a.

In general, the postbuckling shapes of the large cylinders are similar to the shapes of the small cylinders, Figures 4.11-4.14, both having a number of circumferential waves, with ridges and valleys arranged in a helical manner around the circumference. Also, like the initial postbuckling shapes of the small circular cylinder, the amplitudes of the ridges and valleys are uniform around the circumference of large circular cylinder. The amplitudes of the ridges and valleys are greater in the flatter part of the cross section than in the more curved regions of the cross section in the large elliptical cylinders, as was seen in the small cylinders.

The magnitudes of the normal deformations among the initial postbuckling configurations of the large cylinders are different, depending on the behavior in the prebuckling range. In the circular cylinder, the amplitudes of the wrinkles for the initial postbuckling response, e.g., Figure 5.11a, are approximately three orders of magnitude greater than the inward deformation at a twist slightly less than the critical value, e.g., Figure 5.7a. For the three elliptical cylinders with aspect ratios of 0.85 in which wrinkles did not develop in the prebuckling range, the amplitudes of the largest deformations, Figures 5.11b, 5.12b, and 5.14b, are about two orders of magnitude greater than before buckling. The amplitude of the normal deformations in the initial postbuckling configuration of the constant-length elliptical cylinder with an aspect ratio of 0.70 and a negative twist, Figure 5.12c, which is the fourth case that did not develop wrinkles in the prebuckling range, are more than one order of magnitude larger than the deformations prior to buckling. With the three cases in which prebuckling wrinkles only develop in the flatter part of the cross section, as was shown in Figures 5.9c, 5.9d, and 5.10d, the increase in the magnitude is approximately one order of magnitude for the constant-twist-sensitivity cylinder with an aspect ratio of 0.70 and a positive twist, Figure 5.13c, and an increase of roughly two to six times for the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and positive and negative twists, Figures 5.13d and 5.14d. In the five cases with the dogleg-shaped torque-twist...
relations, which also develop prebuckling wrinkles with amplitudes of multiple wall thicknesses, the deformations in the initial postbuckling state are of approximately the same magnitude, Figures 5.11c, 5.11d, 5.12d, 5.13b, and 5.14c.

As in the small cylinders, the number of circumferential waves in the large cylinders is determined by cylinder geometry and twist direction. The number of circumferential waves is counted using the normal deformations at the cylinder midlength. There are nine circumferential waves in the circular cylinder for either direction of twist. There are also nine circumferential waves in three cases of cylinders with an aspect ratio of 0.85, Figures 5.11b, 5.12b, and 5.14b, and the fourth case, the constant-twist-sensitivity cylinder with a positive twist, Figure 5.13b, has eight circumferential waves. As in the small cylinders, the effect of cylinder length on the number of circumferential waves for an aspect ratio of 0.85 is illustrated by the shorter constant-length cylinder having one more circumferential wave than the longer constant-twist-sensitivity cylinder, both for positive twist. The constant-twist-sensitivity cylinder under positive and negative twists with eight and nine circumferential waves, respectively, illustrates that twist direction still has an effect on the number of circumferential waves in the large cylinders. The larger number of circumferential waves in the large cylinders, when compared to their respective geometries in the small cylinders, is again due to the larger values of the ratio of the radius of curvature to the wall thickness, \( R/H \). For larger values of \( R/H \), there is less resistance to bending in the cylinder wall, and as a result there are more wrinkles in the initial postbuckling state of the large cylinders.

Similarly in the cylinders with an aspect ratio of 0.70, there are more circumferential waves in the initial postbuckling state in the large cylinders than in the small cylinders. In all four cases of the cylinders with an aspect ratio of 0.70, there are eight circumferential waves in the initial postbuckling state, Figures 5.11c, 5.12c, 5.13c, and 5.14c. Although the postbuckling behavior of the cylinders will be discussed in the following section, it is worth noting that early in the postbuckling response of the constant-length cylinder with an aspect ratio of 0.70 and a negative twist, Figure 5.12c, there is transition to a shape with nine circumferential waves then back to eight circumferential waves. This implies that for a slightly shorter length cylinder, the initial postbuckling state could possibly have nine circumferential waves.

To complete the discussion of the number of circumferential waves in the initial postbuckling state of the large cylinders, the cases of the constant-length cylinder with an aspect ratio of 0.55 and positive and negative twists have eight circumferential waves in the initial
postbuckling state, Figures 5.11d and 5.12d. The constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a positive twist has seven circumferential waves in the initial postbuckling state, Figure 5.13d. This number is only valid when counting the waves at the cylinder midlength, because the waves in the more curved sides of the cylinder are not fully formed along the length of the cylinder. However, early in the postbuckling range the waves do develop along the entire length of the cylinder. The large constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a negative twist is the one case where there are the same number of circumferential waves in the initial postbuckling state as in the small cylinder, Figures 5.14d and 4.14d. There are six circumferential waves in both cases.

A general statement can be made for all five cases that have a dogleg-shaped torque-twist relation in the prebuckling range. All five of those cases have an even number of circumferential waves, Figures 5.11c, 5.11d, 5.12d, 5.13b, and 5.14c.

Finally, because untwisting of cylinders from the initial postbuckling response for the small cylinders and a dogleg-shaped prebuckling torque-twist relation was not seen in the small cylinders, the large constant-length elliptical cylinder with an aspect ratio of 0.70 and a positive twist, which exhibits this behavior, is untwisted from the initial postbuckling state. The resulting torque-twist relation is shown in Figure 5.15. As with the small cylinders, there is a hysteresis-like behavior for this case. The torque-twist relation for untwisting crosses the prebuckling torque-twist relation twice, once near a twist of $\bar{\phi} = 1.00$ and a second time near the 'knee' in the dogleg-shaped prebuckling torque-twist relation. Slightly below the 'knee' in the prebuckling response, the untwisting torque-twist relation returns to the prebuckling path.
Figure 5.11. Deformed shape and normal deformation of initial postbuckling state for large constant-length cylinders, positive twist
Figure 5.12. Deformed shape and normal deformation of initial postbuckling state for large constant-length cylinders, negative twist
Figure 5.13. Deformed shape and normal deformation of initial postbuckling state for large constant-twist-sensitivity cylinders, positive twist.
Figure 5.14. Deformed shape and normal deformation of initial postbuckling state for large constant-twist-sensitivity cylinders, negative twist.
Figure 5.15. Untwisting from initial postbuckling state of large constant-length $b/a = 0.70$ cylinder, positive twist
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5.3 Postbuckling Response

The torque-twist relations including the postbuckling responses are shown in Figure 5.16 for the constant-length cylinders and in Figure 5.17 for the constant-twist sensitivity cylinders. The range of twist is the same as in Figures 4.16 and 4.17, however, as will be discussed in the next section, a range of twist to $\phi = 12$ is needed for some large cylinders to include the twist at which the first fiber failure occurs. The torque-twist relations for this range of twist are shown in Figures 5.18 and 5.19 for the constant-length cylinders and constant-twist-sensitivity cylinders, respectively. Note the torque-twist relation for the circular cylinder with a positive twist is shown in both Figures 5.16a and 5.17a, and also for a negative twist in both Figures 5.16b and 5.17b. This is also true for the relations shown for the extended ranges of twist in Figures 5.18 and 5.19.

The torque-twist relation in the initial postbuckling range of the large circular cylinders has a negative slope for both twist directions, Figures 5.16a and 5.16b. The slope remains negative over the entire range of twist considered. Recall, from Figures 4.16 and 4.17, the slope was also negative for the small circular cylinders. In three cases of the large cylinders with aspect ratios of 0.85, the slope of the torque-twist relation is positive in the initial postbuckling range, Figures 5.16 and 5.17. In the fourth case, the constant-twist-sensitivity cylinder with a positive twist, Figure 5.17a, the slope is negative, but it is also the only case of the cylinders with an aspect ratio of 0.85 that has a dogleg-shaped prebuckling torque-twist relation. Recall, again from Figures 4.16 and 4.17, the slopes were initially negative for all the small cylinders with an aspect ratio of 0.85. Similarly, there are three cases for the large cylinders with an aspect ratio of 0.70 with positive slopes in the torque-twist relations for the initial postbuckling response, Figures 5.16 and 5.17. The fourth case is the constant-twist-sensitivity cylinder with a negative twist, Figure 5.17b, and again it is one of the cases with a dogleg-shaped torque-twist relation in the prebuckling range. In this case, the slope of the initial postbuckling path is approximately zero. For cylinders with an aspect ratio of 0.55, the slope of the torque-twist relation is positive for the initial postbuckling response in all four cases, including those with the dogleg-shaped torque-twist relations.

At a twist of $\phi = 2.5$, the end of the twist range for the small cylinders and in Figures 5.16 and 5.17, the slope of the torque-twist relation for the circular cylinder remains negative. Also in all cases of cylinders with an aspect ratio of 0.85, the slope of the postbuckling response is negative at a twist of $\phi = 2.5$. For the constant-length cylinders with an aspect ratio of 0.70, the slope of the postbuckling response is negative at a twist of $\phi = 2.5$ for both twist directions. The postbuckling response of the constant-length cylin-
der with an aspect ratio of 0.55 has a positive slope at a twist of $\bar{\phi} = 2.5$ for both positive and negative twists. The slope is approximately zero at a twist of $\bar{\phi} = 2.5$ for the constant-twist-sensitivity cylinders with aspect ratios of 0.70 and 0.55 for both twist directions.

Similar to the behavior in the small cylinders, Figures 4.16 and 4.17, the difference between the torque levels for cylinders with different aspect ratios decreases with increasing twist, for both constant-length and constant-twist-sensitivity cylinders and positive and negative twists, Figures 5.16 and 5.17, i.e., the torque-twist relations tend to converge for increasing twist levels. As with the small cylinders, the difference between the torque values for the large constant-length cylinders at a twist of $\bar{\phi} = 2.5$ is less than difference between the torque values of the constant-twist-sensitivity cylinders at the same twist. Again, this can be attributed to the difference in effects of the boundaries between the constant-length and the longer constant-twist-sensitivity cylinders. The slope of the torque-twist relations of the large constant-length cylinders also remains positive over a larger range of twist than the constant-twist-sensitivity cylinders.

As with the small cylinders, instabilities occur in the postbuckling responses of most of the cases. Considering the larger range of twist shown in Figures 5.18 and 5.19, there are no instabilities in the postbuckling responses of the large circular cylinders, but there is at least one instability for each case involving the elliptical cylinders. Again, the instabilities in the postbuckling range are identified by a sharp drop in torque in the torque-twist relation. However, only the instabilities in the range of twist before the first fiber failure occurs are of interest. For example, in the large constant-length cylinder with an aspect ratio of 0.85 and a negative twist, Figure 5.18b, the instability that occurs near a twist of $\bar{\phi} = 11$ is not of interest because the first fiber failure occurs at a twist of $\bar{\phi} = 6.95$. The instabilities are discussed in this section, because even for instabilities that occur after the initial failures occur, the character of the change in shape as a result of the instability is the same whether or not material failure is considered, although as discussed for the small cylinders, the twist at which the instability occurs may be larger when material failure is included, e.g., Figure 4.32.

In addition to the types of instabilities that were seen with the small cylinders, which were either changes in the circumferential location of the wrinkles, Figure 4.18, or a change in the number of circumferential waves, Figure 4.19, there are instabilities in the large cylinders that result in localized changes in the deformation pattern only near the sides of the cylinder. This type of change in the deformation pattern is illustrated in Figure 5.20 for the constant-length cylinder with an aspect ratio of 0.70 and a positive twist and will
be discussed shortly. Illustrations of the changes in the circumferential location of the wrinkles and a change in the number of circumferential waves will not be shown for the large cylinder, because the character of the change in the deformation pattern is the same as illustrated for the small cylinder in Figures 4.18 and 4.19. Also note that deep in the postbuckling range of constant-twist-sensitivity cylinders with aspect ratios of 0.70 and 0.55, Figures 5.19a and 5.19b, that there are several small but sharp drops in torques. For most of these instabilities there is localized buckling occurring, either near the sides of the cylinder or at the ends of the cylinder. Most of these instabilities are out of the range of interest once material failure is included in the analysis, and so will not be discussed in detail.

To catalogue the instabilities with respect to the change in shape that occurs, a brief description of the changes in shape that occur in the range of twist that is of interest in the postbuckling response of each of the cases for the elliptical cylinders will be given. First consider the constant-length cylinders with a positive twist, Figure 5.18a. For the cylinder with an aspect ratio of 0.85, the only instability of interest occurs near a twist of \( \phi = 3.0 \). The change in shape is a localized change near the sides of the cylinder, and no large drop in torque occurs. There are two instabilities in the cylinder with an aspect ratio of 0.70. One, near a twist of \( \phi = 2.2 \) that results in shift in the circumferential location of the wrinkles, and another near a twist of \( \phi = 3.4 \) where there is a localized change in shape near the sides. Only one of the instabilities in the cylinder with an aspect ratio of 0.55 is of interest, and it occurs near a twist of \( \phi = 3.0 \), and results in a shift of the circumferential location of the wrinkles.

Considering the constant-length cylinders with a negative twist, Figure 5.18b, the cylinder with an aspect ratio of 0.85 has one instability in the range of twist that is of interest for this case. The instability occurs near a twist of \( \phi = 3.2 \), and is a localized change in the shape near the sides of the cylinder. For the cylinder with an aspect ratio of 0.70 there are changes in number of circumferential waves that occur near twists of \( \phi = 0.97 \) and \( \phi = 1.05 \) and there is no drop in torque that occurs, but only a change in the slope of the torque-twist relation (see Figure 5.16b). Near a twist of \( \phi = 0.97 \), the change is from eight to nine circumferential waves, and near a twist of \( \phi = 1.05 \), the change is from nine back to eight circumferential waves. For this cylinder an instability occurs near a twist of \( \phi = 2.0 \) that results in a shift of the circumferential location of the wrinkles, and another instability occurs near a twist of \( \phi = 3.4 \) that results in a localized change in shape near the sides of the cylinder. Only one instability occurs in the range of interest for the cylinder with an aspect ratio of 0.55, and it occurs near a twist of \( \phi = 2.6 \) and results in a shift in the
Continuing to describe the instabilities that occur at twists within the range of interest, consider the constant-twist-sensitivity cylinders with a positive twist, Figure 5.19a. For the cylinder with an aspect ratio of 0.85, an instability occurs near a twist of $\bar{\phi} = 4.1$ that results in a localized change in the shape near the sides of the cylinder. There are three instabilities that occur within the range of interest for the cylinder with an aspect ratio of 0.70. The first occurs near a twist of $\bar{\phi} = 1.2$ and is a shift in the circumferential location of the wrinkles. The second occurs near a twist of $\bar{\phi} = 2.5$, and is a change in the number of circumferential waves from a shape with eight waves to one with seven. The third instability occurs near a twist of $\bar{\phi} = 2.8$, and is a localized change in shape near the sides of the cylinder. For the cylinder with an aspect ratio of 0.55, there are two instabilities that occur within the range of interest. One occurs near a twist of $\bar{\phi} = 0.72$. There is only a small drop in torque, and the shape changes from having seven circumferential waves to a shape with six circumferential waves. The second instability occurs near a twist of $\bar{\phi} = 1.2$, and results in a change in the circumferential location of the wrinkles.

Finally, consider the constant-twist-sensitivity cylinders with a negative twist, Figure 5.19b. There are two instabilities in the cylinder with an aspect ratio of 0.85. The first occurs near a twist of $\bar{\phi} = 2.8$ and results in a localized change in shape near the sides of the cylinder. The second occurs near a twist of $\bar{\phi} = 4.0$ and results in a change from eight circumferential waves to 7 circumferential waves. One instability occurs within the range of interest for the cylinder with an aspect ratio of 0.70. The instability occurs near a twist of $\bar{\phi} = 2.6$ and results in a shift in the circumferential location of the wrinkles. Also, for the cylinder with an aspect ratio of 0.55, the only instability that occurs in the range of interest is near a twist of $\bar{\phi} = 1.10$, and results in a change of the circumferential location of the wrinkles.

Because an example has not yet been shown, the type of change in the deformation pattern that is characterized by localized changes in the deformations in the sides of the cylinder is illustrated in Figure 5.20 for the large constant-length cylinder with an aspect ratio of 0.70 and a positive twist. At a twist of approximately $\bar{\phi} = 3.4$ and before the drop in torque, Figure 5.20a, there are eight circumferential waves in the deformation pattern, Figure 5.20b. After the drop in torque, there are still eight circumferential waves in the deformation pattern, Figure 5.20c. Near the flatter crown and keel, there does not appear to be a significant difference in the deformation pattern. The only significant difference appears to be in the valleys that cross the cylinder midlength at circumferential locations.
of $s/C = 0.25$ and $0.75$, the sides of the cylinder. In these valleys there are inward dimples, areas of increased inward normal deformation, that have formed in the bottom of the valleys. As listed above, this type of localized change in the deformation pattern occurs in most of the cases of large cylinders with aspect ratios of $0.85$ and $0.70$.

Having discussed the postbuckling response of the large cylinders, and seen behavior that is similar to what was observed for the small cylinders, the next step is to include material failure in the analysis. The effect of material failure on the response of the large cylinders is discussed in the following section.
Figure 5.16. Torque-twist relations of large constant-length cylinders, including postbuckling paths
Figure 5.17. Torque-twist relations of large constant-twist-sensitivity cylinders, including postbuckling paths
Figure 5.18. Torque-twist relations of large constant-length cylinders, including extended postbuckling paths
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Figure 5.19. Torque-twist relations of large constant-twist-sensitivity cylinders, including extended postbuckling paths.

Torque, \[ T = \left| \frac{T}{T_{cir+}} \right| \]

Twist, \[ \bar{\phi} = \left| \frac{\phi}{L \cdot \phi_{cir+}} \right| \]

(a) positive twist

(b) negative twist
Figure 5.20. Postbuckling change in deformation pattern, localized change near sides, large constant-length $b/a = 0.70$ cylinder, positive twist
5.4 Influence of Material Failure

The response of the large cylinders when material failure is included in the finite element analysis is presented in this section. As with the small cylinders, material failure does not occur until deep into the postbuckling range. Also, as with the small cylinders, the analysis is stopped at the twist at which first fiber failures occur. Using the same format for the figures that was used for the small cylinders, the failure characteristics for each of the large cylinders are illustrated in Figures 5.21-5.34. The failure characteristics are also summarized in Table 5.2 for the twist at which the first failures occur and the twist at which first fiber failures occur.

To provide the baseline to which the responses of the elliptical cylinders are compared, the response of the circular cylinder is discussed first, and the failure characteristics for the large circular cylinder with a positive twist are illustrated in Figure 5.21. Initial failures occur at a twist of $\phi = 5.54$. The locations of the initial failures are in the outermost 0-degree ply, ply number 14, and they occur near the ends of wrinkled area on each of the ridges around the circumference, Figure 5.21b. The initial failures are matrix tensile failures. The axial location of the initial failures in the large circular cylinder are in contrast to the case of the small circular cylinder, Figure 4.21b, where initial failure is at the cylinder midlength. With an increase in twist to $\phi = 6.08$, failures occur near the midlength of the cylinder in each of the valleys and on each ridge, Figure 5.21c. With further increases in twist, the failure locations spread along the more developed regions of the ridges and valleys. The first fiber failures, which are compressive failures, occur at a twist of $\phi = 10.56$, in the innermost +45 degree ply, ply number 1. The location of the fiber failures are marked by the white circle. As with the small cylinders, due to the small number of ply points with fiber failures, only one in this case, failure does not occur at uniform locations around the circumference. However, with an increase in twist beyond the level at which the first fiber failures occur, locations with fiber failures of occur on each of the ridges. At the twist for first fiber failures, 2.1% of the cylinder contains failure, Figure 5.21e. Until the first fiber failures occur, all failures are matrix tensile failures. At a twist of $\phi = 10.56$, failures have occurred in ply numbers 1, 2, 3, 14, 15, and 16, and the largest amount of failure in any element is approximately 20%. Over the range of twist from first failures to first fiber failures, there is only a slight decrease in the torsional stiffness compared to the torsional stiffness when material failure is not considered, as Figure 5.21a depicts.

The failure characteristics of the large circular cylinder with a negative twist are illustrated in Figure 5.22. Initial failures, which are matrix tensile failures, occur at a twist
of $\phi = 4.32$, and occur in the innermost $+45$-degree ply, ply number 1, near the cylinder midlength in each of the valleys of the wrinkles, Figure 5.22b. Except for the number of circumferential waves, the illustration of failure looks much like the small circular cylinder with a negative twist, Figure 4.22b. At a twist of $\phi = 5.21$, Figure 5.22c, failures have spread along the length of the valleys of the wrinkled area. Additional locations of failures have occurred on each of the ridges, and are located near the ends of the more developed region. With increasing twist, Figure 5.22d, failures spread along the length of the ridges. At the twist at which first fiber failures occur, $\phi = 9.40$, failures have occurred along the length the ridges and valleys in the wrinkled area, Figure 5.22e. The first fiber failures are compressive failures, and occur in the innermost $+45$ degree layer near the end of the wrinkled area. The locations of the fiber failures are marked by the white circles. At the twist at which first fiber failures occur, failures have occurred in ply numbers 1, 3, 14, 15, and 16. Also, about 2.1% of the cylinder has failed, and the largest amount of failure in any element is approximately 19%. The decrease in the torsional stiffness due to matrix failures, Figure 5.22a, is very small.

There are similarities in the progression of failure among all of the large elliptical cylinders, particularly those with aspect ratios of 0.70 and 0.55, and for this reason, the progression of failure will be described in detail for only one geometry, the large constant-length cylinder with an aspect ratio of 0.70. What differences there are between cylinders will be discussed in general terms. The failure characteristics of the large constant-length cylinder with an aspect ratio of 0.70 and a positive twist are illustrated in Figure 5.25. When failure is included in the analysis, the change in torsional stiffness is very small when compared to the torsional stiffness when material failure is not considered, Figure 5.25a. Initial failures occur at a twist of $\phi = 3.37$, Figure 5.25b, and are matrix tensile failures in the outermost 0-degree ply, ply number 14. The initial failures are located on ridges near the sides of the cylinder, near the ends of wrinkled area. These failures occur before the second drop in torque that occurs due to the instability. Recall, as shown in Figure 5.20, the change in shape that occurs as a result of the instability is local buckling that results in dimples growing in the valleys of the wrinkles at the sides of the cylinder. After the cylinder settles into its new shape, the low failure count only doubles and there are no new locations with failure, Figure 5.25c. With an increase in twist to $\phi = 4.32$, the failure count increases by an order of magnitude, and failures have spread to other ridges and valleys at the sides of the cylinder, Figure 5.25d. First fiber failures occur at a twist of $\phi = 5.20$, and the first fiber failures are compressive failures, Figure 5.25e. The locations of the fiber failures are marked by the white circles, and for this case there are three locations with fiber failures.
that can be seen in Figure 5.25e. The fiber failures occur on ridges near the end of the wrinkled area and near the sides of the cylinder in the innermost $-45$ degree ply, ply number 2. At this twist only $0.16\%$ of the cylinder has failed, and the largest amount of failure in any element is approximately $19\%$. At the twist at which first fiber failures occur, failures have occurred in layers 2, 3, 14, 15, and 16.

The failure characteristics of the constant-length cylinder with an aspect ratio of 0.70 and a negative twist are illustrated in Figure 5.26. As with the other large cylinders that have been discussed above, there is very little change in the torsional stiffness as a result of material failure, Figure 5.26a. Initial failures occur in a valley near the end of the wrinkled area near the side of the cylinder, and occur near a twist of $\phi = 2.58$, Figure 5.26b. The initial failures are matrix tensile failures, and occur in the innermost ply, ply number 1. With an increase of twist to $\phi = 3.40$, which is just before the second sharp drop in torque, failures have spread within the valley and have also developed on the adjacent ridge, Figure 5.26c. After the drop in torque, where the character of the change in shape is the same as for a positive twist, namely dimples growing in the valleys of the wrinkles at the sides of the cylinder, failures have occurred in the dimple, Figure 5.26d. First fiber failures occur at a twist of $\phi = 4.86$, and are compressive failures in the innermost $+45$ degree ply, ply number 1, Figure 5.26e. Only $0.21\%$ of the cylinder has failed at this twist, and the largest amount of failure in any element is approximately $19\%$.

As stated, in general terms, the initiation and progression of failure in the large elliptical cylinders is similar for all aspect ratios; however, there is less spreading of failure to new locations for cylinders with smaller aspect ratios. The initial failures in the elliptical cylinders occur near the sides of the cylinder, and close to the end of the wrinkled area, Figures 5.23b-5.34b. For a positive twist, the initial failures occur on a ridge of a wrinkle, and for a negative twist, in a valley of a wrinkle. When the twist is increased, failures spread along the ridges and valleys along the side of the cylinder. For the cylinders with an aspect ratio of 0.85, failure locations also spread to ridges and valleys closer to the crown and keel, Figures 5.23e, 5.24e, 5.29e, and 5.30e. After the initial failure, there are almost no new failure locations in the cylinders with an aspect ratio of 0.55, Figures 5.27e, 5.28e, 5.33e, and 5.34e, or the constant-length-sensitivity cylinders with an aspect ratio of 0.70, Figures 5.31e and 5.32e. For all of the large cylinders, the first fiber failures occur on ridges near the end of the wrinkled area. In the elliptical cylinders the first fiber failures are located on the end of the ridge that is closest to the side of the cylinder, Figure 5.23e-5.34e.
When compared to the initiation and progression of failure with the small circular cylinders, the initiation and progression of failure in the large circular cylinder are very similar. In both sizes of circular cylinders, initial failures occur on each of the ridges or in each of the valleys around the circumference. The difference, as noted, is in the axial location. And with increasing twist, these failure locations grow, and at the twist for first fiber failures, failures have spread along the length of the ridges and valleys of the wrinkled area, but not to the boundaries of the cylinder. For the elliptical cylinders, the progression of failure is very different in the small and large cylinders. Initial failures in the small elliptical cylinders occur on ridges and in valleys near the flatter crown and keel often near the midlength, and with increasing twist failure spreads to the ridges and valleys in the more curved sides of the cylinder. However, in the large cylinders, initial failures occur on ridges and in valleys in the more curved sides often toward the end of the wrinkled area, and with increasing twist spread to ridges and valleys in the flatter part of the cross section. As noted, though, spreading of failure after initial failure decreases with decreasing aspect ratio.

Considering the summary in Table 5.2. The twist at which first failures occur is smaller for cylinders with smaller aspect ratios. This is true for constant-length and constant-twist sensitivity cylinders and for both directions of twist. The twist at which first fiber failures occur is also smaller for cylinders with smaller aspect ratios. The percent failure in a cylinder is less for smaller aspect ratios. For example, the percent failure in the constant-length cylinders with a negative twist is 2.11%, 0.56%, 0.16%, and 0.07% for cylinders with aspect ratios of 1.00, 0.85, 0.70, and 0.55. For any cylinder geometry, the twist for both first failures and first fiber failures is larger for a positive twist than for a negative twist. Except for the circular cylinder and the constant-twist-sensitivity cylinder with an aspect ratio of 0.55 and a negative twist, a larger percentage of the cylinder fails under a negative twist than for a positive twist. For any of the three elliptical aspect ratios, and with only one exception, the twist for first failures and first fiber failures is larger for the constant-length cylinder than for the constant-twist-sensitivity cylinders, and this is true for either twist direction. The one exception is the cylinder with an aspect ratio of 0.70 with a negative twist, where the twist for initial failure is smaller for the constant-length cylinder than for the constant-twist-sensitivity cylinder. Also, for any elliptical aspect ratio, the percent failure in the cylinder is greater for the constant-length cylinder than for the constant-twist-sensitivity cylinder. Again, this is for either twist direction. In general, the progression of failure through the thickness is approximately the same for all of the large cylinders, although it is interesting that no failures occur in the innermost −45 degree layer, layer number 2, with a negative twist.
In comparing the failure characteristics of the large and small cylinders in Tables 5.2 and 4.2, the initial failures in the large cylinders occur deeper into the postbuckling range than the first fiber failures in the small cylinders. Except for the circular cylinders with a positive twist, a significantly larger percent failure accumulates in the small cylinders than in the large cylinders. Interestingly, the large cylinders can be twisted further in the positive direction than in the negative direction before first fiber failure occurred, while the small cylinders can generally be twisted further in the negative direction. In some cases the difference in between the percent failure in the small and large cylinders is greater than one order of magnitude. The effect of smaller aspect ratios is greater in the large cylinders than in the small cylinders. For instance, for the small circular cylinder with a positive twist the percent failure is 1.38% and for the constant-length cylinders with an aspect ratios of 0.85, 0.70, and 0.55 and a positive twist, the percent failures are 1.37%, 1.30%, and 0.67%, respectively. However, for the large circular cylinder with a positive twist, the percent failure is 2.1%, and for the constant-length cylinders with aspect ratios of 0.85, 0.70, and 0.55 and a positive twist, the percent failures are 0.56%, 0.16%, and 0.07%. Analogous statements can be made about the constant-length cylinders with a negative twist and the constant-twist-sensitivity cylinders with either twist direction.

In this chapter the behavior of large composite circular and elliptical cylinders has been examined. The aspects of the response that were considered are the prebuckling response, the initial postbuckling state, the postbuckling response, and the postbuckling response when material failure is included in the analysis. The effect of aspect ratio, cylinder length and twist direction was studied, and comparisons were also made with the results for the small cylinders that were presented in Chapter 4. Between this chapter and Chapter 4, a good physical understanding has been developed for how quasi-isotropic composite cylinders with elliptical cross sections respond when one end is twisted relative to the other. A range of geometries has been explored, in addition to the inclusion of material failure. The next chapter offers a brief summary of the results, and conclusions, and provides thoughts on future directions.
### Table 5.2. Summary of failure characteristics for large cylinders

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<th>aspect ratio, $b/a$</th>
<th>length, $L$</th>
<th>twist, twist, $\phi$</th>
<th>layer nos. with failure</th>
<th>layer nos. with fiber failures</th>
<th>% maximum possible failure count</th>
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<tr>
<td>1.00 64.2 (+)</td>
<td></td>
<td>5.54 14$^a$</td>
<td>10.56 1, 2, 3, 14, 15, 16 1</td>
<td>25923 2.11</td>
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<td>1.00 64.2 (−)</td>
<td></td>
<td>4.32 1</td>
<td>9.40 1, 3, 14, 15, 16    1</td>
<td>25400 2.07</td>
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<td>7.11 1, 2, 3, 14, 15, 16 1</td>
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<td>5.82 1, 3, 14, 15, 16    1</td>
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<td>1902 0.16</td>
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<td>4.22 1, 2, 3, 14, 16    2</td>
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<td>3.10 1, 14, 15, 16      1</td>
<td>440 0.024</td>
<td></td>
</tr>
</tbody>
</table>

$^a$ outermost 0-degree layer
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Figure 5.21. Failure progression in large circular cylinder, positive twist (maximum possible failure count: 1,228,800)

(a) torque-twist relation

(b) $\phi = 5.54$, failure count: 68
   (0.0055% of maximum possible failure count)

(c) $\phi = 6.08$, failure count: 1071
   (0.087%)

(d) $\phi = 7.04$, failure count: 5386
   (0.44%)

(e) $\phi = 10.56$, failure count: 25,923
   (2.1%) (fiber failure region)
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Figure 5.22. Failure progression in large circular cylinder, negative twist (maximum possible failure count: 1,228,800)

(a) torque-twist relation

Torque, \[ T = \frac{|T|}{T_{cr}^{cir} + T_{cr}^{cr}} \]

Twist, \[ \phi = \frac{\phi L^{cir}}{L^{cr} \phi_{cr}} \]

(b) \( \bar{\phi} = 4.32 \), failure count: 67  
(0.0054\% of maximum possible failure count)

(c) \( \bar{\phi} = 5.21 \), failure count: 2419  
(0.20\%)

(d) \( \bar{\phi} = 6.16 \), failure count: 6687  
(0.54\%)

(e) \( \bar{\phi} = 9.40 \), failure count: 25,400  
(2.1\%) (shaded fiber failure region)
Figure 5.23. Failure progression in large constant-length $b/a = 0.85$ cylinder, positive twist  
(maximum possible failure count: 1,228,800)
Chapter 5

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Figure 5.24. Failure progression in large constant-length $b/a = 0.85$ cylinder, negative twist (maximum possible failure count: 1,228,800)
Figure 5.25. Failure progression in large constant-length $b/a = 0.70$ cylinder, positive twist (maximum possible failure count: 1,228,800)
Chapter 5 Results for Large Cylinders

Torque, \( T = \left| \frac{T}{T_{\text{cir}+\text{cr}}} \right| \)

Twist, \( \phi = \left| \frac{\phi L_{\text{cir}}}{L_{\phi_{\text{cr}+\text{cr}}}^2} \right| \)

(a) torque-twist relation

(b) \( \phi = 2.58 \), failure count: 24
   (0.0020\% of maximum possible failure count)

(c) \( \phi = 3.40 \), failure count: 336
   (0.027\%)

(d) \( \phi = 3.41 \), failure count: 660
   (0.054\%)

(e) \( \phi = 4.86 \), failure count: 2578
   (0.21\%)

Figure 5.26. Failure progression in large constant-length \( b/a = 0.70 \) cylinder, negative twist
(maximum possible failure count: 1,228,800)
Chapter 5

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Figure 5.27. Failure progression in large constant-length $b/a = 0.55$ cylinder, positive twist
(maximum possible failure count: 1,228,800)
Figure 5.28. Failure progression in large constant-length $b/a = 0.55$ cylinder, negative twist (maximum possible failure count: 1,228,800)
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Figure 5.29. Failure progression in large constant-twist-sensitivity $b/a = 0.85$ cylinder, positive twist (maximum possible failure count: 1,382,400)
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Figure 5.30. Failure progression in large constant-twist-sensitivity $b/a = 0.85$ cylinder, negative twist  
(maximum possible failure count: 1,382,400)
Results for Large Cylinders

Figure 5.31. Failure progression in large constant-twist-sensitivity $b/a = 0.70$ cylinder, positive twist
(maximum possible failure count: 1,561,600)
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Results for Large Cylinders

Figure 5.32. Failure progression in large constant-twist-sensitivity $b/a = 0.70$ cylinder, negative twist (maximum possible failure count: 1,561,600)
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Results for Large Cylinders

Figure 5.33. Failure progression in large constant-twist-sensitivity \( b/a = 0.55 \) cylinder, positive twist (maximum possible failure count: 1,804,800)
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Results for Large Cylinders

Figure 5.34. Failure progression in large constant-twist-sensitivity $b/a = 0.55$ cylinder, negative twist
(maximum possible failure count: 1,804,800)
CHAPTER 6

SUMMARY AND CONCLUSIONS

The prebuckling, buckling, and postbuckling responses of elliptical composite cylindrical shells under torsion have been studied. As stated in Chapter 1, the objectives of the study were to

- Examine the effect of the aspect ratio of the noncircular geometry on the prebuckling, buckling and postbuckling response of geometrically perfect quasi-isotropic composite cylinders under torsion.

- Determine how the response changes when failure initiation and progression are considered.

- Consider the influence of a larger scale on cylinder response.

While acknowledging that a circular cylinder is a more efficient structural shape than a cylinder with a noncircular cross section, the reasons for using a noncircular cylindrical cross section arise not from trying to maximize the load capacity by itself, but in trying to do so in concert with some other design constraint for which the benefits of a noncircular cross section outweigh the loss of load capacity. Also, considering that a cylindrical structure in a contemporary aerospace structure is likely to be constructed from fiber-reinforced composite materials, understanding the stability and material failure characteristics is important. Even though a noncircular cylinder is unlikely to be designed to resist torsion alone, it is likely some combined load condition of which torsion is a component will be important. As a means of understanding what part of a response under a combined load condition may be due to the torsional component, torsion alone has been considered.

In this study four different aspect ratios for elliptical cross sections were considered: 1.00, 0.85, 0.70 and 0.55. For these aspect ratios, two overall cylinder sizes were investigated, so-called small and large cylinders. The small circular cylinder, the baseline
geometry, had a radius of 4.28 in. and a length of 12.85 in. The radii of the elliptical cylinders were determined so that all of the cylinders had the same circumference. For the elliptical cylinders, two lengths were considered. The small constant-length cylinders all had a length of 12.85 in., and the constant-twist-sensitivity cylinders had lengths such that the sensitivity to changes in \( L/R_{\text{max}} \) were the same for all aspect ratios considered. As a result, the constant-twist-sensitivity cylinders had lengths of 14.42, 16.29, and 18.86 in. for cylinders with aspect ratios of 0.85, 0.70, and 0.55, respectively. The material was assumed to be a medium-modulus graphite-epoxy composite with a quasi-isotropic stacking sequence of \([\pm 45/0/90]_S\), with each ply having a thickness of 0.0055 in. Because the \( D_{16} \) and \( D_{26} \) stiffness terms were non-zero, the response under torsion was different for positive and negative twists, so both directions of twist were considered for each cylinder. The large circular cylinder had a length and radius five times larger than the small cylinder, making it 64.2 in. long with a radius of 21.4 in. The wall thickness was increased by a factor of two by doubling the number of plies in the laminate, and the stacking sequence was \([\pm 45/0/90]_{2S}\). The geometries of the large elliptical cylinders were scaled using the same factors as for the large circular cylinder.

A common approach and similar models based on the finite element code STAGS was used to analyze all of the cylinders. The analyses were conducted under displacement control by prescribing the twist angle applied to one end of the cylinder. Four-node quadrilateral shell elements were used. The number of elements in the circumferential direction was the same for all cylinders, and the number of elements in the axial direction was adjusted so the elements were approximately square. First, a nonlinear static analysis was used to determine the critical twist. The analysis was then continued to a twist angle slightly past the critical value and therefore in an unstable equilibrium configuration. With the cylinder in an unstable configuration, a transient restart with a variable time step and the twist angle held constant to move the cylinder from the unstable prebuckling equilibrium configuration to a stable postbuckling equilibrium configuration. Then a second transient restart with a constant time step and a twist angle that was increased linearly over time was used to determine postbuckling equilibrium configurations of the cylinder. When material failure was included in the analysis, the maximum stress failure criterion was used to predict when the material was predicted to fail at ply points throughout the cylinder.

Before studying the response of specific cylinders, different approaches were used to establish the credibility of the finite element results. First, a mesh refinement study was conducted, and it was determined that a mesh with 200 elements in the circumferential direction was sufficient. Second, comparisons were made for selected cases for the response
under slow and fast twist rates in the transient analysis. The differences in the response when a faster twist rate was used were considered acceptable considering the computational savings that were realized. Then, as a means of validating the finite element model, the initial slope of the torque-twist relation was compared to the slope predicted from a theoretical strength-of-materials approach. Very good agreement between the results from the finite element model and the theoretical formulation. Next, using another numerical approach, comparisons were made to validate the predicted critical values for a circular cylinder from the finite element analysis. With acceptable agreement with results from formulations other than the finite element analysis, and finding an acceptable mesh density and twist rate, a level of confidence in the results was established so the cylinders could be analyzed using this approach.

In general, the effect of a noncircular cross section was the same in the large and small cylinders. In the prebuckling range, the torsional stiffness was smaller for cylinders with smaller aspect ratios. Also, the critical torque was smaller for cylinders with smaller aspect ratios. This was also true for the critical twist in the small cylinders, but was not in the large due some of the cylinders having a nonlinear prebuckling response, a so-called dogleg-shaped prebuckling torque-twist relation. This behavior was attributed to the development of wrinkles in the cylinder walls in the prebuckling range. When wrinkling occurred the torque-twist relation became nonlinear, with the torsional stiffness decreasing. There was also a sharp increase in the axial end displacement that coincided with the development and growth of the wrinkles. For the small cylinders, the wrinkles developed in a small range near the critical value. Similar behavior was seen in some of the large cylinders; however in some of the large cylinders, the range over which the wrinkles develop, and the range with the reduced torsional stiffness, was more than 20% of the prebuckling range of twist.

For both large and small cylinders, the critical values for a negative twist are less than the critical values for a positive twist. For the small cylinders, the negative critical values were roughly 20% less than the positive values, and for the large cylinders the negative critical values were roughly 10% less.

In the initial postbuckling state, there was wrinkling in all of the cylinders. The ridges and valleys of the wrinkles were arranged in a helical manner due to the nature of the loading. The amplitudes of the ridges and valleys of the wrinkles in the circular cylinders were uniform around the circumference of the cylinder. For the elliptical cylinder, the amplitudes were greater near the flatter crown and keel than near the more curved sides of the cylinders. The difference between the amplitudes near the flatter crown and keel and the
amplitudes near the sides increased as the aspect ratio of the cylinder decreased. In all cases there were an integer number of wrinkles around the circumference, and the number was dependent on the geometry and the twist direction. The number of wrinkles was smaller for cylinders with smaller aspect ratios and for longer cylinders. For most of the cases when there was a difference in the number of wrinkles for different twist directions, there were fewer wrinkles for a positive twist. There were more wrinkles in the large cylinders than in the small cylinders, due to the larger $R/H$ in the large cylinders.

For the circular cylinders, the maximum torque level was the critical torque. The torsional stiffness for these cylinders was initially negative in the postbuckling range, and remained that way far into the postbuckling range. Some large cylinders with an aspect ratio of 0.85 showed a slight increase from the critical torque in the postbuckling range. Cylinders with aspect ratios of 0.70 and 0.55 had maximum torque levels that were greater than the critical torque levels. For these cylinders the maximum torque occurred in the postbuckling response, so there was some positive stiffness and, essentially, postbuckling torque capacity. Deeper into the postbuckling range the torsional stiffness was negative, or at most approximately zero, for all cylinders.

Instabilities in the postbuckling range occurred in most of the elliptical cylinders. No instabilities were found in the postbuckling paths of the circular cylinders for the range of twists that were considered. The instabilities were characterized by the changes in the deformation pattern that resulted as the cylinder moved from one postbuckling path to another. In general there were three types of changes to the deformation pattern. The most common change in configuration was when the ridges and valleys shifted their circumferential location. A second type of configuration change was characterized by a change in the number of circumferential waves. A third type of change in the deformation pattern were local changes that occurred in the valleys near the sides of the cylinder, near the ends of the major radii. These local changes were small areas of inward normal displacement that grew at a faster rate than the surrounding area within a valley. With each of the changes in the deformation pattern caused by instabilities there was a sudden drop in torque. In general, the drop in torque was greater for the first two types of configuration changes that had an effect on the overall deformation pattern than was seen in the configuration changes that were more localized in nature. It was shown that when a cylinder was untwisted after a drop in torque in the postbuckling range, the untwisting torque-twist relation did not coincide with the twisting torque-twist relation until the twist level was somewhat less than the level where the torque dropped.
When material failure was included in the model, the initial failures occurred within a ply in the direction perpendicular to the fiber direction. These matrix failures were on the ridges and in the valleys in the more developed area of the wrinkles. The change in torsional stiffness as a result of the matrix failures was minimal. In the circular cylinders, the initiation and progression of failure occurred uniformly on each of the ridges and valleys around the circumference. For the small elliptical cylinders, failures began on the ridges or valleys near the flat crown and keel, at the midlength of the cylinder. With increasing twist, failures spread to the ridges and valleys of the more curved region. In contrast, initial failures in the large elliptical cylinders occurred near the sides of the cylinder, and away from the cylinder midlength, closer to the cylinder ends, and for less extreme elliptical cross sections, i.e., cylinders with an aspect ratio of 0.85, locations with failures spread to the flatter crown and keel regions. A very interesting characteristic of failure initiation and failure progression occurs in the large cylinders. Specifically, the smaller the aspect ratio, the less spreading away from the initial failure sites as the damage level increases. This trend is true to a lesser degree for the small cylinders.

Having concluded the present investigation, there are several recommendations for future work. This study was limited to a numerical analysis, and there are not any experimental results for the buckling or the postbuckling response of elliptical composite cylinders. An experimental study is necessary to verify the results of this study. Because cylinders used in experimental analyses will contain imperfections, extending the current study to include the effect of imperfections on the cylinder response would be useful. Only a single quasi-isotropic laminate was considered in this study, so the effect of other lamination sequences have not been studied. Also, further insight into the effect of a noncircular cross-sectional geometry could be gained by considering cylinders with different radii or cross-sectional shapes. Combining torsion with another load condition such as axial compression or internal pressure should also be considered.
REFERENCES


References


References


APPENDIX A

GOVERNING EQUATIONS

The governing equations for an elliptical cylinder using the Donnell-Mushtari-Vlasoz formulation will be presented here. The equations are taken from the equations for a general shell given by Brush and Almroth [A.1] and Sanders [A.2] and are adapted for an elliptical cylinder. The governing equations include the kinematic, equilibrium, and constitutive equations, and also the boundary conditions. Before presenting the governing equations, the parameters needed to describe the surface, namely the Lamé coefficients and the curvature, will be given.

The derivation of the parameters to describe a surface is given in detail in texts by Dym [A.3] and Ventsel and Krauthammer [A.4], but for the purpose of this study only the final equations necessary to determine these parameters for the surface of an elliptical cylinder are presented here. The reference surface for an elliptical cylinder shown in Figure A.1. With the origin at the geometric center of the cylinder, the global coordinate system (X, Y, Z) is defined so the X-axis is aligned with the cylinder axis. The arc length coordinate \( s \) is measured from the crown of the cylinder, as shown. The major diameter \( 2a \) and minor diameter \( 2b \) are used to define the cross-sectional geometry. The circumference of the cylinder, not shown, is denoted as \( C \). The length of the cylinder is \( L \) and the wall thickness is \( H \). The reference surface can be described in terms of the surface coordinates \( x \) and \( s \) by the position vector

\[
\mathbf{r}(x,s) = x\hat{e}_1 + y(s)\hat{e}_2 + z(s)\hat{e}_3, \tag{A.1}
\]

where \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3 \) are the unit vectors parallel to the X, Y, and Z axes, respectively. However, the elliptical cross section is more easily described by the parametric equations

\[
y = a\sin t \quad \text{and} \quad z = b\cos t \quad \text{for} \quad 0 \leq t \leq 2\pi, \tag{A.2}
\]
where parameter $t$ has no physical meaning, but can be related to the arc length coordinate $s$ by the relation
\[ ds = \sqrt{dy^2 + dz^2} = \sqrt{a^2 \cos^2 t + a^2 \sin^2 t} \, dt. \] (A.3)

Then the Lamé coefficients for the surface of the elliptical cylinder are
\[ A = \sqrt{r_x \cdot r_x} = 1 \quad \text{and} \quad B = \sqrt{r_s \cdot r_s} = 1, \] (A.4)

where the notation $[ \cdot \cdot \cdot ]_{(\cdot)}$ is used to denote the partial derivative of $[ \cdot \cdot \cdot ]$ with respect to the variable $(\cdot)$. This notation will be used throughout the remainder of this appendix. Because the $y$ and $z$ components of $r$ are functions of $t$ and not $s$, the chain rule was employed to determine $r_s$. The curvatures and radii of curvatures of the surface are defined.
Appendix A

Governing Equations

as

\[ K_x = \frac{\hat{n}_x \cdot r_x}{A} = \frac{1}{R_x} = 0 \]

\[ K_s(t) = \frac{\hat{n}_s \cdot r_s}{B} = \frac{1}{R_s(t)} = \frac{ab}{(a^2 \cos^2 t + b^2 \sin^2 t)^{3/2}}. \]  \hspace{1cm} (A.5)

Equations (A.4) and (A.5) are used in the subsequent discussion of the governing equations. Also from the second of Equations (A.5), the maximum values of \( R_s \) can be found to occur at \( t = 0 \) and \( \pi \), and the minimum values at \( t = \pi / 2 \) and \( 3\pi / 2 \). The maximum and minimum values are

\[ R_{s,\text{max}} = \frac{a^2}{b} \quad \text{and} \quad R_{s,\text{min}} = \frac{b^2}{a}. \]  \hspace{1cm} (A.6)

The governing equations presented here are an adaptation of the equations for the DMV theory to an elliptical cylinder. This theory is based on the following assumptions:

– The ratio of the wall thickness to the smallest principal radius of curvature is much less than unity, i.e., \( H/R \ll 1 \).

– The strains are small enough that Hooke’s law is valid.

– Transverse shear and normal strains are negligible because straight lines that are normal to the undeformed reference surface remain straight and normal to the deformed reference surface, and their length does not change.

– The transverse normal stress is small and can be neglected, i.e. \( \sigma_z \approx 0 \).

– The displacement components tangent to the reference surface, \( u \) and \( v \), are small compared to the normal deformation of the wall, and their influence on changes of curvature is negligible. This also means the rotations of the reference surface in bending are large enough to be included in the strains of the reference surface.

Using these assumptions, the total potential energy, expressed as the sum of the strain energy and the potential energy from the applied loads, is

\[ \pi = \frac{1}{2} \int \int (N_x \varepsilon_x + N_s \varepsilon_s + N_{xs} \gamma_{xs} + M_x \kappa_x + M_s \kappa_s + 2M_{xs} \kappa_{xs}) \, dxds + \pi_{\text{load}}, \]  \hspace{1cm} (A.7)

where \( N_x, N_s, \) and \( N_{xs} \) are the stress resultants; \( M_x, M_s, \) and \( M_{xs} \) are the moment resultants; \( \varepsilon_x, \varepsilon_s, \) and \( \gamma_{xs} \) the strains of the reference surface; and \( \kappa_x, \kappa_s, \) and \( \kappa_{xs} \) are the curvature
changes and twist of the reference surface. For this problem body forces are not considered, and there are no forces on the surface of the cylinder. The only load that is considered is a torsional load applied at the ends of the cylinder. The potential energy for this applied load is

$$\pi_{load} = \int \left( N_{xs}^-(s) v(-\frac{L}{2}, s) - N_{xs}^+(s) v(+\frac{L}{2}, s) \right) ds,$$

(A.8)

where $N_{xs}^-(s)$ is the applied load at $x = -L/2$, and $N_{xs}^+(s)$ is the applied load at $x = +L/2$.

To be consistent with the description of the surface, the chain rule should be applied to all derivatives with respect to $s$ so the governing equations can be expressed in terms of $t$; however, since the equations are for reference only, the equations will be left in the more familiar forms in terms of $s$.

The kinematic relations for the strains of the reference surface can be expressed as

$$\epsilon_x = e_{xx} + \frac{1}{2} \beta_s^2$$
$$\epsilon_s = e_{ss} + \frac{1}{2} \beta_x^2$$
$$\gamma_{xs} = e_{xs} + \beta_x \beta_s.$$  

(A.9)

The rotations $\beta_x$ and $\beta_s$ are defined as

$$\beta_x = -w_{,x} \quad \text{and} \quad \beta_s = -w_{,s}.$$  

(A.10)

The strains $e_{xx}$, $e_{ss}$, and $e_{xs}$ can be written in terms of the displacements by the relations

$$e_{xx} = u_{,x}$$
$$e_{ss} = v_{,s} + \frac{w}{R(s)}$$
$$e_{xs} = v_{,x} + u_{,s}.$$  

(A.11)

Similarly, the expressions for the curvatures in terms of the rotations are

$$\kappa_x = \beta_{x,x}$$
$$\kappa_s = \beta_{y,s}$$
$$2\kappa_{xs} = \beta_{y,x} + \beta_{x,s}.$$  

(A.12)
Appendix A  Governing Equations

When written in terms of the displacements, these expressions become

\[
\begin{align*}
\kappa_x &= -w_{,xx} \\
\kappa_s &= -w_{,ss} \\
\kappa_{xs} &= -w_{,xs}.
\end{align*}
\] (A.13)

The force and moment results, simplified for balanced \((A_{16} = A_{26} = 0)\) and symmetric \((B_{ij} = 0, \text{ all } i \text{ and } j)\) laminates, are given by

\[
\begin{align*}
N_x &= \int_{-H/2}^{H/2} \sigma_x \, dz = A_{11} \epsilon_x + A_{12} \epsilon_s \\
N_s &= \int_{-H/2}^{H/2} \sigma_s \, dz = A_{12} \epsilon_x + A_{22} \epsilon_s \\
N_{xs} &= \int_{-H/2}^{H/2} \tau_{xs} \, dz = A_{66} \gamma_{ss} \\
M_x &= \int_{-H/2}^{H/2} z \sigma_x \, dz = D_{11} \kappa_x + D_{12} \kappa_s + D_{16} \kappa_{xs} \\
M_s &= \int_{-H/2}^{H/2} z \sigma_s \, dz = D_{12} \kappa_x + D_{22} \kappa_s + D_{26} \kappa_{xs} \\
M_{xs} &= \int_{-H/2}^{H/2} z \tau_{xs} \, dz = D_{16} \kappa_x + D_{26} \kappa_s + D_{66} \kappa_{xs}.
\end{align*}
\] (A.14)

Using Equations (A.7) to (A.14) the equilibrium equations can be derived by applying the principal of stationary total potential energy to Equation (A.7). The equilibrium equations are

\[
\begin{align*}
N_{x,x} + N_{xs,s} &= 0 \\
N_{xs,x} + N_{s,s} &= 0 \\
M_{x,xx} + M_{s,ss} + 2M_{xs,xx} - \frac{N_s}{R_s} &- (N_s \beta_s + N_{ss} \beta_s)_x - (N_s \beta_s + N_{ss} \beta_s)_s = 0.
\end{align*}
\] (A.15)
and the resulting boundary conditions at $x = -L/2$ are

\[
\begin{align*}
\text{Either } & N_x = 0 \text{ or } u \text{ is prescribed,} \\
\text{Either } & N_{xx} = N_{xx}^- \text{ or } v \text{ is prescribed,} \\
\text{Either } & M_{xx,x} + 2M_{xx,s} - \beta_x N_x - \beta_s N_{xs} = 0 \text{ or } w \text{ is prescribed,} \\
\text{Either } & M_x = 0 \text{ or } \beta_x \text{ is prescribed;}
\end{align*}
\]

(A.16)

and at $x = +L/2$ are

\[
\begin{align*}
\text{Either } & N_x = 0 \text{ or } u \text{ is prescribed,} \\
\text{Either } & N_{xx} = N_{xx}^+ \text{ or } v \text{ is prescribed,} \\
\text{Either } & M_{xx,x} + 2M_{xx,s} - \beta_x N_x - \beta_s N_{xs} = 0 \text{ or } w \text{ is prescribed,} \\
\text{Either } & M_x = 0 \text{ or } \beta_x \text{ is prescribed.}
\end{align*}
\]

(A.17)
REFERENCES


Vita

Waddy Thomson Haynie was born in Belton, SC. He graduated from Belton-Honea Path High School, and received a Bachelor of Science degree in Mechanical Engineering from Clemson University in 1999. He continued his studies at Clemson University, and conducting research sponsored by the Center for Advanced Engineering Fibers and Films, received his Master of Science degree in Mechanical Engineering in 2002. In the Fall of 2002 he enrolled in Virginia Tech to pursue a Ph.D. in Engineering Mechanics, and for the last three years has conducted research at NASA Langley Research Center in Hampton, VA, through the graduate program of the National Institute of Aerospace and Virginia Tech.