Chapter 5: 2-1 Stacked Formation

5.1 Introduction

This chapter considers stacked geosynthetic tubes in a 2-1 formation. A single tube resting on two lower tubes will add significant height compared to the single tube analyzed in Chapter 3. The analysis for the 2-1 formation withstanding external water will be discussed in section 5.3.

Chapter 5 will not consider some of the previously discussed parameters. Since these structures will most likely be resting on soil, only the deformable foundation will be considered here. It is also assumed that all tubes will contain a fill with the same specific weight, since most likely the material will come from the same source. Therefore, for all the cases in Chapter 5 $\alpha$ will be equal to 1 and ignored. In addition, it is assumed that all tubes have the same perimeter length.

5.2 2-1 formation without water

Figure 5.1 shows the model used for the 2-1 formation. It is necessary to place small anchored blocks on each side of the structure to keep it from sliding. The weight of the top tube transmits horizontal forces onto the bottom tubes, which have to be counteracted by the blocks. The horizontal length between the inner corners of the blocks in this condition is defined as $\Lambda$, or $\lambda$ nondimensionalized ($\lambda = \Lambda/L_1$).

For this condition, the pressure heads in the bottom tubes are assumed to be equal. Therefore, this model is also symmetric about its center axis. A small part of the bottom tube’s perimeter will be in contact with the opposite bottom tube. This section between points N and M, referred to as segment 9, will be vertical since the pressure heads and specific weight are equal. Also, the pressure head in the top tube is assumed to always be less than the resulting pressure head for the bottom tubes. Again, all the variables used in this formulation are nondimensionalized as in equation (2.1) with $\gamma = \gamma_1$ and $L = L_1$. 
(a) 2-1 formation

(b) Insert from Figure 5.1.a

Figure 5.1 Model of 2-1 formation
5.2.1 Segments

This formulation consists of seven segments shown in Figure 5.1. Because segment 3 is part of the contact area between tubes 1 and 2, it requires a special formulation discussed below. The quantities G, J, and tension that are needed in equation (2.16) are repeated below with the subscript i:

$$\frac{d\theta_i}{ds_i} = \frac{G_i - J_i y_i}{t_i} \quad (5.1)$$

The values for J, G, tension, and the initial and final angles follow in Table 5.1 for segments 1, 2, and 4 - 7.

<table>
<thead>
<tr>
<th>Segment (i)</th>
<th>$G_i$</th>
<th>$J_i$</th>
<th>$t_i$</th>
<th>$\theta_{i\alpha}$</th>
<th>$\theta_{i\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p_1 - y_0$</td>
<td>1</td>
<td>$t_1$</td>
<td>$\theta_{1H}$</td>
<td>$\theta_{1D}$</td>
</tr>
<tr>
<td>2</td>
<td>$p_2 - y_0 - y_{1D}$</td>
<td>1</td>
<td>$t_2$</td>
<td>$\theta_{2D}$</td>
<td>$\theta_{2E}$</td>
</tr>
<tr>
<td>4</td>
<td>$p_1$</td>
<td>1 - k</td>
<td>$t_1$</td>
<td>$\theta_{4A}$</td>
<td>$\theta_{4B}$</td>
</tr>
<tr>
<td>5</td>
<td>$p_1$</td>
<td>1</td>
<td>$t_1$</td>
<td>$\theta_{5A}$</td>
<td>$\theta_{5M}$</td>
</tr>
<tr>
<td>6</td>
<td>$p_1 - y_0 - y_{1D} + y_{3D}$</td>
<td>-1</td>
<td>$t_1$</td>
<td>$\theta_{6C}$</td>
<td>$\theta_{6N}$</td>
</tr>
<tr>
<td>7</td>
<td>$p_2 - y_0 - y_{1D} + y_{3D} + y_{7C}$</td>
<td>1</td>
<td>$t_2$</td>
<td>$\theta_{7U}$</td>
<td>$\theta_{7C}$</td>
</tr>
</tbody>
</table>

Equations (2.21) – (2.23), (2.25), and (2.26) can be used to determine the behavior of these segments. However, equation (2.35) must be used in place of (2.24) for $y_{7C}$, since $G_7$ contains the variable $y_{7C}$. For equation (2.35):

$$Q_7 = p_2 - y_0 - y_{1D} + y_{3D} \quad (5.2)$$

and $R_7 = J_7$.

As in Section 4.2.6.2, a special formulation is necessary for segment 3 because $\alpha = 1$. Defining $M_3$ as:

$$M_3 = \frac{p_2 - p_1}{t_1 + t_2} \quad (5.3)$$

and using the same procedure as in Section 4.2.6.2, the necessary equations are:
\[ x_3 = \frac{\sin \theta_3 - \sin \theta_{3C}}{M_3}, \quad y_3 = \frac{\cos \theta_{3C} - \cos \theta_3}{M_3}, \quad (5.4, 5.5) \]

\[ x_{3D} = \frac{-\sin \theta_{1D} - \sin \theta_{3C}}{M_3}, \quad y_{3D} = \frac{\cos \theta_{3C} + \cos \theta_{1D}}{M_3}, \quad (5.6, 5.7) \]

and

\[ s_{3D} = \frac{\theta_{1D} - \pi - \theta_{3C}}{M_3} \quad (5.8) \]

### 5.2.2 Slope conditions

The slope conditions for this problem due to continuity and symmetry are:

\[ \theta_{5A} = -\theta_{4A}, \quad \theta_{4A} = -\theta_{4B}, \]
\[ \theta_{6C} = \theta_{3C}, \quad \theta_{7C} = \theta_{3C}, \]
\[ \theta_{3D} = \theta_{1D} - \pi, \quad \theta_{2D} = \theta_{1D} - \pi, \]
\[ \theta_{2E} = \pi, \quad \theta_{5M} = \pi/2, \]
\[ \theta_{6N} = \pi/2, \quad \theta_{7U} = 0 \quad (5.9) \]

The seven unknowns for this formulation are \( t_1, t_2, \theta_{4B}, \theta_{3C}, \theta_{1D}, \theta_{1H}, \) and \( \lambda. \)

### 5.2.3 Constraints

Using equations (2.42) – (2.44), the constraints for this problem are:

\[ y_0 + y_{1D} - y_{3D} - y_{6N} - y_{5M} - y_{9N} = 0 \quad (5.10) \]
\[ x_{5M} + x_{4B} + x_0 + x_{1D} - x_{3D} - x_{6N} = 0, \quad (5.11) \]
\[ x_{7C} + x_{3D} + x_{2E} = 0, \quad (5.12) \]
\[ s_0 + s_{1D} + s_{3D} + s_{4B} + s_{6N} + s_{5M} + s_{9N} = 1, \quad (5.13) \]
\[ s_{7C} + s_{3D} + s_{2E} = \frac{1}{2} \quad (5.14) \]

The lengths \( s_{9N} \) and \( y_{9N} \) are equal since segment 9 is a vertical section. Therefore, equations (5.10) and (5.13) can be combined into:
\[ y_0 + y_{1D} - y_{3D} - y_{6N} - y_{5M} + s_0 + s_{1D} + s_{3D} + s_{4B} + s_{6N} + s_{5M} = 1 \] (5.15)

Two other constraints can be defined by using some geometric considerations of the problem:
\[ x_{7C} - x_{6N} = 0, \] (5.16)
\[ x_{5M} + x_{4B} = \frac{\lambda}{2} \] (5.17)

Thus, 6 constraints are defined by equations (5.11), (5.12), and (5.14) – (5.17).

This formulation consists of 7 unknowns and 6 equations. Therefore the designer must generate one unknown. Though theoretically any unknown could be given, the most desirable unknowns to provide would be \( \theta_{1H} \) or \( \lambda \). As discussed later in Section 5.2.4.2, the angle \( \theta_{1H} \) has a large effect on \( \lambda \), and vice versa. For most of the examples in this section, \( \theta_{1H} \) is chosen. However, in most applications, a constant distance between the blocks will be desired for simplicity and for ease of construction. If \( \theta_{1H} \) is given, it is not necessary to use equation (5.17) and solve for \( \lambda \). The length between the blocks can be calculated after computation by use of equation (5.17).

### 5.2.4 Examples and results

Figures 5.2 - 5.7 show results for the 2-1 formation, with some outlines of the tubes plotted in Figures 5.2 and 5.5. All examples and figures in this section will rest on a deformable foundation with \( k = 15 \). Also, blocks with \( x_0 = y_0 = 0.05 \) will be used through the duration of the thesis. The numerical results for Figures 5.2 and 5.5 can be found in Table 5.2.
(a) $h_1 = 0.55$, $h_2 = 0.50$

(b) $h_1 = 0.80$, $h_2 = 0.60$

Figure 5.2 2-1 formation with $k = 15$, $\theta_{1H} = 0.40$, and $x_0 = y_0 = 0.05$
Table 5.2 Numerical results for 2-1 formation, $k = 15$, $x_0 = y_0 = 0.05$

<table>
<thead>
<tr>
<th></th>
<th>Figure</th>
<th>5.2.a</th>
<th>5.2.b</th>
<th>5.5.a</th>
<th>5.5.b</th>
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</thead>
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<td><strong>Givens</strong></td>
<td></td>
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<tr>
<td>$h_1$</td>
<td>0.55</td>
<td>0.80</td>
<td>0.65</td>
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</tr>
<tr>
<td>$h_2$</td>
<td>0.50</td>
<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
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<tr>
<td>$\theta_{1H}$</td>
<td>0.40</td>
<td>0.40</td>
<td>-0.60</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td><strong>Unknowns</strong></td>
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<td></td>
</tr>
<tr>
<td>$t_1$</td>
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<td>0.0749</td>
<td>0.0571</td>
<td>0.0584</td>
<td></td>
</tr>
<tr>
<td>$t_2$</td>
<td>0.0185</td>
<td>0.0269</td>
<td>0.0306</td>
<td>0.0317</td>
<td></td>
</tr>
<tr>
<td>$\theta_{1D}$</td>
<td>3.7453</td>
<td>3.6299</td>
<td>3.7428</td>
<td>3.7431</td>
<td></td>
</tr>
<tr>
<td>$\theta_{3C}$</td>
<td>0.7227</td>
<td>0.7026</td>
<td>0.6668</td>
<td>0.6643</td>
<td></td>
</tr>
<tr>
<td>$\theta_{4B}$</td>
<td>0.7100</td>
<td>0.6947</td>
<td>0.5199</td>
<td>0.7033</td>
<td></td>
</tr>
</tbody>
</table>

**5.2.4.1 Effect of pressure heads**

Figure 5.2.a illustrates a condition with $h_1 = 0.55$, $h_2 = 0.50$, and a given $\theta_{1H}$ of 0.4. Figure 5.2.b shows the same conditions with $h_1 = 0.80$ and $h_2 = 0.60$.

Figures 5.3.a and 5.3.b show tension versus $h_1$ for tubes with these parameters. Figures 5.3 and 5.4 consider three pressure head variances between the top and bottom tubes. The tension increases steadily in both tubes when the pressure is increased. However, the pressure variation has little effect on $t_1$, most likely because the three curves are based on the same values of $h_1$. However, this graph does show that the tension $t_1$ in the bottom tube is hardly affected by the pressure head in the top tube. As shown in Figure 5.3.b, the pressure variation affects $t_2$ significantly. This most likely occurs since the values for $h_2$ are not equivalent at a particular value of $h_1$. 
Figure 5.3 Tension versus $h_1$ for 2-1 formation, $\theta_{1H} = 0.4$, $k = 15$, and $x_0 = y_0 = 0.05$

Figure 5.4.a shows $\lambda$ versus $h_1$. Figure 5.4 uses the same conditions as in Figure 5.3. The nondimensional length between the base of the blocks generally decreases as $h_1$ is increased for a given value of $\theta_{1H}$. In Figure 5.4.b, the height is significantly increased with higher pressure heads. This coincides with the results from the single tube and for two stacked tubes. The structure height for large pressure head variances ($h_2 = h_1 - 0.20$) lags substantially behind the height for structures with similar pressure heads in all tubes. This occurs because the small pressure head in the top tube does not significantly contribute to the structure height.
5.2.4.2 Effect of $\theta_{1H}$

For each condition, a certain range of acceptable values exists for $\theta_{1H}$ (and $\lambda$). This is determined from the solution of the problem. The length of segment 9, the section connecting the lower tubes, is near zero at both ends of this range. Eventually, invalid solutions occur when $s_{9N}$ is negative. Assuming the tubes lose contact, a formulation was developed for this particular case. However, solutions could not be determined from this analysis. Therefore, it is assumed that the structure fails when the tubes lose contact.
Figure 5.5 shows some extreme cases for the predetermined angle $\theta_{1H}$ in a structure with $h_1 = 0.65$ and $h_2 = 0.60$. Figure 5.3.a illustrates a condition giving $\theta_{1H} = -0.60$, resulting in a small $\lambda$ (0.3097). Figure 5.3.b, on the other hand, has $\theta_{1H} = 0.75$ ($\lambda = 0.5432$). Both of these structures have very minimal contact lengths between the bottom tubes ($s_{9N} < 0.002$). A change in $\theta_{1H}$ which would cause $s_{9N}$ to be negative would most likely cause the structure to fail. For example, if the blocks were closer in Figure 5.5.a, they may cause the suspended portion of the bottom tube to become unstable and collapse over the block. Alternatively, if the blocks are separated too much, such as more than in Figure 5.5.b, the force from the top tube may push the bottom tubes apart and result in the top tube falling.

The effect of $\theta_{1H}$ on tension, height, and $\lambda$ are shown in Figures 5.6.a, 5.6.b, and 5.7, respectively. Two conditions are considered: $h_1 = 0.55$, $h_2 = 0.50$, and $h_1 = 0.65$, $h_2 = 0.60$. The acceptable ranges of $\theta_{1H}$ are shown on the figures for those conditions where data is plotted. The angle $\theta_{1H}$ has a larger range with higher pressure heads. The minimum acceptable pressure heads for $\theta_{1H} = 0.4$ are shown by the curves in Figure 5.4. The bottom tubes are not in contact for pressure heads smaller than the ones shown.

Neither the tension nor the height is significantly changed by the given angle $\theta_{1H}$, as shown in Figures 5.6.a and 5.6.b. However, the tension curves are convex with a local minimum at $\theta_{1H} = 0$, and the height curves are concave, with a local maximum at $\theta_{1H} = 0$. 
Figure 5.5  2-1 formation with $h_1 = 0.65$, $h_2 = 0.60$, $k = 15$, and $x_0 = y_0 = 0.05$
(a) Tension versus $\theta_{1H}$

![Tension Graph](image)

(b) Height versus $\theta_{1H}$

![Height Graph](image)

Figure 5.6 Tension and height versus $\theta_{1H}$ for 2-1 formation, $k = 15$, and $x_0 = y_0 = 0.05$

A larger $\lambda$ coincides with a larger $\theta_{1H}$, as shown in Figure 5.7. Though $\theta_{1H}$ is chosen in most of the examples in this section, in most conditions it will be desired to specify the distance between the blocks. Using Figure 5.7 as a guide, $\lambda = 0.5$ was chosen. This distance is valid for most pressure heads and will allow for easy deployment of the tubes. It will be used exclusively in Section 5.3.
5.3 2-1 formation with external water

This section discusses the 2-1 formation with external water on the right side of the structure. This formulation requires four cases as the water rises from ground level to the top of the structure. At water levels below the height of the block, \( y_0 \), no new formulation is required. The block is assumed to be stationary and therefore can withstand any forces.

The basic formulation, Case 1, shown in Figure 5.8, is applicable when the external water is acting on tube 1. A small change in the formulation is necessary for Case 2 when the water rises to tube 2. As the water continues to rise, the horizontal force from the water will gradually overcome the force provided by the right block. This causes tube 1 to gently come off part of the block. Eventually the force provided by the water will replace the force from the block and cause the tube to come completely off the block. This requires two different formulations, Case 3 where the structure is partially supported by the right block, and Case 4 where tube 1 has no contact with the right block.
(a) 2-1 formation with external water on tube 1

(b) Inset from Figure 5.8.a

Figure 5.8 2-1 formation with external water on tube 1
Due to the force of the external water acting on one side, the model is no longer symmetric and all segments in the structure will have to be analyzed. The tube on the left bottom is referred to as tube 3. As in section 5.2, the resulting pressure heads in the bottom tubes are assumed equal and greater than that in the top tube. All quantities used here are nondimensional, as in equation (2.1) with $L = L_1$ and $\gamma = \gamma_1$. Sections 5.3.1 – 5.3.3 refer to the basic condition, Case 1, in which the external water only acts on tube 1.

### 5.3.1 Segments for Case 1 (external water on tube 1)

Using equation (5.1), the values for $G$, $J$, and tension can be determined for each segment. These values and the initial and final angles are summarized in Table 5.3 for the 2-1 formation with external water on tube 1. Segments 3, 9, and 10 require special equations since they involve contact with two tubes. The parameter $\beta$ is defined in equation (3.15).

<table>
<thead>
<tr>
<th>Segment (i)</th>
<th>$G_i$</th>
<th>$J_i$</th>
<th>$t_i$</th>
<th>$\theta_{1i}$</th>
<th>$\theta_{1o}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p_1 - y_0 + \beta(y_0 - h_W)$</td>
<td>1 - $\beta$</td>
<td>$t_1$</td>
<td>$\theta_{1H}$</td>
<td>$\theta_{1W}$</td>
</tr>
<tr>
<td>2</td>
<td>$p_2 - y_0 - y_{1W} - y_{8D}$</td>
<td>1</td>
<td>$t_2$</td>
<td>$\theta_{2D}$</td>
<td>$\theta_{2E}$</td>
</tr>
<tr>
<td>4</td>
<td>$p_1$</td>
<td>1 - $k$</td>
<td>$t_1$</td>
<td>$\theta_{4A}$</td>
<td>$\theta_{4B}$</td>
</tr>
<tr>
<td>5</td>
<td>$p_1$</td>
<td>1</td>
<td>$t_1$</td>
<td>$\theta_{5A}$</td>
<td>$\theta_{5M}$</td>
</tr>
<tr>
<td>6</td>
<td>$p_1 - y_0 - y_{1W} + y_{3D} - y_{8D}$</td>
<td>-1</td>
<td>$t_1$</td>
<td>$\theta_{6C}$</td>
<td>$\theta_{6N}$</td>
</tr>
<tr>
<td>7</td>
<td>$p_2 - y_0 - y_{11E} + y_{10F}$</td>
<td>1</td>
<td>$t_2$</td>
<td>$\theta_{7F}$</td>
<td>$\theta_{7C}$</td>
</tr>
<tr>
<td>8</td>
<td>$p_1 - y_{1W} - y_0$</td>
<td>1</td>
<td>$t_1$</td>
<td>$\theta_{8W}$</td>
<td>$\theta_{8D}$</td>
</tr>
<tr>
<td>11</td>
<td>$p_1 - y_0$</td>
<td>1</td>
<td>$t_3$</td>
<td>$\theta_{11G}$</td>
<td>$\theta_{11E}$</td>
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<td>12</td>
<td>$p_1$</td>
<td>1 - $k$</td>
<td>$t_3$</td>
<td>$\theta_{12P}$</td>
<td>$\theta_{12Q}$</td>
</tr>
<tr>
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<td>$\theta_{13Q}$</td>
<td>$\theta_{13M}$</td>
</tr>
<tr>
<td>14</td>
<td>$p_1 - y_{13M} - y_{9N}$</td>
<td>1</td>
<td>$t_3$</td>
<td>$\theta_{14N}$</td>
<td>$\theta_{14F}$</td>
</tr>
</tbody>
</table>

The equations for segment 3 are given in equations (5.3) – (5.7). For segment 10, define $M_{10}$ as:

$$M_{10} = \frac{p_2 - p_1}{t_2 + t_3}$$  \hspace{1cm} (5.18)
where $t_3$ is the circumferential tension in tube 3. Using the procedure outlined in Section 4.2.6.2, the equations for segment 10 are:

$$x_{10} = \frac{\sin \theta_{10} - \sin \theta_{10E}}{M_{10}}, \quad y_{10} = \frac{\cos \theta_{10E} - \cos \theta_{10}}{M_{10}}, \quad (5.19, 5.20)$$

$$x_{10F} = \frac{\sin \theta_{7F} - \sin \theta_{10E}}{M_{10}}, \quad y_{10F} = \frac{\cos \theta_{10E} - \cos \theta_{7F}}{M_{10}}, \quad (5.21, 5.22)$$

and

$$s_{10F} = \frac{\theta_{7F} - \theta_{10E}}{M_{10}} \quad (5.23)$$

For segment 9, the distance $s_{9N}$ is unknown. Since the specific weight and relative pressure heads are equal in the two bottom tubes, this segment will have no curvature. From the external water load, this section will most likely be rotated at a small angle with $\theta_{13M} < \pi/2$. The equations for $x_{9N}$ and $y_{9N}$ are related to $s_{9N}$ by trigonometry:

$$x_{9N} = s_{9N} \cos \theta_{13M}, \quad y_{9N} = s_{9N} \sin \theta_{13M} \quad (5.24, 5.25)$$

### 5.3.2 Slope conditions for Case 1 (external water on tube 1)

The slope conditions for this problem due to continuity and symmetry of segments 4 and 12 are:

$$\theta_{5A} = -\theta_{4A}, \quad \theta_{4A} = -\theta_{4B}, \quad (5.26)$$

$$\theta_{6C} = \theta_{3C}, \quad \theta_{7C} = \theta_{3C}, \quad (5.27)$$

$$\theta_{2D} = \theta_{8D} - \pi, \quad \theta_{3D} = \theta_{8D} - \pi, \quad (5.28)$$

$$\theta_{10E} = \theta_{2E} - 2\pi, \quad \theta_{11E} = \pi - \theta_{10E}, \quad (5.29)$$

$$\theta_{10F} = \theta_{7F}, \quad \theta_{14F} = \pi + \theta_{7F}, \quad (5.30)$$

$$\theta_{5M} = \pi - \theta_{13M}, \quad \theta_{13M} = \theta_{8N}, \quad \theta_{13M} = \theta_{14N}, \quad (5.31)$$

$$\theta_{13Q} = \theta_{12Q}, \quad \theta_{12Q} = -\theta_{12P}, \quad (5.32)$$

$$\theta_{1W} = \theta_{8W} \quad (5.33)$$
where $\theta_{4A}$, $\theta_{7F}$, and $\theta_{12P}$ are less than 0. In addition to a chosen angle from each of equations (5.26) – (5.33), six more unknowns for this problem are $t_1$, $t_2$, $t_3$, $s_{9N}$, $\theta_{11G}$, and $\theta_{11H}$. For this formulation the nondimensional base length between the blocks, $\lambda$, will be given by the designer.

### 5.3.3 Constraints for Case 1 (external water on tube 1)

Using equations (2.42) – (2.44) the constraints for tube 1 are:

\[
\begin{align*}
y_{5M} + y_{9N} + y_{6N} + y_{3D} - y_{8D} - y_{1W} - y_{0} &= 0, \\
x_{5M} - x_{9N} - x_{6N} - x_{3D} + x_{8D} + x_{1W} + x_{0} + x_{4B} &= 0, \\
s_{5M} + s_{9N} + s_{6N} + s_{3D} + s_{8D} + s_{1W} + s_{0} + s_{4B} &= 1
\end{align*}
\]  

respectively. For tube 2:

\[
\begin{align*}
y_{3D} + y_{2E} + y_{10F} + y_{7C} &= 0, \\
x_{3D} + x_{2E} + x_{10F} + x_{7C} &= 0, \\
s_{3D} + s_{2E} + s_{10F} + s_{7C} &= 1,
\end{align*}
\]

and for tube 3,

\[
\begin{align*}
y_{13M} + y_{9N} + y_{14F} - y_{10F} - y_{11E} - y_{0} &= 0, \\
x_{13M} + x_{9N} + x_{14F} - x_{10F} + x_{11E} + x_{0} + x_{12Q} &= 0, \\
s_{13M} + s_{9N} + s_{14F} + s_{10F} + s_{11E} + s_{0} + s_{12Q} &= 1
\end{align*}
\]

Using geometric considerations for the problem, five more constraints are:

\[
\begin{align*}
y_{1W} + y_{0} - h_{W} &= 0 \\
x_{6N} - x_{14F} - x_{7C} &= 0 \\
y_{14F} + y_{7C} - y_{6N} &= 0 \\
y_{13M} - y_{5M} &= 0 \\
x_{12Q} + x_{13M} + x_{5M} + x_{4B} - \lambda &= 0
\end{align*}
\]

Using these 14 constraints, the 14 unknowns can be determined numerically.
5.3.4 Case 2: External water on tubes 1 and 2

After the water level reaches the maximum value of $y_1$, a few changes must be made to the formulation. As shown in Figure 5.9, segment 8 becomes part of tube 2. The necessary changes to the segments, slope continuity, and constraints for Case 2 are discussed below.

This formulation requires a few minor changes to segments 1, 2, 6, and 8. Table 5.4 gives the new values for $G$, $J$, tension, $\theta_{1\alpha}$, and $\theta_{1\omega}$ for these segments. The values for the remaining segments can be found in Table 5.3.

![Figure 5.9 2-1 formation with water on tube 2](image)

Table 5.4 Changes for Case 2

<table>
<thead>
<tr>
<th>Segment (i)</th>
<th>$G_i$</th>
<th>$J_i$</th>
<th>$t_i$</th>
<th>$\theta_{1\alpha}$</th>
<th>$\theta_{1\omega}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p_1 - y_0 + \beta(y_0 - h_W)$</td>
<td>$1 - \beta$</td>
<td>$t_1$</td>
<td>$\theta_{1H}$</td>
<td>$\theta_{1D}$</td>
</tr>
<tr>
<td>2</td>
<td>$p_2 - y_0 - y_{1D} + \beta(y_0 + y_{1D} - h_W)$</td>
<td>$1 - \beta$</td>
<td>$t_2$</td>
<td>$\theta_{2D}$</td>
<td>$\theta_{2W}$</td>
</tr>
<tr>
<td>6</td>
<td>$p_1 - y_0 + y_{3D} - y_{1D}$</td>
<td>$-1$</td>
<td>$t_1$</td>
<td>$\theta_{6C}$</td>
<td>$\theta_{6N}$</td>
</tr>
<tr>
<td>8</td>
<td>$p_2 - y_{1D} - y_{2W} - y_0$</td>
<td>$1$</td>
<td>$t_2$</td>
<td>$\theta_{8W}$</td>
<td>$\theta_{8E}$</td>
</tr>
</tbody>
</table>

Slope continuity equations (5.28), (5.29), and (5.33) can no longer be used. The replacement equations for this formulation are:

\[
\begin{align*}
\theta_{2D} &= \theta_{1D} - \pi, \\
\theta_{3D} &= \theta_{1D} - \pi, \\
\theta_{10E} &= \theta_{8E} - 2\pi, \\
\theta_{11E} &= \pi - \theta_{10E}, \\
\theta_{2W} &= \theta_{8W}
\end{align*}
\]  

(5.48)  
(5.49)  
(5.50)

The constraints given in equations (5.34) – (5.39) are no longer valid for tubes 1 and 2. The new constraints are:
\begin{align*}
y_{5M} + y_{9N} + y_{6N} + y_{3D} - y_{1D} - y_0 &= 0, \quad (5.51) \\
x_{5M} - x_{9N} - x_{6N} - x_{3D} + x_1 + x_0 + x_{4B} &= 0, \quad (5.52) \\
s_{5M} + s_{9N} + s_{6N} + s_{3D} + s_{1D} + s_0 + s_{4B} &= 1, \quad (5.53) \\
y_{3D} + y_{2W} + y_{8E} + y_{10F} + y_{7C} &= 0, \quad (5.54) \\
x_{3D} + x_{2W} + x_{8E} + x_{10F} + x_{7C} &= 0, \quad (5.55) \\
s_{3D} + s_{2W} + s_{8E} + s_{10F} + s_{7C} &= 1, \quad (5.56) \\
\end{align*}

respectively. Equation (5.43) also changes to:

\[ y_{1D} + y_{2W} + y_0 - h_W = 0 \] (5.57)

This case likewise has 14 equations and unknowns.

\subsection*{5.3.5 Case 3: Partial block contact}

Tube 1 does not physically maintain contact with the entire length of the block, \(s_0\), when the solution for \(\theta_{1H}\) is greater than \(\theta_0\). The exact point of lift-off can be calculated by setting \(\theta_{1H} = \theta_0\) (see Figure 5.10). Assuming \(\theta_0 = \pi/4\), the horizontal and vertical distances the tube is in contact with the block are equal. Let \(x_H\) and \(y_H\) specify these distances. Since \(x_H = y_H\), \(x_H\) can be the unknown replacing \(\theta_{1H}\).

From trigonometry:

\[ s_H = \sqrt{x_H^2 + y_H^2} \] (5.58)

To solve for the point of lift-off, simple substitution can be made in any formulation outlined in this thesis. The substitutions for \(x_0\), \(y_0\), and \(s_0\) must be made all times when the formulation references the particular block where the tube is lifting off. In the 2-1 formation with water, the
substitutions must be made in $G_i$ for segments 1, 2, 6, and 8, in all constraints for tube 1, and in either equation (5.43) or (5.57).

### 5.3.6 Case 4: No contact with right block

As the water height is increased, the force from the external water may eventually push tube 1 completely away from the right block. The horizontal distance between points $P$ and $B$ is defined nondimensionally as $b$. This case is shown in Figure 5.11. This distance must be less than $\lambda$. Assuming this condition will occur when the water is on tube 2, the following few changes are based on the formulation in Section 5.3.4. Table 5.5 contains the changes to segments 1, 2, 6, and 8 from Table 5.4.

![Figure 5.11 2-1 formation with no contact of right block](image)

#### Table 5.5 New values for external water on tube 2, no contact of right block

<table>
<thead>
<tr>
<th>Segment (i)</th>
<th>$G_i$</th>
<th>$J_i$</th>
<th>$t_i$</th>
<th>$\theta_{1B}$</th>
<th>$\theta_{1D}$</th>
<th>$\theta_{2D}$</th>
<th>$\theta_{2W}$</th>
<th>$\theta_{6C}$</th>
<th>$\theta_{6N}$</th>
<th>$\theta_{8W}$</th>
<th>$\theta_{8E}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p_1 - y_0 - \beta h_W$</td>
<td>$1 - \beta$</td>
<td>$t_1$</td>
<td>$\theta_{1B}$</td>
<td>$\theta_{1D}$</td>
<td></td>
<td></td>
<td>$\theta_{6C}$</td>
<td>$\theta_{6N}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$p_2 - y_0 - y_{1D} + \beta(y_{1D} - h_W)$</td>
<td>$1 - \beta$</td>
<td>$t_2$</td>
<td>$\theta_{2D}$</td>
<td>$\theta_{2W}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\theta_{6N}$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$p_1 - y_{1D} + y_{3D}$</td>
<td>$-1$</td>
<td>$t_1$</td>
<td>$\theta_{6C}$</td>
<td>$\theta_{6N}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\theta_{8E}$</td>
</tr>
<tr>
<td>8</td>
<td>$p_2 - y_{2w} - y_{1D}$</td>
<td>$1$</td>
<td>$t_2$</td>
<td>$\theta_{8W}$</td>
<td>$\theta_{8E}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since the tube is no longer in contact with the block, the slope is continuous at point $B$:  

![Diagram](image)
\[ \theta_{4B} = \theta_{1B} \quad (5.59) \]

The constraints in equations (5.51), (5.52), (5.53), and (5.57) are the same except without the block dimensions:

\[ y_{5M} + y_{9N} + y_{6N} + y_{3D} - y_{1D} = 0, \quad (5.60) \]
\[ x_{5M} - x_{9N} - x_{6N} - x_{3D} + x_{1D} + x_{4B} = 0, \quad (5.61) \]
\[ s_{5M} + s_{9N} + s_{6N} + s_{3D} + s_{1D} + s_{4B} = 1 \quad (5.62) \]
\[ y_{1D} + y_{2W} - h_{W} = 0 \quad (5.63) \]

Equations (5.54) - (5.56) are still used for this formulation, while the constraint given in equation (5.47) is no longer valid. The numerical results can be found for this formulation by using the 13 unknowns and 13 equations (\( \theta_{1H} \) is no longer an unknown).

### 5.3.7 Examples and results

Figure 5.12 shows outlines of structures with the same parameters but different water heights. In this example \( h_{1} = 0.65, h_{2} = 0.60, \lambda = 0.5, k = 15, \beta = 0.5, \) and \( x_{0} = y_{0} = 0.05. \) Table 5.6 shows the various water heights and the numerical solution for all of the unknowns. In addition the table also gives the contact condition of the right block and which particular case was used.

In Figure 5.12.b, \( \theta_{1H} > \theta_{0}, \) so the tube is actually lifting off the block. However, the lift-off in this case is minimal and has a negligible effect on the numerical results. The true analysis for this case should be done by Case 3; however, this example is used for illustration of Case 2. Case 4 illustrates a condition where the tube just comes off the block and \( b = 0.5. \)
Figure 5.12 2-1 formation with external water, $h_1 = 0.65$, $h_2 = 0.60$, $\lambda = 0.5$, $k = 15$, $\beta = 0.5$, and $x_0 = y_0 = 0.05$
(c) Case 3: $h_W = 0.35$

(d) Case 4: $h_W = 0.40$

Figure 5.12 (cont’d) 2-1 formation with external water, $h_1 = 0.65$, $h_2 = 0.60$, $\lambda = 0.5$, $k = 15$, $\beta = 0.5$, and $x_0 = y_0 = 0.05$
Table 5.6 Numerical solutions for 2-1 formation with external water, 
h\textsubscript{1} = 0.65, h\textsubscript{2} = 0.60, \lambda = 0.5, k = 15, \beta = 0.5, and x\textsubscript{0} = y\textsubscript{0} = 0.05

<table>
<thead>
<tr>
<th>h\textsubscript{W} on tube #</th>
<th>Block contact</th>
<th>Case</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>h\textsubscript{W} on tube #</td>
<td>Full</td>
<td>1</td>
<td>5.12.a</td>
</tr>
<tr>
<td></td>
<td>Full</td>
<td>2</td>
<td>5.12.b</td>
</tr>
<tr>
<td></td>
<td>Partial</td>
<td>3</td>
<td>5.12.c</td>
</tr>
<tr>
<td></td>
<td>None</td>
<td>4</td>
<td>5.12.d</td>
</tr>
<tr>
<td>t\textsubscript{1}</td>
<td>0.0553</td>
<td>0.0531</td>
<td>0.0511</td>
</tr>
<tr>
<td>t\textsubscript{2}</td>
<td>0.0304</td>
<td>0.0304</td>
<td>0.0301</td>
</tr>
<tr>
<td>t\textsubscript{3}</td>
<td>0.0556</td>
<td>0.0559</td>
<td>0.0550</td>
</tr>
<tr>
<td>s\textsubscript{9N}</td>
<td>0.0181</td>
<td>0.0228</td>
<td>0.0258</td>
</tr>
<tr>
<td>\theta\textsubscript{5A}</td>
<td>0.7198</td>
<td>0.7464</td>
<td>0.7732</td>
</tr>
<tr>
<td>\theta\textsubscript{3C}</td>
<td>0.6533</td>
<td>0.5987</td>
<td>0.5612</td>
</tr>
<tr>
<td>\theta\textsubscript{2D}</td>
<td>0.5855</td>
<td>0.5245</td>
<td>0.4910</td>
</tr>
<tr>
<td>\theta\textsubscript{11E}</td>
<td>3.7443</td>
<td>3.7917</td>
<td>3.8183</td>
</tr>
<tr>
<td>\theta\textsubscript{14F}</td>
<td>2.4728</td>
<td>2.4310</td>
<td>2.4026</td>
</tr>
<tr>
<td>\theta\textsubscript{11G}</td>
<td>0.4850</td>
<td>0.3980</td>
<td>0.1356</td>
</tr>
<tr>
<td>\theta\textsubscript{1H}</td>
<td>0.6382</td>
<td>0.7921*</td>
<td>\pi/4</td>
</tr>
<tr>
<td>\theta\textsubscript{13M}</td>
<td>1.5656</td>
<td>1.5358</td>
<td>1.5121</td>
</tr>
<tr>
<td>\theta\textsubscript{12Q}</td>
<td>0.7108</td>
<td>0.6985</td>
<td>0.6791</td>
</tr>
<tr>
<td>\theta\textsubscript{8W}</td>
<td>1.9585</td>
<td>1.5229</td>
<td>1.8723</td>
</tr>
<tr>
<td>x\textsubscript{H} = y\textsubscript{H}</td>
<td></td>
<td></td>
<td>0.0272</td>
</tr>
</tbody>
</table>

* This case shown for illustration purposes (\theta\textsubscript{1H} > \theta\textsubscript{0})

5.3.7.1 Multiple solutions

Multiple solutions exist for a variety of pressure head and stiffness coefficient combinations for various water heights in Cases 3 and 4. One solution for Case 3 could usually be found along with two solutions for Case 4 at a particular water height. The solution in Case 4 usually includes a solution with a large distance b between points P and B and a positive value of \theta\textsubscript{11G}, and a solution with a smaller value of b and a negative value of \theta\textsubscript{11G}. Multiple solutions for \theta\textsubscript{11G} and \lambda can be found in Table 5.7 for various water heights.
Table 5.7  Multiple solutions for $h_1 = 0.65$, $h_2 = 0.60$, $\lambda = 0.5$, $k = 15$, $x_0 = y_0 = 0.05$

<table>
<thead>
<tr>
<th></th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Solution set 1</td>
<td>Solution set 2</td>
</tr>
<tr>
<td>$h_W$</td>
<td>$\theta_{1G}$</td>
<td>$x_H = y_H$</td>
</tr>
<tr>
<td>0.35</td>
<td>0.135</td>
<td>0.027</td>
</tr>
<tr>
<td>0.38</td>
<td>-0.030</td>
<td>0.016</td>
</tr>
<tr>
<td>0.40</td>
<td>-0.269</td>
<td>0.000</td>
</tr>
<tr>
<td>0.41</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

The solutions for all unknowns in sets 1 and 2 are very similar at $h_W = 0.40$ for the example with $h_1 = 0.65$, $h_2 = 0.60$, and $k = 15$. Here, solution set 2 has $b \geq \lambda$, suggesting that the tube is losing contact with the block and moving away. However, solution set 2 examples with higher water levels give values of $b$ greater than $\lambda$. Instead of tube 1 moving away from the right block, the structure may become unstable without its support. Therefore, it is unclear if the 2-1 formation is stable without contact of the right block, though solutions exist for a wide range of data.

Small deviations occur at the transition between the solutions in Cases 3 and 4 (sets 1 and 2) because of the angle between segments 4 and 1. Solutions in Case 3 which have $x_H = y_H = 0$ still allow an angle discontinuity between $\theta_{4B}$ and $\theta_{1H}$. Case 4 assumes that these angles are equal.

It appears that solution set 3 is invalid since $b$ is not less than $\lambda$. However, the solutions for $t_1$, $t_2$, $t_3$, $h_T$, and $d_1$ are very similar to those in set 2. Typically only two solutions could be found in Case 4, though more could exist. It is believed that if any of the solutions from Case 4 are stable and physically expected to occur, it would be the set which matches the solution for a partial block contact length near zero (solution set 2).

Figure 5.13 shows a case for $h_1 = 0.55$, $h_2 = 0.50$, $k = 15$, and $h_W = 0.36$ with multiple solutions. In Figure 5.13.a, $\theta_{11G} = 0.440$ and $b = 0.588$. In Figure 5.15.b, $\theta_{11G} = -0.355$ and $b = 0.522$. The distance $\lambda$ is not specified in this example.
(a) First solution: $\theta_{11G} = 0.440$, $b = 0.588$

(b) Second solution: $\theta_{11G} = -0.355$, $b = 0.522$

Figure 5.13  2-1 formation with multiple solutions,
$h_1 = 0.55$, $h_2 = 0.50$, $h_w = 0.36$, $k = 15$, $x_0 = y_0 = 0.05$
5.3.7.2 Water height effect on tension

Figures 5.14 and 5.15 show how the external water level affects circumferential tension, ground deflection, and the height of the structure, $h_T$. The same parameters are used in these figures as given in Figure 5.12 and Table 5.7: $h_1 = 0.65$, $h_2 = 0.60$, $\lambda = 0.5$, $k = 15$, and $x_0 = y_0 = 0.05$. All three solution sets are shown in the figures. As the water level increases, the appropriate analysis was used to calculate the data points. Figures 5.14 and 5.15 also include an example with internal pressure heads $h_1 = 0.55$ and $h_2 = 0.50$ for Cases 1 and 2. The partial block analysis, Case 3, for $h_1 = 0.65$ and $h_2 = 0.60$ begins at $h_W = 0.26$.

Figure 5.14 considers the circumferential tension of all tubes as a function of the water height. Because the tubes are symmetric initially (when $h_W = 0$), the tensions in tubes 1 and 3 are equal (see Figure 5.13.a). When the nondimensional height of the water reaches about 0.15, the tension values begin to diverge. The tension in tube 3, $t_3$, decreases very slowly compared with $t_1$ as the water level increases. However, the change in tension is not substantial in either tube. The tension gradually increases in both tubes as the contact length with the right block in Case 3 approaches zero.

Figure 5.14 also considers the tension in all tubes for Case 4, where the tube is not in contact with the right block. For $h_1 = 0.65$, $h_2 = 0.60$, the tension decreases substantially from $h_W = 0.34$ to $h_W = 0.415$. Nondimensional heights below 0.34 for this case resulted in a negative value for the distance $s_{9N}$.

The circumferential tension in tube 2 decreases very gradually as the height of the water increases (see Figure 5.14.b). As with the tensions in the other tubes, $t_2$ increases slightly as the partial block length approaches zero. Once again, $t_2$ decreases substantially for Case 4.
(a) Tension versus water height, $h_W$

![Graph showing tension versus water height with various lines and markers for different cases and set numbers.]

(b) Tension, $t_2$, versus water height, $h_W$

![Graph showing tension $t_2$ versus water height with different cases and set numbers.]

Figure 5.14  Tension versus water height $h_W$ for 2-1 formation, $h_1 = 0.65$, $h_2 = 0.60$, $\lambda = 0.5$, $k = 15$, $\beta = 0.5$, and $x_0 = y_0 = 0.05$
(a) Height, $h_T$, versus water height, $h_W$

(b) Ground deflection versus water height, $h_W$

Figure 5.15  Height and ground deflection versus water height, $h_W$, for 2-1 formation, $h_1 = 0.65$, $h_2 = 0.60$, $\lambda = 0.5$, $k = 15$, $\beta = 0.5$, and $x_0 = y_0 = 0.05$
5.3.7.3 Water height effect on height $h_T$ and ground deflection

Compared to the circumferential tension, the height of the water has the opposite effect on the structure height (see Figure 5.15.a). The structure height increases gradually until the partial block contact approaches zero, where a swift decrease occurs. At $h_W = 0.40$, the total height in Case 3 is very close to $h_T$ in Case 4. Overall, for Cases 1 - 3, $h_W$ has very little effect on $h_T$.

The maximum ground deflections under tubes 1 and 3, $d_1$ and $d_3$, are equal before external water is applied to the tubes (Figure 5.15.b). At nondimensional water levels slightly above 0.10, the ground deflection $d_1$ begins to increase as $d_3$ decreases. This occurs because the angle $\theta_{11G}$ decreases as $h_W$ increases. Therefore, more of tube 3 is suspended over the block, which decreases $x_{12Q}$ (see Figure 5.8.b) and the downward force of tube 3 on the ground. A larger value of $x_{4B}$ accompanies a smaller value of $x_{12Q}$, and therefore tube 1 applies more force on the ground and there is a greater ground deflection for tube 1. Except for $d_3$ in solution set 3, the downward ground deflection increases for both tubes in Case 4 as $h_W$ increases. The ground deflection $d_3$ decreases because $x_{4B}$ decreases as $b$ increases.