Chapter 4: Two Stacked Tubes

4.1 Introduction

This chapter considers two stacked geosynthetic tubes resting on rigid and deformable foundations. Some new terms and formulations will be introduced here for the special conditions concerning the contact of the tubes.

4.2 Rigid foundation

The model of the stacked tubes, as seen in Figure 4.1, is symmetric about its center axis. For simplicity, only the right side of the drawing will be analyzed. All data points can be copied to the left side, as a mirror image, for a full view of the structure. This analysis is divided into three segments, as shown below. It is assumed that point C lies above point A (i.e., \( y_{3D} < y_{1D} \)), since it would be undesirable for the top tube to rest on the ground.

Figure 4.1 Model for stacked tubes on a rigid foundation
The bottom tube is defined as tube 1, and the top tube is number 2. Some ratios are defined for the different attributes of the top and bottom tubes:

\[
\alpha = \frac{\gamma_2}{\gamma_1}, \quad \ell = \frac{L_2}{L_1} \quad (4.1, 4.2)
\]

\[
h_1 = p_1, \quad h_2 = \frac{p_2}{\alpha} \quad (4.3, 4.4)
\]

where \(\gamma_2\) and \(L_2\) are the specific weight and perimeter length of the top tube, respectively, and \(\gamma_1\) and \(L_1\) are the specific weight and perimeter length of the bottom tube, respectively. The quantities \(h_1\) and \(h_2\) are the nondimensional final pressure heads of the bottom and top tubes, respectively. The head \(h_1\) must be greater than the nondimensional height of the bottom tube and \(h_2\) must be greater than the nondimensional total height \(h_T\) of the entire structure. All variables have been nondimensionalized as in equation (2.1).

It is important to note that \(h_1\) is not the initial pressure head in tube 1, but the resulting pressure due to the internal material and any external forces, including the weight of the top tube or the force of external water. Referring to Namias (1985) for a tube resting on a rigid foundation, the final bottom pressure is the initial bottom pressure plus the ratio of the weight of the upper tube divided by the contact length of the bottom tube with the foundation.

### 4.2.1 Segment 1

Segment 1 is solely part of the bottom tube. For segment 1:

\[
\frac{d\theta_1}{ds_1} = \frac{p_1 - y_1}{t_1} \quad (4.5)
\]

Therefore:

\[
G_1 = p_1, \quad J_1 = 1, \quad t_1 = t_1 \quad (4.6)
\]

Also, the initial and final angles in section 1 are:

\[
\theta_{1a} = 0, \quad \theta_{1o} = \theta_{1D} \quad (4.7)
\]

These are the necessary values to use in equations (2.21) - (2.26) for \(i = 1\).
4.2.2 Segment 2

Segment 2 forms the section between points D and E, and is only part of the upper tube. For segment 2:

\[
\frac{d\theta_2}{ds_2} = \frac{p_2 - \alpha(y_2 + y_{1D})}{t_2}
\]  

(4.8)

Therefore:

\[
G_2 = p_2 - \alpha y_{1D}, \quad J_2 = \alpha, \quad t_2 = t_2, \\
\theta_{2\alpha} = \theta_{2D}, \quad \theta_{2\omega} = \pi,
\]  

(4.9)

Equations (2.21) - (2.26) can be applied with \( i = 2 \).

4.2.3 Segment 3

The section of the structure where the tubes are in contact is segment 3. Using the symmetry of the tube, the initial angle at point C is 0. For segment 3:

\[
\frac{d\theta_3}{ds_3} = \frac{p_2 - p_1 - (\alpha - 1)(y_3 + y_{1D} - y_{3D})}{t_1 + t_2}
\]  

(4.10)

Hence:

\[
G_3 = p_2 - p_1 - (\alpha - 1)(y_{1D} - y_{3D})
\]  

(4.11)

and

\[
J_3 = \alpha - 1, \quad t_3 = t_1 + t_2, \\
\theta_{3\alpha} = 0, \quad \theta_{3\omega} = \theta_{3D},
\]  

(4.12)

For \( i = 3 \), equations (2.21) - (2.23), (2.25), and (2.26) are used along with (2.35), since \( G_3 \) involves \( y_{3D} \) and \( R_3 = J_3 \). In equation (2.35),

\[
Q_3 = p_2 - p_1 - (\alpha - 1)y_{1D}
\]  

(4.13)

Due to mathematical constraints, this section will require special formulas when \( \alpha = 1 \). Section 4.2.6 discusses these special conditions.
4.2.4 Slope continuity

When segments join at a point in Figure 4.1, all segments are assumed to have equivalent slopes. The continuities to be used in this condition are:

\[
\begin{align*}
\theta_{1B} &= 0, & \theta_{1D} &= \theta_{2D} + \pi, & \theta_{3D} &= \theta_{2D}, \\
\theta_{2E} &= \pi, & \theta_{3C} &= 0 
\end{align*}
\]

(4.14)

Therefore, the unknowns for this problem are \( t_1, t_2, \) and \( \theta_{2D} \).

4.2.5 Constraints

Using equations (2.43) and (2.44) for the upper tube, respectively:

\[
\begin{align*}
x_{3D} + x_{2E} &= 0 \\
s_{3D} + s_{2E} &= \frac{\ell}{2}
\end{align*}
\]

(4.15) (4.16)

Using equation (2.47) for the bottom tube:

\[
x_{3D} - x_{1D} + s_{1D} + s_{3D} = \frac{1}{2}
\]

(4.17)

The three unknowns given above can be obtained numerically by solving these three conditions.

4.2.6 Cases for the rigid foundation

Three cases are outlined for two stacked tubes. One case is necessary when the pressure heads and specific weights of the fill are equal in both tubes. Another case is needed when the specific weights are equal, but not the pressure heads. Finally, a case is required when the specific weights are not equal.
4.2.6.1 Case 1: $\alpha = 1$, $p_1 = p_2$

In cases where $\gamma_1 = \gamma_2$, the denominator of equations (2.21) and (2.24) for $y_3$ is 0. Therefore, these equations cannot be used. However, in this case $p_1 = p_2$ also, so $y_3$ is constant, and $y_{3D} = 0$. Due to this condition and slope continuity, $\theta_{2D} = 0$, and the resulting unknowns are only $t_1$ and $t_2$. In this case, just two constraints are necessary. Since $y_{3D} = 0$,

$$s_{3D} = x_{3D}$$

From equation (4.15),

$$x_{3D} = -x_{2E}$$

By substituting equation (4.18) into (4.19),

$$s_{3D} = -x_{2E}$$

Using this in equations (4.16) and (4.17), the two constraints are:

$$-x_{2E} + s_{2E} = \frac{\ell}{2}$$

$$s_{1D} - 2x_{2E} - x_{1D} = \frac{1}{2}$$

In this case with $\alpha = 1$ and $h_1 = h_2$, section 3 will be horizontal due to the equivalent opposing forces on the top and bottom tubes. Figure 4.2 shows a diagram with $\ell = 0.7$. The length of the perimeter of the upper tube is 70% of that of the bottom tube. Though values of $\ell$ slightly above 1 may give meaningful results, it is assumed that $\ell$ will always be equal to or less than 1.

![Figure 4.2 Stacked tubes with $h_1 = h_2 = 0.25$, $\alpha = 1$, $\ell = 0.7$](image)
However, this analysis will mainly focus on tubes with equal perimeters. Figure 4.3 shows a variety of plots with $\ell = 1$, varying $h_1$, where $h_1 = h_2$. As $h_1$ gets very large, the shapes of the tubes become almost circular with very little contact area, as seen in Figure 4.3.d. The tubes are much flatter with low internal pressures. The numerical solutions from Mathematica for the nondimensional tensions for these structures are given in Table 4.1.

Table 4.1 Solutions for $h_1 = h_2$ where $\alpha = 1$ and $\ell = 1$

<table>
<thead>
<tr>
<th>Unknowns</th>
<th>$h_1 = h_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td>$t_1$</td>
<td>0.00921</td>
</tr>
<tr>
<td>$t_2$</td>
<td>0.00639</td>
</tr>
</tbody>
</table>

Figure 4.4 plots $h_1$ versus the total structure height $h_T$ for $h_1 = h_2$. For lower $h_1$ values, both $\ell = 0.7$ and 1.0 have nearly the same heights. Higher pressures result in the total height for $\ell = 1.0$ exceeding that for $\ell = 0.7$ by over 20%. Figure 4.5 shows the circumferential tension in both tubes versus $h_1$ for $h_1 = h_2$. From the graph, the tension is higher in the lower tube for both cases. For $\ell = 1$, $t_1$ is only slightly higher than $t_2$, while it is significantly higher for $\ell = 0.7$. 
Figure 4.3 Comparison of $h_1$ for stacked tubes where $h_1 = h_2$, $\alpha = 1$, and $\ell = 1$
Figure 4.4 Total structure height, $h_T$, versus $h_1$ where $h_1 = h_2$

Figure 4.5 Circumferential tension versus $h_1$, where $h_1 = h_2$
4.2.6.2 Case 2: $\alpha = 1$, $p_1 \neq p_2$

Case 2 is much like Case 1, where equations (2.21) and (2.35) cannot be used for segment 3. From equation (4.10) with $\alpha = 1$:

$$\frac{d\theta_3}{ds_3} = \frac{p_2 - p_1}{t_1 + t_2}$$

(4.23)

for segment 3. Define $M_3$ as:

$$M_3 = \frac{p_2 - p_1}{t_1 + t_2}$$

(4.24)

Integration gives:

$$\theta_3 = M_3 s_3 + c$$

(4.25)

where the constant $c$ equals 0 because $\theta_3 = 0$ when $s_3 = 0$. Substituting (4.25) into equation (2.3) gives:

$$\frac{dx_3}{ds_3} = \cos(M_3 s_3)$$

(4.26)

or

$$x_3 = \int_0^{s_3} \cos(M_3 s_3) ds_3$$

(4.28)

Integration of equation (4.28) gives:

$$x_3 = \frac{\sin(M_3 s_3)}{M}$$

(4.29)

Using the same procedure for $y$ in equation (2.4):

$$\frac{dy_3}{ds_3} = \sin(M_3 s_3)$$

(4.30)

$$y_3 = \int_0^{s_3} \sin(M_3 s_3) ds_3$$

(4.31)

$$y_3 = \frac{1 - \cos(M_3 s_3)}{M}$$

(4.32)

Likewise for the equations for $x_{3D}$ and $y_{3D}$:

$$x_{3D} = \frac{\sin(M_{3D} s_3)}{M}$$

(4.33)
\[ y_{3D} = \frac{1 - \cos(Ms_{3D})}{M} \]  
\( (4.34) \)

Recall from equation (4.14):
\[ \theta_{3D} = \theta_{2D} = \theta_{1D} - \pi \]  
\( (4.35) \)

From equation (4.25),
\[ \theta_{3D} = Ms_{3D} \]  
\( (4.36) \)

Substitution of (4.35) into (4.36) yields:
\[ s_{3D} = \frac{\theta_{1D} - \pi}{M} \]  
\( (4.37) \)

Therefore, substituting (4.37) into equations (4.33) and (4.34) yields:
\[ x_{3D} = -\frac{\sin \theta_{1D}}{M}, \quad y_{3D} = \frac{1 + \cos \theta_{1D}}{M} \]  
\( (4.38, 4.39) \)

These new formulas for segment 3 can be used along with those for segments 1 and 2 given above. The unknowns in this case are \( t_1, t_2, \) and \( \theta_{1D} \). They can be determined by using the constraints from section 4.2.5.

For the case with \( \alpha = 1, \theta_{3D} \) will be greater than \( \pi \) when \( h_2 > h_1 \) and less than \( \pi \) when \( h_2 < h_1 \). This affects the direction of the curvature of segment 3. Figure 4.6 demonstrates two examples when the final resulting pressure of the bottom tube is greater than that of the top. Figure 4.7 shows two examples when the top pressure exceeds the pressure in the bottom tube.
Figure 4.6 Comparison of stacked tubes for $\alpha = 1$ and $\ell = 1$, $h_1 > h_2$

(a) $h_1 = 0.6$, $h_2 = 0.4$

(b) $h_1 = 0.6$, $h_2 = 0.5$

Figure 4.7 Comparison of stacked tubes for $\alpha = 1$ and $\ell = 1$, $h_2 > h_1$

(a) $h_1 = 0.4$, $h_2 = 0.6$

(b) $h_1 = 0.5$, $h_2 = 0.6$
4.2.6.3 Case 3: $\alpha \neq 1$

The direction of the curvature in Case 3 is a function of $\alpha$, $p_1$, and $p_2$, represented by $G_3$ in equation (4.11). For this case a special formulation is not needed for segment 3; however, equations (2.36) - (2.41) must be used for negative values of $G_3$. Though initially unknown, this sign can be determined through trial and error.

Figure 4.8 illustrates examples of structures with tubes possessing equal or unequal pressure heads but different fills (i.e., $\gamma_1 \neq \gamma_2$). When the material in the upper tube has a lower specific weight ($\alpha < 1$), the contact region of the tubes is convex with respect to the bottom tube.

Figure 4.9 shows $\alpha$ versus total height when $h_1 = h_2$. Three different values of $h_1$ are shown, including the condition in Figure 4.8 where $h_1 = 0.5$. Due to an earlier constraint that $y_{1D} > y_{3D}$, the plots for $h_1 = 0.3$ terminate at $\alpha = 1.3$. From these curves, it can be assessed that higher values of $\alpha$ cause a decrease in the total structure height.

Figure 4.10 shows $\alpha$ versus the circumferential tension in the bottom tube for $h_1 = h_2$. Figure 4.11 illustrates this for the tension in the top tube. The tension in tube 2 increases steadily as $\alpha$ increases. This increased tension is explained by the excess pressure and specific weight in the upper tube when $\alpha$ increases. However, the tension in the lower tube decreases slightly as $\alpha$ increases due to the lower specific weight of the bottom tube with respect to the top tube.
Figure 4.8 Comparison of stacked tubes with $h_1 = h_2 = 0.5$ and $\ell = 1$
Figure 4.9  Height $h_T$ versus $\alpha$, where $h_1 = h_2$

Figure 4.10  Circumferential tension of the bottom tube versus $\alpha$, where $h_1 = h_2$

Figure 4.11  Circumferential tension of the top tube versus $\alpha$, where $h_1 = h_2$
4.3 Deformable foundation

Figure 4.12 shows the model for two stacked tubes resting on a deformable foundation. This model is also symmetric about its center axis, and only the right side of the drawing will be analyzed. This analysis is divided into four sections, with segment 4 consisting of the section in contact with the ground. Segments 2 and 3 are identical to the ones described for the rigid foundation. The new formulations for segments 1 and 4 are described below.

\begin{equation}
G_1 = p_1, \quad J_1 = 1, \quad t_1 = t_1, \\
\theta_{1a} = \theta_{1B}, \quad \theta_{1o} = \theta_{1D}.
\end{equation} 

Equations (2.21) - (2.26) can be applied for \( i = 1 \).
4.3.2 Segment 4

Segment 4 stretches from the bottom center of the lower tube to the ground level. For segment 4:

\[
\frac{d\theta_4}{ds_4} = \frac{p_1 - y_4 + y_{4B} - ky_{4B} + ky_4}{t_1}
\]  

(4.41)

Therefore:

\[
G_4 = p_1 + y_{4B} - ky_{4B}
\]

(4.42)

and

\[
J_4 = 1 - k,
\]

\[
t_4 = t_1,
\]

\[
\theta_{4A} = 0,
\]

\[
\theta_{4B} = \theta_{4B}
\]

(4.43)

For \( i = 4 \), equations (2.21) - (2.23), (2.25), and (2.26) are used along with (2.35), since \( G_4 \) involves \( y_{4B} \) and \( R_4 = J_4 \). In equation (2.35),

\[
Q_4 = p_1
\]

(4.44)

4.3.3 Slope continuity

The slope continuities in this problem are:

\[
\theta_{4A} = 0, \quad \theta_{1D} = \theta_{2D} + \pi, \quad \theta_{3D} = \theta_{2D},
\]

\[
\theta_{4B} = \theta_{1B}, \quad \theta_{2E} = \pi, \quad \theta_{3C} = 0
\]

(4.45)

Therefore, the unknowns in this case are \( t_1, t_2, \theta_{1B}, \) and \( \theta_{2D} \).

4.3.4 Constraints

Using equations (2.43) and (2.44) for the upper tube, respectively:

\[
x_{3D} + x_{2E} = 0
\]

(4.46)

\[
s_{3D} + s_{2E} = \frac{\ell}{2}
\]

(4.47)

The bottom tube also utilizes equations (2.43) and (2.44):
\[
x_{4B} + x_{1D} - x_{3D} = 0 \quad (4.48)
\]
\[
s_{4B} + s_{1D} + s_{3D} = \frac{1}{2} \quad (4.49)
\]

Then \(t_1, t_2, \theta_{1B},\) and \(\theta_{2D}\) can be obtained by solving equations (4.46) - (4.49).

### 4.3.5 Cases for the deformable foundation

Case 1 for the deformable foundation is very similar to Case 1 for the rigid foundation. As before when \(\alpha = 1\) and \(p_1 = p_2\), the unknown \(\theta_{2D}\) is equal to 0, and one constraint is lost. From equation (4.46):

\[
s_{3D} = x_{3D} = -x_{2E} \quad (4.50)
\]

Substituting equation (4.50) into equations (4.47), (4.48), and (4.49):

\[
s_{2E} = x_{2E} = \frac{\ell}{2} \quad (4.51)
\]
\[
x_{4B} + x_{1D} + x_{2E} = 0 \quad (4.52)
\]
\[
s_{1D} - x_{2E} + s_{4B} = \frac{1}{2} \quad (4.53)
\]

The unknowns \(t_1, t_2,\) and \(\theta_{1B}\) can be determined by using equations (4.51), (4.52), and (4.53). Figure 4.13 shows examples of two stacked tubes in Case 1 with the foundation stiffness \(k\) equal to 10 and 25. In this figure, \(h_1 = h_2 = 0.6, \alpha = 1,\) and \(\ell = 1\). The numeric solutions for these figures are shown in Table 4.2.

| Table 4.2 Solutions for \(h_1 = h_2 = 0.6\) where \(\alpha = 1\) and \(\ell = 1\) |
|-----------------------------|--------|--------|
| \textbf{Unknows}          | \textbf{10} | \textbf{25} |
| \(t_1\)                  | 0.05076 | 0.04851 |
| \(t_2\)                  | 0.03999 | 0.03756 |
| \(\theta_{1B}\)          | 0.90600 | 0.56253 |
Case 2 ($\alpha = 1, p_1 \neq p_2$) and Case 3 ($\alpha \neq 1$) are very similar to those for the rigid foundation. Segments 1 and 4 given above are used along with segments 2 and 3 from before. For the deformable foundation, the constraints listed in equations (4.46) - (4.49) are used instead of those given previously.

### 4.3.6 Results for the deformable foundation

As the foundation stiffness increases, the downward deflection of the ground $d_1$ decreases. Figure 4.14 displays the effect $k$ has on the ground deflection at the central point A ($y_{4B}$ from the analysis). For the condition when $h_1 = h_2$, it appears that a change in the specific weight ratio $\alpha$ has little effect on ground deflection. For stiffness coefficients above 100, the depths are minimal.

Figure 4.13 Comparison of varying $k$ for stacked tubes with $h_1 = h_2 = 0.6$, $\alpha = 1$, and $\ell = 1$
Figure 4.15 illustrates the circumferential tensions in the tubes versus the foundation stiffness for $h_1 = h_2 = 0.5$. The tensions in the tubes decrease substantially as $k$ increases to 10. The value of $k$ has almost no effect on tension when $k$ is greater than 25.

The total height $h_T$ of the structure above ground level increases as $k$ increases. Figure 4.16 illustrates this point. As before, the height increases substantially as $k$ increases to 10, but $k$ values above 25 have little effect on the total height. This is due to the decreasing ground deflection which allows more of the structure height to be above ground.

The structures shown in Figure 4.17 complement the values in Figures 4.14, 4.15, and 4.16. Figures 4.17.a, 4.17.b, and 4.17.c show two stacked tubes on a deformable foundation with stiffness coefficient $k = 10$, for $\alpha = 0.6$, 1.0, and 1.5, respectively. Figures 4.17.d - f illustrate $k = 25$ and Figures 4.17.g - i show two stacked tubes resting on a rigid foundation.

![Figure 4.14](image)

**Figure 4.14** Ground deflection, $d_1$, versus foundation stiffness, $k$, for $h_1 = h_2 = 0.5$
Figure 4.15  Circumferential tension versus foundation stiffness, $k$, for $h_1 = h_2 = 0.5$

Figure 4.16  Structure height $h_T$ versus foundation stiffness for $h_1 = h_2 = 0.5$
Figure 4.17  Comparison of stacked tubes with $h_1 = h_2 = 0.5, \ell = 1$