4.0 Non Uniform B-splines

4.1 Introduction to Parametric Curves

A curve segment is a point-bound collection of points whose coordinates are given by continuous, one-parameter, single-valued, mathematical functions of the form [Mort85]

\[ x = x(u) \quad y = y(u) \quad z = z(u) \]  \hspace{1cm} (4.1)

where, \( u \) is the parametric variable, \( u \in [0,1] \)

If we consider the coordinates of any point on a parametric curve as the components of a vector \( P(u) \), then \( P'(u) \) is the tangent vector to the curve at the same point. \( P'(u) \) is obtained by differentiating \( P(u) \). Figure 4.1 illustrates how the vectors \( P(0), P(1), P'(0), \) and \( P'(1) \) form the boundary conditions of the curve. In the above description, \( u \) is the parametric variable, and each value of \( u \) generates a distinct point on the curve.
4.2 B-spline Curves

The word spline comes from the loftman’s spline, which is a long, narrow strip of wood used to fit curves passing through a specified series of points. Mathematical B-splines were first used by Schoenberg for statistical data smoothing [Fari97]. The original description of B-spline curves made use of divided differences and is discussed in the subsequent sections of this research.

Essentially, mathematical splines are piece-wise polynomials having degree k with $C^{k-1}$ continuity at the points common to adjacent sections (called junction point). Each point on the curve is a weighted combination of a number of control points in the proximity of that point. These control points form a control polygon enclosing the curve. In addition, the curve is a result of mapping elements from the knot sequence onto Cartesian space. Figure (4.2) shows a B-spline curve along with its knot sequence and control polygon.

Figure 4.1: Parametric curve $P(u)$ shown with the boundary conditions $P(0)$, $P(1)$, $P'(0)$, and $P'(1)$. 

Figure 4.2: A B-spline curve with its knot sequence and control polygon.
In fact, B-splines can be completely specified by the control points, the order of the curve, and the basis function, as shown in the equation below:

Here, $P(u)$ is a point along the curve as a function of the parameter $u$

$$P(u) = \sum_{i=1}^{n+1} b_i N_{i,k}(u)$$

$b_i$ are the control points

$N_{i,k}$ are the basis functions

and $k$ is order of the curve

For a spline of order $k$, the $i$th basis function is defined as:

$$N_{i,1}(u) = \begin{cases} 1 & \text{if } (u_i \leq u \leq u_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

(4.6)

Recursively, we have the relation

$$N_{i,k}(u) = \frac{(u-u_i)N_{i,k-1}(u)}{(u_{i+k-1}-u_i)} + \frac{(u_{i+1}-u)N_{i+1,k-1}(u)}{(u_{i+k}-u_{i+1})}$$

(4.7)
where, $u_i$ is an element of the knot sequence, satisfying the relation $u_i \leq u_{i+1}$

As discussed earlier, the curve results from mapping the knot sequence onto Cartesian space. The spacing between adjacent knots is an important consideration that determines the shape of a curve. A number of techniques have been developed to determine the knot sequence most suited to fit a curve through a given number of points. For example, the uniform knot spacing technique uses the same spacing for each segment of the curve.

Chord length and centripetal parametrization are two examples of non-uniform knot spacing schemes. With chord length parameterization, the knot spacing is proportional to the distance between points (or control points), while with centripetal parameterization, it is proportional to the square root of the distance between points. Uniform knot sequencing fails to cope with all but the most simplistic of geometries [Fari97]. Due to this reason, the research in this thesis uses non-uniform B-spline.
4.3 B-Spline Surfaces

In general, B-spline surfaces are the two dimensional analogues of B-spline curves and may be represented as:

\[ Q(u, v) = \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} b_{i,j} N_{i,k}(u) M_{j,l}(v) \]  

where,

\[ Q(u, v) \] are points on the surface as functions of the parameter \( u \) and \( v \),

\( b_{i,j} \) are the control points, and

\( N_{i,k}(u) \) and \( M_{j,l}(v) \) are the basis functions.

The \( i^{th} \) and \( j^{th} \) basis functions respectively are described as:

\[ N_{i,1}(u) = \begin{cases} 
1 & \text{if } (u_i \leq u \leq u_{i+1}) \\
0 & \text{otherwise} 
\end{cases} \]  

\[ M_{j,1}(v) = \begin{cases} 
1 & \text{if } (v_j \leq v \leq v_{j+1}) \\
0 & \text{otherwise} 
\end{cases} \]

\( u_i \) and \( v_j \) are elements of knot sequences satisfying the relations \( u_i \leq u_{i+1} \) and \( v_j \leq v_{j+1} \).

In general, using the recursive relation, we get:

\[ N_{i,k}(u) = \frac{(u-u_i) N_{i,k-1}(u)}{(u_{i+k-1}-u_i)} + \frac{(u_{i+k}-u) N_{i+1,k-1}(u)}{(u_{i+k}-u_{i+1})} \]  

\[ M_{j,l}(v) = \frac{(v-v_j) M_{j,l-1}(v)}{(v_{j+l-1}-v_j)} + \frac{(v_{j+l}-v) M_{j+1,l-1}(v)}{(v_{j+l}-v_{j+1})} \]
The B-spline surface has a control net as the analogue of the control polygon. Figure 4.3 shows the control net, the control points and the B-spline surface. The B-spline surface is composed of B-spline curves, which means that individual isoparametric curves in either parametric direction are themselves B-spline curves. Also, each control point is associated with a unique basis function. For a curve of degree $n$, the portion affected by a control point, spans $n+1$ curve segments. This results in local controllability of B-splines. The above two properties have been extensively used in this research.

Figure 4.3: B-spline surface along with the control net and the control points.
4.4 Mathematical Formulation of Error

In this thesis, the error plot is formulated by sampling the position data across the two B-spline surfaces under consideration and then computing their differences. This data is sampled at \( n \) points along each of the \( m \) isoparametric curves. Since these are B-spline curves, this section will discuss a formulation for the error between two B-spline curves. This will then be extended to the generation of error plot and visualization of the shape of the error plot for a pair of closely matching surface patches.

Given:

Two cubic B-spline curves, \( P(u) \) and \( Q(u) \) with \( n \) control points \( b_i \) and \( c_i \) known on each curve. Chord length parameterization is used to obtain the knot sequence.

\[
P(u) = \sum_{i=1}^{n+1} b_i N_{i,k}(u)
\]

(4.13)

where, \( u \) is the parameter

\( b_i \) are the control points

\( k \) is the order of the curve (\( k = 4 \))

\( N_{i,k}(u) \) or rather \( N_{i,4}(u) \) are the basis functions

The curve \( Q(u) \) can be similarly described. Instead of the recursive procedure, this section uses a divided differences approach to calculate the basis functions as follows:
\[
N_{i,4}(u) = \begin{cases}
0 & (u \leq u_i) \\
(u_i + 4 - u_i) \left\{ \frac{(u_i + 1 - u)^3}{(u_i + 1 - u) (u_i + 1 - u + 2) (u_i + 1 - u + 3) (u_i + 1 - u + 4)} + \frac{(u_i + 2 - u)^3}{(u_i + 2 - u) (u_i + 2 - u + 1) (u_i + 2 - u + 3) (u_i + 2 - u + 4)} + \frac{(u_i + 3 - u)^3}{(u_i + 3 - u) (u_i + 3 - u + 1) (u_i + 3 - u + 2) (u_i + 3 - u + 4)} + \frac{(u_i + 4 - u)^3}{(u_i + 4 - u) (u_i + 4 - u + 1) (u_i + 4 - u + 2) (u_i + 4 - u + 3)} \right\} & (u_i < u \leq u_i + 1) \\
(u_i + 4 - u_i) \left\{ \frac{(u_i + 2 - u)^3}{(u_i + 2 - u) (u_i + 2 - u + 1) (u_i + 2 - u + 3) (u_i + 2 - u + 4)} + \frac{(u_i + 3 - u)^3}{(u_i + 3 - u) (u_i + 3 - u + 1) (u_i + 3 - u + 2) (u_i + 3 - u + 4)} + \frac{(u_i + 4 - u)^3}{(u_i + 4 - u) (u_i + 4 - u + 1) (u_i + 4 - u + 2) (u_i + 4 - u + 3)} \right\} & (u_i + 1 < u \leq u_i + 2) \\
(u_i + 4 - u_i) \left\{ \frac{(u_i + 3 - u)^3}{(u_i + 3 - u) (u_i + 3 - u + 1) (u_i + 3 - u + 2) (u_i + 3 - u + 4)} + \frac{(u_i + 4 - u)^3}{(u_i + 4 - u) (u_i + 4 - u + 1) (u_i + 4 - u + 2) (u_i + 4 - u + 3)} \right\} & (u_i + 2 < u \leq u_i + 3) \\
(u_i + 4 - u_i) \left\{ \frac{(u_i + 4 - u)^3}{(u_i + 4 - u) (u_i + 4 - u + 1) (u_i + 4 - u + 2) (u_i + 4 - u + 3)} \right\} & (u_i + 3 < u \leq u_i + 4) \\
0 & (u \geq u_i)
\end{cases}
\]

(4.14)
From chord length parametrization

\[ u_i = 0 \]

\[ u_k = u_{i-1} + \|b_i - b_{i+1}\| \]

\[ u_{i+1} - u_i = \|b_{i+1} - b_i\| \]

\[ u_{i+2} - u_i = \|b_{i+2} - b_{i+1}\| + \|b_{i+1} - b_i\| \]

\[ u_{i+3} - u_i = \|b_{i+3} - b_{i+2}\| + \|b_{i+2} - b_{i+1}\| + \|b_{i+1} - b_i\| \]

\[ u_{i+4} - u_i = \|b_{i+4} - b_{i+3}\| + \|b_{i+3} - b_{i+2}\| + \|b_{i+2} - b_{i+1}\| + \|b_{i+1} - b_i\| \]

Similarly, we can express the other differences in terms of distances between the control points.
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\[
N_{i+4}(u) = \begin{cases}
0 & (u \leq u_i) \\
\left[ b_{i+4} - b_{i+3} \right] + \left[ b_{i+3} - b_{i+2} \right] + \left[ b_{i+2} - b_{i+1} \right] + \left[ b_{i+1} - b_i \right] & (u_i < u \leq u_{i+1}) \\
\left( u - u_i + \left[ b_{i+4} - b_{i+3} \right] \right)^3 & (u_{i+1} < u \leq u_{i+2}) \\
\left( u - u_i + \left[ b_{i+4} - b_{i+3} \right] \right)^3 & (u_{i+2} < u \leq u_{i+3}) \\
\left( u - u_i + \left[ b_{i+4} - b_{i+3} \right] \right)^3 & (u_{i+3} < u \leq u_{i+4}) \\
\end{cases}
\]

continued
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\[ N_{i+4}(u) = \begin{cases} 
\frac{(u - u + |b_{i+4} - b_{i+3}| + |b_{i+3} - b_{i+2}| + |b_{i+2} - b_{i+1}|)}{(|b_{i+4} - b_{i+3}| + |b_{i+3} - b_{i+2}| + |b_{i+2} - b_{i+1}| + |b_{i+1} - b_{i}|)} \left( |b_{i+4} - b_{i+3}| + |b_{i+3} - b_{i+2}| + |b_{i+2} - b_{i+1}| + |b_{i+1} - b_{i}| \right), \\
\frac{(u - u + |b_{i+4} - b_{i+3}| + |b_{i+3} - b_{i+2}| + |b_{i+2} - b_{i+1}| + |b_{i+1} - b_{i}|)}{(|b_{i+4} - b_{i+3}| + |b_{i+3} - b_{i+2}| + |b_{i+2} - b_{i+1}| + |b_{i+1} - b_{i}|)} \right) \\
0 & \text{if } u \geq w_{i+4} 
\end{cases} \] (4.15)
To summarize, the B-spline curve at any point can be expressed in terms of the control points and the value of the parameter. The second curve can be similarly expressed.

Since the control points in Cartesian space are known for both the curves, the points on the curves can be evaluated in terms of the components in cartesian space \((x, y, \text{and} \ z\ \text{component respectively})\), say \(P_x(u), Q_x(u), P_y(u), Q_y(u), P_z(u), \text{and} \ Q_z(u)\).

The error between two curves at the parameter value \(v_i\), or rather the points corresponding to the same parameter value on both the curves, is given by:

\[
E(u) = P(u) - Q(u) \quad (4.16)
\]

\[
E(u) = \sum_{i=1}^{n+1} b_i N_{i-4}(u) - \sum_{i=1}^{n+1} c_i M_{i-4}(u) \quad (4.17)
\]

Since the curves are evaluated in terms of the \(x, y, \text{and} \ z\) components, the error can also be calculated in terms of the \(x, y, \text{and} \ z\) component, called \(E_x(u), E_y(u), E_z(u)\) respectively.

The total error is calculated as a vector sum of the individual components:

\[
E(u) = \sqrt{E_x(u)^2 + E_y(u)^2 + E_z(u)^2} \quad (4.18)
\]

Once the error values are calculated for all the points along an isoparametric curve, linear regression is used to fit a straight line to the error data. The fit has the form

\[
E(u) = a \times u + b \quad (4.19)
\]
The use of linear least squares fitting minimizes the sum of the squared deviations of the data from the fit. The least squares fit gives us two independent parameters, the slope and the ordinate intercept. However, consider the case where the error data is shaped like a hemisphere. In such a case, the slope of the fit would be zero. Hence, out of the two, the ordinate intercept value is a better error measure and the ordinate intercept value plot is a better error visualization tool. The ordinate intercept plot not only replicates the shape of the error surface, but also provides a high statistical correlation with points of maximum error on the error plot. The slope and ordinate intercept values are calculated using the relations:

\[
\begin{align*}
a &= \frac{\sum E(u_i) \sum (u_i)^2 - \sum u_i \sum E(u_i) \cdot u_i}{n \sum u_i^2 - (\sum u_i)^2} \quad (4.20) \\
b &= \frac{n \sum u_i E(u_i) - \sum u_i \sum E(u_i)}{n \sum u_i^2 - (\sum u_i)^2} \quad (4.21)
\end{align*}
\]

In this manner, the ordinate intercept values are calculated for a number of isoparametric curves in the other direction (compared to the linear fit direction). If we refer to these values as \(a_j\), then a plot of the ordinate intercept values versus the parameter \((a_j v/s v)\) has been found to duplicate the shape of the error surface. The importance of this kind of data reduction is that the designer can still visualize a small magnitude of localized error, which is not so obvious to the human eye. Figures 4.4, 4.5, and 4.6 show the error plot, the linear fit (along with the data) for an isoparametric curve on the error surface, and the ordinate intercept plot, respectively.
Figure 4.4: Parametric plot of error.
Figure 4.5: Plot of error along each isoparametric curve in the \( u \) direction and linear fit for each curve.
Figure 4.6: Plot of error along each isoparametric curve in the \( v \) direction and linear fit for each curve.
Figure 4.7: Ordinate plot visualizing error in the $u$ parametric direction.
Figure 4.8: Ordinate plot visualizing error in the $v$ parametric direction.