Chapter 4

Forced Motions of a Point-Mass Breakwater

4.1 Introduction

Now that a simple analysis has been performed and checked, a more complicated and realistic analysis will be conducted. For this problem the same formulation and geometry are used as in the first problem investigated, with the breakwater still modeled as a point mass. However, in this problem the point-mass breakwater (PMBW) is subjected to horizontal and vertical sinusoidal forcing used to model natural wave forcing (Fig. 4.1). This will cause the PMBW to respond more like an object that is actually subjected to the natural conditions of being in the ocean.

4.2 Wave Forcing

To model the horizontal and vertical sinusoidal forcing, consider the forces $F_x$ acting in the $x$ direction and $F_y$ acting in the $y$ direction in the $x$-$y$ plane. These forces are acting on the PMBW as seen in Fig. 4.1. The forces shall be harmonic to simulate the periodic nature of wave particle motions in the ocean.

Fig. 4.1. Breakwater with Horizontal and Vertical Forces
4.2.1 Forcing Equations

The mathematical equations used to simulate the wave motions are as follows:

\[ F_x = F_0 \cos(\omega (T - T_x)) \] (4.1)
\[ F_y = \nu F_0 \cos(\omega (T - T_y)) \] (4.2)

where

- \( T \) = independent variable, time
- \( F_0 \) = amplitude of forcing
- \( T_x = 0 \)
- \( \nu \) = amplitude reduction factor for \( F_y \)
- \( \omega = \frac{\pi}{2T_{xy}} \)
- \( \omega \) = forcing frequency

Notice that the only difference between the forcing in the x direction and the forcing in the y direction is the amplitude reduction factor, \( \nu \), and a phase angle, \( T_y \). In nondimensional form,

\[ x = \frac{X}{S}, \; y = \frac{Y}{S}, \; t = T \sqrt{\frac{g}{S}}, \; \Omega = \omega \sqrt{\frac{S}{g}}, \; f_x = \frac{F_x}{(mg)}, \; f_y = \frac{F_y}{(mg)}, \; f_0 = \frac{F_0}{(mg)} \] (4.9)

Equations 4.1 and 4.2 are plotted in nondimensional form against time in Figs. 4.2 and 4.3, where \( \nu = 0.5 \). When Equations 4.1 and 4.2 are combined, they form an ellipse (Fig. 4.4). This elliptical forcing is a mathematical model of natural wave forcing. These forces are simply added into the ODE’s (Equations 2.14 and 2.15) in the formulation in first-order form to be analyzed. The EOM’s for this problem are

\[ m \frac{d^2X}{dT^2} = F_x \] (4.10)
\[ m \frac{d^2Y}{dT^2} = -mg + F_y \] (4.11)

Summarizing, the force in the x direction is:

1. \( F_x \), forcing in x direction by wave action

Forces in the y direction are:

1. \( g \), forcing in the y direction by gravity
2. \( F_y \), forcing in the y direction by wave action
Fig. 4.2. Horizontal Sinusoidal Forcing

Fig. 4.3. Vertical Sinusoidal Forcing

Fig. 4.4. Combined Elliptical Forcing
Nondimensionalizing the forcing equation gives the final form

\[
\frac{d^2x}{dt^2} = f_x \quad (4.12)
\]

\[
\frac{d^2y}{dt^2} = -1 + f_y \quad (4.13)
\]

with

\[f_x = f_o \cos[\Omega(t-t_x)] \quad [t_x = 0] \quad (4.14, 4.15)\]

\[f_y = \nu f_o \cos[\Omega(t-t_y)] \quad [t_y = \pi/(2\Omega)] \quad (4.16, 4.17)\]

Therefore

\[x = f_o \cos[\Omega(t-t_x)] \quad (4.18)\]

\[y = -1 + \nu f_o \cos[\Omega(t-t_y)] \quad (4.19)\]

### 4.2.2 Analytical Solution

Now that the forcing functions have been formulated, the extra forces may be added into the analytical solution by integrating Equations 4.12 and 4.13 to obtain the x and y position and velocity equations similar to those obtained in the free vibration formulation. In this problem only a few modifications were applied to the previous method of solution. A numerical solution was developed for this case; however, it was found that an analytical solution may be determined for solving the forced case. It was felt that using an analytical solution would give better results than an approximate numerical solution. Thus, the solution procedure was modified to use the following analytical solution for the forced problem:

\[x = c_1 + c_2 t - \frac{f_o}{\Omega^2} \cos[\Omega(t-t_x)] \quad (4.20)\]

\[x = c_2 + \frac{f_o}{\Omega} \sin[\Omega(t-t_x)] \quad (4.21)\]

\[y = c_3 + c_4 t - \frac{1}{2} t^2 - \frac{\nu f_o}{\Omega^2} \cos[\Omega(t-t_y)] \quad (4.22)\]
\[
\dot{y} = c_4 - t + \frac{\nu_0}{\Omega} \sin[\Omega(t-t_y)]
\]  \hspace{1cm} (4.23)

where

\[
c_1 = x_i - c_2 t_i + \frac{f_0}{\Omega^2} \cos[\Omega(t_i-t_x)]
\]  \hspace{1cm} (4.24)

\[
c_2 = x_i - \frac{f_0}{\Omega} \sin[\Omega(t_i-t_x)]
\]  \hspace{1cm} (4.25)

\[
c_3 = y_i - c_4 t_i + \frac{1}{2} t_i^2 + \frac{\nu f_0}{\Omega^2} \cos[\Omega(t_i-t_y)]
\]  \hspace{1cm} (4.26)

\[
c_4 = y_i + t_i - \frac{\nu_0}{\Omega} \sin[\Omega(t_i-t_y)]
\]  \hspace{1cm} (4.27)

4.3 Analyzed Cases

These solutions were used and several different cases were analyzed to investigate how the varying of different parameters affected the responses of the breakwater. For this problem, the only additional parameters added to this solution are the forcing amplitude, \(f_0\), the amplitude reduction factor, \(\nu\), and the forcing frequency, \(\Omega\). The parameters in this problem were varied to see any characteristic responses and the parameters and initial conditions considered in this problem are summarized in Table 4.1 where \(\dot{x}_0 = x(0)\) and \(\dot{y}_0 = y(0)\). This table shows the case number, which was used to identify each case analyzed. This number is a series of numbers taken from the parameters and initial conditions. In this problem, the case number uses \(r, e, x(0), y(0), \dot{x}(0), \dot{y}(0) - f_0, \nu, \Omega\) in this order to identify the case. For example, the standard case has a case number of 190010-559. The standard case will be explained in detail later and is denoted by the shaded cells in Table 4.1. Further, the parameters and initial conditions shown in this table are grouped into series. These groups were used to see how variation of only one parameter would affect the breakwater’s response.
Table 4.1. Parameters and Initial Conditions for Forced Motions of a Point-Mass Breakwater

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<th>y</th>
<th>v_y</th>
<th>f_o</th>
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4.3.1 Standard Case

A standard case was developed for this problem as a starting point for the varying of parameters. The standard parameters and initial conditions for this PMBW under forced motions are:

\[
\begin{align*}
    r &= 1.5 \\
    e &= 0.9 \\
    f_o &= 0.5 \\
    x(0) &= 0.0 \\
    x(0) &= 0.0 \\
    v &= 0.5 \\
    y(0) &= 0.1 \\
    y(0) &= 0.0 \\
    \Omega &= 0.9
\end{align*}
\]

For the forced PMBW problem, the cases studied have initial x and y positions near the bottom of the region (i.e., near the highest point of a floating breakwater). These cases are unlike the free vibration cases where the PMBW is released from the top of the region. This difference in position is due to the fact that the forced case has forces acting in several directions on the mass and it may move more freely than in the free motion case. The free motion case only has gravity acting down on the PMBW, hence the mass is released from a higher position to gain more energy (potential to kinetic) from falling a farther distance in order to cause the PMBW to go through a longer period of motion. Further, waves possess more energy near the surface of a body of water, thus to effectively attenuate wave energy, a breakwater would have to be near the surface of the water where its mooring cables are stretched to their extreme length. Thus, in practical usage a breakwater would not be near the ocean floor; it would be near the surface so it might effectively break up the waves passing by. So, it is better to model the breakwater at a point near the bottom of the region being investigated because this more accurately represents the location of an actual breakwater.

4.4 Analysis of Data

Cases were analyzed and data was collected to see what types of behavior the PMBW exhibited under forced motions. Again, several types of graphs were used to evaluate the responses of the PMBW.
4.4.1 Observations

There does not seem to be any noticeable characteristic motion changes between the two radii investigated. The only major difference is that the 1.5 case has a sharper point at the region’s bottom than the 2.5 case, as seen in Fig. 4.5. Further, as seen in Fig. 4.5, the nondimensional area of the region with a radius of 1.5 is 0.774 and the nondimensional area of the region with a radius of 2.5 is 4.954, which is more than six times larger than that of the r=1.5 case. With a larger region and shallower boundary angles, the r=2.5 case permits larger motions than that of the r=1.5 case (Figs. 4.6 and 4.7). Note that these figures are not drawn to scale as in Fig. 4.5. Though the PMBW moves around the larger region and exhibits a larger range of movements, the r=2.5 region is not attributed to any new motion phenomena. Thus, the radius r=1.5 will be the primary focus of this investigation.

As seen in Figs. 4.8-4.12, as the coefficient of restitution, e, was varied from 1.0 to 0.95 to 0.9 to 0.8 to 0.7, the motions of the PMBW became less active as more energy was dissipated through the impacts occurring as the cables became taut. It must be noted that in Figs. 4.9-4.12 and subsequent trajectory plots, the lower portion of the region (0≤y≤0.1) has been focused in on to see the motions of the breakwater in more detail. This behavior is further illustrated in Figs. 4.13-4.17 where the corresponding $v_n$ vs. t plots show that the energy tends to decrease if e<1. Note that Fig. 4.13 and Figs. 4.14-4.17 are not to the same time scale. The chaotic nature of the case where e=1.0 may be attributed to the forcing adding energy to the system while gravity tries to draw the PMBW down. As e is decreased, more and more energy is lost from the system, causing the PMBW to move around the region less and in turn produces less snap loading. As time goes on, more energy is dissipated and gravity starts to control the motions of the breakwater and draws it down to the equilibrium state at the bottom of the region. Thus, the lower the e value, the less time the breakwater will move around the region before it settles to the bottom of the region because of the influence of gravity.
Fig. 4.5. Comparison of $r=1.5$ and $r=2.5$ Areas

$A_{1.5} = 0.774$

$A_{2.5} = 4.954$

Fig. 4.6. $y$ vs. $x$, Case 190010-559

(a.) $0 \leq y \leq h_{1.5} = 1.118$

(b.) $0 \leq y \leq 0.1$

Fig. 4.7. $y$ vs. $x$, Case 290010-559

$0 \leq y \leq h_{2.5} = 2.291$
Fig. 4.8. $y$ vs. $x$, Case 110010-559 ($e=1.0$)

Fig. 4.9. $y$ vs. $x$, Case 1950010-559 ($e=0.95$)

Fig. 4.10. $y$ vs. $x$, Case 190010-559 ($e=0.9$)

Fig. 4.11. $y$ vs. $x$, Case 180010-559 ($e=0.8$)

Fig. 4.12. $y$ vs. $x$, Case 170010-559 ($e=0.7$)
Fig. 4.13. $v_n$ vs. $t$, Case 110010-559 (e=1.0)

Fig. 4.14. $v_n$ vs. $t$, Case 1950010-559 (e=0.95)

Fig. 4.15. $v_n$ vs. $t$, Case 190010-559 (e=0.9)

Fig. 4.16. $v_n$ vs. $t$, Case 180010-559 (e=0.8)

Fig. 4.17. $v_n$ vs. $t$, Case 170010-559 (e=0.7)
The forcing parameter $\Omega$ was varied to see its effect on the response of the PMBW under forced motions. Figures 4.18-4.22 show the trajectory plots of the data obtained from the cases which were run with $\Omega$ varying from 0.5 to 0.75 to 0.9 to 1.5 to 2.0. Characteristic features do not noticeably appear from these plots; however, Figs. 4.23-4.27 show some interesting phenomena in the $v_n^-$ vs. time plots. One interesting feature is seen in Fig. 4.23 around the time $t=8$. This is where three parallel lines of o’s stick out. By looking at the trajectory and impact data, it can be deduced that this is caused by the PMBW bouncing down the $g_1=0$ boundary and then striking the $g_2=0$ boundary at the lowest point in the region with a large normal velocity. Then the PMBW rebounds and bounces down the $g_1=0$ boundary and again strikes the $g_2=0$ boundary, and the process repeats until the PMBW settles to the bottom of the region due to the effect of gravity.

As seen in the plots of various normal velocity just before impact versus time ($v_n^-$ vs. t) throughout this chapter, a pattern to the x’s and o’s may be noticed which is different than the patterns seen in the PMBW free motion chapter. Instead of having the normal velocities consistently higher on one boundary than the other, as seen in Fig. 3.2, in this problem they seem to have two types of patterns. The first pattern shows the boundary normal velocities converging and then diverging, which is referred to as a serpentine pattern and is seen in Figs. 4.23 and 4.26. The other pattern is one where the boundary normal velocities intertwine and seem to braid in and out of each other. This may be seen in Figs. 4.24, 4.25, and 4.27. These patterns are most likely due to the elliptical forcing in this problem which is continually changing direction. For example, say the PMBW is headed towards the right boundary and the forcing is acting in the same direction; then the normal velocity at impact will be high. Then after the rebound the PMBW is headed towards the left boundary but the forcing is acting in the opposite direction; then the normal velocity will be low when it strikes the left boundary, showing up on the graph as a separation in the braids. If both boundaries are struck in succession with similar conditions and the forcing has switched direction, the normal velocities may have the same value, indicated on the $v_n^-$ vs. t graph as a point of convergence or overlapping.
Fig. 4.18. y vs. x, Case 190010-555 ($\Omega = 0.5$)

Fig. 4.19. y vs. x, Case 190010-5575 ($\Omega = 0.75$)  
Fig. 4.20. y vs. x, Case 190010-559 ($\Omega = 0.9$)

Fig. 4.21. y vs. x, Case 190010-5515 ($\Omega = 1.5$)  
Fig. 4.22. y vs. x, Case 190010-5520 ($\Omega = 2.0$)
Fig. 4.23. $v_n^-$ vs. $t$, Case 190010-555 ($\Omega=0.5$)

Fig. 4.24. $v_n^-$ vs. $t$, Case 190010-5575 ($\Omega=0.75$)

Fig. 4.25. $v_n^-$ vs. $t$, Case 190010-559 ($\Omega=0.9$)

Fig. 4.26. $v_n^-$ vs. $t$, Case 190010-5515 ($\Omega=1.5$)

Fig. 4.27. $v_n^-$ vs. $t$, Case 190010-5520 ($\Omega=2.0$)
4.4.2 Critical Force

As the amplitude of the forcing, $f_o$, increases, the motions tend to become larger and last longer. This behavior is seen in Figs. 4.28-4.32 as $f_o$ increases from 0.3 to 0.5 to 0.75 to 1.0 to 1.5. As seen, the amplitude of the forcing increases and this causes the breakwater to climb the sides of the region because there is more energy introduced to the system to cause this behavior. At some point the motions of the breakwater becomes less dependent upon the gravity component of the force and more dependent on the wave forcing component. This can be seen in Fig. 4.30, the $f_o=0.75$ case, where the motions reach a higher value of $y$ than in Fig. 4.29, the $f_o=0.5$ case, but still die down towards the bottom because the gravity force is controlling. However, in the $f_o=1.5$ case the forcing ultimately controls, and this is seen in Fig. 4.32 because the breakwater travels all around the region instead of staying near the bottom. It should be noted that trajectory plots of Figs. 4.28-4.32 are not plotted to the same scale. Figures 4.28–4.30 are plotted up to $y=0.1$, while Fig. 4.31 goes to $y=0.4$, and Fig. 4.32 to $y=h$.

Due to this occurrence, a new term will be introduced known as critical force, $f_{cr}$. The critical force is defined as the forcing amplitude which would cause the breakwater to hit the upper boundary ($y=h$), which indicates that the forcing was so large that it controlled the motions of the breakwater and caused it to hit the sea floor. Plots of the critical force versus the coefficient of restitution, $e$, and the wave frequency, $\Omega$, were created (Figs. 4.33 and 4.34). These plots were produced by fixing all of the standard case conditions except $e$ in Fig. 4.33 and $\Omega$ in Fig. 4.34, and increasing $f_o$ until the calculated $y$ value reached the height $h$. It is seen in Figs. 4.33 and 4.34 that the critical force for the standard case is around 1.17. This means that when $f_o=1.17=f_{cr}$ for the standard case conditions, the PMBW will hit the upper boundary $y=h$. This can be seen in Fig. 4.32 where $f_o=1.5$ which is greater than $f_{cr}=1.17$, and thus the breakwater goes above $y=h$ at the point denoted by A. Further, Figs. 4.33 and 4.34 show a decreasing nature of the critical force as the variables $e$ and $\Omega$ increase. However, the critical force solutions are not monotonic in nature; they contain some local maxima and minima.
Fig. 4.28. $y$ vs. $x$, Case 190010-359 ($f_o=0.3$)

Fig. 4.29. $y$ vs. $x$, Case 190010-559 ($f_o=0.5$)

Fig. 4.30. $y$ vs. $x$, Case 190010-7559 ($f_o=0.75$)

Fig. 4.31. $y$ vs. $x$, Case 190010-1059 ($f_o=1.0$)  Fig. 4.32. $y$ vs. $x$, Case 190010-1559 ($f_o=1.5$)
Fig. 4.33. Critical Force vs. $e$

Fig. 4.34. Critical Force vs. $\Omega$
4.4.3 Norms

Norm plots were also created for a set of given parameters. Norms for this problem are either maximum data points or summation of data. Four norms were investigated; three dealt with the normal velocity just before impact, \( v_n^- \), and are as follows:

\[
\rho_1 = \max_t(v_n^-) \tag{4.29}
\]

\[
\rho_2 = \sum_{i=0}^{10} v_n^- \tag{4.30}
\]

\[
\rho_3 = \sum_{i=0}^{\infty} v_n^- \tag{4.31}
\]

The last was the maximum height, \( y_{\max} \), which the PMBW attained during its period of motion. The data used in the determination of the norms was the same data used to produce the trajectory and normal velocity vs. time plots. The norms are plotted versus the magnitude of the forcing, \( f_o \), for a given set of parameters (Figs. 4.35-4.37). The standard case parameters and initial conditions were used to create these plots with the exception of \( f_o \) which was the parameter varied. As seen in Fig. 4.35, the phenomenon of a critical forcing is reinforced. It can be seen that as the forcing is increased, \( y_{\max} \) increases to a point where the PMBW is just below the boundary height \( h \), and the forcing is just below the predetermined value of \( f_{cr} \). When the forcing is low, the initial height of \( y_o = 0.1 \) is \( y_{\max} \); this is due to gravity controlling the motions at this low forcing, and as the forcing increases, the response becomes more and more controlled by the forcing. Further, as the forcing amplitude, \( f_o \), is increased, the norm \( \rho_1 \) (Fig. 4.36) increases because more energy is introduced into the system and this causes the normal velocity just before impact to increase because the PMBW strikes the boundaries with more force. Finally, for this same reason, the cumulative norms \( \rho_2 \) and \( \rho_3 \) (Figs. 4.36 and 4.37) increase because more energy is moving the PMBW about. The chaotic nature increases as the forcing increases towards the critical force and may be attributed to the highly nonlinear nature of the problem. An increase in the forcing amplitude, \( f_o \), does not necessarily cause the maximum height, \( y_{\max} \), or the other norms to increase.
Fig. 4.35. $y_{\text{max}}$ vs. $f_o$, Norm Plot

Fig. 4.36. $\rho_1$ and $\rho_2$ vs. $f_o$, Norm Plots
4.5 Special Cases

Several special cases were developed in order to observe the effects of varying the forcing parameters and to see if some special situations may arise under certain specific conditions.

4.5.1 No-Gravity Case

A special case where gravity and/or buoyant forces were neglected was investigated. This case simulates the property of a “neutrally buoyant” object in a fluid. This means that the gravity forces pulling down on an object exactly equal the buoyant forces that are making the object float up. This eliminates the gravitational force and leaves only the wave forces to act on the object. Several trials were conducted to see how varying the forcing parameters affected the motions of the PMBW, and these parameters are summarized in Table 4.1.
4.5.1.1 No-Gravity Case Formulation

The EOM’s used in this analysis are similar to those in section 4.2.2; however, the effects of gravity are neglected. The resulting solutions are summarized here:

\[
x = c_1 + c_2 t - \frac{f_o}{\Omega^2} \cos[\Omega(t - t_x)]
\]  
\[
\cdot x = c_2 + \frac{f_o}{\Omega} \sin[\Omega(t - t_x)]
\]  
\[
y = c_3 + c_4 t - \frac{vf_o}{\Omega^2} \cos[\Omega(t - t_y)]
\]  
\[
\cdot y = c_4 + \frac{vf_o}{\Omega} \sin[\Omega(t - t_y)]
\]

where

\[
c_1 = x_i - c_2 t_i + \frac{f_o}{\Omega^2} \cos[\Omega(t_i - t_x)]
\]  
\[
c_2 = x_i - \frac{f_o}{\Omega} \sin[\Omega(t_i - t_x)]
\]  
\[
c_3 = y_i - c_4 t_i + \frac{vf_o}{\Omega^2} \cos[\Omega(t_i - t_y)]
\]  
\[
c_4 = y_i - \frac{vf_o}{\Omega} \sin[\Omega(t_i - t_y)]
\]

However, for these equations to produce motions which resemble the applied wave forcing, the initial values of the terms \(c_2\) and \(c_4\) need to be zero. Thus, initial x and y positions, \(f_o\), \(v\), and \(\Omega\) were chosen to satisfy these initial conditions. To accomplish this, the Equations 4.40 and 4.41 were used to get the initial x and y velocities. The values of these parameters and initial conditions may be seen in Table 4.1. It must be noted that the no-gravity cases investigated here were mistakenly given \(t_x = 5.0\) during the analysis, contrary to the other cases analyzed. The results of this analysis were not significantly impacted because the \(t_x\) value was not equal to zero since this parameter is a phase shift. The only effect on the no-gravity cases would be that the forcing would start at a different point on the forcing ellipse. Thus, since the no-gravity motions follow the forcing, the
only effect would be that the motions in the trajectory plots would be shifted. Normally \( \dot{x}_o \) would be zero, but with \( t_x \neq 0 \), the values of \( \dot{x}_o \) do not equal zero as seen in Table 4.1. Since this does not affect the desired conclusion of the varying of the forcing parameters, it was deemed acceptable for the analysis. The values \( \dot{x}_o \) and \( \dot{y}_o \) are

\[
\begin{align*}
\dot{x}_o &= \frac{f_o}{\Omega} \sin[-\Omega t_x] \\
\dot{y}_o &= \frac{vf_o}{\Omega} \sin[-\Omega t_y]
\end{align*}
\]

(4.40) (4.41)

4.5.1.2 No-Gravity Case Results

Table 4.1 shows the initial conditions used to produce the responses investigated. The initial conditions were produced from the analytical solution with the gravity terms neglected. In Figs. 4.38-4.42 the results of varying \( f_o \) can be seen, with \( f_o \) increasing from 0.01 to 0.02 to 0.03 to 0.04 and 0.05. As expected, when the amplitude of the forcing increases, the size of the elliptical motion increases. The elliptical path occurs because the only forces affecting the motions are the periodic forces which cause the PMBW to travel in an elliptical path produced by the elliptical forces previously discussed. Also, during this analysis, both the left and right mooring cables are slack. In Figs. 4.38-4.40, the size of the ellipse increases until it reaches a boundary where the impact (i.e., cable becoming taut) causes the motion to change direction, thus causing the loss of the elliptical pattern (Figs. 4.41 and 4.42). The ellipses occur off center because of the way the forcing functions start and how the initial conditions were chosen. As seen in Fig. 4.43 and Table 4.1, initial conditions can be chosen to make the trajectory occur about the axis \( x=0 \). The asymmetry seen in the previous figures arises from terms \( c_2 \) and \( c_4 \) which come about from the integral derivation of the EOM’s used in the solution. By setting these constants to zero and staying zero throughout the analysis, an ellipse with its small axis along the line \( x=0 \) may be obtained (Fig. 4.43).
Fig. 4.38. y vs. x, Case 110010-01575ng ($f_o=0.01$)

Fig. 4.39. y vs. x, Case 110010-02575ng ($f_o=0.02$)

Fig. 4.40. y vs. x, Case 110010-03575ng ($f_o=0.03$)

Fig. 4.41. y vs. x, Case 110010-04575ng ($f_o=0.04$)

Fig. 4.42. y vs. x, Case 110010-05575ng ($f_o=0.05$)

Fig. 4.43. y vs. x, Case 115302-04575ng (Centered Ellipse)
In Figs. 4.44-4.47, the amplitude reduction factor, \( \nu \), varied from 0.1 to 0.25 to 0.5 to 1.0 and this causes the height-to-width ratio of the forcing ellipse to vary. Further initial conditions and parameters may be seen in Table 4.1. The parameter \( \nu \) is the constant that determines the ratio of \( f_y \) to \( f_x \) as seen in Equations 4.14 and 4.16. As \( \nu \) increases toward unity, the more circular the force ellipse becomes; this is because the \( x \) and \( y \) forcing amplitudes are becoming more equal as \( \nu \) gets closer to one. This trend is seen in Figs. 4.44-4.47 where a low value of \( \nu \) produces a flatter ellipse and a higher value of \( \nu \) produces a more circular ellipse.
4.5.2 Harmonic Motion Case

The special case of harmonic motion of the forced motion problem was investigated to see if a periodic solution existed in which the motions of the PMBW trajectory would become harmonic.

4.5.2.1 Harmonic Motion Case Formulation

Constraints on the parameters and conditions were imposed in the formulation of this special case to see if a periodic solution could be induced. The following parameters were used in the derivation of the equations to find the solution and used to determine suitable initial conditions. This case included the effects of gravity. No energy was to be lost at impacts. Vertical forcing was inhibited and only horizontal forcing was allowed; thus, to induce this,

\[ e = 1.0 \quad t_s = 0.0 \quad t_i = 0.0 \quad \nu = 0.0 \rightarrow f_y = 0.0 \]  \hspace{1cm} (4.42)

By imposing the above restrictions and making the initial time \( t_o = 0 \) and the time at the first impact \( t_1 = \pi/\Omega \) (i.e., half the forcing period), further conditions at the first impact may be specified as

\[ x_1 = -x_o \quad \dot{x}_1 = x_o \quad y_1 = y_o \quad \dot{y}_1 = y_o \]  \hspace{1cm} (4.43)

with the mass starting on the left boundary.

Thus, the following equations were derived to obtain the initial conditions, which would induce a periodic solution from the previously stated solutions and restrictive conditions:

\[ x_o = 1 - \sqrt{r^2 - (h - y_o)^2} \]  \hspace{1cm} (4.44)

\[ f_o = -x_o \Omega^2 - \frac{\pi^2 (1 - x_o)}{4\sqrt{r^2 - (1 - x_o)^2}} \]  \hspace{1cm} (4.45)

\[ \dot{x}_o = -\frac{2x_o \Omega}{\pi} - \frac{2f_o}{\pi \Omega} \]  \hspace{1cm} (4.46)

\[ \dot{y}_o = \frac{\pi}{2\Omega} \]  \hspace{1cm} (4.47)
By choosing an initial height, \(y_o\), and frequency, \(\Omega\), then using Equations 4.44-4.47, the other initial conditions may be calculated in order to obtain a periodic solution. Table 4.2 shows the initial conditions used in the analysis of the periodic solution, and the figures in the following section show how the trajectories vary according to the varying of independent variables \(y_o\) and \(\Omega\).

### 4.5.2.2 Harmonic Motion Case Results

A standard case was developed for the special harmonic forcing case utilizing the restrictive parameters. The values of the parameters and initial conditions for the standard case may be seen in Table 4.2 (shaded). As seen in Fig. 4.48, as time increases, the small errors in the rounding and estimations of calculated values accumulate and the trajectories lose their nice periodic motion. Because a periodic solution is the desired outcome, plots such as seen in Fig. 4.49 were produced from the plots with the accumulated errors. The only difference between Figs. 4.48 and 4.49 is that Fig. 4.49 has had the data points cut out starting from the time at which the solution stopped being periodic, to show just the trajectory of the initial periodic solution without the accumulated errors. Figure 4.50 shows a nice example of this with the \(v_n\) vs. time graphs initially having the same \(v_n\) values for a while, as expected, but then after some time the velocities start to diverge as the error increases.

A standard case was run but with \(e=0.95\) instead of \(1.0\) to see the effects of \(e\) on the trajectory. As seen in Fig. 4.51, the first loop is similar to that of the periodic solution when \(e=1.0\), but after the first impact the loop shape is lost due to the loss of energy. When the coefficient of restitution was allowed to be \(0.99\), the PMBW followed a similar pattern but went above the upper boundary \(y=h\). Thus, a periodic solution will exist when \(e=1.0\); however, because energy is lost from the system when \(e\neq1.0\), the PMBW will not continue along this periodic path.
When $\Omega$ is varied, the periodic trajectory of the PMBW decreases from a loop to an arc (Figs. 4.52-4.56). These figures show $\Omega$ varying from 1.5 to 2.0 to 3.0 to 4.0 to 5.0, and the other parameters and initial conditions may be seen in Table 4.2. The length of the path between the boundaries decreases as the forcing frequency increases, thus, the mass is moving faster.

As the initial height on the left boundary, $y_o$, is increased, the trajectory just moves up, keeping the same type of shape (Figs. 4.57-4.60). The variable $y_o$ is varied from 0.1 to 0.2 to 0.3 to 0.4 in the figures. There do not seem to be any notable characteristics, except that the trajectory is moving up. As $y_o$ is increased, the trajectory will eventually strike the upper boundary $y=h$.

The harmonic motion case gives a good opportunity to show an impact Poincaré plot. The impact Poincaré plots used here show the value of velocity before impact versus position for either the vertical or horizontal direction. In Figs. 4.61 and 4.62 the x and y impact Poincaré plots are shown for the fspo3 case from Table 4.2 (Fig. 4.54). These plots put a dot corresponding to the impact velocity versus the position in the x and y directions. According to the formulation, the criteria of Equation 4.43 from Section 4.4.2 must be met for the harmonic motion solution to exist. Thus, only two dots, which are mirror images of one another, should show up on the x plot, and one dot would show up on the y plot. These criteria are met, as seen in Figs. 4.61 and 4.62.
Table 4.2. Initial Conditions for Harmonic Motions of a Point-Mass Breakwater

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<th>r</th>
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<th>v_x</th>
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<th>f_o</th>
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Fig. 4.48. y vs. x, Standard Case with Errors (Case fspstd)

Fig. 4.49. y vs. x, Standard Case (Case fspstd)

Fig. 4.50. $v_n^*$ vs. t, Case fspo3

Fig. 4.51. y vs. x, Case fspe95 (e=0.95)
Fig. 4.52. y vs. x, Case fspstd ($\Omega=1.5$)

Fig. 4.53. y vs. x, Case fspo2 ($\Omega=2.0$)

Fig. 4.54. y vs. x, Case fspo3 ($\Omega=3.0$)

Fig. 4.55. y vs. x, Case fspo4 ($\Omega=4.0$)

Fig. 4.56. y vs. x, Case fspo5 ($\Omega=5.0$)
Fig. 4.61. x Impact Poincaré Plot, Case fspo3

Fig. 4.62. y Impact Poincaré Plot, Case fspo3