6.0 Frequency Determination

6.1 Fourier Analysis

Any measurement is dependent on one or more parameters. Typically, time is the parameter used most frequently to describe variations in measurands. However, the magnitude of the measurand may or may not change with time. Nevertheless, the study of the variation of a quantity with change in the parameter(s) is important. Typically, computational geometry has measurands that are dynamic in nature. The magnitude of curves and surfaces changes with one or more parameters.

Fourier analysis has been described as the process of determining the frequency spectrum of a known waveform [Beck95]. The task involves measuring a complex waveform and determining which frequencies are present in it. However, not all measurands are periodic in nature. In such cases, it is helpful to think of the non-periodic relation as one cycle of a periodic relation; with all the other cycles being fictitious. Using this assumption, non-periodic functions may be analyzed in exactly the same manner as periodic functions.
A simple harmonic function is one whose second derivative is proportional to the function, but of opposite sign. Many variables in electrical and mechanical engineering are harmonic functions of time. However, it is possible for any two variables to be related harmonically. In other words any function may be expressed as a combination of simple harmonic components. Each of these components will have their own amplitude and frequency and phase relations will be used to describe their combination fully. Mathematically, the above statement may be expressed as:

\[ y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( A_n \cos(n\omega t) \pm B_n \sin(n\omega t) \right) \]  

where,

- \( A_0, A_n, \) and \( B_n \) are the harmonic coefficients
- \( n = 1 \) to \( \infty \) are integers called the harmonic order
- \( \omega = 2\pi/T \) is fundamental frequency of the fictitious non-periodic wave.

For the case \( n = 1 \), the corresponding terms are called fundamentals. The above equation is referred to as a Fourier series.

The harmonic coefficients are actually integrals of the waveform \( y(t) \) and can be obtained as:

\[ A_n = \frac{\omega}{\pi} \int_{0}^{2\pi/\omega} y(t) \cos(\omega nt) \, dt \quad n=0,1,2,\ldots \]  

\[ B_n = \frac{\omega}{\pi} \int_{0}^{2\pi/\omega} y(t) \sin(\omega nt) \, dt \quad n=0,1,2,\ldots \]  

[4.6]
In most practical cases, however, \( y(t) \) is known only at discrete points. The analog signal is therefore reduced to a series of measurements, \( y(\Delta t), y(2 \Delta t), \ldots, y(N \Delta t) \). To perform the Fourier analysis of a discrete signal, the integrals need to be replaced by summations. The continuously varying parameter also needs to be replaced by the discrete parameter values. The harmonic coefficients of a discretely sampled waveform are:

\[
A_n = \frac{2}{N} \sum_{r=1}^{N} y(r \Delta t) \cos \left( \frac{2\pi r n}{N} \right) \quad n=0,1,\ldots, \frac{N}{2} \quad (6.4)
\]

\[
B_n = \frac{2}{N} \sum_{r=1}^{N} y(r \Delta t) \sin \left( \frac{2\pi r n}{N} \right) \quad n=0,1,\ldots, \frac{N}{2} \quad (6.5)
\]

The above equations are called the Discrete Fourier Transform (DFT) of \( y(t) \). The reader may please note that unlike the ordinary Fourier series, the DFT yields only harmonic components up to \( n = N/2 \). This is a direct consequence of the discrete sampling process.

### 6.2 Frequency Spectrum

Typically, a combination of known waveforms produces complex waveforms. Such a complex waveform is easily analyzed using a frequency spectrum. A frequency spectrum is a plot with frequency as the abscissa and the amplitude of each frequency component as the ordinate. For example, Figure 6.1 shows two examples of frequency spectra, for the waveforms \( y = 10 \sin \omega t + 5 \sin 2\omega t \) and \( y = 10 \sin \omega t + 5 \sin 4\omega t \), respectively. The
frequency spectrum is useful as it facilitates the identification of the frequencies present in a signal at a glance.

In this research, the frequency characteristics of the error surface are investigated to gain a better understanding of the characteristics of the matching surface. By just looking at the frequency spectrum of the error surface, the designer can tell whether low or high frequencies are predominant. A preponderance of high frequency components suggests a poor match. By the same token, a good match would yield minimal frequency components and that too having low amplitudes. Ideally, a perfect match would result in an error surface with zero height all along the parametric interval. Figure 6.2 shows an example of a error surface that yields high frequency components on Fourier analysis. Figures 6.3 and 6.4 show the fourier coefficients for the above surface in either direction. This is an exaggerated case but serves to demonstrate the point that fourier analysis can be used to discern whether a match is a good one or not.
Figure 6.2: Exaggerated error surface to demonstrate feasibility of Fourier analysis as an error visualization tool.
Figure 6.3: Fourier coefficients (first six) for error data collected along isoparametric curves in v direction.
Figure 6.4: Fourier coefficients (first six) for error data collected along isoparametric curves in u direction.