Elongational Flows in Polymer Processing

by
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(ABSTRACT)

The production of long, thin polymeric fibers is a main objective of the textile industry. Melt-spinning is a particularly simple and effective technique. In this work, we shall discuss the equations of melt-spinning in viscous and viscoelastic flow. These quasilinear hyperbolic equations model the uniaxial extension of a fluid thread before its solidification.

We will address the following topics: first we shall prove existence, uniqueness, and regularity of solutions. Our solution strategy will be developed in detail for the viscous case. For non-Newtonian and isothermal flows, we shall outline the general ideas. Our solution technique consists of energy estimates and fixed-point arguments in appropriate Banach spaces. The existence result for a simple transport equation is the key to understanding the quasilinear case. The second issue of this exposition will be the stability of the unforced frost line formation. We will give a rigorous justification that, in the viscous regime, the linearized equations obey the "Principle of Linear Stability". As a consequence, we are allowed to relate the stability of the associated C_0 semigroup to the numerical resolution of the spectrum of its generator. By using a spectral collocation method, we shall derive numerical results on the eigenvalue distribution, thereby confirming prior results on the stability of the steady-state solution.

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To my beloved parents

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"Es un hombre de categoría."
— Mexican fortune cookie
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Blacksburg, December 1998 Thomas Hagen

Foreword

Fiber spinning is an important industrial process used in the manufacture of synthetic fibers such as nylon. In this process, a fluid is withdrawn vertically from a capillary and then stretched into a thin fiber. Due to cooling, the fluid eventually solidifies, and the solidified fiber is then wound up onto a roller. Closely related, but more complex problems are film casting where the fluid is drawn into a thin film and film blowing, where an annular film is inflated from the inside.

It has long been known that instabilities known as draw resonance can occur in fiber spinning when the "draw ratio" (i.e. the ratio of take-up speed to extrusion speed) exceeds a critical value. This phenomenon is well studied and analyzed for isothermal spinning of a Newtonian fiber; the result of the instability is a Hopf bifurcation which leads to oscillations.

Mathematical studies of fiber spinning have generally ignored the possibility that the fiber might solidify along the spin line. This possibility leads to a free boundary problem, since the solidification point is a priori unknown. The freezing of the fiber can lead to the absence of draw resonance in situations where it would otherwise be expected. On the other hand, recent experimental studies of film blowing show that oscillations of the freeze line can play an important role in instabilities (no satisfactory theoretical explanations of these experiments seems to exist).

This thesis undertakes a systematic mathematical study of the fiber spinning problem which takes into account the free (and potentially moving) boundary created by the freezing point. The analysis, like most studies of fiber spinning, is based on one-dimensional models, which can be derived from an asymptotic theory assuming thinness of the fiber. The fundamental question of well-posedness of the governing equations is resolved for Newtonian as well as viscoelastic fibers. In addition, the linear stability of steady fiber spinning is investigated. Beyond the study of the eigenvalue spectrum, a rigorous proof is given to show that linear stability really is determined by the location of the eigenvalues.

Blacksburg, December 1998 Michael Renardy

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CHAPTER 1

Introduction

The uniform extension of viscous and viscoelastic fluids with the objective of forming long, thin polymeric fibers is a fundamental production process in the polymer industry. Production techniques were successfully developed for the first fifty years of this century on grounds of experimental data and empirical observations. In the past forty years, the search for faster, more reliable, and more effective manufacturing procedures, the enormous progress in the development of new polymeric liquids, and the high demand in versatile, low-cost products initiated and sustained theoretical interest in the dynamical properties of the various manufacturing processes and in the rheological properties of the spun liquids (cf. [62]).

The exact spinning method is usually dictated by the chemical and physical properties of the polymer. For polymers with thermally stable melts, melt-spinning is a particularly simple and effective technique (cf. [47], [62]). In this nonisothermal process, fibers are formed by extruding the molten polymer from a pressurized reservoir through a small circular orifice (spinneret), stretching and cooling the liquid jet, and winding the solidified filament on a take-up device (spool). The melt-spinning process is schematically depicted in Fig. 1.1. Solidification is solely caused by heat transfer to the nonreactive ambient gas. The solidified filament is wound up at higher speeds than the extrusion velocity, ensuring that the threadline is stretched (cf. [12]). The polymer fiber is afterwards subjected to additional processing steps. Normally, a large number of solidified fibers is assembled together in a yarn.

Other spinning procedures include dry-spinning and wet-spinning. Both procedures rely on chemical rather than thermal properties of the polymer. Moreover, both spinning techniques involve a mass transfer and often also chemical reactions, leading therefore to more complex dynamical aspects than melt-spinning. The theoretical understanding of wet-spinning in particular is at best scanty.

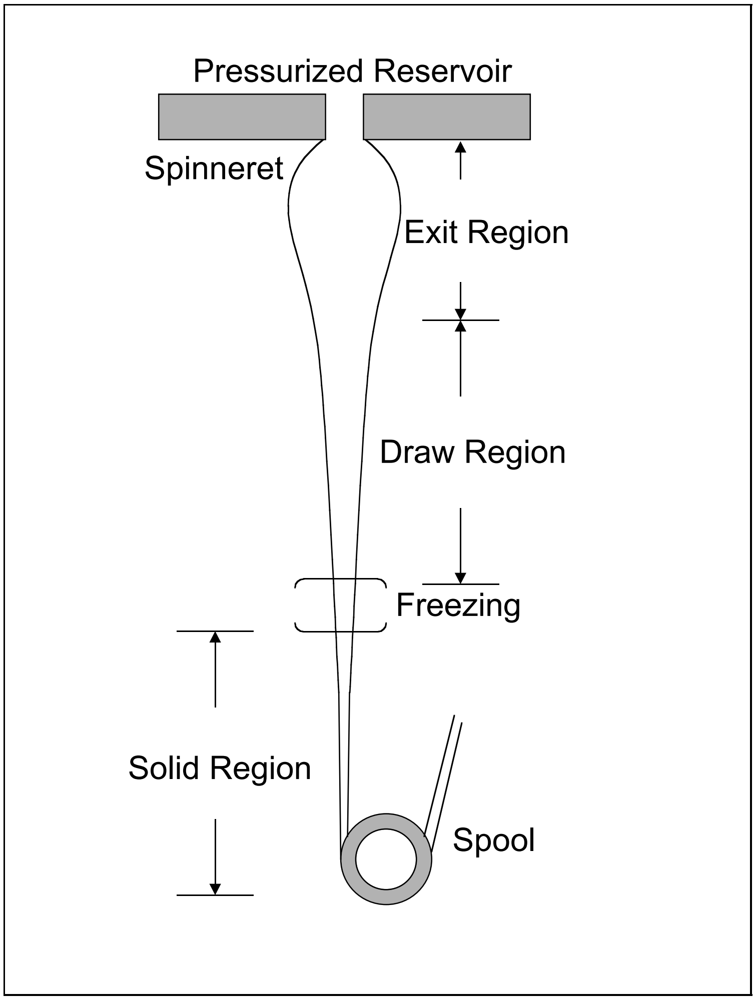

Figure 1.1: The melt-spinning process

Textile fibers must conform to prescribed dimensions and physical properties. Therefore, even small oscillations in the filament diameter or other undesirable fluctuations in the dimensions of the thread have to be avoided or at least kept to a minimum to allow for acceptable production conditions. The avoidance of flow instabilities requires not only a sound understanding of the underlying dynamics, but also an appropriate description of the mechanical and rheological properties of the fluid.

Filament breakup is the most severe restriction on a production process. This failure has been reported for both high- and low-viscosity fluids and seems to be connected to necking (cf. [12], [62]). Less catastrophic, yet undesirable, is draw resonance. Draw resonance refers to oscillatory variations in the jet diameter in melt-spinning or related processes (such as film casting and film blowing), even though extrusion and take-up speed remain constant (cf. [4]). This instability was first observed when the polymer was drawn without cooling and then suddenly chilled to its solidification point. The cyclic variation of the jet diameter occurs when the ratio of wind-up velocity to extrusion velocity is of order twenty in viscous flow (cf. [13] and [43]). This critical draw ratio was determined in numerical simulations (cf. [9]) and was qualitatively corroborated in experiments (cf. [11] and [28]). Although draw resonance is usually not observed under the commercial manufacturing conditions of melt-spinning, it has been reported in the related process of extrusion coating (cf. [12]). Theoretical studies predict the onset of draw resonance also for other flow conditions. The exact mechanism causing draw resonance is still an issue of debate, although it was recently proposed that draw resonance is caused by a mechanism in which smaller cross-sectional areas of the filament are stretched to a larger extent than larger cross sections (cf. [8]).

In this work, we shall primarily discuss the dynamics of the melt-spinning process for viscous fluids. Our interest in this particular problem is twofold.

1. The one-dimensional equations of melt-spinning are fundamental to the understanding of elongational flows in general. The essential questions of existence, regularity, and uniqueness of solutions have not been answered. The simpler case of isothermal fiber spinning has been studied mainly under steady-state conditions and in the linear regime. In [15] and [16], isothermal fiber spinning of viscoelastic fluids was investigated with means of characteristic analysis. Rigorous results, however, are not provided, and it is unclear whether the ansatz chosen there can be used to corroborate the given formal developments. We shall use techniques of nonlinear analysis to treat the problem of fiber spinning in a mathematically concise way. Our analysis will be guided by the physical features significant for the flow.

2. Numerical studies yield that isothermal spinning of viscous fluids can become unstable (draw resonance). If the process contains continuous temperature gradients (the non-isothermal case), the process has been predicted to be unconditionally stable in the linear regime (cf. [14], [46]). This result is based on the assumption that the eigenvalues of the linearized model equations determine the stability. However, several counterexamples to this assumption are known. Therefore, we will furnish a rigorous proof that validates the connection between the eigenvalues and the stability properties for melt-spinning of viscous fluids.

This work is organized as follows: in Chapter 2, we derive the equations governing the fiber spinning process. In Chapter 3, we solve an auxiliary problem which forms the basis of the existence results of Chapters 4 and 5 for various flow regimes. The nonisothermal viscous case is discussed in detail, while the results for non-Newtonian fluids and viscous isothermal flow are briefly sketched. In Chapter 6, we perform the linear analysis for the equations. In particular, we use semigroup theory and spectral analysis to prove the spectral determinacy of the semigroup that corresponds to the fiber spinning equations. The computational results of Chapter 7 complement our stability study. In Appendix A, we list the relevant function spaces and norms that are used throughout the text. Appendix B summarizes the constitutive equations for viscous and viscoelastic fluid models.

CHAPTER 2

The Model Equations

In this chapter, we shall derive the equations of fiber spinning in the nonisothermal regime. Our approach is axiomatic: First we impose a priori conditions on solutions to the flow, following mainly the lead of [3] and [29]. Then we use these assumptions to model the flow with the techniques of continuum mechanics in a way similar to [51].

2.1 Derivation of the Model

We assume, as illustrated in Fig. 2.1, that the flow is axisymmetric and vertically downward. Hence a cylindrical coordinate system, centered at the spinneret exit, is reduced to a radial (r) and an axial (z) component. The flow is completely described by the fluid velocity $\mathbf{v}$, the fluid radius R, the fluid temperature T, the hydrodynamic pressure p, and the extra stress tensor $\bar{\tau}$. The fluid density ρ is considered as constant.

The polymer melt exits the spinneret at $z = 0$ with radius R_E, velocity v_E and temperature T_E. The velocity v_S of the take-up spool is larger than the exit velocity, so that the thread is actually stretched. In the flow region between extrusion and take-up, the liquid jet is cooled by the ambient gas until the polymer reaches its solidification temperature T_S ($< T_E$) and freezes. The heat transfer is caused predominantly by convection. We normalize the temperature of the environment to 0, so that T_S is positive. As soon as the polymer solidifies, it is no longer extended and moves with the wind-up velocity v_S to the spool.

Although the process is intended to be steady-state (in the Eulerian sense), we allow time-dependence (t) of all quantities. Following [29, p. 2545] and [51, pp. 97–98], we suppose a thin

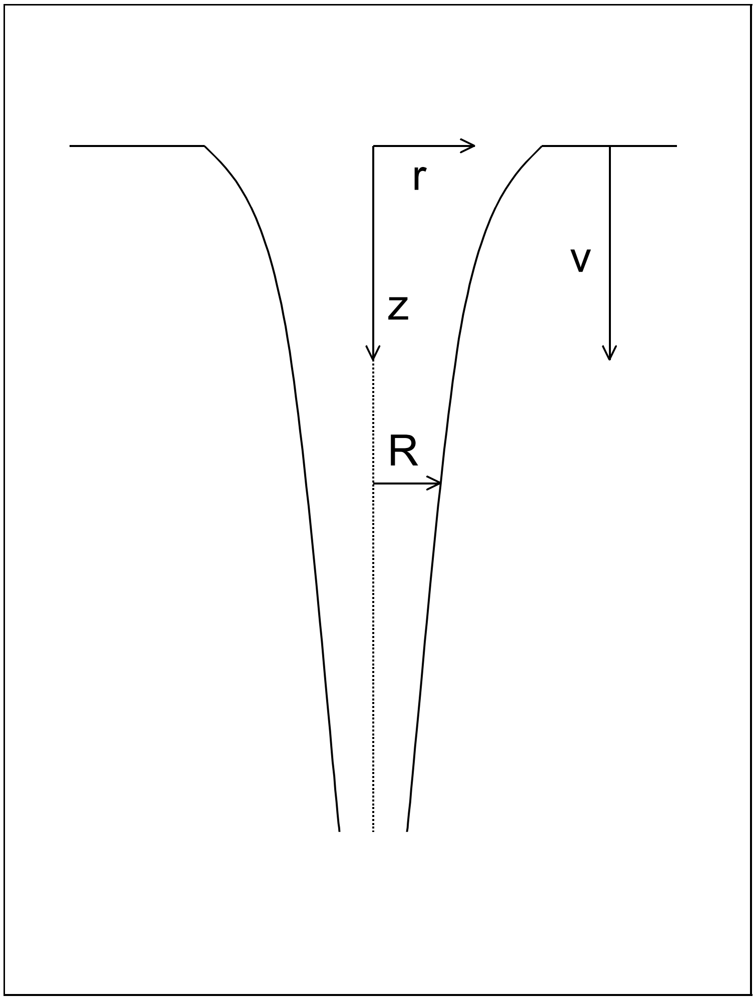

Figure 2.1: The flow variables

filament approximation: the dynamic quantities describing the flow do not vary considerably across the radius of the filament when compared to the change in the axial direction. Hence, at leading order, the flow variables depend on the axial coordinate z and time t only. This assumption is reasonable if the fluid radius is small compared to the length l of the draw-down region. The thin filament assumption can be summarized in the somewhat informal statement

$$\left|\frac{\partial}{\partial z}R\right| \ll 1. \tag{2.1}$$

Due to the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0, \tag{2.2}$$

the radial and axial velocity components, u and v, are related through

$$u \equiv -\frac{r\,v_z}{2}. \tag{2.3}$$

Hence, by (2.1), the velocity $\mathbf{v}$ satisfies at leading order

$$\mathbf{v} = (0, 0, v). \tag{2.4}$$

The assumption of radial independence allows us to interpret the dynamic variables as averages over the cross section.

Remark 2.1 Eq. (2.3) yields, at leading order, the velocity gradient

$$\mathbf{D} = \begin{pmatrix} -\frac{1}{2}\,v_z & 0 & 0 \\ 0 & -\frac{1}{2}\,v_z & 0 \\ 0 & 0 & v_z \end{pmatrix}. \tag{2.5}$$

Hence the flow is classified as an elongational flow (cf. [6]).

An elementary mass balance yields the equation of mass conservation

$$\frac{\partial}{\partial t}\left(R^2(t,z)\right) + \frac{\partial}{\partial z}\left(v(t,z)\,R^2(t,z)\right) = 0. \tag{2.6}$$

Following [29, pp. 2542–2545] and [47, p. 56], we shall assume that the viscous forces are the dominating ones. In particular, surface tension, aerodynamical friction, inertia, and gravity will be neglected. On the filament boundary, the radial component of the extra stress, τ_{rr}, satisfies then

$$\tau_{rr} = p. \tag{2.7}$$

The force acting over the threadline cross section is

$$\pi\,R^2\,\left(\tau_{zz} - p\right). \tag{2.8}$$

τ_{zz} is the axial normal stress component. By conservation of linear momentum, this force must be constant in the axial direction. Therefore, using Eq. (2.7), we derive the momentum balance

$$\frac{\partial}{\partial z}\left(R^2(t,z)\,\left(\tau_{zz} - \tau_{rr}\right)\right) = 0. \tag{2.9}$$

Finally, we assume that the cooling process obeys Newton's law of cooling (cf. [2, p. 47]) and that heat production and axial heat conduction are of no importance. Then the energy equation takes the form (cf. [3, p. 252], [43, p. 429])

$$\frac{\partial}{\partial t}T(t,z) + v(t,z)\,\frac{\partial}{\partial z}T(t,z) + \beta\,\frac{T(t,z)}{R(t,z)} = 0. \tag{2.10}$$

β is a positive heat transfer coefficient.

Eqs. (2.6), (2.9), (2.10) are the equations of change for the uniaxial extension of the fluid before it solidifies. These equations are complemented by the boundary and initial conditions

$$\text{at } z = 0: \qquad v = v_E, \quad R = R_E, \quad T = T_E, \tag{2.11}$$
$$\text{at } T = T_S: \qquad v = v_S, \tag{2.12}$$
$$\text{at } t = 0: \qquad R = R^0, \quad T = T^0. \tag{2.13}$$

We remark that the boundary condition (2.12) is given implicitly, and that initial and boundary conditions have to satisfy certain compatibility conditions. Instead of prescribing the exit velocity v_E, it will also be necessary to prescribe the stresses at the spinneret:

$$\text{at } z = 0: \qquad \tau_{rr} = \tau_E^*, \quad \tau_{\theta\theta} = \hat{\tau}_E, \quad \tau_{zz} = \tau_E. \tag{2.14}$$

τ_{rr}, $\tau_{\theta\theta}$, and τ_{zz} refer to the radial, azimuthal, and axial normal stress components of the extra stress tensor $\bar{\tau}$. This approach will be taken when the constitutive equations relating deformation rate and stresses are complicated.

Remark 2.2 We have made the thin filament assumption a priori. This assumption is motivated by heuristic descriptions of the flow. Several theories have been proposed to circumvent the thin filament assumption (cf. [22], [39], [51], [56], and [57]). In lieu of imposing radial independence of the flow variables from the beginning, lubrication scaling with a small "slenderness ratio" ϵ has been used to derive radially independent equations from the full axisymmetric equations of change. Under appropriate flow conditions, the first order expansion of these approximate equations in the expansion parameter ϵ yields one-dimensional (in space) equations, similar to the ones we have stated above. This lubrication approach has the advantage of allowing estimates on the validity of the one-dimensional model by clarifying the significance of assumption (2.1) (cf. [5] and [57]).

2.2 Quality of the Model

The one-dimensional equations of the preceding section form an idealized model for the dynamics of fiber spinning. Nonetheless, this model provides a reasonable qualitative description of the flow behavior from the standpoint of continuum mechanics. Our assumption that the flow is uniaxial and extensional is not true close to the orifice, since the fluid, before exiting the spinneret, is exposed to a strong shear stress. As soon as the fluid leaves the orifice, the velocity profile undergoes a transition from shear flow to extensional flow. This velocity rearrangement is accompanied by a change of the fluid jet diameter (die swell or contraction). However, for Newtonian fluids this effect is negligible if the draw-down region is long enough (cf. [5] and [61, p. 43]). To accomodate die swell, we could consider the flow farther downstream where a fully extensional behavior has developed, and impose appropriate upstream conditions on the flow. For non-Newtonian fluids, die swell is more problematic. The fluid diameter can swell by several orders of magnitude, even causing a loss of axisymmetry. Since viscoelastic fluids have a memory, the upstream disturbances propagate downstream, rendering our modeling assumptions invalid. Therefore, with the exception of Chapter 5, we shall mainly consider Newtonian flows throughout this work.

In the momentum balance, Eq. (2.9), we have left out inertia, gravity, aerodynamical friction, and surface tension. Surface tension is usually small for melt-spinning, thus being dropped (cf. [14], [47]). For fluids that remain viscous, even at large temperatures, it is reasonable to neglect inertia as well. However, we can expect an exponentially decreasing tensile viscosity η^* with increasing temperatures (Arrhenius equation), so that inertial forces may dominate at least throughout the high-temperature region of the flow. On the other hand, fluids of low viscosity may not be spinnable at all. In general, our momentum equation (2.9) performs well enough for industrial melt-spinning conditions of viscous fluids.

The heat equation (2.10) assumes a constant temperature over the jet cross section. This assumption has been criticized for its failure to model the difference between the surface temperature and the centerline temperature of the filament (cf. [26] and [59]). It has been suggested that radial temperature gradients are necessary to model the fiber attenuation adequately. Consequently, several two-dimensional models have been put forward to address this issue. Qualitatively, however, these models do not differ substantially from the one-dimensional model we shall study, even though they model the solidification process more appropriately.

Our model assumes that the solidification occurs as soon as the fluid reaches its freezing point. Also, if the temperature rises, the solidified polymer has to liquify again. In reality, the process is more complicated, involving a complex phase transition and hysteresis. The solidification particulars have also been linked to a necking instability (cf. [32]). However, since we are mainly interested in the overall dynamics, we shall not refine our model to take the specifics of the frost line formation into account.

For more details on the validity of the thin filament approximation and related topics, we refer to [29], [43], [47], [61], and [62].

CHAPTER 3

The Fundamental Transport Equation

In this chapter, we study the linear transport equation

$$u_t(t,x) = p(t,x)\,u_x(t,x) + f(t,x), \quad t \in [0,t_0], x \in [a,b], \tag{3.1}$$
$$u(0,x) = u^0(x), \quad x \in [a,b], \tag{3.2}$$
$$u(t,b) = u^b(t), \quad t \in [0,t_0]. \tag{3.3}$$

The initial-boundary value problem (3.1)–(3.3) is fundamental for our existence results in Chapters 4 and 5. In the following, we shall discuss existence, uniqueness, and regularity of solutions for (3.1)–(3.3). Our approach is based on energy estimates and weak* compactness arguments. Similar strategies can be found elsewhere (cf. [34], [37], [38], [53]).

The time $t_0 > 0$ is assumed as given. The interval $[a,b]$ shall be chosen once and for all. Thus any dependence with respect to (w.r.t.) the quantities a and b will usually be suppressed. Also, we will frequently omit the arguments of functions. All function spaces will be real.

We shall use several abbreviations: Let $r_1 < r_2$, $s_1 < s_2$, $t_1 > 0$, and m, n, $k \in \mathbb{N}_0$. Then we write

1. $||\cdot||_p$ for the norm on the Lebesgue space $L^p(r_1,r_2)$, $1 \le p \le \infty$,
2. $||\cdot||_{H^k}$ for the norm on the Sobolev space $H^k(r_1,r_2)$,
3. $||\cdot||_{m,n}$ for the norm on the Sobolev space $W^{m,\infty}([r_1,r_2];H^n(s_1,s_2))$,
4. $||\cdot||_{H^{m,n}}$ for the norm on the Sobolev space $H^m([r_1,r_2];H^n(s_1,s_2))$.

We shall also use the notation

1. $||\cdot||_{p,[t]}$ for the seminorms on the space $L^p(0,t_1)$, $1 \le p \le \infty$, defined for $0 \le t \le t_1$ by

$$||f||_{p,[t]} \stackrel{\text{def}}{=} ||f|_{[0,t]}||_p, \tag{3.4}$$

2. $||\cdot||_{H^k[t]}$ for the seminorms on the space $H^k(0,t_1)$, defined for $0 \le t \le t_1$ by

$$||f||_{H^k[t]} \stackrel{\text{def}}{=} ||f|_{[0,t]}||_{H^k}, \tag{3.5}$$

3. $||\cdot||_{m,n,[t]}$ for the seminorms on the space $W^{m,\infty}([0,t_1];H^n(s_1,s_2))$, defined for $0 \le t \le t_1$ by

$$||f||_{m,n,[t]} \stackrel{\text{def}}{=} ||f|_{[0,t]}||_{m,n}, \tag{3.6}$$

4. $||\cdot||_{H^{m,n}[t]}$ for the seminorms on the space $H^m([0,t_1];H^n(s_1,s_2))$, defined for $0 \le t \le t_1$ by

$$||f||_{H^{m,n}[t]} \stackrel{\text{def}}{=} ||f|_{[0,t]}||_{H^{m,n}}. \tag{3.7}$$

By convention, the preceding norms amd seminorms will always be understood w.r.t. the entire domain of the particular function (unless noted otherwise), so that the numbers r_1, r_2, s_1, and s_2 are clear from the context. For the definition of the used function spaces, see Appendix A.

3.1 Statement of the Main Result

Definition 3.1 *We shall call a function u a solution of (3.1)–(3.3) if and only if*

$$u \in W^{1,\infty}([0,t_0];L^2(a,b)) \cap L^\infty([0,t_0];H^1(a,b)), \tag{3.8}$$

$$u \text{ satisfies Eq. (3.1)}, \tag{3.9}$$

$$u \text{ satisfies Eqs. (3.2) and (3.3) pointwise.} \tag{3.10}$$

Remark 3.2 Condition (3.10) for a function u satisfying (3.8) requires that $u(t,b)$ exist for each $t \in [0,t_0]$. This requirement is indeed satisfied according to the following lemma and the Sobolev imbedding theorem.

Lemma 3.3 *Let the Hilbert spaces V and H be given such that V is continuously and densely imbedded in H. Let V^* be the dual space of V, paired through the inner product $(\cdot,\cdot)$ on H. Suppose that $h \in L^\infty([0,t_0];V) \cap C([0,t_0];H)$. Then $h(t) \in V$ for every $t \in [0,t_0]$, and the map $t \mapsto (h(t),\phi)$ is continuous on $[0,t_0]$ for every $\phi \in V^*$.*

Proof. See [54, p. 389].

□

Definition 3.4 *We shall call a function h on $[0,t_0]\times[a,b]$ boundary-regular if and only if*

$$h \in W^{1,\infty}([0,t_0];H^1(a,b)) \cap L^\infty([0,t_0];H^2(a,b)), \tag{3.11}$$
$$h_x(\cdot,a), h_x(\cdot,b) \in H^1(0,t_0). \tag{3.12}$$

Theorem 3.5 *For given functions f and p on $[0,t_0]\times[a,b]$, u^0 on $[a,b]$, and u^b on $[0,t_0]$ such that*

$$p \text{ and } f \text{ are boundary-regular,} \tag{3.13}$$
$$p > 0 \text{ on } [0,t_0]\times[a,b], \tag{3.14}$$
$$u^0 \in H^2(a,b), \tag{3.15}$$
$$u^b \in H^2(0,t_0), \tag{3.16}$$
$$u^0(b) = u^b(0), \tag{3.17}$$
$$u^b_t(0) = p(0,b)\,u^0_x(b) + f(0,b), \tag{3.18}$$

the boundary-initial value problem (3.1)–(3.3) has a boundary-regular solution u such that

$$u \in C^1([0,t_0];H^1(a,b)) \cap C([0,t_0];H^2(a,b)), \tag{3.19}$$
$$u \text{ is unique in } W^{1,\infty}([0,t_0];L^2(a,b)) \cap L^\infty([0,t_0];H^1(a,b)). \tag{3.20}$$

Remark 3.6

(a) Theorem 3.5 yields the existence and uniqueness of classical solutions for problem (3.1)–(3.3).

(b) If u and w are any two solutions of the boundary-initial value problem (3.1)–(3.3) in $W^{1,\infty}([0,t_0];L^2(a,b)) \cap L^\infty([0,t_0];H^1(a,b))$, then they satisfy

$$\int_a^b (u(t,x)-w(t,x))^2\,dx \le C\,||p||_{0,2}\int_0^t\int_a^b (u(s,x)-w(s,x))^2\,dx\,ds \tag{3.21}$$

with some constant C. By Gronwall's inequality, $u \equiv w$. Hence the uniqueness claim (3.20) is obvious.

(c) Conditions (3.17)–(3.18) are obviously necessary for the existence of classical solutions. Similar compatibility conditions will be studied later on.

(d) The condition of boundary-regularity (3.13) is the analogue to the condition

$$p, f \in W^{1,\infty}([0, t_0]; H^1(\mathbb{R})) \cap L^\infty([0, t_0]; H^2(\mathbb{R})) \tag{3.22}$$

for the pure initial value problem, stated on the whole horizon (cf. [30]).

The proof of Theorem 3.5 will be split up into several parts. In Sections 3.2 and 3.3, we shall obtain smooth solutions for C^∞-functions f and p. Section 3.4 treats the approximation of the coefficients f and p, satisfying (3.11)–(3.12), through C^∞-coefficients. In Section 3.5, we shall derive energy estimates for the solutions of Section 3.3 . These estimates allow the application of weak* compactness arguments, which yield solutions to (3.1)–(3.3) (Section 3.6). Finally, in Section 3.7 we shall discuss the regularity of solutions.

3.2 A Preliminary Result

We consider the following modified problem:

$$u_t(t,x) = p(t,x)\, u_x(t,x) + q(t,x)\, u(t,x) + f(t,x), \;\; t \in [0,t_0], x \in [a,b], \tag{3.23}$$

$$u(0,x) = u^0(x), \quad x \in [a,b], \tag{3.24}$$

$$u(t,b) = u^b(t), \quad t \in [0,t_0]. \tag{3.25}$$

Definition 3.7 *We shall call a function u a continuous solution of (3.23)–(3.25) if and only if*

$$u \in C^1([0,t_0]; L^2(a,b)) \cap C([0,t_0]; H^1(a,b)), \tag{3.26}$$

$$u \textit{ satisfies Eq. (3.23)}, \tag{3.27}$$

$$u \textit{ satisfies Eqs. (3.24) and (3.25) pointwise.} \tag{3.28}$$

Theorem 3.8 *For given functions f, p, and q on $[0,t_0] \times [a,b]$, u^0 on $[a,b]$, and u^b on $[0,t_0]$ such that*

$$p \in C^1([0,t_0]; L^2(a,b)) \cap C([0,t_0]; C^1([a,b])), \tag{3.29}$$

$$p > 0 \textit{ on } [0,t_0] \times [a,b], \tag{3.30}$$

$$q \in C^1([0,t_0]; L^2(a,b)) \cap C([0,t_0] \times [a,b]), \tag{3.31}$$

$$f \in C^1([0,t_0]; L^2(a,b)), \tag{3.32}$$

$$u^0 \in H^1(a,b), \tag{3.33}$$

$$u^b \in C^2([0,t_0]), \tag{3.34}$$

$$u^0(b) = u^b(0), \tag{3.35}$$

the boundary-initial value problem (3.23)–(3.25) has a unique continuous solution u.

Proof. It suffices to assume $u^b \equiv 0$. Let

$$D = \left\{ v \in H^1(a,b) \;\mid\; v(b) = 0 \right\}. \tag{3.36}$$

Then D is dense in $L^2(a,b)$. Define a family $\{\mathcal{A}(t)\}_{0 \le t \le t_0}$ of operators on D by

$$[\mathcal{A}(t)\, v]\,(x) = p(t,x)\, v_x(x) + q(t,x)\, v(x). \tag{3.37}$$

Since

$$\langle v, \mathcal{A}(t)\, v \rangle_2 \le \int_a^b \left(q(t,x) - \frac{1}{2}\, p_x(t,x) \right) v^2(x)\, dx, \tag{3.38}$$

each operator $\mathcal{A}(t)$ is quasidissipative in $L^2(a,b)$ (cf. [54]). Moreover, the theory of ordinary differential equations yields immediately that the spectrum of $\mathcal{A}(t)$, $\sigma(\mathcal{A}(t))$, is empty. Hence, by the Lumer-Philips theorem, each operator $\mathcal{A}(t)$ generates a quasicontraction semigroup, $\{\mathcal{T}_t(s)\}_{s \ge 0}$, on $L^2(a,b)$ (cf. [42]). Since p_x and q are bounded on $[0,t_0] \times [a,b]$, there exists $\omega \in \mathbb{R}$ such that, for every $v \in L^2(a,b)$,

$$||\mathcal{T}_t(s)\, v||_2 \le e^{\omega s}\, ||v||_2 \quad \text{for } s \ge 0 \text{ and } 0 \le t \le t_0. \tag{3.39}$$

Hence the theory of evolution systems (cf. [42, pp. 139–149]) yields the claim.

□

Remark 3.9

(a) A continuous solution of (3.23)–(3.25) belongs to $C([0,t_0] \times [a,b])$, hence its name is well deserved. Moreover, the notions of "continuous solution" of (3.23)–(3.25) and "solution" of (3.1)–(3.3) are compatible if $q \equiv 0$: A continuous solution will also satisfy Definition 3.1. In addition, the uniqueness of a continuous solution is equivalent to its uniqueness in the larger class $W^{1,\infty}([0,t_0]; L^2(a,b)) \cap L^\infty([0,t_0]; H^1(a,b))$. The correctness of this statement is an obvious consequence of Gronwall's inequality.

(b) A result similar to Theorem 3.8 can be proved by using the classical technique of characteristics. However, for this approach, we must study the impact of certain compatibility conditions on solutions. We will shed light on this issue in the next section.

3.3 The Compatibility Conditions

Definition 3.10 *For given functions f and p on $[0,t_0]\times[a,b]$, u^0 on $[a,b]$, and u^b on $[0,t_0]$ such that*

$$f, p \in C^\infty([0,t_0]\times[a,b]), \tag{3.40}$$
$$p > 0, \tag{3.41}$$
$$u^0 \in C^\infty([a,b]), \tag{3.42}$$
$$u^b \in C^\infty([0,t_0]), \tag{3.43}$$

we shall say that, for $N \in \mathbb{N}_0$, the functions u^0 and u^b are compatible of order N w.r.t. f and p if and only if the sequences (V^n) and (W^n), defined for $n \geq 0$ by

$$V^n(x) \overset{\text{def}}{=} [D_x^n u^0](x) \tag{3.44}$$

and

$$W^0(t) \overset{\text{def}}{=} u^b(t), \tag{3.45}$$
$$\begin{aligned} W^{n+1}(t) \overset{\text{def}}{=} {} & (p(t,b))^{-1}\,(W_t^n(t) - [D_x^n f](t,b)) - \\ & (p(t,b))^{-1}\left(\sum_{k=0}^{n-1}\binom{n}{k} W^{k+1}(t)\,[D_x^{n-k}p](t,b)\right), \end{aligned} \tag{3.46}$$

satisfy the compatibility conditions

$$V^n(b) = W^n(0) \quad \text{for } 0 \leq n \leq N. \tag{3.47}$$

If the compatibility conditions (3.47) hold for every $N \in \mathbb{N}_0$, we shall say that u^0 and u^b are infinitely compatible w.r.t. f and p.

Lemma 3.11 *Suppose that the functions f, p, u^0, and u^b satisfy hypotheses (3.40)–(3.43). If u^0 and u^b are compatible of order N w.r.t. f and p for some $N \in \mathbb{N}_0$, then the boundary-initial value problem (3.1)–(3.3) has a unique solution u such that*

$$u \in C^1([0,t_0]; H^N(a,b)) \cap C([0,t_0]; H^{N+1}(a,b)). \tag{3.48}$$

Proof. We shall proceed by induction over the order of compatibility N. To this end, we consider the continuous solutions $u^{[n]}$, $0 \leq n \leq N$, of the problems

$$u_t^{[n]}(t,x) = p(t,x)\,u_x^{[n]}(t,x) + n\,p_x(t,x)\,u^{[n]}(t,x) + f^n(t,x), \tag{3.49}$$
$$u^{[n]}(0,x) = V^n(x), \tag{3.50}$$
$$u^{[n]}(t,b) = W^n(t), \tag{3.51}$$

where V^n and W^n are defined by (3.44)–(3.46), and f^n is given by

$$f^n(t,x) \stackrel{\text{def}}{=} [D_x^n f](t,x) + \sum_{k=0}^{n-2} \binom{n}{k} [D_x^{n-k} p](t,x)\, u^{[k+1]}(t,x). \tag{3.52}$$

We shall prove the following claim:
For each $n \in \mathbb{N}_0$, $0 \le n \le N$, the boundary-initial value problem (3.49)–(3.51) has a unique continuous solution $u^{[n]}$ in $C^1([0,t_0];H^{N-n}(a,b)) \cap C([0,t_0];H^{N+1-n}(a,b))$ such that

$$u^{[n]} \equiv D_x^n u^{[0]}. \tag{3.53}$$

The case $N = 0$ was proven in Theorem 3.8. Assume that the claim holds for $N \ge 0$. Then (3.52) well-defines f^{N+1} as an element of $C^1([0,t_0];L^2(a,b))$. Hence, by Theorem 3.8, the boundary-initial value problem (3.49)–(3.51) with $n = N+1$ has a unique continuous solution $u^{[N+1]}$ in $C^1([0,t_0];L^2(a,b)) \cap C([0,t_0];H^1(a,b))$. Let

$$U(t,x) \stackrel{\text{def}}{=} W^N(t) + \int_b^x u^{[N+1]}(t,y)\,dy. \tag{3.54}$$

Then we have

$$\begin{aligned} &U(0,x) = V^N(x),\\ &U(t,b) = W^N(t),\\ &U \in C^1([0,t_0];H^1(a,b)) \cap C([0,t_0];H^2(a,b)). \end{aligned} \tag{3.55}$$

Moreover, (3.46), (3.49), and (3.52) yield that

$$\begin{aligned} U_t(t,x) \;&=\; p(t,x)\,U_x(t,x) + N\,p_x(t,x)\,U(t,x) + \\ &\quad f^N(t,x) + \int_b^x p_{xx}(t,y)\,\left(u^{[N]}(t,y) - U(t,y)\right)\,dy. \end{aligned} \tag{3.56}$$

Define

$$W \stackrel{\text{def}}{=} U - u^{[N]}. \tag{3.57}$$

Then $W(0,x) = W(t,b) = 0$, and

$$\begin{aligned} W_t(t,x) \;&=\; p(t,x)\,W_x(t,x) + N\,p_x(t,x)\,W(t,x) - \\ &\quad \int_b^x p_{xx}(t,y)\,W(t,y)\,dy. \end{aligned} \tag{3.58}$$

We multiply (3.58) by W and integrate over $[a,b]$. If $M > 0$ is a constant that is sufficiently large, we obtain

$$
\begin{aligned}
\frac{1}{2}\,\frac{d}{dt}\int_a^b W^2\,dy \;&=\; \frac{p}{2}\,W^2\Big|_a^b - \int_a^b \frac{p_x}{2}\,W^2\,dy + \int_a^b N\,p_x\,W^2\,dy + \\
&\quad \int_a^b W \int_b^z p_{xx}\,W\,dy\,dz \qquad (3.59)\\
&\le\; M\int_a^b W^2\,dy + M\left(\int_a^b |W|\,dy\right)^2 \\
&\le\; M\,(b-a+1)\int_a^b W^2\,dy.
\end{aligned}
$$

For this estimate, we have used (3.41) and the Cauchy-Schwarz inequality. By Gronwall's inequality, it follows from (3.59) that $W \equiv 0$. Hence $u^{[N+1]} \equiv D_x u^{[N]}$. The induction hypothesis (3.53) yields

$$
u^{[n]} \in C^1([0,t_0]; H^{N+1-n}(a,b)) \cap C([0,t_0]; H^{N+2-n}(a,b)) \quad \text{for } 0 \le n \le N+1. \tag{3.60}
$$

Hence the claim is proven.

□

Theorem 3.12 *Suppose that the functions f, p, u^0, and u^b satisfy hypotheses (3.40)–(3.43). Then, for $N \in \mathbb{N}_0$, the boundary-initial value problem (3.1)–(3.3) has a unique solution u, satisfying*

$$
u \in \bigcap_{k=0}^{N+1} C^k([0,t_0]; H^{N+1-k}(a,b)), \tag{3.61}
$$

if and only if u^0 and u^b are compatible of order N w.r.t. f and p.

Corollary 3.13 *Suppose that the functions f, p, u^0, and u^b satisfy hypotheses (3.40)–(3.43).*

(i) For $N \in \mathbb{N}_0$, the boundary-initial value problem (3.1)–(3.3) has a unique solution u of the class $C^N([0,t_0] \times [a,b])$ if and only if u^0 and u^b are compatible of order N w.r.t. f and p.

(ii) The boundary-initial value problem (3.1)–(3.3) has a unique solution u of the class $C^\infty([0,t_0] \times [a,b])$ if and only if u^0 and u^b are infinitely compatible w.r.t. f and p.

Proof of Theorem 3.12 Let V^n and W^n be given by (3.44)–(3.46). Firstly, assume that u is a solution that satisfies (3.61) for some $N \in \mathbb{N}_0$. If $N = 0$, then necessarily

$$u^0(b) = V^0(b) = W^0(0) = u^b(0). \tag{3.62}$$

In case $N > 0$, we define

$$\omega^n \stackrel{\text{def}}{=} D_x^n u \quad \text{for } 0 \leq n \leq N. \tag{3.63}$$

Then we have

$$\omega^n(0,x) = V^n(x) \text{ and } \omega^0(t,b) = W^0(t). \tag{3.64}$$

Apply D_x^n, $0 \leq n \leq N-1$, to (3.1) to get

$$\omega_t^n(t,x) = p(t,x)\,\omega^{n+1}(t,x) + [D_x^n f](t,x) + \sum_{k=0}^{n-1} \binom{n}{k}\,\omega^{k+1}(t,x)\,[D_x^{n-k} p](t,x). \tag{3.65}$$

Hence, comparing (3.65) with (3.46), we conclude that

$$\omega^n(t,b) = W^n(t) \quad \text{for } 0 \leq n \leq N. \tag{3.66}$$

Therefore, by regularity of u,

$$V^n(b) = \omega^n(0,b) = W^n(0) \quad \text{for } 0 \leq n \leq N. \tag{3.67}$$

Secondly, assume that u^0 and u^b are compatible of order N w.r.t. f and p. By Lemma 3.11, the boundary-initial value problem (3.1)–(3.3) has a unique solution u which belongs to $C^1([0,t_0]; H^N(a,b)) \cap C([0,t_0]; H^{N+1}(a,b))$. Hence (3.61) is obvious for $N = 0$. Assume $N > 0$. By Eq. (3.1) and regularity of u,

$$D_t D_x^n u(t,x) = D_x^n \left(p(t,x)\,u_x(t,x) + f(t,x)\right) \quad \text{for } 0 \leq n \leq N. \tag{3.68}$$

Thus, if $D = D_t^m D_x^n$ with $0 \leq m+n \leq N+1$, Du takes on the form

$$[Du](t,x) = \sum_{k=0}^{m+n} \beta_k(t,x)\,[D_x^k u](t,x) \tag{3.69}$$

with some functions $\beta_k \in C^\infty([0,t_0] \times [a,b])$, depending on D. The operator D is understood as the identity if $m = n = 0$. Since the right-hand side of Eq. (3.69) is a function in $C([0,t_0]; L^2(a,b))$ for every choice of D, (3.61) is proven.

$\square$

3.4 Approximating Sequences

Lemma 3.14 *For a given boundary-regular function h on $[0,t_0]\times[a,b]$, let M be a constant such that*

$$M \geq ||h||_{1,1} + ||h||_{0,2}. \tag{3.70}$$

Then there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of functions on $[0,t_0]\times[a,b]$ such that, for $n\in\mathbb{N}$,

$$h_n \in C^\infty([0,t_0]\times[a,b]), \tag{3.71}$$

$$h_n \longrightarrow h \text{ in } C([0,t_0];H^1(a,b)) \text{ as } n\to\infty, \tag{3.72}$$

$$||h_n||_{H^{1,1}} \leq \sqrt{2\,t_0}\,(M+1), \tag{3.73}$$

$$||h_n||_{0,2} \leq M+1. \tag{3.74}$$

Moreover, for each $t\in(0,t_0]$, there exists $N\in\mathbb{N}$ such that

$$||h_n||_{H^{1,1}[t]} \leq \sqrt{2\,t}\,(M+1) \quad \textit{for } n\geq N. \tag{3.75}$$

Proof. Define $\hat{h}$ by

$$\hat{h}(t) \stackrel{\text{def}}{=} \begin{cases} h(t_0) & \text{for } t>t_0, \\ h(t) & \text{for } 0\leq t\leq t_0, \\ h(0) & \text{for } t<0. \end{cases} \tag{3.76}$$

Then it is readily seen that $\hat{h}\in W^{1,\infty}(\mathbb{R};H^1(a,b))\cap L^\infty(\mathbb{R};H^2(a,b))$. Now let $h_a \stackrel{\text{def}}{=} \hat{h}_x(\cdot,a)$, $h_b \stackrel{\text{def}}{=} \hat{h}_x(\cdot,b)$, and define H by

$$H(t,x) \stackrel{\text{def}}{=} \begin{cases} h_b(t)\,(x-b)+\hat{h}(t,b) & \text{for } t\in\mathbb{R},\ x>b, \\ \hat{h}(t,x) & \text{for } t\in\mathbb{R},\ a\leq x\leq b, \\ h_a(t)\,(x-a)+\hat{h}(t,a) & \text{for } t\in\mathbb{R},\ x<a. \end{cases} \tag{3.77}$$

It follows immediately that, for every compact interval I in $\mathbb{R}$, $H|_{\mathbb{R}\times I}\in L^\infty(\mathbb{R};H^2(I))$. Moreover, by boundary-regularity of h, the restriction of H to any compact rectangle $I_1\times I_2$ in $\mathbb{R}^2$ belongs to $H^1(I_1;H^1(I_2))$. Let J be a nonnegative function in $C^\infty(\mathbb{R})$, having compact support in $[-1,1]$ and unit integral. We define the sequence $(J_n)_{n\in\mathbb{N}}$ of Friedrichs mollifiers by

$$J_n(x) \stackrel{\text{def}}{=} n\,J(n\,x) \quad \text{for } x\in\mathbb{R},\ n\in\mathbb{N}. \tag{3.78}$$

Forming the convolution with the sequence (J_n), we define the sequence $(H_n)_{n\in\mathbb{N}}$ by

$$H_n(t,x) \stackrel{\text{def}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} J_n(t-s)\,J_n(x-y)\,H(s,y)\,dy\,ds \quad \text{for } t,\ x\in\mathbb{R}. \tag{3.79}$$

It follows that $H_n \in C^\infty(\mathbb{R}^2)$. For s, $y \in \mathbb{R}$, let L_s, R_y be the shift operators, defined for functions f on $\mathbb{R}^2$ by

$$L_s f(t,y) = f(t-s,x), \tag{3.80}$$
$$R_y f(t,x) = f(t,x-y). \tag{3.81}$$

Let $[t_1,t_2]$ be an arbitrary interval, and let $\epsilon > 0$ be given. Since, for every compact interval I in $\mathbb{R}$, $H|_{\mathbb{R}\times I} \in C(\mathbb{R}; H^1(I))$ and since $[t_1,t_2]$ is compact, there exists $m \in \mathbb{N}$ such that $|y| \le m^{-1}$ implies

$$\int_a^b |R_y H(t,x) - H(t,x)|^2 \, dx < \epsilon \quad \text{for } t \in [t_1,t_2], \tag{3.82}$$
$$\int_a^b |R_y H_x(t,x) - H_x(t,x)|^2 \, dx < \epsilon \quad \text{for } t \in [t_1,t_2]. \tag{3.83}$$

Also, there exists $k \in \mathbb{N}$ such that $|s| \le k^{-1}$ implies

$$\int_a^b |L_s H(t,x) - H(t,x)|^2 \, dx < \epsilon \quad \text{for } t \in [t_1,t_2], \tag{3.84}$$
$$\int_a^b |L_s H_x(t,x) - H_x(t,x)|^2 \, dx < \epsilon \quad \text{for } t \in [t_1,t_2]. \tag{3.85}$$

In the remainder of the proof, we denote the norm in $H^k(a,b)$, $k \in \mathbb{N}_0$, by $||\cdot||_{k,(a,b)}$. Since we find

$$\begin{aligned} ||H_n(t) - H(t)||_{1,(a,b)} &\le \int_{\mathbb{R}}\int_{\mathbb{R}} J_n(s)\, J_n(y)\, ||L_s\, R_y H(t) - H(t)||_{1,(a,b)}\, dy\, ds \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} J_n(s)\, J_n(y)\, ||L_s H(t) - H(t)||_{1,(a,b)}\, dy\, ds + \\ &\quad \int_{\mathbb{R}}\int_{\mathbb{R}} J_n(s)\, J_n(y)\, ||L_s\, R_y H(t) - L_s H(t)||_{1,(a,b)}\, dy\, ds, \end{aligned} \tag{3.86}$$

estimates (3.82)–(3.85) prove that $H_n(\cdot)$ (restricted to $[a,b]$) converges to $H(\cdot)$ (restricted to $[a,b]$) in $C([0,t_0]; H^1(a,b))$ as $n \to \infty$. The estimates

$$\begin{aligned} \int_a^b |R_y H(t,x)|^2 \, dx &\le \int_{a-|y|}^a \Big(h_a(t)\,(x-a) + \hat{h}(t,a) \Big)^2 \, dx + \\ &\quad \int_b^{b+|y|} \Big(h_b(t)\,(x-b) + \hat{h}(t,b) \Big)^2 \, dx + \\ &\quad \int_a^b \hat{h}^2(t,x)\, dx, \end{aligned} \tag{3.87}$$

$$\int_a^b |R_y H_x(t,x)|^2\,dx \;\leq\; \int_{a-|y|}^a h_a^2(t)\,dx + \int_b^{b+|y|} h_b^2(t)\,dx + \int_a^b \hat{h}_x^2(t,x)\,dx, \tag{3.88}$$

$$\int_a^b |R_y H_{xx}(t,x)|^2\,dx \;\leq\; \int_a^b \hat{h}_{xx}^2(t,x)\,dx \tag{3.89}$$

imply that

$$||R_y H(t)||_{2,(a,b)} \leq M+1 \quad \text{for all } t \in \mathbb{R} \text{ if } |y| \text{ is sufficiently small.} \tag{3.90}$$

It follows from the inequality

$$||H_n(t)||_{2,(a,b)} \leq \int_{\mathbb{R}}\int_{\mathbb{R}} J_n(t-s)\,J_n(y)\,||R_y H(s)||_{2,(a,b)}\,dy\,ds \tag{3.91}$$

that

$$||H_n(t)||_{2,(a,b)} \leq M+1 \quad \text{for all } t \in \mathbb{R} \text{ if } n \text{ is large enough.} \tag{3.92}$$

Next, for large n, we deduce from (3.92) that

$$\int_0^{t_0} ||H_n(t)||_{1,(a,b)}^2\,dt \leq t_0\,(M+1)^2. \tag{3.93}$$

Moreover,

$$\begin{aligned}\int_0^{t_0}\int_a^b |L_s\,R_y H_t(t,x)|^2\,dx\,dt \;\leq\; & \int_0^{t_0}\int_a^b \hat{h}_t^2(t-s,x)\,dx\,dt + \\ & \int_0^{t_0}\int_{a-|y|}^a \Big(\dot{h}_a(t-s)\,(x-a) + \hat{h}_t(t-s,a)\Big)^2\,dx\,dt + \\ & \int_0^{t_0}\int_b^{b+|y|} \Big(\dot{h}_b(t-s)\,(x-b) + \hat{h}_t(t-s,b)\Big)^2\,dx\,dt,\end{aligned} \tag{3.94}$$

$$\begin{aligned}\int_0^{t_0}\int_a^b |L_s\,R_y H_{xt}(t,x)|^2\,dx\,dt \;\leq\; & \int_0^{t_0}\int_a^b \hat{h}_{xt}^2(t-s,x)\,dx\,dt + \\ & \int_0^{t_0}\int_{a-|y|}^a \dot{h}_a^2(t-s)\,dx\,dt + \int_0^{t_0}\int_b^{b+|y|} \dot{h}_b^2(t-s)\,dx\,dt.\end{aligned} \tag{3.95}$$

Since we have

$$||H_n||_{H^1([0,t_0];H^1(a,b))} \leq \int_{\mathbb{R}}\int_{\mathbb{R}} J_n(s)\,J_n(y)\,||L_s R_y H||_{H^1([0,t_0];H^1(a,b))}\,dy\,ds, \tag{3.96}$$

estimates (3.93)–(3.95) show that

$$||H_n||_{H^1([0,t_0];H^1(a,b))} \leq \sqrt{2\,t_0}\,(M+1) \quad \text{if } n \text{ is large enough.} \tag{3.97}$$

The same calculation also holds with $t' \in (0, t_0]$ replacing t_0. Hence, for sufficiently large $K \in \mathbb{N}$, the sequence $(H_{n+K})_{n\in\mathbb{N}}$, restricted to $[0, t_0] \times [a, b]$, has the required properties (3.71)–(3.75).

□

Remark 3.15 Lemma 3.14 is crucial for the following developments. Boundary-regularity plays the key role in its proof.

3.5 Energy Estimates

Lemma 3.16 *Let the function $\tilde{u}^b \in C^3([0, t^*])$ be given for some $t^* > 0$. Let $t_0 \in (0, t^*]$. Define $u^b \stackrel{\text{def}}{=} \tilde{u}^b|_{[0,t_0]}$. Suppose the boundary-initial value problem (3.1)–(3.3) has a solution u in $C^3([0, t_0] \times [a, b])$ with $f,\ p \in C^2([0, t_0] \times [a, b])$ and $u^0 \in C^3([a, b])$. Let $p^a \stackrel{\text{def}}{=} p(\cdot, a)$ and $p^b \stackrel{\text{def}}{=} p(\cdot, b)$. Then there exists a positive constant C depending only on t^* such that, for $0 \leq t \leq t_0$, the solution u obeys the estimates:*

$$\begin{aligned}
||u(t)||^2_{H^1} \;&\leq\; \big(||u^0||^2_{H^1} + C\,||p||_{0,1}\,\big(||u^b||^2_{2,[t]} + ||u_x(\cdot,b)||^2_{2,[t]}\big) + t\,||f||^2_{0,1}\big)\cdot \\
&\quad\ \exp\{(C\,||p||_{0,2}+1)\,t\}, && (3.98)\\
||u(t)||^2_{H^2} \;&\leq\; \big(||u^0||^2_{H^2} + C\,||p||_{0,1}\,\big(||u^b||^2_{2,[t]} + ||u_x(\cdot,b)||^2_{2,[t]} + \\
&\quad\ ||u_{xx}(\cdot,b)||^2_{2,[t]}\big) + t\,||f||^2_{0,2}\big)\,\exp\{(C\,||p||_{0,2}+1)\,t\}, && (3.99)\\
||u||^2_{1,1,[t]} \;&\leq\; \Big(||u^0||^2_{H^1} + ||p(0)\,u^0_x + f(0)||^2_{H^1} + C\,||u||_{0,2}\,||p||^2_{H^{1,1}[t]} + \\
&\quad\ ||f||^2_{H^{1,1}[t]} + C\,||p||_{0,1}\,\Big(||u^b||^2_{H^1[t]} + ||u_x(\cdot,b)||^2_{H^1[t]}\Big)\Big)\cdot && (3.100)\\
&\quad\ \exp\{(C\,||p||_{0,2} + C\,||u||_{0,2}+1)\,t\},
\end{aligned}$$

where

$$\begin{aligned}
||u_x(\cdot,b)||_{2,[t]} \;&\leq\; ||(p^b)^{-1}||_\infty\,(||u^b||_{H^1[t]} + C\,\sqrt{t}\,||f||_{0,1}), && (3.101)\\
||u_x(\cdot,b)||_{H^1[t]} \;&\leq\; C\,||(p^b)^{-1}||_\infty\,(||u^b||_{H^2[t]} + ||f||_{H^{1,1}[t]} + \\
&\quad\ ||(p^b)^{-1}||_\infty\,(||\tilde{u}^b||_{H^2} + ||f||_{0,1})\,||p||_{H^{1,1}[t]}), && (3.102)\\
||u_{xx}(\cdot,b)||_{2,[t]} \;&\leq\; ||(p^b)^{-1}||_\infty\,(||u_x(\cdot,b)||_{H^1[t]} + \\
&\quad\ C\,||p||_{0,2}\,||u_x(\cdot,b)||_{2,[t]} + C\,\sqrt{t}\,||f||_{0,2}). && (3.103)
\end{aligned}$$

Moreover

$$||u_x(\cdot,a)||_{H^1}^2 \le \Big(||u^0||_{H^1}^2 + ||p(0)\,u_x^0 + f(0)||_{H^1}^2 + C\,||u||_{0,2}\,||p||_{H^{1,1}}^2 + ||f||_{H^{1,1}}^2 + C\,||p||_{0,1}\,\big(||u^b||_{H^1}^2 + ||u_x(\cdot,b)||_{H^1}^2\big)\Big)\cdot ||(p^a)^{-1}||_\infty \,\exp\{(C\,||p||_{0,2} + C\,||u||_{0,2} + 1)\,t_0\}. \tag{3.104}$$

Proof. The constant C of the theorem will be a generic constant that absorbs the imbedding constants of various Sobolev estimates of functions over $[a,b]$ and $[0,t^*]$. Let t_0 be in $(0,t^*]$. Since we find

$$u_x(t,b) = \big(p^b(t)\big)^{-1}\,\big(u_t^b(t) - f(t,b)\big), \tag{3.105}$$
$$u_{xt}(t,b) = \big(p^b(t)\big)^{-1}\,\big(u_{tt}^b(t) - p_t^b(t)\,u_x(t,b) - f_t(t,b)\big), \tag{3.106}$$
$$u_{xx}(t,b) = \big(p^b(t)\big)^{-1}\,\big(u_{xt}(t,b) - p_x(t,b)\,u_x(t,b) - f_x(t,b)\big), \tag{3.107}$$

estimates (3.101)–(3.103) follow immediately if we note in (3.102) that

$$||u_x(\cdot,b)||_\infty \le C\,||(p^b)^{-1}||_\infty\,\big(||\tilde{u}^b||_{H^2} + ||f||_{0,1}\big). \tag{3.108}$$

Differentiation of (3.1) yields the equations

$$u_{tx} = p\,u_{xx} + p_x\,u_x + f_x, \tag{3.109}$$
$$u_{txx} = p\,u_{xxx} + 2\,p_x\,u_{xx} + p_{xx}\,u_x + f_{xx}, \tag{3.110}$$
$$u_{tt} = p\,u_{tx} + p_t\,u_x + f_t, \tag{3.111}$$
$$u_{ttx} = p\,u_{txx} + p_x\,u_{tx} + p_t\,u_{xx} + p_{tx}\,u_x + f_{tx}. \tag{3.112}$$

We multiply (3.1) by u and (3.109) by u_x, integrate over $[a,b]$, and add the two equations to obtain

$$\frac{1}{2}\frac{d}{dt}\int_a^b \left(u^2 + u_x^2\right)\,dx \le \frac{p^b(t)}{2}\left(\left(u^b(t)\right)^2 + u_x^2(t,b)\right) + \frac{1}{2}\,||f(t)||_{H^1}^2 + \left(\frac{C}{2}\,||p||_{0,2} + \frac{1}{2}\right)\int_a^b \left(u^2 + u_x^2\right)\,dx. \tag{3.113}$$

Integration in t yields

$$||u(t)||_{H^1}^2 \le ||u^0||_{H^1}^2 + ||p^b||_\infty\,\big(||u^b||_{2,[t]}^2 + ||u_x(\cdot,b)||_{2,[t]}^2\big) + t\,||f||_{0,1}^2 + (C\,||p||_{0,2} + 1)\int_0^t ||u(s)||_{H^1}^2\,ds. \tag{3.114}$$

We apply Gronwall's inequality to (3.114) to deduce estimate (3.98). Next we multiply (3.110) by u_{xx} and integrate again. Proceeding as before leads to the estimate

$$||u(t)||_{H^2}^2 \le ||u^0||_{H^2}^2 + t\,||f||_{0,2}^2 + (C\,||p||_{0,2} + 1)\int_0^t ||u(s)||_{H^2}^2\,ds + ||p^b||_\infty\,\big(||u^b||_{2,[t]}^2 + ||u_x(\cdot,b)||_{2,[t]}^2 + ||u_{xx}(\cdot,b)||_{2,[t]}^2\big). \tag{3.115}$$

By Gronwall's inequality, estimate (3.99) follows. For (3.115), we have used that $H^1(a,b)$ is a Banach algebra. To derive estimate (3.100), we observe that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_a^b \left(u^2+u_x^2+u_t^2+u_{tx}^2\right)\,dx \;\leq\; & \frac{1}{2}\,||f(t)||_{H^1}^2+\frac{1}{2}\,||f_t(t)||_{H^1}^2+ \\
& \frac{p^b(t)}{2}\,\left(\left(u^b(t)\right)^2+u_x^2(t,b)+u_t^2(t,b)+u_{xt}^2(t,b)\right)+ \\
& \int_a^b p_t\,u_x\,u_t\,dx+\int_a^b u_{tx}\,\frac{\partial}{\partial x}\left(p_t\,u_x\right)\,dx+ \\
& \left(\frac{C}{2}\,||p||_{0,2}+\frac{1}{2}\right)\int_a^b \left(u^2+u_x^2+u_t^2+u_{tx}^2\right)\,dx- \\
& \frac{p^a(t)}{2}\,\left(u_x^2(t,a)+u_{tx}^2(t,a)\right).
\end{aligned}
\tag{3.116}
$$

In this estimate, we have intentionally included the boundary terms at a. We integrate (3.116) in t, noting that

$$\int_a^b p_t\,u_x\,u_t\,dx \;\leq\; C\,||u||_{0,2}\,\left(||u_t(t)||_2^2+||p_t(t)||_2^2\right), \tag{3.117}$$

$$\int_a^b u_{tx}\,\frac{\partial}{\partial x}\left(p_t\,u_x\right)\,dx \;\leq\; C\,||u||_{0,2}\,\left(||u_{tx}(t)||_2^2+||p_t(t)||_{H^1}^2\right). \tag{3.118}$$

Gronwall's inequality now yields estimate (3.100) when the boundary terms at a in (3.116) are neglected. Having shown (3.100), we obtain the bound at a, (3.104), from estimate (3.116).

□

3.6 Proof of the Main Result – Part I

Theorem 3.17 *Suppose that the functions f, p, u^0, and u^b satisfy hypotheses (3.13)–(3.18). Then the boundary-initial value problem (3.1)–(3.3) has a unique boundary-regular continuous solution u such that*

$$u\in W^{1,\infty}([0,t_0];H^1(a,b))\cap L^\infty([0,t_0];H^2(a,b)). \tag{3.119}$$

Proof. Let $M \stackrel{\text{def}}{=} ||p||_{1,1}+||p||_{0,2}+||f||_{1,1}+||f||_{0,2}$. By Lemma 3.14, for p and f there are C^∞-sequences $\left(p^k\right)$ and $\left(f^k\right)$, resp., with properties (3.71)–(3.74). By density, there exist sequences $\left(u_k^b\right)$ and (u_k^0) such that, for $k \in \mathbb{N}$,

$$u_k^b \in C^\infty([0,t_0]), \tag{3.120}$$
$$u_k^0 \in C^\infty([a,b]), \tag{3.121}$$
$$u_k^0 \text{ and } u_k^b \text{ are compatible of order 3 w.r.t. } f^k,\ p^k, \tag{3.122}$$
$$u_k^b \longrightarrow u^b \text{ in } H^2(0,t_0) \text{ as } k \to \infty, \tag{3.123}$$
$$u_k^0 \longrightarrow u^0 \text{ in } H^2(a,b) \text{ as } k \to \infty. \tag{3.124}$$

The boundary-initial value problems

$$\begin{aligned} u_t^{[k]}(t,x) &= p^k(t,x)\, u_x^{[k]}(t,x) + f^k(t,x), \\ u^{[k]}(0,x) &= u_k^0(x), \\ u^{[k]}(t,b) &= u_k^b(t) \end{aligned} \tag{3.125}$$

have solutions $u^{[k]}$ in $C^3([0,t_0]\times[a,b])$ by Corollary 3.13. For n, $m \in \mathbb{N}$, define

$$w^{n,m} \overset{\text{def}}{=} u^{[n]} - u^{[m]}. \tag{3.126}$$

Then $w^{n,m}$ solves

$$\begin{aligned} w_t^{n,m}(t,x) &= p^n(t,x)\, w_x^{n,m}(t,x) + (p^n(t,x) - p^m(t,x))\; u_x^{[m]}(t,x) + \\ &\quad\; f^n(t,x) - f^m(t,x), \\ w^{n,m}(0,x) &= u_n^0(x) - u_m^0(x), \\ w^{n,m}(t,b) &= u_n^b(t) - u_m^b(t). \end{aligned} \tag{3.127}$$

By Lemma 3.16, $w^{n,m}$ obeys the estimate

$$\begin{aligned} ||w^{n,m}(t)||_{H^1}^2 \;\leq\; & (||u_n^0 - u_m^0||_{H^1}^2 + C\,(1 + ||p^n||_{0,1} + \\ & ||p^n||_{0,1}\, ||(p^n(\cdot,b))^{-1}||_\infty^2)\,(||u_n^b - u_m^b||_{H^1} + \\ & \sqrt{t_0}\,(||p^n - p^m||_{0,1}\, ||u^{[m]}||_{0,2} + ||f^n - f^m||_{0,1}))^2) \cdot \\ & \exp\{(C\, ||p^n||_{0,2} + 1)\, t_0\}. \end{aligned} \tag{3.128}$$

The quantities $||p^n||_{0,2}$, $||f^n||_{0,2}$, $||p^n||_{1,1}$, and $||f^n||_{1,1}$ are bounded according to (3.73)–(3.74). Hence $||u^{[m]}||_{0,2}$ is bounded according to (3.99). It follows that

$$w^{n,m} \longrightarrow 0 \text{ in } C([0,t_0];H^1(a,b)) \text{ as } n,\ m \to \infty. \tag{3.129}$$

Thus $\left(u^{[n]}\right)$ is Cauchy in $C([0,t_0];H^1(a,b)) \cap C^1([0,t_0];L^2(a,b))$ with limit u. u is necessarily a continuous solution of (3.1)–(3.3). On the other hand, the sequence $\left(u^{[n]}\right)$ is bounded in $W^{1,\infty}([0,t_0];H^1(a,b)) \cap L^\infty([0,t_0];H^2(a,b))$ by estimates (3.99)–(3.100). Both $W^{1,\infty}([0,t_0];H^1(a,b))$ and $L^\infty([0,t_0];H^2(a,b))$ are the conjugates of separable Banach spaces, hence both spaces are weak* sequentially compact by Alaoglu's theorem (cf. [27]). Thus $\left(u^{[n]}\right)$ contains a subsequence, say (v^n), that is weak* convergent both in $W^{1,\infty}([0,t_0];H^1(a,b))$ and in $L^\infty([0,t_0];H^2(a,b))$. By boundedness according to (3.102) and (3.104), we may also assume that the sequences $(v_x^n(\cdot,a))$, $(v_x^n(\cdot,b))$ are weakly convergent in $H^1(0,t_0)$. Otherwise, we extract a subsequence of (v^n). Weak* convergence in $L^\infty([0,t_0];H^2(a,b))$ and in $W^{1,\infty}([0,t_0];H^1(a,b))$ implies strong convergence in $L^2([0,t_0]\times[a,b])$ by Rellich's theorem. Hence the sequence (v^n) converges strongly in $L^2([0,t_0]\times[a,b])$ to its unique weak* limit $v\in W^{1,\infty}([0,t_0];H^1(a,b))\cap L^\infty([0,t_0];H^2(a,b))$. Since the sequence $\left(u^{[n]}\right)$ also converges strongly in $L^2([0,t_0]\times[a,b])$, we have $u\equiv v$. Thus we have shown (3.119). Suppose v_a is the weak limit of $(v_x^n(\cdot,a))$ in $H^1(0,t_0)$. By Rellich's theorem, $(v_x^n(\cdot,a))$ converges strongly in $L^2(0,t_0)$ to its weak limit v_a in $H^1(0,t_0)$. However, (v_x^n) is weak* convergent to u_x in $L^\infty([0,t_0];H^1(a,b))$, hence weakly convergent to u_x in $L^2([0,t_0];H^1(a,b))$. Since the linear map $h\mapsto h(a)$ from $L^2([0,t_0];H^1(a,b))$ to $L^2(0,t_0)$ is bounded, the sequence $(v_x^n(\cdot,a))$ converges weakly to $u_x(\cdot,a)$ in $L^2(0,t_0)$. Weak and strong limits must coincide, hence necessarily $u_x(\cdot,a)\equiv v_a$. Since the same argument works for $u_x(\cdot,b)$, we have shown that u is boundary-regular. Finally, uniqueness was already proven in (3.21).

$\square$

Corollary 3.18 *Let the function $\tilde{u}^b\in H^2(0,t^*)$ be given for some $t^*>0$. Let $t_0\in(0,t^*]$. Define $u^b \stackrel{\text{def}}{=} \tilde{u}^b|_{[0,t_0]}$. Suppose that the functions f, p, u^0, and u^b satisfy (3.13)–(3.18). Let M satisfy*

$$M\geq ||p||_{1,1}+||p||_{0,2}+||f||_{1,1}+||f||_{0,2}, \tag{3.130}$$

and let $p^a\stackrel{\text{def}}{=}p(\cdot,a)$, $p^b\stackrel{\text{def}}{=}p(\cdot,b)$. Then there exists a constant $C>0$ and continuous functions F on $\mathbb{R}^5$ and G on $\mathbb{R}^6$ such that, for $x_i\in\mathbb{R}$, $2\leq i\leq 6$,

$$F(0,0,x_3,x_4,x_5)=x_3=G(0,0,x_3,x_4,x_5,x_6), \tag{3.131}$$

and, for $0\leq t\leq t_0$, the solution u of the boundary-initial value problem (3.1)–(3.3) obeys the estimates:

$$\begin{aligned} ||u||^2_{0,2,[t]} &\leq F(t,||\tilde{u}^b||_{H^2[t]},||u^0||^2_{H^2},||(p^b)^{-1}||_\infty,M)\stackrel{\text{def}}{=}\Phi(t), && (3.132)\\ ||u||^2_{1,1,[t]} &\leq G(t,||\tilde{u}^b||_{H^2[t]},||u_0||^2_{H^1}+||p(0)\,u^0_x+f(0)||^2_{H^1}, \\ &\qquad ||(p^b)^{-1}||_\infty,\Phi(t^*),M), && (3.133)\end{aligned}$$

and

$$\begin{aligned} ||u_x(\cdot,a)||^2_{H^1} &\leq ||(p^a)^{-1}||_\infty\, G(t_0,||\tilde{u}^b||_{H^2[t_0]},||u_0||^2_{H^1}+||p(0)\,u^0_x+f(0)||^2_{H^1}, \\ &\qquad ||(p^b)^{-1}||_\infty,\Phi(t^*),M), && (3.134)\\ ||u_x(\cdot,b)||_{H^1} &\leq C\,||(p^b)^{-1}||_\infty\,\big(||\tilde{u}^b||_{H^2[t_0]}+ \\ &\qquad \big(1+(M+2)\,||(p^b)^{-1}||_\infty\big)\,(M+1)\sqrt{t_0}\big)\,. && (3.135)\end{aligned}$$

The constant C and the functions F and G can be chosen to depend on $||\tilde{u}^b||_{H^2}$ and t^* only.

Proof. We fix t_0 in $(0, t^*]$ and choose C^∞-sequences $\left(p^k\right)$, $\left(f^k\right)$, $\left(u_k^b\right)$, and (u_k^0) as in the proof of Theorem 3.17 . Next we construct the sequence $\left(u^{[k]}\right)$ by solving the boundary-initial value problem (3.125) for each k. We may assume that $\left(u^{[k]}\right)$ is weak* convergent in the space $W^{1,\infty}([0,t_0]; H^1(a,b)) \cap L^\infty([0,t_0]; H^2(a,b))$ with weak* limit u. By Lemma 3.14, the quantities $||f^k||_{0,2}$ and $||p^k||_{0,2}$ obey the uniform bound in (3.74), whereas the quantities $||f^k||_{H^{1,1}[t]}$ and $||p^k||_{H^{1,1}[t]}$ obey the estimate in (3.75) if k is large enough. Lemma 3.16 yields bounds for $||u^{[k]}||_{0,2,[t]}$ and $||u^{[k]}||_{1,1,[t]}$ in terms of t, $||u_k^b||_{H^2[t]}$, and the quantities $||u_k^0||_{H^1}$, $||(p^k(\cdot,b))^{-1}||_\infty$, and M. Similar bounds for the boundary quantities $||u^{[k]}(\cdot,a)||_{H^1}$ and $||u^{[k]}(\cdot,b)||_{H^1}$ follow from estimates (3.102) and (3.104). As we take the lim inf as $k \to \infty$, we obtain estimates of the form (3.132)–(3.135). The indicated dependencies of F, G, and C are immediately seen in Lemma 3.16.

□

3.7 Proof of the Main Result – Part II

Lemma 3.19 *Suppose that the conditions of Theorem 3.17 hold. Then the solution u of the boundary-initial value problem (3.1)–(3.3) is a strongly right-continuous map into $H^2(a,b)$, i.e. u satisfies*

$$\lim_{t \downarrow t'} ||u(t) - u(t')||_{H^2} = 0 \quad \text{for every } t' \in [0, t_0). \tag{3.136}$$

Proof. By Corollary 3.18,

$$\limsup_{t\downarrow 0} ||u(t)||_{H^2} \leq ||u^0||_{H^2}. \tag{3.137}$$

Hence the claim is true for $t' = 0$ by the weak continuity result of Lemma 3.3 (cf. [31, p. 253]). To prove the claim for any other $t' \in [0, t_0)$, we consider $w(t) \stackrel{\text{def}}{=} u(t + t')$ on $[0, t_0 - t']$. Then Corollary 3.18 holds for w. Therefore, the claim follows from estimate (3.137), applied to w.

□

Theorem 3.20 *Suppose that the conditions of Theorem 3.17 hold. Then the solution u of the boundary-initial value problem (3.1)–(3.3) has the regularity property (3.19).*

Proof. We shall show that u is a strongly left-continuous map on $(0, t_0]$ into $H^2(a, b)$. Then we can deduce from Lemma 3.19 and Eq. (3.1) that (3.19) holds. To this end, define the functions v, $\tilde{p}$ and $\tilde{f}$ on $[0, t_0] \times [a, b]$ by

$$v(t,x) \stackrel{\text{def}}{=} u(t_0 - t, a + b - x), \tag{3.138}$$

$$\tilde{p}(t,x) \stackrel{\text{def}}{=} p(t_0 - t, a + b - x), \tag{3.139}$$

$$\tilde{f}(t,x) \stackrel{\text{def}}{=} -f(t_0 - t, a + b - x), \tag{3.140}$$

and define the functions v^0 and v^b by

$$v^0(x) \stackrel{\text{def}}{=} u(t_0, a + b - x), \quad x \in [a, b], \tag{3.141}$$

$$v^b(t) \stackrel{\text{def}}{=} u(t_0 - t, a), \quad t \in [0, t_0]. \tag{3.142}$$

Then v is the unique solution of the boundary-initial value problem

$$v_t(t,x) = \tilde{p}(t,x)\, v_x(t,x) + \tilde{f}(t,x), \quad t \in [0, t_0], x \in [a, b], \tag{3.143}$$

$$v(0,x) = v^0(x), \quad x \in [a, b], \tag{3.144}$$

$$v(t,b) = v^b(t), \quad t \in [0, t_0]. \tag{3.145}$$

$\tilde{f}$ and $\tilde{p}$ are boundary-regular, and $\tilde{p}$ satisfies (3.14). Also, (3.15) holds for v^0. Lemma 3.3 implies that

$$u_t(t,a) = p(t,a)\, u_x(t,a) + f(t,a) \quad \text{pointwise in } t. \tag{3.146}$$

By Theorem 3.17, u is boundary-regular. Hence Eq. (3.146) proves that v^b satisfies (3.16). Finally, the compatibility conditions (3.17)–(3.18) are valid for $\tilde{f}$, $\tilde{p}$, v^0, and v^b by virtue of Eq. (3.146) and continuity of u. Therefore, by Lemma 3.19, v is strongly right-continuous from $[0, t_0)$ into $H^2(a, b)$.

$\square$

CHAPTER 4

Solvability in the Viscous Regime

In this chapter, we shall prove existence, uniqueness, and regularity of solutions for the equations of fiber spinning of viscous fluids. The main result of this chapter is Theorem 4.3 . Rather than solving the equations of fiber spinning directly, we shall cast them in a form more accessible to mathematical analysis. Our existence results justify this reformulation afterwards.

Throughout, we shall use the notation of Chapter 3 (see also Appendix A). In addition, we shall assume that the exit and solidification temperatures T_E and T_S ($T_E > T_S > 0$) are fixed, so that any dependence on these quantities may be omitted.

4.1 The Equations for Viscous Fluids

Using the stress-strain relation for viscous fluids, Eq. (B.3), in Eq. (2.9), we can list the equations of change governing melt-spinning of viscous fluids:

Balance of Mass

$$\frac{\partial}{\partial t}\big(R^2(t,z)\big) + \frac{\partial}{\partial z}\big(v(t,z)\,R^2(t,z)\big) = 0, \tag{4.1}$$

Balance of Momentum

$$\frac{\partial}{\partial z}\left(R^2(t,z)\,\frac{\partial}{\partial z}v(t,z)\right) = 0, \tag{4.2}$$

Balance of Energy

$$\frac{\partial}{\partial t}T(t,z) + v(t,z)\,\frac{\partial}{\partial z}T(t,z) + \beta\,\frac{T(t,z)}{R(t,z)} = 0. \tag{4.3}$$

The heat transfer coefficient β in Eq. (4.3) is considered as positive and constant. The tensile viscosity is also assumed as constant and hence factors out. Eqs. (4.1)–(4.3) are complemented by the boundary and initial conditions

$$\text{at } z=0: \quad v = v_E, \quad R = R_E, \quad T = T_E, \tag{4.4}$$
$$\text{at } T=T_S: \quad v = v_S, \tag{4.5}$$
$$\text{at } t=0: \quad R = R^0, \quad T = T^0. \tag{4.6}$$

We have specified the exit velocity v_E in (4.4) ($v_E < v_S$). It is also possible to prescribe the stresses at the spinneret. This approach will be taken in Chapter 5.

4.2 Reformulation of the Equations

Eqs. (4.1)–(4.3) are to be satisfied as long as the temperature T does not reach the freezing point T_S. This condition like the boundary condition (4.5) is stated implicitly and complicates any further analysis. Hence it is useful to replace the free variable z by the temperature T. To this end, we formally assume $\frac{\partial}{\partial z}T \neq 0$. Thus for $T_S \leq \tilde{T} \leq T_E$, the equation

$$T(t,z) = \tilde{T} \tag{4.7}$$

has a unique solution $z = \tilde{z}(t,\tilde{T})$. Now we redefine:

$$\tilde{R}(t,\tilde{T}) = R(t,\tilde{z}(t,\tilde{T})), \tag{4.8}$$
$$\tilde{v}(t,\tilde{T}) = v(t,\tilde{z}(t,\tilde{T})). \tag{4.9}$$

In general, if we let $\tilde{y}(t,\tilde{T}) = y(t,\tilde{z}(t,\tilde{T}))$, then we obtain:

$$\tilde{y}_t(t,\tilde{T}) = y_t(t,z) + y_z(t,z)\,\tilde{z}_t(t,\tilde{T}), \tag{4.10}$$
$$\tilde{y}_{\tilde{T}}(t,\tilde{T}) = y_z(t,z)\,\tilde{z}_{\tilde{T}}(t,\tilde{T}), \tag{4.11}$$
$$\tilde{y}_{\tilde{T}\tilde{T}}(t,\tilde{T}) = y_z(t,z)\,\tilde{z}_{\tilde{T}\tilde{T}}(t,\tilde{T}) + y_{zz}(t,z)\,\tilde{z}^2_{\tilde{T}}(t,\tilde{T}). \tag{4.12}$$

We also remark that

$$T_z(t,z)\,\tilde{z}_{\tilde{T}}(t,\tilde{T}) = 1, \tag{4.13}$$
$$T_t(t,z) + T_z(t,z)\,\tilde{z}_t(t,\tilde{T}) = 0. \tag{4.14}$$

Replacing y, $\tilde{y}$ by R^2, $\tilde{R}^2$ and v, $\tilde{v}$, resp., in Eqs. (4.10)–(4.12), inserting the resultant equations together with Eqs. (4.13)–(4.14) into Eqs. (4.1)–(4.3), dropping all tildes, and finally defining $A \stackrel{\text{def}}{=} \pi R^2$, $\alpha \stackrel{\text{def}}{=} \sqrt{\pi}\,\beta$, we recover the quasilinear hyperbolic equations

$$A_t(t,T) = \frac{\alpha\, T}{\sqrt{A(t,T)}}\, A_T(t,T) - \frac{v_T(t,T)}{z_T(t,T)}\, A(t,T), \tag{4.15}$$

$$\frac{\partial}{\partial T}\left(\frac{v_T(t,T)}{z_T(t,T)}\, A(t,T)\right) = 0, \tag{4.16}$$

$$z_t(t,T) = \frac{\alpha\, T}{\sqrt{A(t,T)}}\, z_T(t,T) + v(t,T). \tag{4.17}$$

Eqs. (4.15)–(4.17) are accompanied by the corresponding boundary conditions

$$A(t,T_E) = A_E(t), \tag{4.18}$$

$$z(t,T_E) = 0, \tag{4.19}$$

$$v(t,T_E) = v_E(t), \tag{4.20}$$

$$v(t,T_S) = v_S(t) \tag{4.21}$$

and the initial conditions

$$A(0,T) = A^0(T), \tag{4.22}$$

$$z(0,T) = z^0(T). \tag{4.23}$$

Here we have defined $A_E \stackrel{\text{def}}{=} \pi\, R_E^2$.

Remark 4.1

(a) To accomodate die swell, we could replace the boundary condition (4.19) by

$$z(t,T_E) = z_E(t) \tag{4.24}$$

with some prescribed function z_E. Our existence results will hold with only minor modifications for this boundary condition.

(b) The assumption of a constant tensile viscosity is obviously incorrect in a nonisothermal regime. If we included a temperature-dependent viscosity η^*, the momentum balance, Eq. (4.16), would change to

$$\frac{\partial}{\partial T}\left(\frac{v_T(t,T)}{z_T(t,T)}\, \eta^*(T)\, A(t,T)\right) = 0. \tag{4.25}$$

Since Eq. (4.25) is qualitatively not different from Eq. (4.16) for sufficiently smooth η^*, and since we prefer to avoid yet more technicalities in this already detail-laden exposition, we shall assume the tensile viscosity η^* (and also the heat-transfer coefficient α) as constant. However, we point out that straightforward, but tedious calculations would show that our results remain correct if this assumption is dropped.

In lieu of discussing Eqs. (4.1)–(4.6), we shall analyze the boundary-initial value problem posed in Eqs. (4.15)–(4.23). If a solution z of the transformed problem is continuously differentiable in both arguments, the formal transformation (4.8)–(4.9) is indeed justified as long as z_T does not vanish. To this end, we should obviously assume $z_T^0 < 0$.

4.3 Statement of the Main Result

Definition 4.2 *We shall call a vector field (A, z, v), defined on $[0, t_0] \times [T_S, T_E]$, a solution of (4.15)–(4.23) if and only if*

$$A, z, v \in W^{1,\infty}([0, t_0]; H^1(T_S, T_E)) \cap L^\infty([0, t_0]; H^2(T_S, T_E)), \tag{4.26}$$

$$A,\ z,\ v \textit{ satisfy Eqs. (4.15)–(4.17)}, \tag{4.27}$$

$$A \textit{ satisfies Eqs. (4.18), (4.22) pointwise}, \tag{4.28}$$

$$z \textit{ satisfies Eqs. (4.19), (4.23) pointwise}, \tag{4.29}$$

$$v \textit{ satisfies Eqs. (4.20), (4.21) pointwise.} \tag{4.30}$$

Theorem 4.3 *Let the initial values A^0, z^0 on $[T_S, T_E]$ and the boundary values A_E, v_E, v_S on $[0, t^*]$, $t^* > 0$, be given such that*

$$A^0, z^0 \in H^2(T_S, T_E), \tag{4.31}$$

$$A_E \in H^2(0, t^*), \tag{4.32}$$

$$v_E, v_S \in W^{1,\infty}(0, t^*), \tag{4.33}$$

$$A^0 > 0 \textit{ on } [T_S, T_E], \tag{4.34}$$

$$z_T^0 < 0 \textit{ on } [T_S, T_E]. \tag{4.35}$$

Suppose that the compatibility conditions

$$A^0(T_E) = A_E(0), \quad z^0(T_E) = 0, \tag{4.36}$$

$$\dot{A}_E(0) = \frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, A_T^0(T_E) + (v_S(0) - v_E(0)) \left(\int_{T_S}^{T_E} \frac{z_T^0(T)}{A^0(T)}\, dT \right)^{-1}, \tag{4.37}$$

$$\frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, z_T^0(T_E) + v_E(0) = 0 \tag{4.38}$$

hold. Then there exists $t_0 \in (0, t^*]$ *such that the boundary-initial value problem (4.15)–(4.23) has a unique solution* (A, z, v) *on* $[0, t_0] \times [T_S, T_E]$. *This solution* (A, z, v) *has the properties:*

$$A, z \in C^1([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)), \tag{4.39}$$

$$A, z \in W^{2,\infty}([0, t_0]; L^2(T_S, T_E)), \tag{4.40}$$

$$v \in W^{1,\infty}([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)), \tag{4.41}$$

$$A,\ z,\ v \text{ are boundary-regular.} \tag{4.42}$$

Moreover, if

$$v_E, v_S \in C^1([0, t_0]), \tag{4.43}$$

then the solution (A, z, v) *has the additional properties:*

$$A, z \in \bigcap_{k=0}^{2} C^k([0, t_0]; H^{2-k}(T_S, T_E)), \tag{4.44}$$

$$v \in C^1([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)). \tag{4.45}$$

Remark 4.4

(a) Conditions (4.34)–(4.35) are in agreement with the expected physical behavior of the flow.

(b) Theorem 4.3 asserts the existence of classical solutions to the equations of fiber spinning.

(c) Solving the momentum balance (4.16) for v yields

$$v(t, T) = v_S(t) + (v_E(t) - v_S(t)) \int_{T_S}^{T} \frac{z_T(t, \tau)}{A(t, \tau)}\, d\tau \left(\int_{T_S}^{T_E} \frac{z_T(t, T)}{A(t, T)}\, dT \right)^{-1}. \tag{4.46}$$

Hence if we differentiate Eq. (4.46) w.r.t. T, we find

$$\frac{v_T(t, T)}{z_T(t, T)} A(t, T) = (v_E(t) - v_S(t)) \left(\int_{T_S}^{T_E} \frac{z_T(t, T)}{A(t, T)}\, dT \right)^{-1}. \tag{4.47}$$

Therefore, the compatibility conditions (4.36)–(4.38) are necessary for (4.39) to be true.

(d) We will prove (4.39), (4.40), and (4.42) for A and z, while eliminating Eq. (4.16) by (4.46) and (4.47) in Eqs. (4.15) and (4.17).

(e) Theorem 4.3 and its proof can be modified in an obvious fashion to hold for the boundary condition (4.24). It can also be extended to the case of a nonconstant tensile viscosity η^* (see Eq. (4.25)).

From now on, we shall assume that t^*, A^0, z^0, A_E, v_E, and v_S are given once and for all such that hypotheses (4.31)–(4.38) hold.

4.4 The Solution Operator

Definition 4.5 *Let $h = h(t, T)$ be a boundary-regular function with domain $[0, t'] \times [T_S, T_E]$. Then define the energy functional $\mathcal{E}$ by*

$$\mathcal{E}(h) \stackrel{\text{def}}{=} \left(||h||_{0,2}^2 + ||h||_{1,1}^2 + +||h_T(\cdot, T_S)||_{H^1}^2 + ||h_T(\cdot, T_E)||_{H^1}^2\right)^{\frac{1}{2}}. \tag{4.48}$$

Definition 4.6 *For $L > 0$ and $t' \in (0, t^*]$, let $S(t', L)$ be the set of vector fields $(B, \xi)^T$ on $[0, t'] \times [T_S, T_E]$ such that*

$$B \text{ and } \xi \text{ are boundary-regular on } [0, t'] \times [T_S, T_E], \tag{4.49}$$

$$\mathcal{E}(B)^2 + \mathcal{E}(\xi)^2 \leq L^2, \tag{4.50}$$

$$B(0, T) = A^0(T) \text{ and } B(t, T_E) = A_E(t), \tag{4.51}$$

$$\xi(0, T) = z^0(T) \text{ and } \xi(t, T_E) = 0. \tag{4.52}$$

Remark 4.7 For large L, $S(t^*, L)$ is clearly not empty. Hence in this case, $S(t', L) \neq \emptyset$ for every $t' \in (0, t^*]$.

Definition 4.8 *We shall say that $t' \in (0, t^*]$ is admissible if and only if the set $S(t', L)$ is nonempty and all pairs $(B, \xi)^T \in S(t', L)$ have the properties*

$$B(t, T) > 0 \quad \text{and} \quad \int_{T_S}^{T_E} \frac{\xi_T(t, \tau)}{B(t, \tau)}\, d\tau \neq 0 \quad \text{for } t \in [0, t'],\ T \in [T_S, T_E]. \tag{4.53}$$

Lemma 4.9 *Suppose $S(t^*, L)$ is nonempty. Then there exists $t_0 \in (0, t^*]$ such that every $t \in (0, t_0]$ is admissible.*

Proof. Let $(B, \xi)^T \in S(t', L)$ be given for some $t' \in (0, t^*]$. Denote the imbedding constant of the imbedding $H^1(T_S, T_E) \hookrightarrow L^\infty(T_S, T_E)$ by C_1. Let $A_{\min}$ and $A_{\max}$ be the minimum and maximum values, resp., of A^0 over $[T_S, T_E]$. Both values are positive by (4.34). Fix $\mu \in (0, A_{\min})$. Since

$$B(t,T) - A^0(T) = \int_0^t B_t(s,T)\,ds, \tag{4.54}$$

we have

$$|B(t,T) - A^0(T)| \le C_1\,L\,t'. \tag{4.55}$$

Hence we find

$$|B(t,T) - A^0(T)| \le \mu \quad \text{for } t \in [0,t'],\ T \in [T_S, T_E], \tag{4.56}$$

and also

$$\frac{\displaystyle\int_{T_S}^{T_E} |z_T^0|\,dT}{A_{\max} + \mu} > \frac{t'\,\sqrt{T_E - T_S}\,L}{A_{\min} - \mu}, \tag{4.57}$$

if t' is small enough. Thus we derive from the estimate

$$\int_{T_S}^{T_E} |\xi_T(t,T) - z_T^0(T)|\,dT \le \int_0^t \int_{T_S}^{T_E} |\xi_{Tt}(s,T)|\,dT\,ds \le t'\,\sqrt{T_E - T_S}\,L \tag{4.58}$$

that

$$\left|\int_{T_S}^{T_E} \frac{\xi_T}{B}\,dT\right| \ge \frac{\displaystyle\int_{T_S}^{T_E} |z_T^0|\,dT}{A_{\max} + \mu} - \frac{t'\,\sqrt{T_E - T_S}\,L}{A_{\min} - \mu} > 0. \tag{4.59}$$

Therefore, (4.53) is proven. The claim follows.

□

Definition 4.10 *Let $t' \in (0, t^*]$ be admissible. Define the solution operator $\Sigma_{t',L}$ on $S(t', L)$ by*

$$\Sigma_{t',L} : \begin{pmatrix} B \\ \xi \end{pmatrix} \longmapsto \begin{pmatrix} Y \\ \zeta \end{pmatrix}, \tag{4.60}$$

where $Y = Y(t,T)$ and $\zeta = \zeta(t,T)$ are the solutions (in the sense of Definition 3.1) of the boundary-initial value problems, stated on $[0,t'] \times [T_S, T_E]$,

$$Y_t = \frac{\alpha\,T}{\sqrt{B}}\,Y_T + (v_S - v_E)\left(\int_{T_S}^{T_E} \frac{\xi_T}{B}\,dT\right)^{-1}, \tag{4.61}$$

$$Y(0,T) = A^0(T), \tag{4.62}$$

$$Y(t,T_E) = A_E(t) \tag{4.63}$$

and

$$\zeta_t = \frac{\alpha T}{\sqrt{B}}\,\zeta_T + v_S + (v_E - v_S)\int_{T_S}^{T}\frac{\xi_T}{B}\,d\tau\left(\int_{T_S}^{T_E}\frac{\xi_T}{B}\,dT\right)^{-1}, \tag{4.64}$$

$$\zeta(0,T) = z^0(T), \tag{4.65}$$

$$\zeta(t,T_E) = 0. \tag{4.66}$$

Remark 4.11 The coefficients and inhomogeneities in (4.61) and (4.64) satisfy conditions (3.13)–(3.14), resp. By Theorem 3.5, $\Sigma_{t',L}$ is a well-defined map from $S(t',L)$ into $\left(C^1([0,t'];H^1(T_S,T_E))\cap C([0,t'];H^2(T_S,T_E))\right)^2$ if t' is admissible.

4.5 Proof of the Main Result

Lemma 4.12 *Suppose $S(t^*,L)$ is nonempty. Then there exists an admissible $t_0 \in (0,t^*]$ such that, for all $t \in (0,t_0]$, the range of the solution operator $\Sigma_{t,L}$ is contained in $S(t,L)$.*

Proof. Throughout, we shall use the quantities C_1, $A_{\min}$, $A_{\max}$, and μ of Lemma 4.9 . Let $t_0 < t_1$ be two distinct values in $(0,t^*]$. Suppose t_1 is chosen such that $C_1\,L\,t_1 \le \mu$ and such that estimate (4.57) holds with $t' = t_1$. If $(B,\xi)^T \in S(t_\#,L)$ for some $t_\# \in (0,t_0]$, it follows that, for $t \in [0,t_\#]$,

$$|B(t,T) - A^0(T)| \le C_1\,L\,t_1 \le \mu \tag{4.67}$$

and

$$\left|\int_{T_S}^{T_E}\frac{\xi_T}{B}\,dT\right| \ge \frac{\displaystyle\int_{T_S}^{T_E}|z_T^0|\,dT}{A_{\max}+\mu} - \frac{t_1\sqrt{T_E-T_S}\,L}{A_{\min}-\mu} > 0. \tag{4.68}$$

Now define

$$\mathcal{P}(t,T) \stackrel{\text{def}}{=} \frac{\alpha T}{\sqrt{B(t,T)}}, \tag{4.69}$$

$$\mathcal{F}_1(t,T) \stackrel{\text{def}}{=} (v_S(t) - v_E(t))\left(\int_{T_S}^{T_E}\frac{\xi_T(t,T)}{B(t,T)}\,dT\right)^{-1}, \tag{4.70}$$

$$\mathcal{F}_2(t,T) \stackrel{\text{def}}{=} v_S(t) + \frac{v_E(t)-v_S(t)}{\displaystyle\int_{T_S}^{T_E}\frac{\xi_T(t,T)}{B(t,T)}\,dT}\int_{T_S}^{T}\frac{\xi_T(t,\tau)}{B(t,\tau)}\,d\tau. \tag{4.71}$$

Then it is readily seen that $\mathcal{P}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ are boundary-regular (this observation has already been made in Remark 4.11). Moreover, by assumption, both sets of functions, $\mathcal{P}$, $\mathcal{F}_1$, A^0, A_E and $\mathcal{P}$, $\mathcal{F}_2$, z^0, $z_E \equiv 0$, satisfy compatibility conditions on $[0, t_\#] \times [T_S, T_E]$, analogous to (3.17)–(3.18). (4.67) yields

$$|(\mathcal{P}(t,T))^{-1}| \leq (\alpha\, T_S)^{-1}\, \sqrt{A_{\max} + \mu}. \tag{4.72}$$

We also observe that, by virtue of estimates (4.67) and (4.68), the quantities $\mathcal{E}(\mathcal{P})$, $\mathcal{E}(\mathcal{F}_1)$, $\mathcal{E}(\mathcal{F}_2)$, $||\mathcal{P}(0)\, B_T(0) + \mathcal{F}_1(0)||_{H^1}$, and $||\mathcal{P}(0)\, \xi_T(0) + \mathcal{F}_2(0)||_{H^1}$ are bounded by expressions involving only t_1, L, and the boundary/initial values. Since these estimates hold for every $(B,\xi) \in S(t_\#, L)$ if $t_\# \in (0, t_0]$, Corollary 3.18 implies the claim if only t_0 is chosen small enough relative to L.

□

Lemma 4.13 *Suppose $S(t,L)$ is nonempty. Then the metric $d(\cdot,\cdot)$, defined for $(B,\xi)^T$, $(\hat{B},\hat{\xi})^T \in S(t,L)$ by*

$$d\left((B,\xi)^T, (\hat{B},\hat{\xi})^T\right) \stackrel{\text{def}}{=} \left(\left\|B - \hat{B}\right\|_{0,1}^2 + \left\|\xi - \hat{\xi}\right\|_{0,1}^2\right)^{\frac{1}{2}}, \tag{4.73}$$

renders $S(t,L)$ a complete metric space.

Proof. d is clearly a well-defined metric on $S(t,L)$. Suppose the sequence $(v^n) = (B^n, \xi^n)$ is Cauchy in this metric. Then v^n converges strongly to a point v_0 in $\left(L^2([0,t] \times [T_S, T_E])\right)^2$ as $n \to \infty$. On the other hand, since $\mathcal{E}(B^n)^2 + \mathcal{E}(\xi^n)^2 \leq L^2$ for all n, a subsequence of (v^n), say (w^n), is weak* convergent in $\left(W^{1,\infty}([0,t]; H^1(T_S,T_E)) \cap L^\infty([0,t]; H^2(T_S,T_E))\right)^2$ with weak* limit $w_0 \in S(t,L)$. The convergence is strong in $\left(L^2([0,t] \times [T_S, T_E])\right)^2$ by Rellich's theorem. Hence we deduce that $v_0 \equiv w_0$.

□

Theorem 4.14 *Suppose $S(t^*, L)$ is nonempty. Then there exists an admissible $t_0 \in (0, t^*]$ such that the solution operator $\Sigma_{t_0,L}$ is a contraction map on $S(t_0, L)$ w.r.t. the metric d given in (4.73).*

Proof. By Lemma 4.12, for large L, the range of $\Sigma_{t_0,L}$ is contained in $S(t_0, L)$ for all $t_0 \in (0, t^*]$ that are sufficiently small relative to L. Throughout $\kappa = \kappa(t_0, L)$ will be a continuous, positive generic function that stays bounded as $t_0 \to 0$. The arguments of κ will be omitted. Given $(B,\xi)^T$, $(\hat{B},\hat{\xi})^T \in S(t_0, L)$, define

$$(Y,\zeta)^T \stackrel{\text{def}}{=} \Sigma_{t_0,L}\left((B,\xi)^T\right), \quad (\hat{Y},\hat{\zeta})^T \stackrel{\text{def}}{=} \Sigma_{t_0,L}\left((\hat{B},\hat{\xi})^T\right), \tag{4.74}$$

and

$$\gamma \stackrel{\text{def}}{=} \left(\int_{T_S}^{T_E} \frac{\xi_T}{B}\, dT\right)^{-1}, \quad \hat{\gamma} \stackrel{\text{def}}{=} \left(\int_{T_S}^{T_E} \frac{\hat{\xi}_T}{\hat{B}}\, dT\right)^{-1}. \tag{4.75}$$

Estimates (4.56)–(4.59) yield that

$$|\gamma(t) - \hat{\gamma}(t)| \leq \kappa \left(||\xi_T(t) - \hat{\xi}_T(t)||_2 + ||B(t) - \hat{B}(t)||_2\right). \tag{4.76}$$

Therefore, we find that

$$\begin{aligned}
&\left|\int_{T_S}^{T_E}\int_{T_S}^{T}\left(\frac{\xi_T}{B}\,\gamma - \frac{\hat{\xi}_T}{\hat{B}}\,\hat{\gamma}\right) d\tau\, \left(\zeta - \hat{\zeta}\right)\, dT\right| \\
&\leq \kappa\, ||\zeta(t) - \hat{\zeta}(t)||_2 \int_{T_S}^{T_E}\left|\frac{(\xi_T - \hat{\xi}_T)\,\gamma\,\hat{B} + (\hat{B} - B)\,\hat{\xi}_T\,\hat{\gamma} + (\gamma - \hat{\gamma})\,\hat{B}\,\hat{\xi}_T}{B\,\hat{B}}\right| dT \\
&\leq \kappa\, ||\zeta(t) - \hat{\zeta}(t)||_2 \left(||\xi_T(t) - \hat{\xi}_T(t)||_2 + ||B(t) - \hat{B}(t)||_2\right).
\end{aligned} \tag{4.77}$$

Next we obtain the estimate

$$\begin{aligned}
&\left|\int_{T_S}^{T_E} T\left(\frac{\zeta_T}{\sqrt{B}} - \frac{\hat{\zeta}_T}{\sqrt{\hat{B}}}\right)\left(\zeta - \hat{\zeta}\right)\, dT\right| \\
&\leq \int_{T_S}^{T_E}\left|T\,\frac{\zeta_T - \hat{\zeta}_T}{\sqrt{B}}\,\left(\zeta - \hat{\zeta}\right)\right| dT + \int_{T_S}^{T_E}\left|T\,\frac{\hat{\zeta}_T}{\sqrt{B\,\hat{B}}}\,\frac{\hat{B} - B}{\sqrt{B} + \sqrt{\hat{B}}}\,\left(\zeta - \hat{\zeta}\right)\right| dT \\
&\leq \kappa\, ||\zeta(t) - \hat{\zeta}(t)||_2 \left(||B(t) - \hat{B}(t)||_2 + ||\zeta_T(t) - \hat{\zeta}_T(t)||_2\right).
\end{aligned} \tag{4.78}$$

Combining estimates (4.77) and (4.78), we derive

$$\begin{aligned}
&\frac{d}{dt}\int_{T_S}^{T_E}\left(\zeta - \hat{\zeta}\right)^2 dT \\
&\quad\leq \kappa\left(||\zeta(t) - \hat{\zeta}(t)||_{H^1}^2 + ||\xi(t) - \hat{\xi}(t)||_{H^1}^2 + ||B(t) - \hat{B}(t)||_2^2\right).
\end{aligned} \tag{4.79}$$

We observe that

$$\begin{aligned}
&\left|\int_{T_S}^{T_E} T\left(\frac{\zeta_{TT}}{\sqrt{B}} - \frac{\hat{\zeta}_{TT}}{\sqrt{\hat{B}}}\right)\left(\zeta_T - \hat{\zeta}_T\right)\, dT\right| \\
&\leq \int_{T_S}^{T_E}\left|T\,\frac{\zeta_{TT} - \hat{\zeta}_{TT}}{\sqrt{B}}\,\left(\zeta_T - \hat{\zeta}_T\right)\right| dT + \\
&\quad \int_{T_S}^{T_E}\left|T\,\frac{\hat{\zeta}_{TT}}{\sqrt{B\,\hat{B}}}\,\frac{\hat{B} - B}{\sqrt{B} + \sqrt{\hat{B}}}\,\left(\zeta_T - \hat{\zeta}_T\right)\right| dT \\
&\leq \kappa\left(||\zeta_T(t) - \hat{\zeta}_T(t)||_2^2 + ||B(t) - \hat{B}(t)||_{H^1}\, ||\zeta_T(t) - \hat{\zeta}_T(t)||_2\right).
\end{aligned} \tag{4.80}$$

Here we have used Lemma 4.12 . Hence (after a few calculations similar to (4.77)–(4.80)) we obtain an estimate of the form

$$\begin{aligned}&\frac{d}{dt}\,||\zeta(t)-\hat{\zeta}(t)||^2_{H^1}\\ &\quad\leq\ \kappa\left(||\zeta(t)-\hat{\zeta}(t)||^2_{H^1}+||\xi(t)-\hat{\xi}(t)||^2_{H^1}+||B(t)-\hat{B}(t)||^2_{H^1}\right). \end{aligned}\tag{4.81}$$

We integrate (4.81) over $[0,t_0]$ to deduce the estimate

$$\left\|\zeta-\hat{\zeta}\right\|^2_{0,1}\leq\kappa\,t_0\,\left(d\left((B,\xi)^T,(\hat{B},\hat{\xi})^T\right)\right)^2. \tag{4.82}$$

In a completely analogous fashion, we can derive estimate (4.82) with Y, $\hat{Y}$ taking the place of ζ, $\hat{\zeta}$. Hence if t_0 is chosen small enough, the solution operator $\Sigma_{t_0,L}$ is a contraction in the metric d.

□

Proof of Theorem 4.3 By Theorem 4.14, the solution operator $\Sigma_{t_0,L}$ has a unique fixed point (A,z) in $S(t_0,L)$ if L is large and t_0 is small relative to L. It is readily seen from Eqs. (4.61) and (4.64) that (A,z,v) (with v given by (4.46)) is the unique solution of the boundary-initial value problem (4.15)–(4.23). Moreover, A and z are boundary-regular and satisfy the regularity condition (4.39) (see Theorem 3.5). The remaining results for v, (4.41)–(4.42), follow now immediately from the explicit representation (4.46). Also, if hypothesis (4.43) holds, the additional regularity results, (4.44)–(4.45), are obvious consequences of Eq. (4.46) and the model equations (4.15) and (4.17).

□

CHAPTER 5

Existence Results for Other Flow Regimes

We shall demonstrate in this chapter how the mathematical theory developed in the preceding sections applies to more complicated flow regimes. Our goal will be to state the existence results and to sketch the basic ideas of their proofs. We shall forego with laying out every detail since the proofs of Chapter 4 carry over to a large extent. When the situation is clear, we omit the proofs entirely.

Throughout, we shall use τ_{rr}, $\tau_{\theta\theta}$, and τ_{zz} to refer to the radial, azimuthal, and axial normal stress components of the extra stress tensor $\bar{\tau}$, resp. (see Appendix B).

5.1 Solvability for the Upper Convected Maxwell Fluid and the Giesekus Fluid

The constitutive relation (B.7) for an upper convected Maxwell fluid reads in componentwise form

$$\lambda \frac{\partial}{\partial t}\tau_{rr} + \tau_{rr} + \lambda \left(v \frac{\partial}{\partial z}\tau_{rr} + \tau_{rr} \frac{\partial}{\partial z} v \right) = -\mu \frac{\partial}{\partial z} v, \tag{5.1}$$

$$\lambda \frac{\partial}{\partial t}\tau_{\theta\theta} + \tau_{\theta\theta} + \lambda \left(v \frac{\partial}{\partial z}\tau_{\theta\theta} + \tau_{\theta\theta} \frac{\partial}{\partial z} v \right) = -\mu \frac{\partial}{\partial z} v, \tag{5.2}$$

$$\lambda \frac{\partial}{\partial t}\tau_{zz} + \tau_{zz} + \lambda \left(v \frac{\partial}{\partial z}\tau_{zz} - 2\,\tau_{zz} \frac{\partial}{\partial z} v \right) = 2\,\mu \frac{\partial}{\partial z} v. \tag{5.3}$$

Similarly, for a Giesekus fluid the constitutive relation (B.8) reads

$$\lambda \frac{\partial}{\partial t}\tau_{rr} + \tau_{rr} + \lambda \left(v \frac{\partial}{\partial z}\tau_{rr} + \tau_{rr} \frac{\partial}{\partial z} v \right) + \kappa \, \tau_{rr}^2 = -\mu \frac{\partial}{\partial z} v, \tag{5.4}$$

$$\lambda \frac{\partial}{\partial t}\tau_{\theta\theta} + \tau_{\theta\theta} + \lambda \left(v \frac{\partial}{\partial z}\tau_{\theta\theta} + \tau_{\theta\theta} \frac{\partial}{\partial z} v \right) + \kappa \, \tau_{\theta\theta}^2 = -\mu \frac{\partial}{\partial z} v, \tag{5.5}$$

$$\lambda \frac{\partial}{\partial t}\tau_{zz} + \tau_{zz} + \lambda \left(v \frac{\partial}{\partial z}\tau_{zz} - 2\, \tau_{zz} \frac{\partial}{\partial z} v \right) + \kappa \, \tau_{zz}^2 = 2\, \mu \frac{\partial}{\partial z} v. \tag{5.6}$$

μ denotes the zero-shear-rate viscosity, λ the viscoelastic relaxation time, and κ a positive molecular constant (see Section B.2 in Appendix B).

We exchange temperature T and displacement z, using (4.10)–(4.14). Also, we define

$$\tau \stackrel{\text{def}}{=} \tau_{zz} \quad \text{and} \quad N \stackrel{\text{def}}{=} \tau_{zz} - \tau_{rr}. \tag{5.7}$$

Then, together with Eqs. (2.6)–(2.10), we obtain the flow equations

$$A_t(t,T) = \frac{\alpha\, T}{\sqrt{A(t,T)}}\, A_T(t,T) - \frac{v_T(t,T)}{z_T(t,T)}\, A(t,T), \tag{5.8}$$

$$\frac{\partial}{\partial T}\left(N(t,T)\, A(t,T)\right) = 0, \tag{5.9}$$

$$z_t(t,T) = \frac{\alpha\, T}{\sqrt{A(t,T)}}\, z_T(t,T) + v(t,T), \tag{5.10}$$

$$\begin{aligned} N_t(t,T) = {} & \frac{\alpha\, T}{\sqrt{A(t,T)}}\, N_T(t,T) + \left(3\, \tau(t,T) - N(t,T) + \frac{3\, \mu}{\lambda} \right) \frac{v_T(t,T)}{z_T(t,T)} + \\ & \left(\frac{\kappa}{\lambda}\, N(t,T) - \frac{2\, \kappa}{\lambda}\, \tau(t,T) - \frac{1}{\lambda} \right) N(t,T), \end{aligned} \tag{5.11}$$

$$\begin{aligned} \tau_t(t,T) = {} & \frac{\alpha\, T}{\sqrt{A(t,T)}}\, \tau_T(t,T) + \left(2\, \tau(t,T) + \frac{2\, \mu}{\lambda} \right) \frac{v_T(t,T)}{z_T(t,T)} - \\ & \left(\frac{\kappa}{\lambda}\, \tau(t,T) + \frac{1}{\lambda} \right) \tau(t,T). \end{aligned} \tag{5.12}$$

A is the cross-sectional area, and α is the transformed heat transfer coefficient (see Section 4.2). For $\kappa = 0$, Eqs. (5.8)–(5.12) correspond to the upper convected Maxwell fluid; for $\kappa > 0$, to the Giesekus fluid. We have left out the constitutive relations for the azimuthal stresses, Eqs. (5.2) and (5.5), since they are equal to the relations for the radial stresses and do not enter the equations of change. In addition to Eqs. (5.8)–(5.12), we have the boundary conditions

$$A(t, T_E) = A_E(t), \tag{5.13}$$
$$z(t, T_E) = 0, \tag{5.14}$$
$$N(t, T_E) = N_E(t), \tag{5.15}$$
$$\tau(t, T_E) = \tau_E(t), \tag{5.16}$$
$$v(t, T_S) = v_S(t) \tag{5.17}$$

and the initial conditions

$$A(0, T) = A^0(T), \tag{5.18}$$
$$z(0, T) = z^0(T), \tag{5.19}$$
$$N(0, T) = N^0(T), \tag{5.20}$$
$$\tau(0, T) = \tau^0(T). \tag{5.21}$$

Instead of specifying the exit velocity v_E, we prescribe the stresses at the spinneret in Eqs. (5.15)–(5.16).

Definition 5.1 *We shall call a vector field (A, z, N, τ, v), defined on $[0, t_0] \times [T_S, T_E]$, a solution of (5.8)–(5.21) if and only if*

$$A, z, N, \tau, v \in W^{1,\infty}([0, t_0]; H^1(T_S, T_E)) \cap L^\infty([0, t_0]; H^2(T_S, T_E)), \tag{5.22}$$

A, z, N, τ, v satisfy Eqs. (5.8)–(5.12), (5.23)

A satisfies Eqs. (5.13), (5.18) pointwise, (5.24)

z satisfies Eqs. (5.14), (5.19) pointwise, (5.25)

N satisfies Eqs. (5.15), (5.20) pointwise, (5.26)

τ satisfies Eqs. (5.16), (5.21) pointwise, (5.27)

v satisfies Eq. (5.17) pointwise, and $v|_{T=T_E} = v_E < v_S$. (5.28)

Definition 5.2 *We shall say that the boundary and initial values of problem (5.8)–(5.21) obey the stretch condition if and only if*

$$3\,\lambda\,\tau^0 - 2\,\lambda\,N^0 + 3\,\mu \neq 0 \ \text{on}\ [T_S, T_E], \tag{5.29}$$

and if the stretch function S_G, defined on $[T_S, T_E]$ by

$$S_G \stackrel{\text{def}}{=} \frac{\lambda\,N_E(0)\,\dot{A}_E(0) + \lambda\,A_E(0)\,\dot{N}_E(0) + N_E(0)\,A_E(0)\,\left(2\,\kappa\,\tau^0 - \kappa\,N^0 + 1\right)}{3\,\lambda\,\tau^0 - 2\,\lambda\,N^0 + 3\,\mu}, \tag{5.30}$$

satisfies

$$S_G > 0 \ \text{on}\ [T_S, T_E]. \tag{5.31}$$

Remark 5.3

(a) The stretch condition is motivated by the following observation: we can replace v in Eqs. (5.8)–(5.12) by the other quantities since

$$A \frac{v_T}{z_T} = \frac{\lambda N_E \dot{A}_E + \lambda A_E \dot{N}_E + N_E A_E (2\kappa\tau - \kappa N + 1)}{3\lambda\tau - 2\lambda N + 3\mu}. \tag{5.32}$$

Thus we find

$$v = v_S + \int_{T_S}^{T} \frac{z_T}{A} \frac{\lambda N_E \dot{A}_E + \lambda A_E \dot{N}_E + N_E A_E (2\kappa\tau - \kappa N + 1)}{3\lambda\tau - 2\lambda N + 3\mu} ds. \tag{5.33}$$

The initial velocity v^0 must necessarily satisfy

$$v^0 = v_S(0) + \int_{T_S}^{T} \frac{z_T^0(s)}{A^0(s)} S_G(s)\, ds. \tag{5.34}$$

To ensure that the exit velocity v_E, determined by (5.33), is smaller than the take-up velocity v_S for some time span starting at $t = 0$, we impose condition (5.31) on the initial and boundary values. The stretch condition is a sufficient condition. At this moment, we do not know if it can be relaxed.

(b) It is illuminating to discuss the viscous limit $\kappa = 0$, $\lambda \to 0$. The stretch function S_G is reduced to the quantity

$$\frac{N_E(0)\, A_E(0)}{3\mu}. \tag{5.35}$$

Noting Eq. (5.32), we find

$$S_G \equiv A^0 \frac{v_T^0}{z_T^0}, \tag{5.36}$$

where v^0 is the initial velocity. In the original variables, the term $v_T^0\,(z_T^0)^{-1}$ reads v_z^0. Hence the stretch condition (5.31) demands that, in the viscous limit, the initial velocity be increasing along the threadline. For the viscoelastic case, this conclusion can readily be drawn from Eq. (5.34).

(c) The velocity relations (5.32) and (5.33) will be used to replace the momentum balance (5.9).

Theorem 5.4 *Let the initial values A^0, z^0, N^0, τ^0 on $[T_S, T_E]$ and the boundary values A_E, N_E, τ_E, v_S on $[0, t^*]$, $t^* > 0$, be given such that*

$$A^0, z^0, N^0, \tau^0 \in H^2(T_S, T_E), \tag{5.37}$$
$$A_E, N_E \in W^{2,\infty}(0, t^*), \tag{5.38}$$
$$\tau_E \in H^2(0, t^*), \tag{5.39}$$
$$v_S \in W^{1,\infty}(0, t^*), \tag{5.40}$$
$$A^0 > 0 \text{ on } [T_S, T_E], \tag{5.41}$$
$$z_T^0 < 0 \text{ on } [T_S, T_E], \tag{5.42}$$
$$\frac{\partial}{\partial T}\left(N^0 A^0\right) = 0 \text{ on } [T_S, T_E]. \tag{5.43}$$

Assume that the boundary and initial values obey the stretch condition of Definition 5.2, and that the compatibility conditions

$$A^0(T_E) = A_E(0), \quad z^0(T_E) = 0, \quad N^0(T_E) = N_E(0), \quad \tau^0(T_E) = \tau_E(0), \tag{5.44}$$
$$\dot{A}_E(0) = \frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, A_T^0(T_E) - S_G(T_E), \tag{5.45}$$
$$\frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, z_T^0(T_E) + v_S(0) + \int_{T_S}^{T_E} \frac{z_T^0(T)}{A^0(T)}\, S_G(T)\, dT = 0, \tag{5.46}$$
$$\begin{aligned}\dot{N}_E(0) = {}& \frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, N_T^0(T_E) + \left(\frac{\kappa}{\lambda}\, N_E(0) - \frac{2\,\kappa}{\lambda}\, \tau_E(0) - \frac{1}{\lambda}\right) N_E(0) + \\ & \left(3\,\tau_E(0) - N_E(0) + \frac{3\,\mu}{\lambda}\right) \frac{S_G(T_E)}{A_E(0)},\end{aligned} \tag{5.47}$$
$$\dot{\tau}_E(0) = \frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, \tau_T^0(T_E) - \left(\frac{\kappa}{\lambda}\, \tau_E(0) + \frac{1}{\lambda}\right) \tau_E(0) + \left(2\,\tau_E(0) + \frac{2\,\mu}{\lambda}\right) \frac{S_G(T_E)}{A_E(0)} \tag{5.48}$$

hold. Then there exists $t_0 \in (0, t^*]$ such that the boundary-initial value problem (5.8)–(5.21) has a unique solution (A, z, N, τ, v) on $[0, t_0] \times [T_S, T_E]$. This solution (A, z, N, τ, v) has the properties:

$$A, z, N, \tau \in C^1([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)), \tag{5.49}$$
$$A, z, N, \tau \in W^{2,\infty}([0, t_0]; L^2(T_S, T_E)), \tag{5.50}$$
$$v \in W^{1,\infty}([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)), \tag{5.51}$$
$$A,\ z,\ N,\ \tau,\ v \text{ are boundary-regular.} \tag{5.52}$$

Moreover, if

$$v_S \in C^1([0, t_0]) \text{ and } A_E, N_E \in C^2([0, t_0]), \tag{5.53}$$

then the solution (A, z, N, τ, v) has the additional properties:

$$A, z, N, \tau \in \bigcap_{k=0}^{2} C^k([0,t_0]; H^{2-k}(T_S, T_E)), \tag{5.54}$$

$$v \in C^1([0,t_0]; H^1(T_S,T_E)) \cap C([0,t_0]; H^2(T_S,T_E)). \tag{5.55}$$

Outline of the Proof. We use the energy functional $\mathcal{E}$, given in Definition 4.5 . For $L > 0$ and $t' \in (0, t^*]$, let $S(t', L)$ be the set of vector fields $(B, \xi, M, \sigma)^T$ on $[0,t'] \times [T_S, T_E]$ such that

$$B,\ \xi,\ M, \text{ and } \sigma \text{ are boundary-regular on } [0,t'] \times [T_S, T_E], \tag{5.56}$$

$$\mathcal{E}(B)^2 + \mathcal{E}(\xi)^2 + \mathcal{E}(M)^2 + \mathcal{E}(\sigma)^2 \le L^2, \tag{5.57}$$

$$B(0,T) = A^0(T) \text{ and } B(t,T_E) = A_E(t), \tag{5.58}$$

$$\xi(0,T) = z^0(T) \text{ and } \xi(t,T_E) = 0, \tag{5.59}$$

$$M(0,T) = N^0(T) \text{ and } M(t,T_E) = N_E(t), \tag{5.60}$$

$$\sigma(0,T) = \tau^0(T) \text{ and } \sigma(t,T_E) = \tau_E(t). \tag{5.61}$$

If t' is small enough relative to L, $S(t', L)$ is nonempty and, for $t \in [0,t']$, $T \in [T_S, T_E]$, its elements $(B, \xi, M, \sigma)^T$ satisfy

$$B > 0, \tag{5.62}$$

$$3\,\lambda\,\sigma - 2\,\lambda\,M + 3\,\mu \neq 0, \tag{5.63}$$

$$\frac{\lambda\,N_E(0)\,\dot{A}_E(0) + \lambda\,A_E(0)\,\dot{N}_E(0) + N_E(0)\,A_E(0)\,\left(2\,\kappa\,\sigma - \kappa\,M + 1\right)}{3\,\lambda\,\sigma - 2\,\lambda\,M + 3\,\mu} > 0. \tag{5.64}$$

Using the velocity relations (5.32) and (5.33), we can define the solution operator $\Sigma_{t',L}$ on $S(t', L)$ by

$$\Sigma_{t',L} : \begin{pmatrix} B \\ \xi \\ M \\ \sigma \end{pmatrix} \longmapsto \begin{pmatrix} Y \\ \zeta \\ K \\ \rho \end{pmatrix}, \tag{5.65}$$

where $Y = Y(t,T)$, $\zeta = \zeta(t,T)$, $K = K(t,T)$, and $\rho = \rho(t,T)$ are the solutions (in the sense of Definition 3.1) of the boundary-initial value problems, stated on $[0,t'] \times [T_S, T_E]$,

$$Y_t = \frac{\alpha\,T}{\sqrt{B}}\,Y_T - \frac{\lambda\,N_E\,\dot{A}_E + \lambda\,A_E\,\dot{N}_E + N_E\,A_E\,\left(2\,\kappa\,\sigma - \kappa\,M + 1\right)}{3\,\lambda\,\sigma - 2\,\lambda\,M + 3\,\mu} \tag{5.66}$$

$$Y(0,T) = A^0(T), \tag{5.67}$$

$$Y(t,T_E) = A_E(t), \tag{5.68}$$

$$\zeta_t = \frac{\alpha\, T}{\sqrt{B}}\, \zeta_T + v_S + \int_{T_S}^{T} \frac{\xi_T}{B}\, \frac{\lambda\, N_E\, \dot{A}_E + \lambda\, A_E\, \dot{N}_E + N_E\, A_E\, (2\,\kappa\,\sigma - \kappa\, M + 1)}{3\,\lambda\,\sigma - 2\,\lambda\, M + 3\,\mu}\, ds \tag{5.69}$$

$$\zeta(0, T) = z^0(T), \tag{5.70}$$

$$\zeta(t, T_E) = 0, \tag{5.71}$$

$$K_t = \frac{\alpha\, T}{\sqrt{B}}\, K_T + \left(3\,\sigma - M + \frac{3\,\mu}{\lambda}\right) \frac{\lambda\, N_E\, \dot{A}_E + \lambda\, A_E\, \dot{N}_E + N_E\, A_E\, (2\,\kappa\,\sigma - \kappa\, M + 1)}{B\, (3\,\lambda\,\sigma - 2\,\lambda\, M + 3\,\mu)} + \left(\frac{\kappa}{\lambda}\, M - \frac{2\,\kappa}{\lambda}\,\sigma - \frac{1}{\lambda}\right) M, \tag{5.72}$$

$$K(0, T) = N^0(T), \tag{5.73}$$

$$K(t, T_E) = N_E(t), \tag{5.74}$$

and

$$\rho_t = \frac{\alpha\, T}{\sqrt{B}}\, \rho_T + \left(2\,\sigma + \frac{2\,\mu}{\lambda}\right) \frac{\lambda\, N_E\, \dot{A}_E + \lambda\, A_E\, \dot{N}_E + N_E\, A_E\, (2\,\kappa\,\sigma - \kappa\, M + 1)}{B\, (3\,\lambda\,\sigma - 2\,\lambda\, M + 3\,\mu)} - \left(\frac{\kappa}{\lambda}\,\sigma + \frac{1}{\lambda}\right) \sigma, \tag{5.75}$$

$$\rho(0, T) = \tau^0(T), \tag{5.76}$$

$$\rho(t, T_E) = \tau_E(t). \tag{5.77}$$

For sufficiently small $t_0 \in (0, t^*]$, $\Sigma_{t_0, L}$ is a contraction on $S(t_0, L)$ w.r.t. the metric d, defined by

$$d\left((B, \xi, M, \sigma)^T, (\hat{B}, \hat{\xi}, \hat{M}, \hat{\sigma})^T\right) \stackrel{\text{def}}{=} \left(\left\|B - \hat{B}\right\|_{0,1}^2 + \left\|\xi - \hat{\xi}\right\|_{0,1}^2 + \left\|M - \hat{M}\right\|_{0,1}^2 + \|\sigma - \hat{\sigma}\|_{0,1}^2\right)^{\frac{1}{2}}. \tag{5.78}$$

Since $S(t_0, L)$ is complete in the metric d, the claim follows from arguments analogous to the proof of Theorem 4.3 .

□

Remark 5.5 A straightforward calculation yields that the solution afforded by the proof of Theorem 5.4 satisfies Eq. (5.9).

5.2 Solvability for the Phan-Thien–Tanner Fluid

The components of the constitutive relation (B.9) for a Phan-Thien–Tanner fluid read

$$\lambda \frac{\partial}{\partial t}\tau_{rr} + \tau_{rr} + \lambda \left(v \frac{\partial}{\partial z}\tau_{rr} + \tau_{rr} \frac{\partial}{\partial z} v \right) + \kappa\, \tau_{rr}\, (\tau_{rr} + \tau_{\theta\theta} + \tau_{zz}) = -\mu \frac{\partial}{\partial z} v, \tag{5.79}$$

$$\lambda \frac{\partial}{\partial t}\tau_{\theta\theta} + \tau_{\theta\theta} + \lambda \left(v \frac{\partial}{\partial z}\tau_{\theta\theta} + \tau_{\theta\theta} \frac{\partial}{\partial z} v \right) + \kappa\, \tau_{\theta\theta}\, (\tau_{rr} + \tau_{\theta\theta} + \tau_{zz}) = -\mu \frac{\partial}{\partial z} v, \tag{5.80}$$

$$\lambda \frac{\partial}{\partial t}\tau_{zz} + \tau_{zz} + \lambda \left(v \frac{\partial}{\partial z}\tau_{zz} - 2\,\tau_{zz} \frac{\partial}{\partial z} v \right) + \kappa\, \tau_{zz}\, (\tau_{rr} + \tau_{\theta\theta} + \tau_{zz}) = 2\,\mu \frac{\partial}{\partial z} v. \tag{5.81}$$

μ is the zero-shear-rate viscosity, λ the viscoelastic relaxation time, and κ a positive model constant (see Section B.2 in Appendix B).

We proceed as before: first, with the help of Eqs. (4.10)–(4.14), we exchange the roles of temperature T and displacement z. Then we define

$$\tau \overset{\text{def}}{=} \tau_{zz}, \quad \hat{\tau} \overset{\text{def}}{=} \tau_{\theta\theta}, \quad N \overset{\text{def}}{=} \tau_{zz} - \tau_{rr}. \tag{5.82}$$

Together with Eqs. (2.6)–(2.10), this transformation yields the flow equations

$$A_t(t,T) = \frac{\alpha\, T}{\sqrt{A(t,T)}}\, A_T(t,T) - \frac{v_T(t,T)}{z_T(t,T)}\, A(t,T), \tag{5.83}$$

$$\frac{\partial}{\partial T}\left(N(t,T)\, A(t,T)\right) = 0, \tag{5.84}$$

$$z_t(t,T) = \frac{\alpha\, T}{\sqrt{A(t,T)}}\, z_T(t,T) + v(t,T), \tag{5.85}$$

$$\begin{aligned} N_t(t,T) = {} & \frac{\alpha\, T}{\sqrt{A(t,T)}}\, N_T(t,T) + \left(3\,\tau(t,T) - N(t,T) + \frac{3\,\mu}{\lambda} \right) \frac{v_T(t,T)}{z_T(t,T)} + \\ & \left(\frac{\kappa}{\lambda}\, N(t,T) - \frac{2\,\kappa}{\lambda}\, \tau(t,T) - \frac{\kappa}{\lambda}\, \hat{\tau}(t,T) - \frac{1}{\lambda} \right) N(t,T), \end{aligned} \tag{5.86}$$

$$\begin{aligned} \tau_t(t,T) = {} & \frac{\alpha\, T}{\sqrt{A(t,T)}}\, \tau_T(t,T) + \left(2\,\tau(t,T) + \frac{2\,\mu}{\lambda} \right) \frac{v_T(t,T)}{z_T(t,T)} + \\ & \left(\frac{\kappa}{\lambda}\, N(t,T) - \frac{2\,\kappa}{\lambda}\, \tau(t,T) - \frac{\kappa}{\lambda}\, \hat{\tau}(t,T) - \frac{1}{\lambda} \right) \tau(t,T), \end{aligned} \tag{5.87}$$

$$\begin{aligned} \hat{\tau}_t(t,T) = {} & \frac{\alpha\, T}{\sqrt{A(t,T)}}\, \hat{\tau}_T(t,T) - \left(\hat{\tau}(t,T) + \frac{\mu}{\lambda} \right) \frac{v_T(t,T)}{z_T(t,T)} + \\ & \left(\frac{\kappa}{\lambda}\, N(t,T) - \frac{2\,\kappa}{\lambda}\, \tau(t,T) - \frac{\kappa}{\lambda}\, \hat{\tau}(t,T) - \frac{1}{\lambda} \right) \hat{\tau}(t,T). \end{aligned} \tag{5.88}$$

A and α have the same meaning as in the previous section. Eqs. (5.83)–(5.88) are complemented by the boundary conditions

$$A(t, T_E) = A_E(t), \tag{5.89}$$
$$z(t, T_E) = 0, \tag{5.90}$$
$$N(t, T_E) = N_E(t), \tag{5.91}$$
$$\tau(t, T_E) = \tau_E(t), \tag{5.92}$$
$$\hat{\tau}(t, T_E) = \hat{\tau}_E(t), \tag{5.93}$$
$$v(t, T_S) = v_S(t) \tag{5.94}$$

and the initial conditions

$$A(0, T) = A^0(T), \tag{5.95}$$
$$z(0, T) = z^0(T), \tag{5.96}$$
$$N(0, T) = N^0(T), \tag{5.97}$$
$$\tau(0, T) = \tau^0(T), \tag{5.98}$$
$$\hat{\tau}(0, T) = \hat{\tau}^0(T). \tag{5.99}$$

Again we have specified the stresses at the spinneret in lieu of the exit velocity v_E.

Definition 5.6 *We shall call a vector field $(A, z, N, \tau, \hat{\tau}, v)$, defined on $[0, t_0] \times [T_S, T_E]$, a solution of (5.83)–(5.98) if and only if*

$$A, z, N, \tau, \hat{\tau}, v \in W^{1,\infty}([0, t_0]; H^1(T_S, T_E)) \cap L^\infty([0, t_0]; H^2(T_S, T_E)), \tag{5.100}$$

A, z, N, τ, $\hat{\tau}$, v satisfy Eqs. (5.83)–(5.88), (5.101)

A satisfies Eqs. (5.89), (5.95) pointwise, (5.102)

z satisfies Eqs. (5.90), (5.96) pointwise, (5.103)

N satisfies Eqs. (5.91), (5.97) pointwise, (5.104)

τ satisfies Eqs. (5.92), (5.98) pointwise, (5.105)

$\hat{\tau}$ satisfies Eqs. (5.93), (5.99) pointwise, (5.106)

v satisfies Eq. (5.94) pointwise, and $v|_{T=T_E} = v_E < v_S$. (5.107)

Definition 5.7 *We shall say that the boundary and initial values of problem (5.83)–(5.99) obey the stretch condition if and only if*

$$3\,\lambda\,\tau^0 - 2\,\lambda\,N^0 + 3\,\mu \neq 0 \text{ on } [T_S, T_E], \tag{5.108}$$

and if the stretch function S_P, defined on $[T_S, T_E]$ by

$$S_P \stackrel{\text{def}}{=} \frac{\lambda\,N_E(0)\,\dot{A}_E(0) + \lambda\,A_E(0)\,\dot{N}_E(0) + N_E(0)\,A_E(0)\ (2\,\kappa\,\tau^0 + \kappa\,\hat{\tau}^0 - \kappa\,N^0 + 1)}{3\,\lambda\,\tau^0 - 2\,\lambda\,N^0 + 3\,\mu}, \tag{5.109}$$

satisfies

$$S_P > 0 \text{ on } [T_S, T_E]. \tag{5.110}$$

Remark 5.8 The stretch function S_P, defined in (5.109), plays a role analogous to the case of a Giesekus fluid. It is related to the axial velocity since a straightforward calculation yields

$$v = v_S + \int_{T_S}^{T} \frac{z_T}{A} \frac{\lambda\, N_E\, \dot{A}_E + \lambda\, A_E\, \dot{N}_E + N_E\, A_E\, (2\,\kappa\,\tau + \kappa\,\hat{\tau} - \kappa\, N + 1)}{3\,\lambda\,\tau - 2\,\lambda\, N + 3\,\mu}\, ds. \tag{5.111}$$

Theorem 5.9 *Let the initial values A^0, z^0, N^0, τ^0, $\hat{\tau}^0$ on $[T_S, T_E]$ and the boundary values A_E, N_E, τ_E, $\hat{\tau}_E$, v_S on $[0, t^*]$, $t^* > 0$, be given such that*

$$A^0, z^0, N^0, \tau^0, \hat{\tau}^0 \in H^2(T_S, T_E), \tag{5.112}$$
$$A_E, N_E \in W^{2,\infty}(0, t^*), \tag{5.113}$$
$$\tau_E, \hat{\tau}_E \in H^2(0, t^*), \tag{5.114}$$
$$v_S \in W^{1,\infty}(0, t^*), \tag{5.115}$$
$$A^0 > 0 \text{ on } [T_S, T_E], \tag{5.116}$$
$$z_T^0 < 0 \text{ on } [T_S, T_E], \tag{5.117}$$
$$\frac{\partial}{\partial T}\left(N^0\, A^0\right) = 0 \text{ on } [T_S, T_E]. \tag{5.118}$$

Assume that the boundary and initial values obey the stretch condition of Definition 5.7, and that the compatibility conditions

$$A^0(T_E) = A_E(0), \quad z^0(T_E) = 0, \quad N^0(T_E) = N_E(0), \tag{5.119}$$
$$\tau^0(T_E) = \tau_E(0), \quad \hat{\tau}^0(T_E) = \hat{\tau}_E(0), \tag{5.120}$$
$$\dot{A}_E(0) = \frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, A_T^0(T_E) - S_P(T_E), \tag{5.121}$$
$$\frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, z_T^0(T_E) + v_S(0) + \int_{T_S}^{T_E} \frac{z_T^0(T)}{A^0(T)}\, S_P(T)\, dT = 0, \tag{5.122}$$
$$\begin{aligned} \dot{N}_E(0) = {} & \frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, N_T^0(T_E) + \left(\frac{\kappa}{\lambda}\, N_E(0) - \frac{2\,\kappa}{\lambda}\, \tau_E(0) - \frac{\kappa}{\lambda}\, \hat{\tau}_E(0) - \frac{1}{\lambda}\right) N_E(0) + \\ & \left(3\,\tau_E(0) - N_E(0) + \frac{3\,\mu}{\lambda}\right) \frac{S_P(T_E)}{A_E(0)}, \end{aligned} \tag{5.123}$$

$$\dot{\tau}_E(0) = \frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, \tau_T^0(T_E) + \left(\frac{\kappa}{\lambda}\, N_E(0) - \frac{2\,\kappa}{\lambda}\, \tau_E(0) - \frac{\kappa}{\lambda}\, \hat{\tau}_E(0) - \frac{1}{\lambda}\right) \tau_E(0) + \left(2\, \tau_E(0) + \frac{2\,\mu}{\lambda}\right) \frac{S_P(T_E)}{A_E(0)}, \tag{5.124}$$

$$\dot{\hat{\tau}}_E(0) = \frac{\alpha\, T_E}{\sqrt{A^0(T_E)}}\, \hat{\tau}_T^0(T_E) + \left(\frac{\kappa}{\lambda}\, N_E(0) - \frac{2\,\kappa}{\lambda}\, \tau_E(0) - \frac{\kappa}{\lambda}\, \hat{\tau}_E(0) - \frac{1}{\lambda}\right) \hat{\tau}_E(0) - \left(\hat{\tau}_E(0) + \frac{\mu}{\lambda}\right) \frac{S_P(T_E)}{A_E(0)} \tag{5.125}$$

hold. Then there exists $t_0 \in (0, t^*]$ such that the boundary-initial value problem (5.83)–(5.98) has a unique solution $(A, z, N, \tau, \hat{\tau}, v)$ on $[0, t_0] \times [T_S, T_E]$. This solution $(A, z, N, \tau, \hat{\tau}, v)$ has the properties:

$$A, z, N, \tau, \hat{\tau} \in C^1([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)), \tag{5.126}$$
$$A, z, N, \tau, \hat{\tau} \in W^{2,\infty}([0, t_0]; L^2(T_S, T_E)), \tag{5.127}$$
$$v \in W^{1,\infty}([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)), \tag{5.128}$$
$$A,\ z,\ N,\ \tau,\ \hat{\tau},\ v \ \textit{are boundary-regular.} \tag{5.129}$$

Moreover, if

$$v_S \in C^1([0, t_0]) \ \textit{and}\ A_E, N_E \in C^2([0, t_0]), \tag{5.130}$$

then the solution $(A, z, N, \tau, \hat{\tau}, v)$ has the additional properties:

$$A, z, N, \tau, \hat{\tau} \in \bigcap_{k=0}^{2} C^k([0, t_0]; H^{2-k}(T_S, T_E)), \tag{5.131}$$
$$v \in C^1([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)). \tag{5.132}$$

We will skip the proof of Theorem 5.9 . The general line of ideas has already been developed in the proof of Theorem 5.4 .

5.3 Isothermal Flows

In the isothermal case, the equations of change are reduced to the mass balance (2.6) and the momentum balance (2.9). The flow domain is a priori known, i.e. the fluid exits the spinneret at $z = 0$ and solidifies at $z = l$ for some given $l > 0$. The solidification occurs instantaneously, either through a chilling roll, a cooling bath, or a similar device.

Since we are merely interested in motivating the use of the general mathematical theory for isothermal flows, we shall treat the viscous case only. See Appendix B for the constitutive equation. The isothermal flow of a viscoelastic fluid can be analyzed with similar techniques.

If we let A denote the cross-sectional area of the jet, we have to solve the initial-boundary value problem

$$A_t(t,z) = -v(t,z)\,A_z(t,z) - v_z(t,z)\,A(t,z), \tag{5.133}$$
$$\frac{\partial}{\partial z}\,(A(t,z)\,v_z(t,z)) = 0, \tag{5.134}$$
$$A(t,0) = A_E(t), \tag{5.135}$$
$$v(t,0) = v_E(t), \tag{5.136}$$
$$v(t,l) = v_S(t), \tag{5.137}$$
$$A(0,z) = A^0(z). \tag{5.138}$$

As always, it is tacitly understood that $v_E < v_S$. Instead of specifying the exit velocity v_E, we could prescribe the stresses in a way similar to the preceding sections.

Definition 5.10 *We shall call a vector field (A,v), defined on $[0,t_0]\times[0,l]$, a solution of (5.133)–(5.138) if and only if*

$$A, v \in W^{1,\infty}([0,t_0];H^1(0,l)) \cap L^\infty([0,t_0];H^2(0,l)), \tag{5.139}$$
$$A,\ v \textit{ satisfy Eqs. (5.133)–(5.134)}, \tag{5.140}$$
$$A \textit{ satisfies Eqs. (5.135), (5.138) pointwise}, \tag{5.141}$$
$$v \textit{ satisfies Eqs. (5.136), (5.137) pointwise.} \tag{5.142}$$

Theorem 5.11 *Let the initial value A^0 on $[0,l]$ and the boundary values A_E, v_E, v_S on $[0,t^*]$, $t^*>0$, be given such that*

$$A^0 \in H^2(0,l), \tag{5.143}$$
$$A_E \in H^2(0,t^*), \tag{5.144}$$
$$v_E, v_S \in W^{1,\infty}(0,t^*), \tag{5.145}$$
$$A^0 > 0 \textit{ on } [0,l]. \tag{5.146}$$

Also suppose that the compatibility conditions

$$A^0(0) = A_E(0), \tag{5.147}$$

$$\dot{A}_E(0) = -\left(v_E(0) + (v_S(0) - v_E(0)) \frac{\displaystyle\int_0^z \left(A^0(\zeta)\right)^{-1} d\zeta}{\displaystyle\int_0^l \left(A^0(\zeta)\right)^{-1} d\zeta} \right) A_z^0(0) + \frac{v_E(0) - v_S(0)}{\displaystyle\int_0^l \left(A^0(\zeta)\right)^{-1} d\zeta} \tag{5.148}$$

hold. Then there exists $t_0 \in (0, t^*]$ such that the boundary-initial value problem (5.133)–(5.138) has a unique solution (A, v) on $[0, t_0] \times [0, l]$. This solution (A, v) has the properties:

$$A \in C^1([0, t_0]; H^1(0, l)) \cap C([0, t_0]; H^2(0, l)), \tag{5.149}$$
$$A \in W^{2,\infty}([0, t_0]; L^2(0, l)), \tag{5.150}$$
$$v \in W^{1,\infty}([0, t_0]; H^1(0, l)) \cap C([0, t_0]; H^2(0, l)), \tag{5.151}$$
$$A,\ v \text{ are boundary-regular.} \tag{5.152}$$

Moreover, if

$$v_E, v_S \in C^1([0, t_0]), \tag{5.153}$$

then the solution (A, v) has the additional properties:

$$A \in \bigcap_{k=0}^{2} C^k([0, t_0]; H^{2-k}(0, l)), \tag{5.154}$$
$$v \in C^1([0, t_0]; H^1(0, l)) \cap C([0, t_0]; H^2(0, l)). \tag{5.155}$$

Outline of the Proof. Since we have

$$v = v_E + (v_S - v_E) \frac{\displaystyle\int_0^z A^{-1}(\zeta)\, d\zeta}{\displaystyle\int_0^l A^{-1}(\zeta)\, d\zeta}, \tag{5.156}$$

we can define the solution operator $\Sigma_{t_0,L}$ on the space $S(t_0, L)$ of all boundary-regular functions B on $[0, t_0] \times [0, l]$, satisfying

$$\mathcal{E}(B) \le L, \tag{5.157}$$
$$B(0, z) = A^0(z) \text{ and } B(t, 0) = A_E(t), \tag{5.158}$$

by

$$\Sigma_{t_0,L} : B \longmapsto Y, \tag{5.159}$$

where Y is the solution of the boundary-initial value problem, stated on $[0, t_0] \times [0, l]$,

$$Y_t = -\left(v_E + (v_S - v_E) \frac{\displaystyle\int_0^z B^{-1}\, d\zeta}{\displaystyle\int_0^l B^{-1}\, d\zeta} \right) Y_z + \frac{v_E - v_S}{\displaystyle\int_0^l B^{-1}\, d\zeta}, \tag{5.160}$$

$$Y(0, z) = A^0(z), \tag{5.161}$$

$$Y(t, 0) = A_E(t). \tag{5.162}$$

If t_0 is small enough relative to L, $\Sigma_{t_0,L}$ is well-defined and acts as a contraction on $S(t_0, L)$. The claim follows.

□

CHAPTER 6

The Linearized Equations of Fiber Spinning

In this chapter, we shall discuss the linearized equations of melt-spinning of viscous fluids. Our ultimate goal is to shed light on the stability properties of steady-state solutions of Eqs. (4.15)–(4.23). However, this issue will not be resolved before Chapter 7.

Throughout, we shall assume that the flow parameters, in particular the tensile viscosity, are constant. We intend to investigate whether the primary flow without secondary thermal effects is stable. If the flow parameters depend strongly on the temperature gradients, necking instabilities can be expected.

Previous numerical findings have indicated that, for temperature-independent parameters, the steady-state solution is unconditionally stable (cf. [23], [45], [46], [58]). For this latter situation, we shall prove that the stability of the steady-state solution is really governed by the eigenvalues of the corresponding differential operator, thereby justifying the numerical conclusions.

6.1 Motivation

The solution to the linear differential equation

$$\dot{\mathbf{y}}(t) = A\,\mathbf{y}(t), \quad \mathbf{y}(0) = \mathbf{y}_0 \tag{6.1}$$

with $A \in \mathbb{R}^{n\times n}$, $\mathbf{y}_0 \in \mathbb{R}^n$ is asymptotically stable if all eigenvalues of the matrix A lie in the left half-plane. For infinite-dimensional spaces and unbounded operators, the situation is more complicated.

We define the growth rate (or type) of a semigroup $\{\mathcal{T}(t)\}$ with generator $\mathcal{A}$ by

$$\omega(\mathcal{A}) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{\ln \|\mathcal{T}(t)\|}{t} \tag{6.2}$$

and the spectral bound of its generator $\mathcal{A}$ by

$$s(\mathcal{A}) \stackrel{\text{def}}{=} \sup_{\lambda\in\sigma(\mathcal{A})} \operatorname{Re}\lambda. \tag{6.3}$$

Here we have written $\sigma(\mathcal{A})$ for the spectrum of $\mathcal{A}$. In numerical studies, it is usually assumed that

$$\omega(\mathcal{A}) = s(\mathcal{A}). \tag{6.4}$$

This result is sometimes referred to as the "Principle of Linear Stability". In the finite-dimensional case, it is clearly valid. Several important examples emphasize, however, that Eq. (6.4) does not hold true in general (cf. [27], [42], [50], and [60]). In particular, hyperbolic equations provide a source for problems in which the growth of the semigroup is not determined by its generator. Eq. (6.4) has to be replaced by

$$\omega(\mathcal{A}) \geq s(\mathcal{A}). \tag{6.5}$$

While this inequality is used to deduce unstable growth by resolving the spectrum of the generator $\mathcal{A}$, stability cannot be proven. However, if the semigroup has certain smoothing properties, Eq. (6.4) is true after all. One particular property ensuring the spectral determinacy of the semigroup is eventual compactness:

Definition 6.1 *A strongly continuous semigroup $\{\mathcal{T}(t)\}$ of bounded linear operators is eventually compact if and only if there exists a $t_0 > 0$ such that the operator $\mathcal{T}(t_0)$ is compact.*

An abstract discussion of the validity of the principle of linear stability is given in [52]; a complete characterization of the spectrum of a strongly continuous semigroup in Hilbert space through properties of a related abstract differential equation is furnished in [49].

In the following sections, we shall give a rigorous justification for the validity of (6.4) in the case of fiber spinning of viscous fluids. We establish the eventual compactness for the semigroup of a reduced system and extend the result to the full set of equations. A similar comparison principle for strongly continuous semigroups has been employed elsewhere (cf. [25] and [41]). The developments there are based on the notion of the "essential spectrum" of a bounded linear operator (cf. [19]).

6.2 Nondimensionalization

We shall assume that the quantities A_E, v_E, and v_S in Eqs. (4.15)–(4.17) are constant and that $v_S > v_E$. To nondimensionalize, we define a dimensionless temperature T^* and a dimensionless time t^* by

$$T^*(T) \stackrel{\text{def}}{=} 1 - \frac{\ln T - \ln T_E}{\ln T_S - \ln T_E}, \tag{6.6}$$

$$t^*(t) \stackrel{\text{def}}{=} \frac{\alpha}{\sqrt{A_E}\,(\ln T_E - \ln T_S)}\, t. \tag{6.7}$$

Next we introduce the dimensionless quantities

$$A^*(t^*, T^*) \stackrel{\text{def}}{=} A_E^{-1}\, A(t, T), \tag{6.8}$$

$$v^*(t^*, T^*) \stackrel{\text{def}}{=} v_E^{-1}\, v(t, T), \tag{6.9}$$

$$z^*(t^*, T^*) \stackrel{\text{def}}{=} \frac{\alpha}{v_E \sqrt{A_E}\,(\ln T_E - \ln T_S)}\, z(t, T). \tag{6.10}$$

Finally, we define the "draw number"

$$\delta \stackrel{\text{def}}{=} \sqrt{\frac{v_S}{v_E}} - 1 \tag{6.11}$$

and the "draw ratio"

$$D \stackrel{\text{def}}{=} \frac{v_S}{v_E}. \tag{6.12}$$

Observe that, by assumption, $\delta > 0$ or equivalently $D > 1$. In the following developments, it will be convenient to use the draw number δ rather than the draw ratio D (in contrast to the published literature).

Inserting the dimensionless quantities (6.6)–(6.10) into Eqs. (4.15)–(4.17) and dropping the asterisk, we recover the equations of melt-spinning in dimensionless form:

$$A_t(t, T) = \frac{A_T(t, T)}{\sqrt{A(t, T)}} - \frac{v_T(t, T)}{z_T(t, T)}\, A(t, T), \tag{6.13}$$

$$\frac{\partial}{\partial T}\left(\frac{v_T(t, T)}{z_T(t, T)}\, A(t, T)\right) = 0, \tag{6.14}$$

$$z_t(t, T) = \frac{z_T(t, T)}{\sqrt{A(t, T)}} + v(t, T). \tag{6.15}$$

The boundary conditions (4.18)–(4.21) read

$$A(t,1) = 1, \tag{6.16}$$
$$z(t,1) = 0, \tag{6.17}$$
$$v(t,1) = 1, \tag{6.18}$$
$$v(t,0) = (\delta+1)^2. \tag{6.19}$$

In the following, we will employ techniques of semigroup theory to analyze the linearization of Eqs. (6.13)–(6.19).

6.3 Linearization

The steady-state solutions $\hat{A}_\delta$, $\hat{z}_\delta$, $\hat{v}_\delta$ of Eqs. (6.13)–(6.19) are readily found as

$$\hat{A}_\delta(T) = \frac{(\delta T+1)^2}{(\delta+1)^2}, \tag{6.20}$$
$$\hat{z}_\delta(T) = \frac{\delta+1}{\delta} \ln \frac{\delta+1}{\delta T+1}, \tag{6.21}$$
$$\hat{v}_\delta(T) = \frac{(\delta+1)^2}{(\delta T+1)^2}. \tag{6.22}$$

We determine the infinitesimal perturbations $B_\delta = B_\delta(t,T)$, $\zeta_\delta = \zeta_\delta(t,T)$, $\phi_\delta = \phi_\delta(t,T)$ of $\hat{A}_\delta$, $\hat{z}_\delta$, $\hat{v}_\delta$ in such a way that $\hat{A}_\delta + B_\delta$, $\hat{z} + \zeta_\delta$, $\hat{v}_\delta + \phi_\delta$ solve Eqs. (6.13)–(6.15) to the first order and that the boundary conditions

$$B_\delta(t,1) = \zeta_\delta(t,1) = \phi_\delta(t,1) = \phi_\delta(t,0) = 0, \quad t \geq 0, \tag{6.23}$$

are satisfied. This ansatz leads to the equations

$$\begin{aligned} \frac{\partial}{\partial t} B_\delta = {} & \frac{1}{\sqrt{\hat{A}_\delta}} \frac{\partial}{\partial T} B_\delta - \frac{\hat{A}'_\delta}{2\,\hat{A}_\delta^{3/2}}\, B_\delta + \\ & \frac{\hat{v}'_\delta}{\hat{z}'_\delta}\, \hat{A}_\delta \left(\frac{1}{\hat{z}'_\delta} \frac{\partial}{\partial T}\zeta_\delta - \frac{1}{\hat{v}'_\delta} \frac{\partial}{\partial T}\phi_\delta - \frac{1}{\hat{A}_\delta}\, B_\delta \right), \end{aligned} \tag{6.24}$$
$$\frac{\partial}{\partial T} \left(\frac{1}{\hat{z}'_\delta} \frac{\partial}{\partial T}\zeta_\delta - \frac{1}{\hat{v}'_\delta} \frac{\partial}{\partial T}\phi_\delta - \frac{1}{\hat{A}_\delta}\, B_\delta \right) = 0, \tag{6.25}$$
$$\frac{\partial}{\partial t}\zeta_\delta = \frac{1}{\sqrt{\hat{A}_\delta}} \frac{\partial}{\partial T}\zeta_\delta - \frac{\hat{z}'_\delta}{2\,\hat{A}_\delta^{3/2}}\, B_\delta + \phi_\delta. \tag{6.26}$$

Now letting

$$\gamma_\delta \stackrel{\text{def}}{=} \left(\frac{1}{\hat{z}'_\delta} \frac{\partial}{\partial T} \zeta_\delta - \frac{1}{\hat{v}'_\delta} \frac{\partial}{\partial T} \phi_\delta - \frac{1}{\hat{A}_\delta} B_\delta \right), \tag{6.27}$$

we obtain from Eqs. (6.20)–(6.22)

$$\gamma_\delta(t) = -\frac{1}{\delta+2} \left(\int_0^1 \frac{2\,(\delta+1)}{(\delta\,\theta+1)^2} \frac{\partial}{\partial T} \zeta_\delta(t,\theta) + \frac{2\,(\delta+1)^4}{(\delta\,\theta+1)^5} B_\delta(t,\theta)\, d\theta \right), \tag{6.28}$$

hence

$$\begin{aligned} \phi_\delta(t,T) = & \int_0^T \frac{2\,\delta\,(\delta+1)}{(\delta\,\theta+1)^2} \frac{\partial}{\partial T} \zeta_\delta(t,\theta) + \frac{2\,\delta\,(\delta+1)^4}{(\delta\,\theta+1)^5} B_\delta(t,\theta)\, d\,\theta + \\ & \frac{2\,(\delta+1)^3}{\delta+2} \left(\frac{1-(\delta\,T+1)^2}{(\delta\,T+1)^2} \right) \left(\int_0^1 \frac{1}{(\delta\,\theta+1)^2} \frac{\partial}{\partial T} \zeta_\delta(t,\theta) + \right. \\ & \left. \frac{(\delta+1)^3}{(\delta\,\theta+1)^5} B_\delta(t,\theta)\, d\,\theta \right). \end{aligned} \tag{6.29}$$

Inserting (6.28)–(6.29) in Eqs. (6.24)–(6.26) yields

$$\begin{aligned} \frac{\partial}{\partial t} B_\delta(t,T) &= \frac{\delta+1}{\delta\,T+1} \frac{\partial}{\partial T} B_\delta(t,T) - \frac{\delta\,(\delta+1)}{(\delta\,T+1)^2} B_\delta(T,T) - \\ & \quad \frac{4\,\delta}{\delta+2} \left(\int_0^1 \frac{1}{(\delta\,\theta+1)^2} \frac{\partial}{\partial T} \zeta_\delta(t,\theta) + \right. \\ & \quad \left. \frac{(\delta+1)^3}{(\delta\,\theta+1)^5} B_\delta(t,\theta)\, d\,\theta \right), \end{aligned} \tag{6.30}$$

$$\begin{aligned} \frac{\partial}{\partial t} \zeta_\delta(t,T) &= \frac{\delta+1}{\delta\,T+1} \frac{\partial}{\partial T} \zeta_\delta(t,T) + \frac{(\delta+1)^4}{2\,(\delta\,T+1)^4} B_\delta(t,T) + \\ & \quad \frac{2\,(\delta+1)^3}{\delta+2} \left(\frac{1-(\delta\,T+1)^2}{(\delta\,T+1)^2} \right) \left(\int_0^1 \frac{1}{(\delta\,\theta+1)^2} \frac{\partial}{\partial T} \zeta_\delta(t,\theta) + \right. \\ & \quad \left. \frac{(\delta+1)^3}{(\delta\,\theta+1)^5} B_\delta(t,\theta)\, d\,\theta \right) + \\ & \quad \int_0^T \frac{2\,\delta\,(\delta+1)}{(\delta\,\theta+1)^2} \frac{\partial}{\partial T} \zeta_\delta(t,\theta) + \frac{2\,\delta\,(\delta+1)^4}{(\delta\,\theta+1)^5} B_\delta(t,\theta)\, d\,\theta. \end{aligned} \tag{6.31}$$

In addition, the boundary conditions

$$B_\delta(t,1) = \zeta_\delta(t,1) = 0 \tag{6.32}$$

are to hold. For the following discussions, we will need differentiated versions of the linearized equations of melt-spinning, Eqs. (6.30)–(6.32). To this end, we define the differentiated quantities

$$\beta_\delta(t,T) \stackrel{\text{def}}{=} \frac{\delta+1}{\delta\,T+1}\,\frac{\partial}{\partial T}\left(\frac{\delta+1}{\delta\,T+1}\,B_\delta(t,T)\right), \tag{6.33}$$

$$\xi_\delta(t,T) \stackrel{\text{def}}{=} \frac{\partial}{\partial T}\zeta_\delta(t,T) \tag{6.34}$$

and the measures

$$d\mu_\delta(\theta) \stackrel{\text{def}}{=} \left(\delta\,\theta+1-\frac{1}{(\delta\,\theta+1)^2}\right)d\theta, \tag{6.35}$$

$$d\nu_\delta(\theta) \stackrel{\text{def}}{=} \frac{1}{(\delta\,\theta+1)^2}\,d\theta, \tag{6.36}$$

$$d\kappa_\delta(\theta) \stackrel{\text{def}}{=} d\mu_\delta(\theta)+d\nu_\delta(\theta). \tag{6.37}$$

The measures $d\mu_\delta$, $d\nu_\delta$, $d\kappa_\delta$ are positive multiples of Lebesgue measure. The differentiation of Eqs. (6.30)–(6.31) w.r.t. T yields then

$$\begin{aligned}
\frac{\partial}{\partial t}\beta_\delta(t,T) &= \frac{\delta+1}{\delta\,T+1}\,\frac{\partial}{\partial T}\beta_\delta(t,T)+\frac{4\,\delta^2\,(\delta+1)^2}{(\delta+2)\,(\delta\,T+1)^3}\int_0^1 \xi_\delta(t,\theta)\,d\nu_\delta(\theta)- \\
&\quad \frac{4}{3}\,\frac{\delta\,(\delta+1)^3}{(\delta+2)\,(\delta\,T+1)^3}\int_0^1 \beta_\delta(t,\theta)\,d\mu_\delta(\theta),
\end{aligned} \tag{6.38}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\xi_\delta(t,T) &= \frac{\delta+1}{\delta\,T+1}\,\frac{\partial}{\partial T}\xi_\delta(t,T)+\frac{\delta\,(\delta+1)}{(\delta\,T+1)^2}\,\xi_\delta(t,T)+ \\
&\quad \frac{(\delta+1)^2}{2\,(\delta\,T+1)^2}\,\beta_\delta(t,T)-\frac{4\,\delta\,(\delta+1)^3}{(\delta+2)\,(\delta\,T+1)^3}\int_0^1 \xi_\delta\,d\nu_\delta(\theta)+ \\
&\quad \frac{\delta\,(\delta+1)^2}{2\,(\delta\,T+1)^4}\int_1^T \beta_\delta(t,\theta)\,d\kappa_\delta(\theta)+ \\
&\quad \frac{4}{3}\,\frac{(\delta+1)^4}{(\delta+2)\,(\delta\,T+1)^3}\int_0^1 \beta_\delta(t,\theta)\,d\mu_\delta(\theta).
\end{aligned} \tag{6.39}$$

The corresponding boundary conditions read

$$\beta_\delta(t,1) = \frac{4\,\delta}{\delta+2}\int_0^1 \xi_\delta(t,\theta)\,d\nu_\delta(\theta)-\frac{4}{3}\,\frac{\delta+1}{\delta+2}\int_0^1 \beta_\delta(t,\theta)\,d\mu_\delta(\theta), \tag{6.40}$$

$$\xi_\delta(t,1) = 0. \tag{6.41}$$

6.4 Statement of the Main Result

We proceed by expressing the linearized equations of fiber spinning, Eqs. (6.38)–(6.41), in abstract operator form. This approach allows us to discuss the well-definedness of a semigroup through the differential equations. Moreover, we will be able to use well-known results of spectral analysis to describe the spectrum of the semigroup generator.

Using the measures given in (6.35)–(6.36), we define the spaces

$$H \overset{\text{def}}{=} H^1(0,1), \tag{6.42}$$

$$H_0 \overset{\text{def}}{=} \{u \in H \mid u(1) = 0\}, \tag{6.43}$$

$$\mathcal{H}_\delta \overset{\text{def}}{=} \left\{(u,v) \in H \times H_0 \mid u(1) = \frac{4\,\delta}{\delta+2} \int_0^1 v(\theta)\, d\nu_\delta(\theta) - \frac{4}{3}\,\frac{\delta+1}{\delta+2} \int_0^1 u(\theta)\, d\mu_\delta(\theta)\right\}. \tag{6.44}$$

H_0 and $\mathcal{H}_\delta$ inherit their Hilbert space structure from the underlying Sobolev space $H^1(0,1)$.

Now we define the operator

$$\mathcal{K}_\delta \begin{pmatrix} u \\ v \end{pmatrix}(T) \overset{\text{def}}{=} \begin{pmatrix} -\dfrac{4}{3}\,\dfrac{\delta\,(\delta+1)^3}{(\delta+2)\,(\delta\,T+1)^3} \displaystyle\int_0^1 u(\theta)\, d\mu_\delta(\theta) \\[2ex] \dfrac{4}{3}\,\dfrac{(\delta+1)^4}{(\delta+2)\,(\delta\,T+1)^3} \displaystyle\int_0^1 u(\theta)\, d\mu_\delta(\theta) \end{pmatrix} + \begin{pmatrix} 0 \\[2ex] \dfrac{\delta\,(\delta+1)^2}{2\,(\delta\,T+1)^4} \displaystyle\int_1^T u(\theta)\, d\kappa_\delta(\theta) \end{pmatrix} + \begin{pmatrix} \dfrac{4\,\delta^2\,(\delta+1)^2}{(\delta+2)\,(\delta\,T+1)^3} \displaystyle\int_0^1 v(\theta)\, d\nu_\delta(\theta) \\[2ex] -\dfrac{4\,\delta\,(\delta+1)^3}{(\delta+2)\,(\delta\,T+1)^3} \displaystyle\int_0^1 v(\theta)\, d\nu_\delta(\theta) \end{pmatrix} \tag{6.45}$$

$$\text{with domain } D(\mathcal{K}_\delta) \overset{\text{def}}{=} \left(L^2(0,1)\right)^2. \tag{6.46}$$

$\mathcal{K}_\delta$ involves only integral terms, hence it is a compact operator on the space $\left(L^2(0,1)\right)^2$.

Finally, we define the operators

$$
\begin{aligned}
\mathcal{A}^r_\delta \begin{pmatrix} u \\ v \end{pmatrix}(T) &\overset{\text{def}}{=} \begin{pmatrix} \dfrac{\delta+1}{\delta\, T+1}\, u_T(T) \\[2ex] \dfrac{\delta+1}{\delta\, T+1}\, v_T(T) \end{pmatrix} + \\
&\qquad \begin{pmatrix} 0 \\[2ex] \dfrac{\delta\,(\delta+1)}{(\delta\, T+1)^2}\, v(T) + \dfrac{(\delta+1)^2}{2\,(\delta\, T+1)^2}\, u(T) \end{pmatrix} && (6.47)\\
\text{with domain } D(\mathcal{A}^r_\delta) &\overset{\text{def}}{=} \mathcal{H}_\delta, \text{ and} && (6.48)\\
\mathcal{A}_\delta \begin{pmatrix} u \\ v \end{pmatrix}(T) &\overset{\text{def}}{=} \mathcal{A}^r_\delta \begin{pmatrix} u \\ v \end{pmatrix}(T) + \mathcal{K}_\delta \begin{pmatrix} u \\ v \end{pmatrix}(T) && (6.49)\\
\text{with domain } D(\mathcal{A}_\delta) &\overset{\text{def}}{=} \mathcal{H}_\delta. && (6.50)
\end{aligned}
$$

Theorem 6.2 *The operator $\mathcal{A}_\delta$ is the infinitesimal generator of an eventually compact C_0 semigroup $\{\mathcal{T}_\delta(t)\}$ of bounded linear operators on $\left(L^2(0,1)\right)^2$. In particular, $\mathcal{T}_\delta(t)$ is a compact operator for every $t > \frac{\delta+2}{2\,(\delta+1)}$. Moreover, the spectrum of $\mathcal{A}_\delta$ consists of isolated eigenvalues with finite multiplicities.*

Corollary 6.3 *If $\sigma(\mathcal{A}_\delta)$ is the spectrum of the semigroup generator $\mathcal{A}_\delta$ and $\sigma(\mathcal{T}_\delta(t))$ the spectrum of the semigroup operator $\mathcal{T}_\delta(t)$, then*

$$\sigma(\mathcal{T}_\delta(t)) = e^{\sigma(\mathcal{A}_\delta)\, t} \cup \{0\} \quad \textit{for every } t > 0. \tag{6.51}$$

In particular, every nonzero point in $\sigma(\mathcal{T}_\delta(t))$ is an isolated eigenvalue of $\mathcal{T}_\delta(t)$ if $t > 0$.

Corollary 6.4 *The type and spectral bound of the semigroup $\{\mathcal{T}_\delta(t)\}$ and its generator $\mathcal{A}_\delta$ satisfy*

$$\omega(\mathcal{A}_\delta) = s(\mathcal{A}_\delta). \tag{6.52}$$

Remark 6.5

(a) Eventual compactness guarantees the spectral determinacy of the semigroup $\{\mathcal{T}_\delta(t)\}$ (cf. [10] and [21]). Hence Corollaries 6.3 and 6.4 are immediate consequences of Theorem 6.2 .

(b) One might suppose that every generator of an eventually compact semigroup on a Banach space has a discrete spectrum. However, this notion is false (cf. [21, p. 41]).

(c) The semigroup $\{\mathcal{T}_\delta(t)\}$ induces an eventually compact C_0 semigroup on $(H_0)^2$ for Eqs. (6.30)–(6.31) with the boundary conditions (6.32).

Theorem 6.2 will be proven in several steps. In Sections 6.5 and 6.6, we will show that $\mathcal{A}_\delta^r$ generates a semigroup of the kind described in Theorem 6.2. A straightforward argument yields then the sought-for result (see Section 6.7).

6.5 The Related Semigroup

Lemma 6.6 *Let X and Y be Banach spaces such that Y is continuously and densely imbedded in X. Suppose that Λ is a nonzero bounded linear functional on Y. Then the kernel of Λ, $\operatorname{Ker}\Lambda$, is dense in X if and only if Λ cannot be extended continuously to all of X.*

Proof. $\operatorname{Ker}\Lambda$ is of codimension 1 in Y. Hence there exists a nonzero element a in Y such that

$$Y = \langle a \rangle \oplus_Y \operatorname{Ker}\Lambda. \tag{6.53}$$

If $\operatorname{Ker}\Lambda$ is dense in X, the only possible candidate for a bounded linear extension of Λ to all of X is the zero functional, contradicting $\Lambda\, a \neq 0$. If $\operatorname{Ker}\Lambda$ is not dense in X, then the Hahn-Banach theorem yields, by density of Y in X,

$$X = \langle a \rangle \oplus_X \overline{(\operatorname{Ker}\Lambda)}^X. \tag{6.54}$$

The operator Ω, defined by

$$\Omega(x_1 + x_2) \overset{\text{def}}{=} \Lambda x_1 \quad \text{for } x_1 \in \langle a \rangle,\ x_2 \in \overline{(\operatorname{Ker}\Lambda)}^X, \tag{6.55}$$

is a bounded linear extension of Λ to X.

□

Lemma 6.7 *The space $\mathcal{H}_\delta$ is dense in $(L^2(0,1))^2$.*

Proof. Let $X \overset{\text{def}}{=} (L^2(0,1))^2$, $Y \overset{\text{def}}{=} H \times H_0$, and define the functional Λ on Y by

$$\Lambda \begin{pmatrix} u \\ v \end{pmatrix} \overset{\text{def}}{=} \frac{4\,\delta}{\delta+2} \int_0^1 v(\theta)\, d\nu_\delta(\theta) - \frac{4}{3}\, \frac{\delta+1}{\delta+2} \int_0^1 u(\theta)\, d\mu_\delta(\theta) - u(1). \tag{6.56}$$

The desired result follows from Lemma 6.6.

□

Lemma 6.8 *The operator $\mathcal{A}_\delta^r$ is quasidissipative.*

Proof. For given $(u,v)^T \in \mathcal{H}_\delta$, we find by standard convexity arguments that

$$\begin{aligned}
&\int_0^1 \frac{\delta+1}{\delta\,\theta+1}\,(u(\theta)\,u_T(\theta)+v(\theta)\,v_T(\theta))\;d\theta + \\
&\int_0^1 \frac{(\delta+1)^2}{2}\,u(\theta)+\delta\,(\delta+1)\,v(\theta)\,d\nu_\delta(\theta) \\
&\qquad\le \frac{1}{2}\left(\frac{4\,\delta}{\delta+2}\int_0^1 v(\theta)\,d\nu_\delta(\theta)-\frac{4}{3}\,\frac{\delta+1}{\delta+2}\int_0^1 u(\theta)\,d\mu_\delta(\theta)\right)^2 + \\
&\qquad\quad \int_0^1 \frac{\delta\,(\delta+1)}{2}\,u^2(\theta)\,d\nu_\delta(\theta)+\int_0^1 \frac{3\,\delta\,(\delta+1)}{2}\,v^2(\theta)\,d\nu_\delta(\theta) + \\
&\qquad\quad \int_0^1 \frac{(\delta+1)^2}{2}\,u(\theta)\,v(\theta)\,d\nu_\delta(\theta) \\
&\qquad\le \omega_\delta\left(\int_0^1 u^2(\theta)\,d\,\theta+\int_0^1 v^2(\theta)\,d\theta\right)
\end{aligned} \tag{6.57}$$

for some number ω_δ in $\mathbb{R}$.

□

Lemma 6.9 *The spectrum of the operator $\mathcal{A}_\delta^r$ is discrete. In particular, the resolvent operator $R(\lambda;\mathcal{A}_\delta^r)=(\lambda-\mathcal{A}_\delta^r)^{-1}$ is compact for every λ in the resolvent set of $\mathcal{A}_\delta^r$.*

Proof. Let f, g belong to $L^2(0,1)$. Suppose there is a pair $(u,v)^T$ in $D(\mathcal{A}_\delta^r)$ such that

$$\mathcal{A}_\delta^r\begin{pmatrix} u \\ v \end{pmatrix}(T)=\begin{pmatrix} f(T) \\ g(T) \end{pmatrix}. \tag{6.58}$$

Hence it necessarily follows that

$$\begin{aligned}
u(T) &= u(1)+\int_1^T \frac{\delta\,\theta+1}{\delta+1}\,f(\theta)\,d\theta \\
&= \frac{4\,\delta}{\delta+2}\int_0^1 v(\theta)\,d\nu_\delta(\theta)-\frac{4}{3}\,\frac{\delta+1}{\delta+2}\int_0^1 u(\theta)\,d\mu_\delta(\theta) + \\
&\quad \underbrace{\int_1^T \frac{\delta\,\theta+1}{\delta+1}\,f(\theta)\,d\theta}_{F(T)},
\end{aligned} \tag{6.59}$$

$$v(T) = -\int_1^T \frac{\delta}{\delta\,\theta+1}\,v(\theta)\,d\theta - \int_1^T \frac{\delta+1}{2\,(\delta\,\theta+1)}\,u(\theta)\,d\theta + \underbrace{\int_1^T \frac{\delta\,\theta+1}{\delta+1}\,g(\theta)\,d\theta}_{G(T)}. \tag{6.60}$$

Thus $(u,v)^T$ satisfies an equation of the form

$$\begin{pmatrix} u(T) \\ v(T) \end{pmatrix} = \mathcal{C}\begin{pmatrix} u \\ v \end{pmatrix}(T) + \begin{pmatrix} F(T) \\ G(T) \end{pmatrix} \tag{6.61}$$

with the compact operator $\mathcal{C}$ given in Eqs. (6.59)–(6.60). We have supressed the dependence on δ. If we proved that $1 \in \rho(\mathcal{C})$, then we could evidently conclude that $0 \in \rho(\mathcal{A}^r_\delta)$. $\rho(\mathcal{C})$ and $\rho(\mathcal{A}^r_\delta)$ are the resolvent sets of the operators $\mathcal{C}$ and $\mathcal{A}^r_\delta$, resp. Since compactness of $R(0;\mathcal{A}^r_\delta)$ is a consequence of H being compactly imbedded in $L^2(0,1)$, the claim of the lemma would follow from well-known results on operators with compact resolvent (cf. [31, p. 187]). By the Fredholm alternative, $1 \in \rho(\mathcal{C})$ if and only if the null space of $I-\mathcal{C}$, $N(I-\mathcal{C})$, is trivial, i.e. $N(I-\mathcal{C}) = \{0\}$. For $(u,v)^T$ to belong to $N(I-\mathcal{C})$, according to Eq. (6.59), it is necessary that

$$u \equiv \text{const.} \tag{6.62}$$

In this case, we find

$$v(T) = u\,\frac{\delta+1}{2}\,\frac{1-T}{\delta\,T+1}. \tag{6.63}$$

Since, by assumption on $\mathcal{H}_\delta$, we find

$$u \equiv \frac{4\,\delta}{\delta+2}\int_0^1 v(\theta)\,d\nu_\delta(\theta) - u\,\frac{4}{3}\,\frac{\delta+1}{\delta+2}\int_0^1 d\mu_\delta(\theta), \tag{6.64}$$

we obtain the equation

$$u = -\frac{\delta\,(2\,\delta+3)}{3\,(\delta+2)}\,u. \tag{6.65}$$

δ was assumed positive. Hence Eqs. (6.65) and (6.63) yield immediately that

$$u \equiv 0 \equiv v. \tag{6.66}$$

Thus the claim follows.

□

Theorem 6.10 *The operator $\mathcal{A}^r_\delta$ is the infinitesimal generator of a strongly continuous semi-group $\{\mathcal{S}_\delta(t)\}$ of bounded linear operators on $\left(L^2(0,1)\right)^2$.*

Proof. By the preceding lemmas, the Lumer-Philips theorem applies (cf. [42], [54]).

□

6.6 Eventual Compactness of the Related Semigroup

Theorem 6.11 *The semigroup $\{\mathcal{S}_\delta(t)\}$ generated by $\mathcal{A}_\delta^r$ is eventually compact. In particular, $\mathcal{S}_\delta(t_0)$ is a compact operator on $\left(L^2(0,1)\right)^2$ for $t_0 > \frac{\delta+2}{2\,(\delta+1)}$.*

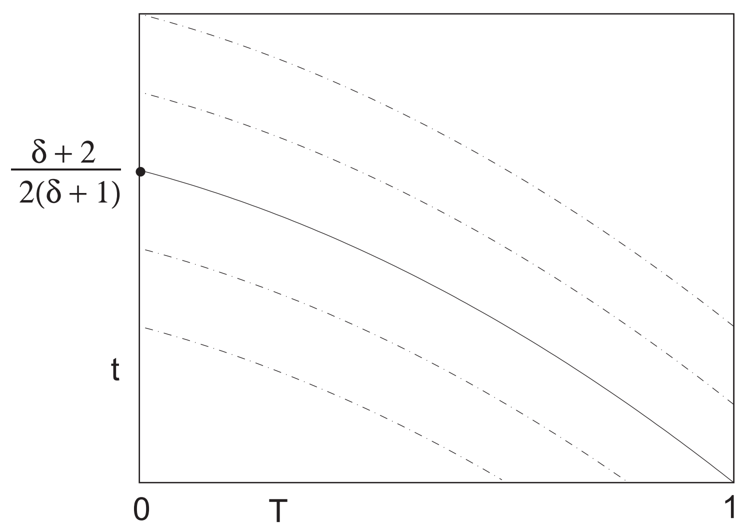

Figure 6.1: The characteristics of the differential equation

Remark 6.12 The following proof is based on the important observation that, after time $t = \frac{\delta+2}{2\,(\delta+1)}$, the boundary values govern the flow. The characteristic corresponding to this particular point in time separates the characteristics that are determined by the initial values from those determined by the boundary values (see Fig. 6.1).

Proof of Theorem 6.11 Suppose $t_0 > \frac{\delta+2}{2\,(\delta+1)}$. It suffices to show that $\mathcal{S}_\delta(t_0)$ is compact on a dense subset of $\left(L^2(0,1)\right)^2$. To this end, we remark that every sequence in $\mathcal{H}_\delta$ that is bounded in $\left(L^2(0,1)\right)^2$ contains a subsequence $\{(u_n, v_n)^T\}$ such that

$$(u_n, v_n)^T \text{ converges weakly in } \left(L^2(0,1)\right)^2. \tag{6.67}$$

Define

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{S}_\delta(\cdot) \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{6.68}$$

We may assume that

$$(U_n, V_n)^T \text{ converges weakly in } L^2\left([0, t_0]; \left(L^2(0,1)\right)^2\right). \tag{6.69}$$

The characteristics corresponding to Eqs. (6.38)–(6.39) are given by

$$(t_\tau(T), T) = (\tau - \frac{\delta\, T^2}{2\,(\delta+1)} - \frac{T}{\delta+1}, T),\ T \in [0,1],\ \tau \in \mathbb{R}. \tag{6.70}$$

By assumption on t_0, the map $T \mapsto \tau(T)$, defined on $[0,1]$ by

$$\tau(T) \stackrel{\text{def}}{=} t_0 + \frac{\delta\, T^2}{2\,(\delta+1)} + \frac{T}{\delta+1}, \tag{6.71}$$

has the following properties:

(a) $t_{\tau(T)}(T) = t_0$ for every $T \in [0,1]$,

(b) $s \mapsto t_{\tau(T)}(s)$ is a diffeomorphism from $[T,1]$ to a subset of $[0,t_0]$ for every $T \in [0,1]$.

By using the boundary conditions, we find, when integrating along the characteristics, that

$$\begin{aligned} U_n(t_0,T) &= U_n(t_{\tau(T)}(T),T) \\ &= U_n(t_{\tau(T)}(1),1) \\ &= \frac{4\,\delta}{\delta+2} \int_0^1 V_n(t_{\tau(T)}(1),\theta)\, d\nu_\delta(\theta) - \qquad (6.72) \\ &\quad \frac{4}{3}\,\frac{\delta+1}{\delta+2} \int_0^1 U_n(t_{\tau(T)}(1),\theta)\, d\mu_\delta(\theta), \\ V_n(t_0,T) &= V_n(t_{\tau(T)}(T),T) \\ &= -\int_1^T \frac{\delta}{\delta\,\theta+1}\, V_n(t_{\tau(T)}(\theta),\theta)\, d\theta - \qquad (6.73) \\ &\quad \int_1^T \frac{\delta+1}{2\,(\delta\,\theta+1)}\, U_n(t_{\tau(T)}(\theta),\theta)\, d\theta. \end{aligned}$$

Assumption (6.67) guarantees that $U_n(t_0,\cdot)$ converges pointwise on $[0,1]$ as $n \to \infty$. Moreover, the right-hand side of Eq. (6.72) obeys a bound, independent of n and T. Hence, by the Lebesgue dominated convergence theorem (cf. [55]), $U_n(t_0,\cdot)$ converges strongly in $L^2(0,1)$ to its pointwise limit. On the other hand, to obtain strong convergence of $V_n(t_0,\cdot)$ in $L^2(0,1)$, we can use a similar argument. We note that functions of the form

$$(t,T) \mapsto \phi(t)\,\psi(T)$$

with $\phi \in C^\infty([0,t_0])$, $\psi \in C^\infty([0,1])$ are dense in $L^2\left((0,t_0)\times(0,1)\right)$. This observation allows us now to estimate the the right-hand side of Eq. (6.73). We find that

$$
\begin{aligned}
&\left|\int_1^T \phi(t_{\tau(T)}(\theta))\, \psi(\theta)\, d\theta\right| \\
&\quad\le \left(\int_T^1 \left(\phi(t_{\tau(T)}(\theta))\right)^2\, d\theta\right)^{\frac{1}{2}} ||\psi||_{L^2(0,1)} \\
&\quad\le \sup_{s\in[0,1]} |(t'_{\tau(T)}(s))^{-1}| \left(-\int_T^1 \left(\phi(t_{\tau(T)}(\theta))\right)^2\, t'_{\tau(T)}(\theta)\, d\theta\right)^{\frac{1}{2}} ||\psi||_{L^2(0,1)} \\
&\quad\le (\delta+1)\left(\int_{t_{\tau(T)}(1)}^{t_0} \phi^2(s)\, ds\right)^{\frac{1}{2}} ||\psi||_{L^2(0,1)} \qquad (6.74) \\
&\quad\le (\delta+1)\, ||\phi\,\psi||_{L^2((0,t_0)\times(0,1))}.
\end{aligned}
$$

Therefore, we can conclude that the right-hand side of Eq. (6.73) is also bounded independent of n and T. Pointwise convergence of $V_n(t_0,\cdot)$ on $[0,1]$ is a consequence of assumption (6.69) and the observation that, for fixed $T\in[0,1]$, the right-hand side of Eq. (6.73) corresponds to a bounded linear functional on $L^2\left([0,t_0];(L^2(0,1))^2\right)$. Hence, again by the Lebesgue dominated convergence theorem, $V_n(t_0,\cdot)$ converges strongly in $L^2(0,1)$.

□

6.7 Proof of the Main Result

Proof of Theorem 6.2 The operators $\mathcal{A}_\delta$ and $\mathcal{A}_\delta^r$ differ only by the bounded operator $\mathcal{K}_\delta$ (see (6.45)). Hence $\mathcal{A}_\delta$ generates a strongly continuous semigroup $\{\mathcal{T}_\delta(t)\}$ of bounded linear operators on $(L^2(0,1))^2$ (cf. [42]). Since, for $x\in\mathcal{H}_\delta$,

$$\mathcal{T}_\delta(t)x = \mathcal{S}_\delta(t)x + \int_0^t \mathcal{S}_\delta(t-s)\,\mathcal{K}_\delta\,\mathcal{T}_\delta(s)x\, ds \qquad (6.75)$$

and since $\mathcal{K}_\delta$ is compact, the first part of Theorem 6.2 is proven. To see that the spectrum of $\mathcal{A}_\delta$ is discrete, we remark that the resolvent set of $\mathcal{A}_\delta$ is nonempty. The result follows therefore from the compactness of the resolvent operator (cf. [31, p. 187]).

□

CHAPTER 7

Numerical Studies

Our results in Chapter 6 enable us to resolve the spectrum of the semigroup $\{\mathcal{T}_\delta(t)\}$ by resolving the spectrum of its generator $\mathcal{A}_\delta$. We have shown that the stability properties of the semigroup are completely determined by the spectrum of $\mathcal{A}_\delta$. In this chapter, we shall compute the eigenvalues of the operator $\mathcal{A}_\delta$, thus giving a positive answer to the question of the stability of melt-spinning in viscous flow.

Several numerical findings have been reported previously (cf. [14], [23], [44], [45], [46], [58]). These results indicate that the unhindered formation of the frost line along the polymer thread has a stabilizing effect on the process in viscous flow. The numerical techniques used, however, did not take advantage of the explicitly stated linear equations of fiber spinning. In addition to determining growth rates numerically, we shall display the location of the eigenvalues in the complex plane for various draw numbers.

7.1 Spectral Collocation

To study the spectrum of the operator $\mathcal{A}_\delta$, we will employ a spectral collocation technique (cf. [7] and [20]). This method is known to yield acceptable results for linear equations on simple domains (cf. [35], [36], [20]). Hence we can expect a reasonable resolution of the eigenvalue distribution for $\mathcal{A}_\delta$. The flow variables will be approximated by their projection onto the $N+1$-dimensional space of Chebyshev polynomials $\{\mathsf{T}_n\}_0^N$ of order 0 through N, given by

$$\mathsf{T}_n(x) \stackrel{\text{def}}{=} \cos\left(n \arccos x\right). \tag{7.1}$$

It seems useful to determine the spectrum of $\mathcal{A}_\delta$ indirectly by using the simpler undifferentiated equations of melt-spinning, Eqs. (6.24)–(6.26), with the boundary conditions (6.23). The quantities B_δ, ϕ_δ, ζ_δ will be approximated by

$$B_\delta^N(T) = \sum_{n=0}^{N} b_n \, \mathsf{T}_n(2T-1), \tag{7.2}$$

$$\phi_\delta^N(T) = \sum_{n=0}^{N} f_n \, \mathsf{T}_n(2T-1), \tag{7.3}$$

$$\zeta_\delta^N(T) = \sum_{n=0}^{N} z_n \, \mathsf{T}_n(2T-1). \tag{7.4}$$

The coefficients b_n, f_n, z_n are unknown. Their dependence on δ has been supressed. Now we define the collocation points

$$x_j \stackrel{\text{def}}{=} \cos\frac{\pi j}{N}, \quad 0 \le j \le N, \tag{7.5}$$

$$T_j \stackrel{\text{def}}{=} \frac{1}{2}(x_j+1), \quad 0 \le j \le N. \tag{7.6}$$

Next we have to find the eigenvalues of the differential operator given by the right-hand sides of the the Eqs. (6.24)–(6.26), subject to the boundary conditions (6.23). When using the approximations (7.2)–(7.4), we obtain the generalized eigenvalue problem in λ

$$0 = \sum_0^N b_n, \tag{7.7}$$

$$0 = \sum_0^N z_n, \tag{7.8}$$

$$0 = \sum_0^N f_n, \tag{7.9}$$

$$0 = \sum_0^N (-1)^n f_n, \tag{7.10}$$

$$\begin{aligned}\lambda \sum_0^N b_n \, \mathsf{T}_n(x_j) = {} & \frac{2(\delta+1)}{\delta T_j + 1} \sum_0^N b_n \, \mathsf{T}_n'(x_j) - \frac{3\delta(\delta+1)}{(\delta T_j+1)^2} \sum_0^N b_n \, \mathsf{T}_n(x_j) - \\ & \frac{4\delta(\delta T_j+1)}{(\delta+1)^2} \sum_0^N z_n \, \mathsf{T}_n'(x_j) + \\ & \frac{2(\delta T_j+1)^3}{(\delta+1)^3} \sum_0^N f_n \, \mathsf{T}_n'(x_j), \quad 1 \le j \le N,\end{aligned} \tag{7.11}$$

$$\begin{aligned}
\lambda \sum_0^N z_n \,\mathsf{T}_n(x_j) &= \frac{2\,(\delta+1)}{\delta\, T_j+1} \sum_0^N z_n \,\mathsf{T}'_n(x_j) + \frac{(\delta+1)^4}{2\,(\delta\, T_j+1)^4} \sum_0^N b_n \,\mathsf{T}_n(x_j) + \\
&\quad \sum_0^N f_n \,\mathsf{T}_n(x_j), \quad 1 \le j \le N, \qquad (7.12) \\
0 &= \frac{2\,\delta\,(\delta+1)^2}{(\delta\, T_j+1)^3} \sum_0^N b_n \,\mathsf{T}_n(x_j) - \frac{2\,(\delta+1)^2}{(\delta\, T_j+1)^2} \sum_0^N b_n \,\mathsf{T}'_n(x_j) - \\
&\quad \frac{2\,\delta}{\delta+1} \sum_0^N z_n \,\mathsf{T}'_n(x_j) - \frac{4\,(\delta\, T_j+1)}{\delta+1} \sum_0^N z_n \,\mathsf{T}''_n(x_j) + \qquad (7.13) \\
&\quad \frac{3\,(\delta\, T_j+1)^2}{(\delta+1)^2} \sum_0^N f_n \,\mathsf{T}'_n(x_j) + \frac{2\,(\delta\, T_j+1)^3}{\delta\,(\delta+1)^2} \sum_0^N f_n \,\mathsf{T}''_n(x_j), \\
&\quad 1 \le j \le N-1.
\end{aligned}$$

We have to determine all numbers λ such that the system (7.7)–(7.13) has a nontrivial solution for b_n, f_n, z_n, $0 \le n \le N$. Eqs. (7.11)–(7.13) correspond to Eqs. (6.24)–(6.26). The conditions given in Eqs. (7.7)–(7.10) take care of the boundary conditions stated in (6.23). This generalized eigenvalue problem can be solved by standard methods such as the QZ-algorithm (cf. [40]). As $N \to \infty$, we expect convergence of the eigenvalues of the finite dimensional system.

7.2 Numerically Determined Growth Rates

Table 7.1: Growth rates for draw numbers $\delta \le 12$

	$N = 50$	$N = 100$	$N = 150$	$N = 200$
$\delta = 1$	-2.24045509	-2.24045509		
$\delta = 2$	-1.95098195	-1.95098195		
$\delta = 3$	-2.01347514	-2.01347514		
$\delta = 4$	-2.12398890	-2.12398890		
$\delta = 5$	-2.14152408	-2.14152408		
$\delta = 10$		-2.14760712	-2.14760666	-2.14760667
$\delta = 12$		-2.15311956	-2.14520568	-2.14520566

Although spectral collocation methods are known for their excellent convergence properties, they also yield large stray eigenvalues. To filter out spurious values, it is useful to have an upper bound on the expected growth rates. In the current situation, it is quite simple to obtain such an estimate since we have shown that the semigroup $\{\mathcal{T}_\delta(t)\}$ is at least a quasicontraction semigroup. A careful calculation in (6.57) and a straightforward estimate of the norm of the operator $\mathcal{K}_\delta$ (see (6.45)) yields the upper bound

$$\omega(\mathcal{A}_\delta) \leq 10\,(\delta+1)^{\frac{7}{2}}. \tag{7.14}$$

Table 7.2: Growth rates for draw numbers $\delta = 15$, $\delta = 20$

	$N = 200$	$N = 250$	$N = 300$	$N = 350$
$\delta = 15$	-2.14174327	-2.14174337	-2.14174326	-2.14174314
$\delta = 20$	-2.18210495	-2.13771839	-2.13716269	-2.13715555

Table 7.1 contains the approximate growth rates for the semigroup corresponding to values of δ between 1 and 12. Up to the draw number 12, the eigenvalues converge very well. For larger values of δ, the number of Chebyshev polynomials and collocation points needed to

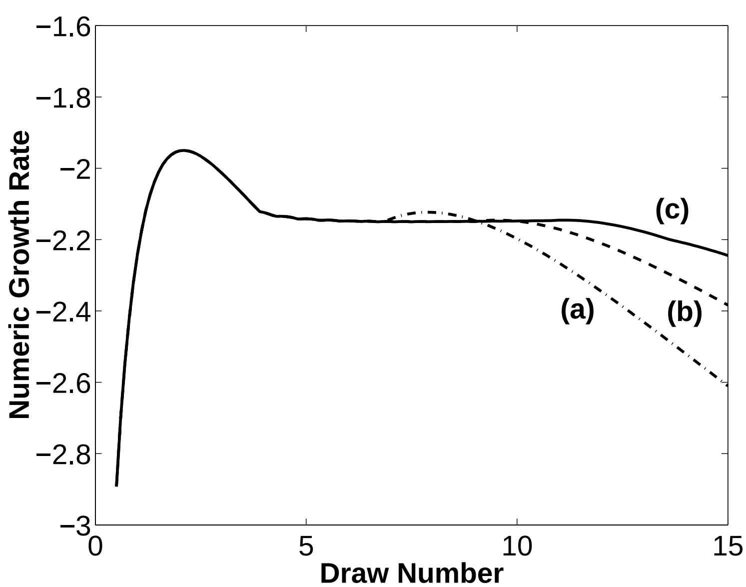

Figure 7.1: Numeric growth rates vs. draw numbers for (a) $N = 50$, (b) $N = 75$, and (c) $N = 100$

approximate the eigenvalues grows considerably (see Table 7.2). Fig. 7.1 confirms this trend: to calculate higher draw numbers, a finer approximation through Chebyshev polynomials is necessary.

Our data suggest that the semigroup of the equations of melt-spinning is asymptotically stable for all practically relevant draw ratios since the real parts of the eigenvalues of the semigroup generator do not exceed -1.

7.3 Numerical Resolution of the Spectrum

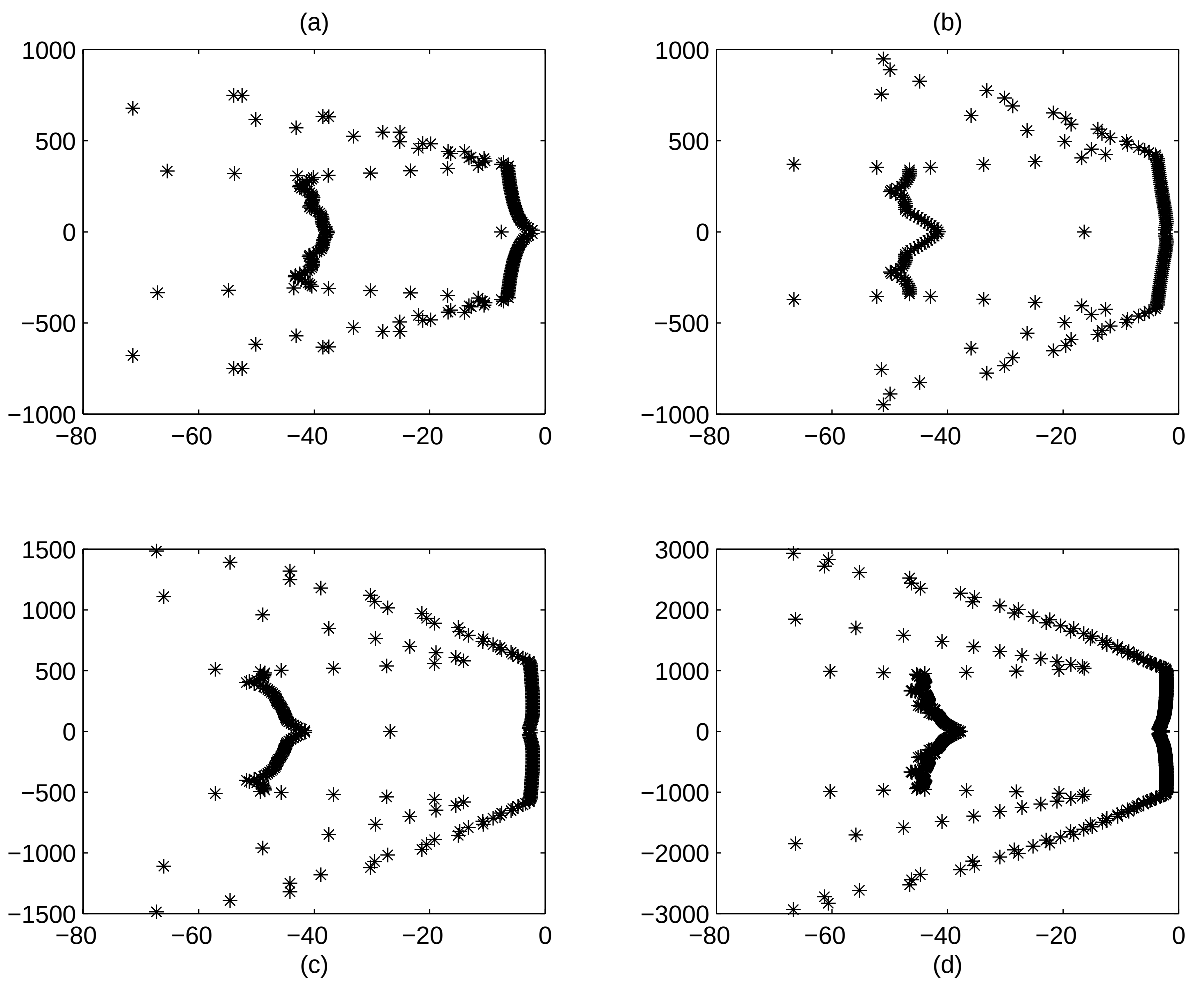

Figure 7.2: Eigenvalue distribution for (a) $\delta = 1$, (b) $\delta = 5$, (c) $\delta = 10$, and (d) $\delta = 20$

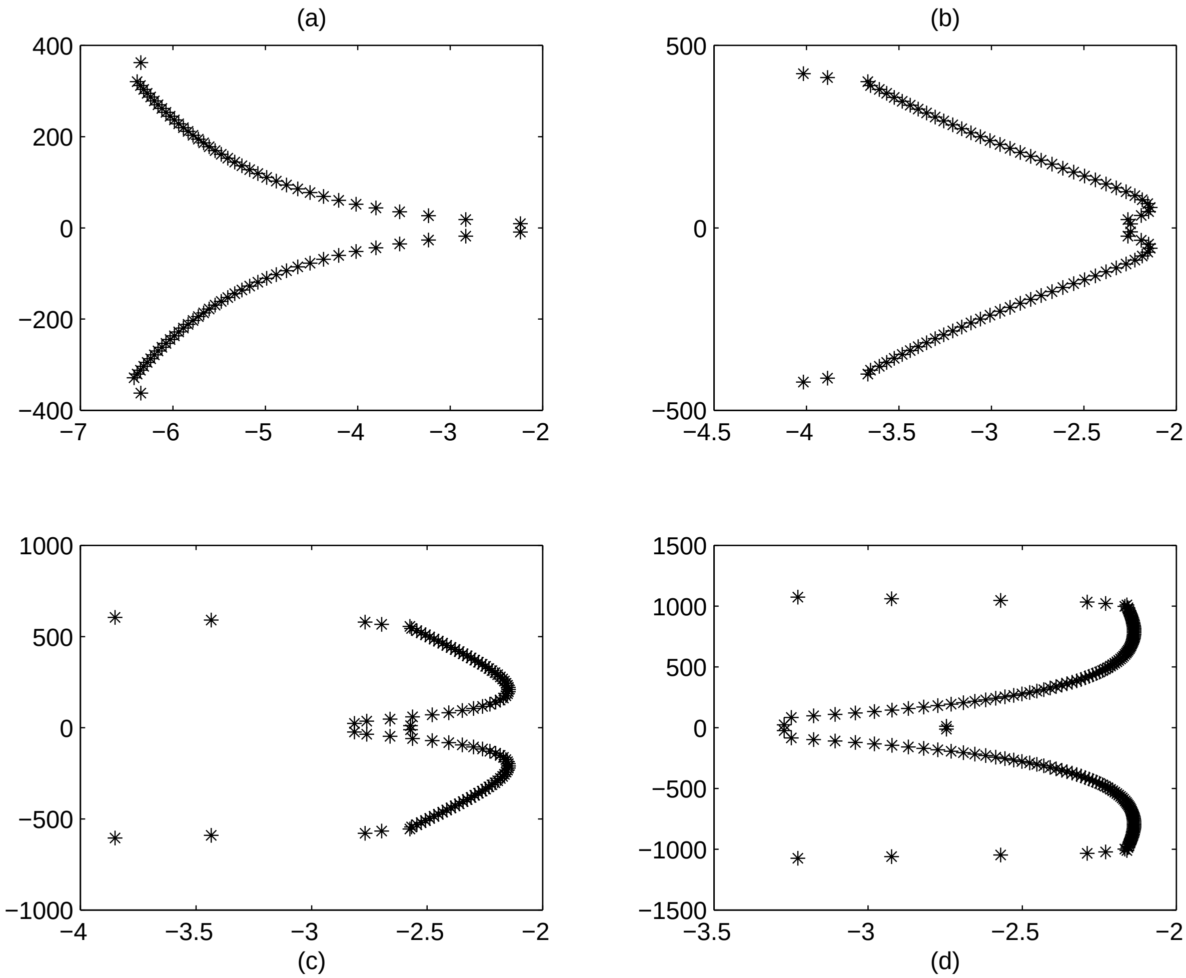

Figure 7.3: Distribution of the eigenvalues closest to the right half-plane for (a) $\delta = 1$, (b) $\delta = 5$, (c) $\delta = 10$, and (d) $\delta = 20$

We close our discussion with some graphic displays of the eigenvalues of the operator $\mathcal{A}_\delta$, corresponding to different draw numbers δ. Fig. 7.2 gives an overview of the eigenvalue distribution in the complex plane. Fig. 7.3 depicts the section closest to the right half-plane.

Figs. 7.2 and 7.3 indicate that the eigenvalues are lined up along certain curves. This observation is quite remarkable. The eigenvalues are the zeroes of an analytic function. However, according to the Weierstraß product theorem (cf. [17]), the zeroes of an analytic function need not appear in a well-ordered fashion. At this time, we do not have an explanation for this phenomenon.

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APPENDIX A

Function Spaces and Norms

In this appendix, we list the basic notations for function spaces and norms that have been used throughout the text.

A.1 Elementary Function Spaces

We denote the set of real numbers by $\mathbb{R}$, the set of natural numbers (positive integers) by $\mathbb{N}$, and the set of nonnegative integers by $\mathbb{N}_0$. For a positive integer k and an arbitrary set M, we denote the set of ordered k-tuples of elements in M by $(M)^k = M^k$. For sets X and Y, we denote the set of ordered pairs (x, y) for elements $x \in X$, $y \in Y$ by $X \times Y$.

Remark A.1

(a) If nothing else is specified, it is tacitly understood that all functions are real-valued.

(b) The restriction of a function ψ on a set X to a set $Y \subset X$ is expressed by $\psi|_Y$.

The following is a list of standard function spaces (cf. [1], [54], [55]). We assume that X is an open or closed, bounded or unbounded rectangle in $\mathbb{R}^n$, $n \in \mathbb{N}$.

1. $C(X)$ is the set of continuous functions ψ on X. For compact X, $C(X)$ is a Banach space when endowed with the norm

$$||\psi||_{C(X)} \stackrel{\text{def}}{=} \sup_{\mathbf{x} \in X} |\psi(\mathbf{x})| \stackrel{\text{def}}{=} ||\psi||_\infty. \tag{A.1}$$

2. $C^m(X)$, $m \in \mathbb{N}$, is the set of m-times continuously differentiable functions ψ on X. For compact X, $C^m(X)$ is a Banach space when endowed with the norm

$$||\psi||_{C^m(X)} \stackrel{\text{def}}{=} \max_{|\alpha| \leq m} ||D^\alpha \psi||_\infty. \tag{A.2}$$

3. $C^\infty(X)$ is the set of continuous functions on X with continuous derivatives of all orders.

4. $L^p(X)$, $1 \leq p < \infty$, is the Lebesgue space of measurable functions ψ on X such that $\int_X |\psi(\mathbf{x})|^p \, d\mathbf{x} < \infty$. $L^p(X)$ is a Banach space when endowed with the norm

$$||\psi||_{L^p(X)} \stackrel{\text{def}}{=} \left(\int_X |\psi(\mathbf{x})|^p \, d\mathbf{x} \right)^{\frac{1}{p}} \stackrel{\text{def}}{=} ||\psi||_p. \tag{A.3}$$

In the special case $p = 2$, the Lebesgue space $L^2(X)$ is a Hilbert space when endowed with the inner product

$$\langle \phi, \psi \rangle_{L^2(X)} \stackrel{\text{def}}{=} \int_X \phi(\mathbf{x}) \, \psi(\mathbf{x}) \, d\mathbf{x} \stackrel{\text{def}}{=} \langle \phi, \psi \rangle_2 \quad (\phi, \psi \in L^2(X)). \tag{A.4}$$

5. $W^{m,p}(X)$, $m \in \mathbb{N}_0$, $1 \leq p < \infty$, is the Sobolev space of measurable functions ψ on X such that ψ and its distributional derivatives $D^\alpha \psi$, $|\alpha| \leq m$, belong to $L^p(X)$. $W^{m,p}(X)$ is a Banach space when endowed with the norm

$$||\psi||_{W^{m,p}(X)} \stackrel{\text{def}}{=} \left(\sum_{|\alpha| \leq m} ||D^\alpha \psi||_p^p \right)^{\frac{1}{p}}. \tag{A.5}$$

$W^{0,p}(X)$ and $L^p(X)$ are identical. In the special case $p = 2$, we set

$$H^m(X) \stackrel{\text{def}}{=} W^{m,2}(X). \tag{A.6}$$

$H^m(X)$ is a Hilbert space when endowed with the inner product

$$\langle \phi, \psi \rangle_{H^m(X)} \stackrel{\text{def}}{=} \sum_{|\alpha| \leq m} \langle D^\alpha \phi, D^\alpha \psi \rangle_2 \quad (\phi, \psi \in H^m(X)). \tag{A.7}$$

6. $L^\infty(X)$ is the Lebesgue space of essentially bounded measurable functions ψ on X. $L^\infty(X)$ is a Banach space when endowed with the norm

$$||\psi||_{L^\infty(X)} \stackrel{\text{def}}{=} \text{ess sup}_{\mathbf{x} \in X} |\psi(\mathbf{x})| \stackrel{\text{def}}{=} ||\psi||_\infty. \tag{A.8}$$

7. $W^{m,\infty}(X)$, $m \in \mathbb{N}_0$, is the Sobolev space of measurable functions ψ on X such that ψ and its distributional derivatives $D^\alpha \psi$, $|\alpha| \leq m$, belong to $L^\infty(X)$. $W^{m,\infty}(X)$ is a Banach space when endowed with the norm

$$||\psi||_{W^{m,\infty}(X)} \stackrel{\text{def}}{=} \max_{|\alpha| \leq m} ||D^\alpha \psi||_\infty. \tag{A.9}$$

$W^{0,\infty}(X)$ and $L^\infty(X)$ are identical.

Remark A.2 If X is an open real interval (a,b) $(-\infty \leq a < b \leq \infty)$, we will write $C(a,b)$, $C^k(a,b)$, and $C^\infty(a,b)$. If X is any real interval with endpoints a and b $(-\infty \leq a < b \leq \infty)$, we will write $L^p(a,b)$, $L^\infty(a,b)$, $W^{m,p}(a,b)$, $W^{m,\infty}(a,b)$, and $H^m(a,b)$.

A.2 Generalized Function Spaces

Let I be an interval in $\mathbb{R}$, possibly all of $\mathbb{R}$, and let X be a real Banach space.

1. $C(I;X)$ is the set of continuous functions ψ on I with values in X. For a compact interval I, $C(I;X)$ is a Banach space when endowed with the norm

$$||\psi||_{C(I;X)} \stackrel{\text{def}}{=} \sup_{t \in I} ||\psi(t)||_X. \tag{A.10}$$

2. $C^m(I;X)$, $m \in \mathbb{N}$, is the set of m-times continuously differentiable functions ψ on I with values in X. For a compact interval I, $C^m(I;X)$ is a Banach space when endowed with the norm

$$||\psi||_{C^m(I;X)} \stackrel{\text{def}}{=} \max_{0 \leq k \leq m} ||D^k \psi||_{C(I;X)}. \tag{A.11}$$

3. $C^\infty(I;X)$ is the set of continuous functions on I with continuous derivatives of all orders, assuming values in X.

4. $L^p(I;X)$, $1 \leq p < \infty$, is the Lebesgue space of measurable functions ψ on I with values in X such that $\int_I ||\psi(t)||_X^p \, dt < \infty$. $L^p(I;X)$ is a Banach space when endowed with the norm

$$||\psi||_{L^p(I;X)} \stackrel{\text{def}}{=} \left(\int_I ||\psi(t)||_X^p \, dt \right)^{\frac{1}{p}}. \tag{A.12}$$

If X is a Hilbert space, the Lebesgue space $L^2(I;X)$ is also a Hilbert space when endowed with the inner product

$$\langle \phi, \psi \rangle_{L^2(I;X)} \stackrel{\text{def}}{=} \int_I \langle \phi(t), \psi(t) \rangle_X \, dt \quad (\phi, \psi \in L^2(I;X)). \tag{A.13}$$

5. $W^{m,p}(I;X)$, $m \in \mathbb{N}_0$, $1 \le p < \infty$, is the Sobolev space of measurable functions ψ on X with values in X such that ψ and its distributional derivatives $D^k \psi$, $0 \le k \le m$, belong to $L^p(I;X)$. $W^{m,p}(I;X)$ is a Banach space when endowed with the norm

$$||\psi||_{W^{m,p}(I;X)} \stackrel{\text{def}}{=} \left(\sum_{k=0}^{m} ||D^k \psi||^p_{L^p(I;X)} \right)^{\frac{1}{p}}. \tag{A.14}$$

$W^{0,p}(I;X)$ and $L^p(I;X)$ are identical. If X is a Hilbert space, $W^{m,2}(I;X)$ is also a Hilbert space when endowed with the inner product

$$\langle \phi, \psi \rangle_{W^{m,2}(I;X)} \stackrel{\text{def}}{=} \sum_{k=0}^{m} \langle D^k \phi, D^k \psi \rangle_{L^2(I;X)} \quad (\phi, \psi \in W^{m.2}(I;X)). \tag{A.15}$$

In this case, we set

$$H^m(I;X) \stackrel{\text{def}}{=} W^{m,2}(I;X). \tag{A.16}$$

6. $L^\infty(I;X)$ is the Lebesgue space of essentially bounded measurable functions ψ on I, assuming values in X. $L^\infty(I;X)$ is a Banach space when endowed with the norm

$$||\psi||_{L^\infty(I;X)} \stackrel{\text{def}}{=} \text{ess sup}_{t \in I} ||\psi(t)||_X. \tag{A.17}$$

7. $W^{m,\infty}(I;X)$, $m \in \mathbb{N}_0$, is the Sobolev space of measurable functions ψ on I with values in X such that ψ and its distributional derivatives $D^k \psi$, $0 \le k \le m$, belong to $L^\infty(I;X)$. $W^{m,\infty}(I;X)$ is a Banach space when endowed with the norm

$$||\psi||_{W^{m,\infty}(I;X)} \stackrel{\text{def}}{=} \max_{0 \le k \le m} ||D^k \psi||_{L^\infty(I;X)}. \tag{A.18}$$

$W^{0,\infty}(I;X)$ and $L^\infty(I;X)$ are identical.

Remark A.3 We shall occasionally interpret a function ψ of the form $\psi = \psi(t,x)$ as a function of t that takes values in a function space X of functions in the variable x. Therefore, we may write $\psi(t)$ in lieu of $\psi(t,\cdot)$ at our convenience.

APPENDIX B

Fluid Models

We list several important constitutive equations, relating the rate-of-strain tensor $\dot{\gamma}$ to the extra stress tensor $\bar{\tau}$. The stress tensor is defined up to an indeterminate isotropic pressure term. Due to a balance of angular momentum, it is a symmetric tensor. The tensor $\mathbf{D}$ refers to the gradient of the velocity field $\mathbf{v}$:

$$\mathbf{D} = \nabla \mathbf{v}. \tag{B.1}$$

The rate-of-strain tensor is defined by

$$\dot{\gamma} \stackrel{\text{def}}{=} \frac{1}{2} \left(\mathbf{D} + \mathbf{D}^T \right). \tag{B.2}$$

Throughout, we shall loosely follow [2], [6], and [33].

B.1 The Newtonian Fluid

The constitutive equation describing purely viscous behavior is Stokes' law of viscosity:

$$\bar{\tau} = 2\,\mu\,\dot{\gamma}. \tag{B.3}$$

μ is the fluid viscosity, depending on whether the flow is a shear flow or a shearfree flow. For Newtonian fluids, there exists a simple relationship between the shear viscosity η and the tensile viscosity η^*:

$$\eta^* = 3\,\eta. \tag{B.4}$$

B.2 Non-Newtonian Fluids

Non-Newtonian fluids are used to model the response of polymeric liquids in shear and shearfree flows. In general, it is found that polymers have elastic properties, resulting in flow phenomena quite different from the viscous case. In addition to the viscosity μ, at least one more parameter, the relaxation time λ, is needed to model time effects (stress relaxation, recoil). Viscoelastic fluids therefore lead to the use of a dimensionless number, measuring the balance between elastic and viscous stresses. This number is referred to as the Weissenberg number (or Deborah number):

$$W \stackrel{\text{def}}{=} \frac{\text{relaxation time}}{\text{characteristic time scale of the flow}}. \tag{B.5}$$

Unfortunately, there is not one single constitutive equation relating stresses and strains as in the purely viscous case. Usually, the particular relation is determined by the polymer in use.

Due to their simplicity, we shall restrict ourselves to quasilinear differential models. μ will always be chosen as the zero-shear-rate viscosity. The shear and tensile viscosities are then implicitly determined by the models.

Remark B.1 Linear viscoelastic fluid models fail to predict important viscoelastic phenomena such as die swell. Moreover, linear viscoelasticity violates the principle of material objectivity (cf. [24]). To remedy this defect, one is forced to include nonlinear terms. This observation leads to the use of convected derivatives.

For the constitutive equations, we shall need the material derivative $\frac{D}{Dt}$, defined by

$$\frac{D}{Dt} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \,. \tag{B.6}$$

The upper convected Maxwell fluid (UCM fluid) is the simplest viscoelastic fluid that obeys the principle of material objectivity. The stress-strain relation is given by

$$\bar{\tau} + \lambda \left(\frac{D\,\bar{\tau}}{Dt} - \mathbf{D} \cdot \bar{\tau} - \bar{\tau} \cdot \mathbf{D}^T \right) = 2\,\mu\,\dot{\gamma}. \tag{B.7}$$

The upper convected Maxwell model predicts unlimited stress growth at high strain rates. Models that do not share this defect are the Giesekus model and the Phan-Thien–Tanner model. If the strain rates stay reasonably small, however, the upper convected Maxwell model can be employed to model the elongation of a polymeric fluid.

The stress-strain relation of the Giesekus fluid is given by

$$\bar{\tau} + \lambda \left(\frac{D\,\bar{\tau}}{Dt} - \mathbf{D} \cdot \bar{\tau} - \bar{\tau} \cdot \mathbf{D}^T \right) + \kappa\,\bar{\tau}^2 = 2\,\mu\,\dot{\gamma}. \tag{B.8}$$

κ is an additional positive parameter associated with anisotropic hydrodynamic friction (cf. [18]). The resultant tensile viscosity stays bounded and approaches a constant value at large rates of extension.

The Phan-Thien–Tanner fluid (PTT fluid) is similar to the Giesekus fluid. The constitutive equation reads

$$\bar{\tau} + \lambda \left(\frac{D\,\bar{\tau}}{Dt} - \mathbf{D} \cdot \bar{\tau} - \bar{\tau} \cdot \mathbf{D}^T \right) + \kappa\,\bar{\tau}\,\mathrm{tr}\,\bar{\tau} = 2\,\mu\,\dot{\gamma}. \tag{B.9}$$

κ is a positive model parameter (cf. [48]). Like the Giesekus model, the Phan-Thien–Tanner model differs from the upper convected Maxwell fluid only by a term quadratic in the stresses. The properties of the Phan-Thien–Tanner fluid resemble the properties of the Giesekus fluid in extensional flow. The tensile viscosity reaches a plateau at high strain rates. There are differences between the Phan-Thien–Tanner fluid and the Giesekus fluid in shear flow, though.

Remark B.2 The viscoelastic fluid models yield nonzero normal stress differences. These stress differences are responsible for a variety of non-Newtonian flow phenomena, in particular the pronounced die swell effect and the Weissenberg effect (cf. [2, pp. 264–265]).

Vita

Thomas Christoph Hagen was born on August 14, 1972 in Ingolstadt, Germany. Thomas studied Mathematics and Computer Science at the Technical University of Munich from November 1991 to July 1994 before transferring to the University of Texas at Dallas. In September 1995, he received his M.S. degree in Applied Mathematics at the University of Texas at Dallas. After a brief return to the Technical University of Munich, Thomas began to work towards a Ph.D. in Mathematics at the Virginia Polytechnic Institute and State University in January 1996. He was awarded his doctorate in December 1998.

Thomas held a Bavarian state scholarship from 1991 to 1996. He was named a William J. Fulbright scholar for the academic year 1994–1995. During his four years of graduate studies, Thomas gave various conference presentations and invited lectures. He has also published several research articles in professional journals.