Kinematic Analysis of Tensegrity Structures

By

William Brooks Whittier

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Dr. Charles F. Reinholtz, Chairman
Dr. Harry H. Robertshaw
Dr. Robert L. West

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Abstract

Tensegrity structures consist of isolated compression members (rigid bars) suspended by a continuous network of tension members (cables). Tensegrity structures can be used as variable geometry truss (VGT) mechanisms by actuating links to change their length. This paper will present a new method of position finding for tensegrity structures that can be used for actuation as VGT mechanisms.

Tensegrity structures are difficult to understand and mathematically model. This difficulty is primarily because tensegrity structures only exist in specific stable tensegrity positions. Previous work has focused on analysis based on statics, dynamics, and virtual work approaches. This work considers tensegrity structures from a kinematic viewpoint. The kinematic approach leads to a better understanding of the conditions under which tensegrity structures exist in the stable positions. The primary understanding that comes from this kinematic analysis is that stable positions for tensegrity structures exist only on the boundaries of nonassembly of the structure. This understanding also allows the tensegrity positions to be easily found. This paper presents a method of position finding based on kinematic constraints and applies that method to several example tensegrity structures.
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Chapter 1, Introduction
Tensegrity structures consist of isolated compression members (rigid bars) suspended by a continuous network of tension members (cables). An example of a tensegrity structure is shown in Figure 1. Tensegrity structures were first developed by artist Kenneth Snelson in 1948 [1996] as well as independently by David Emmerich [1996]. Buckminster Fuller further developed tensegrity structures from Snelson’s ideas and applied engineering principles to the study of these intriguing devices. He also coined the term tensegrity from “tensile integrity” as the integrity or stability of the system comes from the tension members.

Figure 1. Tensegrity Structure Model

As implied by their name, tensegrity structures are stable, statically constrained structures. The mobility of these structures, as predicted by the Kutzbach mobility equation, is always greater than or equal to zero. Yet, the tensegrity structures are prestressed, which indicates a statically indeterminate (overconstrained) structure with a mobility of less than zero. This is one of the unique and interesting characteristics of
tensegrity structures that will be discussed in more detail in the Mobility chapter of this work.

Although tensegrity structures are, by definition, structures, they can invariably be made into mechanisms. These mechanisms can change shape and position by changing the lengths of links. In this respect, tensegrity structures are similar to Variable Geometry Trusses (VGT’s). VGT’s are truss-based mechanisms made from two-force members connected by spherical joints, or their equivalent. These trusses are actuated by changing the length of individual links. Tensegrity structures exist in many of the same geometries as VGT’s and have many of the same applications. This paper will analyze tensegrity structures using a general approach that has previously been applied to VGT mechanisms [Dasgupta and Mruthyunjaya, 2000]. This approach treats tensegrity structures as kinematic devices with their position determined from kinematic constraint equations.

Although tensegrity structures have been known for over 50 years, there has been little use of them as engineering structures or mechanisms. They have been studied and researched from both architectural and engineering viewpoints. Yet, their most common use has been as interesting toys. This lack of use may result from the difficulty in understanding tensegrity structures and an inability to form mathematical models to describe their behavior. Despite the difficulties in modeling tensegrity structures, they have many potential advantages that motivate further study. One of the main advantages is that all the links are subjected to only axial loading, which allows the structure to be light and strong. Other advantages of tensegrity structures are discussed in more detail later in this work.

Tensegrity structures have many potential uses. They may, for example, be adapted for use as towers and space structures. Research has been done on using tensegrity structures as a platform for aircraft simulators [Sultan and Corless, 2000]. They might also be used for robotic manipulators and measurement devices. There are numerous possible uses for tensegrity structures as both trusses and mechanisms if they can be properly modeled.
The purpose of this work is to consider tensegrity structures from a kinematic viewpoint. Previous work has focused on analysis based on statics, dynamics, and virtual work approaches. It will be shown that the kinematic approach leads to a better understanding of the conditions under which statically determinate structures or mechanisms become statically indeterminate tensegrity structures. It is hoped that this understanding will lead to an improved ability to design, analyze, and apply these devices.

**Literature Review**

Although tensegrity structures have been around for 50 years, it is only recently that they have been analyzed from an engineering viewpoint. There are many publications on the geometry and architecture of tensegrity structures, but there is much less literature on the dynamics and mechanics of these devices. The available articles cover the topics of limited shape/form finding for symmetric examples, force displacement relationships, dynamics, and vibration of these tensegrity structures. These papers use statics, Newtonian dynamics, and Lagrangian dynamics to analyze the tensegrity structures. No work was found that uses kinematic constraints to determine equilibrium tensegrity positions. Some of the papers that were most helpful and that are most applicable to this work are detailed below.

Oppenheim and Williams wrote a series of articles on tensegrity prisms. They discuss how tensegrity structures can be used as variable geometry trusses (VGT’s) with position finding incrementally using the kinematics (Jacobian) and statics matrices [1997]. A second article [2000] discusses the force displacement relationship in an elastic tensegrity structure. Oppenheim and Williams also wrote two articles [July, 2001], [November, 2001] regarding vibration in tensegrity structures.

Sultan, Corless, and Skelton [2001] studied finding the equilibrium positions of tensegrity structures where prestress is allowed using a virtual work approach. They also published a paper [2000] on application of a tensegrity structure as a flight simulator. Skelton and colleagues [2001] also wrote a conference paper on the basic mechanics of tensegrity structures.
Stern and Duffy wrote several articles concerning tensegrity prisms that are self-deployable. One article [2001] presents equations governing the prisms including initial tensegrity positions. Pellegrino and Calladine analyzed statically and kinematically indeterminate structures with matrices [1986].

Hanaor suggested that a kinematic method would be useful for position finding [1992]. This was the only mention found of a kinematic method for position finding of tensegrity structures. However, he stated: “Unfortunately, to the best of the author’s knowledge, no functional algorithms are available to date using the kinematic method.”

**Advantages and Disadvantages of Tensegrity Structures**

Tensegrity structures have specific advantages that merit their consideration for use as engineering structures and mechanisms. Tensegrity structures are form finding and deployable. This means that for a structure with flexible (cable) tension members, as the last cable is shortened, the structure will automatically deploy (i.e. raise from its initial collapsed state to the stable tensegrity position). This aspect makes tensegrity structures attractive when pre-use storage space is at a premium or for compact transportation. Additionally, by using elastic, tensegrity structures can be self-erecting or self-deployable. The potential energy in the elastic cables is used to deploy the structure into its stable configuration.

One of the greatest benefits of tensegrity structures is that forces are purely axial and predominantly tension. The tension-only members are not subject to compressive forces that cause buckling and can take advantage of materials that are strong in tension. This feature results in material efficiencies and low structure weight. Tension members also self stabilize and become more stable the larger the load as the tension pulls the structure towards the equilibrium position. To eliminate bending loads in links, trusses must typically be constructed from rigid links connected through spheric joints at each node. Because each link must articulate independently from the other links at a node, mechanical design can become complex. Because the tension members in tensegrity
structures are usually made from flexible cables, discrete joints can be replaced by the inherent cable flexure. Simple joints are particularly important when the tensegrity structures are used as variable geometry truss mechanisms. Another advantage for use as a mechanism is that the lengths of the tension links (cables) can be easily adjusted through pulleys, a cable drum, or shape memory alloy (NiTi) wire.

There are also several disadvantages that must be overcome to make tensegrity structures useful, especially as variable geometry truss (VGT) mechanisms. Tensegrity structures are not conventionally rigid; they always exhibit an infinitesimal flex and must be prestressed to resist deformation in the direction of the flex [Pellegrino and Calladine, 1986]. Tensegrity structures also tend to be susceptible to vibration because of the infinitesimal flex. Some research has been done concerning vibration in tensegrity structures and methods for damping out the vibration. When vibration is of concern for the application of tensegrity structures, this must be considered and analyzed. One of the greatest disadvantages is that tensegrity structures only exist under specific geometries. The nodal positions cannot be specified arbitrarily for a tensegrity structure. Thus, some positions cannot be achieved with a tensegrity structure that could be achieved using a variable geometry truss. The mathematics involved in tensegrity structures is difficult. When in a tensegrity position, the governing statics equations result in singular matrices. Position finding is a difficult process, as will be discussed in the corresponding chapter. Mobility (Kutzbach) and determinacy (Maxwell) equations don’t directly apply due to the special geometry that will be discussed in this work that is inherent in tensegrity structures. Due to these problems, the mathematical models needed to analyze and design tensegrity structures are still being developed and must be further researched before extensive use of tensegrity structures is possible.
Chapter 2, Geometries

Tensegrity Positions

A large number of trusses and mechanisms can move into special positions where they become tensegrity structures. These positions correspond to specific link lengths of the system members that place the device in a singular position. Only in these specific tensegrity positions would the structure not collapse if specified rigid links were replaced with flexible tension members. These tensegrity configurations allow the system to be prestressed, which is required to give the system rigidity. These stable positions will be called tensegrity positions in the rest of this work. The properties of these specific positions will be discussed in more detail in the chapter entitled “Position Finding”.

For a bar system (truss or mechanism) that can become a tensegrity structure but is not yet in a tensegrity position, one link can be adjusted (shortened or lengthened) to move the system into a tensegrity position and create a tensegrity structure. The number of link lengths that must be changed in order to reach a tensegrity position will generally be only one. The reason for this principle and the possible exceptions to it will be discussed in more detail in the Kinematic Analysis chapter. The last link length will be unique for the set of link lengths of the remaining links and will also specify a unique tensegrity geometrical configuration. Due to these properties, tensegrity structures are deployable and form finding. As discussed above, as one last link is shortened from a length greater than its tensegrity length, the structure will raise from its initial collapsed state to a stable tensegrity position determined by the lengths of the other links.

In order to use a tensegrity structure as a VGT mechanism, the structure must move from one tensegrity position to another tensegrity position. If the mechanism will be supporting any weight or must retain its shape while moving between the positions, it must also remain in tensegrity positions between the starting and ending points. This requires a minimum of two links to change in length simultaneously. As discussed above, adjusting the length of only one link in an existing tensegrity structure will move the device out of a tensegrity position, since this is a unique, singular position based on
the lengths of the remaining links. Because of this, at least two link lengths must be changed to reach a new tensegrity position. By adjusting two link lengths simultaneously the structure can remain in a tensegrity position continuously while moving between positions. Thus, a “tensegrity surface” exists of an infinite number of tensegrity positions and the tensegrity structure can move to any point on the surface while still remaining in stable configurations. This was verified both analytically and using a model in which the tension only links were made from string that could be adjusted in length by pulling the strings through a pair of eye screws. Several combinations of two links could be adjusted simultaneously and the structure was moved to many different positions while maintaining its rigidity, thus showing it remained in tensegrity positions.

**Categorization**

Tensegrity structures can be categorized in several ways. The first categorization is based on what is known in some literature [Skelton et al, 2001] as the class number of the tensegrity structure. The class is determined by the number of compression members that meet at a node. A class 1 tensegrity structure has only one compression member terminating at each node. This is the most common type and it is the type studied by Buckminster Fuller. Only class 1 structures will be analyzed in this work.

The next obvious categorization of tensegrity structures is based on the number and type of members in the structure. We will classify tensegrity structures by the number of rigid bars in the structure; this is a common approach used in many papers. The naming convention will follow that used by others including Oppenheim and Williams [1997] with a capital T followed by the number of rigid bars. Thus, a tensegrity structure with three rigid bars will be known as a T-3 structure. The rigid bars in a tensegrity structure can be connected in many different ways.

**Topology Families**

There are many possible topologies (number of members and manner in which they are connected) for which tensegrity structures can be built. The different possible topologies fit into families of tensegrity structures. Tensegrity structures with a specified number of
rigid bars can usually be connected in different ways and thus will fit into different families of tensegrity structures. Thus, we will also define tensegrity structures by the topology family into which they fit.

There is a family of tensegrity structures, the basic structure of which consists of two equal end polygons in parallel planes joined by members at their vertices [Stern and Duffy 2001]. The tensegrity structure shown in Figure 1 is an example of this family with triangles as the end polygons. This family will be called the prism family. The end polygons are not required to remain of equal size or in parallel planes and may even become skew, but they must be geometrically similar. Stern and Duffy show that original configurations that are tensegrity positions can easily be found for these structures. Starting in a configuration with the end polygons lined up such that the sides are parallel, the top polygon is rotated by a twist angle \( \alpha \) with respect to the base polygon. The angle \( \alpha \) can be found from Equation (1):

\[
\alpha = \frac{\pi}{2} - \frac{\pi}{n}
\]

where \( n \) is the number of sides in the end polygons. This result will be used to test the code developed in this thesis for finding tensegrity positions.

Another family of tensegrity structures is based on the truss unit cell polyhedra [Arun, 1990]. The truss unit cell polyhedra are the polyhedra formed from all triangular faces with a mobility of zero. They include the tetrahedron, octahedron, decahedron, dodecahedron, and icosahedron. This family of tensegrity structures will be known by their polyhedra name. The octahedron and icosahedron form the basis for tensegrity structures discussed in this paper. It should also be possible to form more tensegrity structures out of the other truss unit cell polyhedra.
There are also other families of tensegrity structures that are not discussed here; other tensegrity structure types may also remain to be discovered. In addition, the basic tensegrity structures can be joined to create compound tensegrity structures.

Example Topologies

A few basic structure examples are discussed below. A T-3 structure is shown in Figure 2. It has 3 compression members (rigid bars) and 9 tension members (cables). This tensegrity structure is equivalent to an octahedral truss with 8 faces, 12 edges/links, and 6 vertices/joints. This structure fits into both the prism and truss unit cell families. The T-3 structure will be used as an example throughout this thesis. In addition, the code described in the “Position Finding” chapter uses the T-3 structure as one of the examples.
The next higher order structure is a T-4 prism structure. This tensegrity structure is shown in Figure 3. It has 4 bars and 12 cables. This tensegrity structure is not equivalent to any of the platonic solids. If it were built out of rigid members with general link lengths, it would be a mechanism with two degrees of freedom and not a truss structure. The mobility of this example will be discussed in the Mobility chapter.

![Figure 3. T-4 Prism Tensegrity Structure (a) Perspective View (b) Top View](image1)

The T-6 icosahedron structure is the most commonly seen and built tensegrity structure. A diagram of the structure with its 6 bars and 24 cables can be seen in Figure 4. This tensegrity structure is equivalent to an icosahedron with 20 faces, 30 edges/links, and 12 vertices/joints.

![Figure 4. T-6 Icosahedron Tensegrity Structure (a) Perspective View 1 (b) Perspective View 2 (c) Top View](image2)
Topology Rules

There are many tensegrity structure geometries that have been built and studied. After studying many of the existing tensegrity geometries, I submit the following hypothesis on the requirements for forming three-dimensional class 1 tensegrity structures. First, a three-dimensional tensegrity structure must have a minimum number of three compression bars. This requirement is derived from the need for compression components in each of the three principal directions. If there were only two compression members, there would be no compression member to resist forces in the direction orthogonal to the other two members and the structure would collapse into two dimensions. Second, each node must connect at least four members, one of which must be a compression member. This requirement also comes from the need for support in each of the three orthogonal directions. In order to keep the general three-dimensional structure, three tension members are required to oppose the one compression member at each joint. Any bar and cable structure that meets these requirements should be able to be manipulated, by changing link dimensions, into a tensegrity position to form a tensegrity structure.
Chapter 3, Mobility

Although tensegrity structures are, by definition, statically indeterminate (overconstrained) structures, they can invariably be made into mechanisms that can change shape and position by changing the lengths of links.

As discussed above, the mobility of tensegrity structures as calculated by the Kutzbach mobility equation is different from the observed mobility. Several papers discuss the determinacy of tensegrity structures as found by Maxwell’s rule [Pellegrino and Calladine 1986]. This determinacy criterion is roughly the equivalent of mobility for stable trusses. Pellegrino and Calladine attempt to explain the discrepancies in determinacy values that differ from the expected values by differences in the rank of the equilibrium statics matrix [1986]. However, there appears to be no research of the mobility from a kinematic viewpoint.

Two models of the T-4 tensegrity structure shown in Figure 3 and discussed in the Geometry chapter were built and analyzed. The Kutzbach mobility equation predicts that this device will have 2 degrees of freedom as shown in equation (2):

\[
M = 6(n - 1) - 3f_3 - I = 6(14 - 1) - 3 \times 21 - 13 = 2
\]  

(2)

where

- \(M\) = mobility, or number of degrees of freedom
- \(n\) = total number of links including ground
- \(f_3\) = number of three-degree-of-freedom (spheric) joints
- \(I\) = Idle degrees of freedom from links rotating about their own axis

The tensegrity model produced a stable structure that was prestressed. This prestress indicates a mobility of less than zero. A second model was built from all rigid members. This model showed that there were indeed two degrees of freedom in the truss when it is not in a tensegrity position.
An octahedral truss of all rigid members was also built. The octahedral truss is configurationally equivalent to the T-3 tensegrity structure. From this model it could be seen that with an added degree of freedom (link extending) the truss was able to move. This also verified that the octahedral truss did have a mobility of zero as predicted by the mobility equation.
Chapter 4, Kinematic Analysis

Kinematic analysis of mechanisms generally starts with position analysis of the mechanism. There are two types of position analysis, namely, forward kinematics and inverse kinematics. Forward kinematics is finding the position/orientation of the mechanism given the link lengths and any inputs of angles and/or displacements of the joints. This is the basic analysis needed to find stable tensegrity positions from the input of link lengths that is the focus of this work. Inverse kinematics involves finding possible angles/displacements of the joints (link lengths for VGT’s) given the position/orientation of the mechanism. Inverse kinematic position analysis is difficult for tensegrity structures, since these devices only exist in a highly constrained locus of positions. Inverse kinematic analysis for manipulators, in general, and VGT’s, in particular, has some of the same difficulties when certain positions cannot be achieved because the mechanism lacks the needed degrees of freedom. Tensegrity structures have these same constraints and also have the constraint that only specific truss positions are stable tensegrity positions. If stable tensegrity positions are known, it is easy to verify that the structure is in a tensegrity position. Then, determining the link lengths required to give that position is a simple calculation of point-to-point distances.

Mechanism synthesis involves developing a mechanism to perform a specified task such as producing a specified position and/or orientation of a particular link. Tensegrity structure synthesis is a complicated process. First, the type (topology – e.g. T-3 prism or T-6 icosahedron) of tensegrity structure to be used must to be determined. With other mechanisms the type (e.g. 4-bar, RSSR, or VGT) is determined by the type of motion desired. The possible positions and motions achievable with different tensegrity types are not well known, so even that simple decision would be difficult. Once the type of tensegrity structure is determined, the set of link lengths and the changes in those link lengths would have to be determined from the desired positions of the structure. The difficulties involved in inverse kinematic position analysis discussed above would make such a synthesis process difficult. Future work on tensegrity structures could focus on tensegrity mechanism synthesis.
Model Analysis

To gain a greater understanding of tensegrity structures several models were built, some of which have been discussed previously. These models led to an important observation regarding the mobility of tensegrity structures. As discussed above, an octahedral truss similar to the T-3 structure was built of all rigid members. The dimensions of the truss were modified such that it started in a tensegrity position. From this model it could be seen that, when any link was extended, the tensegrity structure gained a freedom and became a VGT. In contrast, it was impossible to contract any link, showing that the tensegrity position is in a special geometric configuration on the boundary of the workspace of the VGT. Models of the T-3 prism, T-4 prism, and T-6 icosahedron tensegrity structures were also built and yielded insights that have been discussed and will be further discussed below.

Code Analysis

Numerical analysis software was developed to analyze the octahedral variable geometry truss as a kinematic device. The code was written based on kinematic constraints that require the distance between points defined by known link lengths to remain invariant. This requirement is expressed mathematically in Equation (3):

$$(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 = L_{AB}^2$$

where,

- $x_A = x$ position of joint A
- $x_B = x$ position of joint B
- $y_A = y$ position of joint A
- $y_B = y$ position of joint B
- $z_A = z$ position of joint A
- $z_B = z$ position of joint B

One such equation is written for each link in the device, and the resulting system of equations is solved for the Cartesian location of each node. The code was run starting
with the truss in a known tensegrity position. Various links were changed in length in order to see what effect this had on the configuration of the truss. The results showed that tension links are allowed to get longer and compression links are allowed to get shorter but not vice-versa. Shortening a tension member or lengthening a compression member with no other changes in geometry led to an insolvable system of equations. This shows that the tensegrity position is a special geometry on the boundary of the workspace where the truss is able to assemble.

**Boundary of Nonassembly**

In any structure or mechanism a set of link lengths can be found that does not allow the system to be assembled. Two examples of planar four-bar mechanisms are shown in Figure 5. In mechanism (a) it is obvious that the chosen set of link lengths allows assembly. The links of mechanism (b) on the other hand have been chosen such that they cannot be assembled. This same principle is true for three-dimensional structures and mechanisms such as tensegrity structures. However, it is more difficult to find and visualize the configurations that produce nonassembly.

![Figure 5. 4-Bar Planar Mechanisms](image)

(a) Normal Configuration (b) Nonassembly Configuration

Bar systems can also be constructed of link lengths such that they exist on the boundary of nonassembly. An example of this is shown in Figure 6. Again mechanism (a) is clearly in a region where it can be assembled. However, mechanism (b) has been built so that it is on the edge of being able to be assembled. If the ground link was lengthened or any of the three other links were shortened, assembly of the mechanism would be impossible.
Stable positions for tensegrity structures lie on the boundary of nonassembly similar to that shown for this simple planar four-bar linkage example. However, the three-dimensional nonassembly case for tensegrity structures is more difficult to visualize and find.

The realization that tensegrity positions lie on the boundary of nonassembly of trusses and mechanisms led to four important findings, creating a better understanding of the mechanics of tensegrity structures. Those findings are that:

1) A configuration on the boundary of nonassembly allows prestress and removes mobility from the structure.

2) The position on the boundary allows infinitesimal flex in the structure.

3) The fact that tensegrity positions exist on the boundary of nonassembly is the reason that changing only one link length generally moves the device to a tensegrity position. This understanding also explains the exceptions to the general rule that only one link length adjustment is necessary.

4) The knowledge that tensegrity positions exist only on the boundary of nonassembly provides a new method of finding stable tensegrity positions.

**Prestress**

Building structures or mechanisms (with degrees of freedom greater or equal to zero) on the edge of the nonassembly region allows them to become prestressed. As discussed above, prestress normally indicates a mobility of less than zero. The possibility for prestress in a device that normally has freedom can be seen in Figure 6(b), where
shortening any link except ground induces stress. This condition also holds true with
tensegrity structures. As the tension members attempt to pull the structure into the
nonassembly area, this provides the extra constraint that allows prestressing of the
structure and removes the extra degrees of freedom.

The boundary of nonassembly also removes degrees of freedom. In Figure 6, mechanism
(a) has one degree of freedom and can be moved by rotating any of the links.
Mechanism (b) however, cannot be moved. Any attempt to rotate one of the links will be
resisted by the other links, as they would have to lengthen to stay connected to the rotated
link. Again, three-dimensional structures would also be affected by this boundary in a
similar manner. As seen from the T-4 structure, this extra constraint can remove more
than one degree of freedom from tensegrity structures.

**Infinitesimal Flex**

Along with the ability to be prestressed at the equilibrium tensegrity positions, all
tensegrity structures have infinitesimal flex from an infinitesimal mechanism. The
infinitesimal mechanism is the ability of the structure to move in a certain direction an
infinitesimal amount without extending (or contracting) the length of any of the links.
This infinitesimal flex was seen in all the tensegrity models that were built. All of the
structures had a direction in which a small (approximately 1/4 inch) movement could be
made without significant change in link lengths and without significant resistance.

A planar example of this infinitesimal mechanism can be seen again in the four-bar
mechanism in Figure 6. Mechanism (a) has the normal finite mechanism movement as
discussed above. If a vertical displacement is attempted on either of the center nodes of
mechanism (b) there is nothing to resist the initial displacement. That initial
displacement is the infinitesimal mechanism that exists. As the mechanism attempts to
displace a finite amount, the displacement will be resisted by the required lengthening of
the members that removes the degrees of freedom as discussed above. This principle also
exists for tensegrity structures when in a tensegrity position on the boundary of a
nonassembly region.
A chief characteristic of the infinitesimal mechanism is that the stiffness (force/displacement relationship) from displacement in the direction of the infinitesimal mechanism has an initial slope of zero. If the mechanism is internally prestressed the prestress will resist the infinitesimal mechanism motion. As can be seen in mechanism (b), because there is no component to resist the force in the vertical direction, the initial stiffness will always be zero. However, prestress changes the slope of the force/displacement curve and eliminates the initial zero slope. A model built of the T-6 icosahedron tensegrity structure showed that prestressing the structure greatly increased its resistance to movement in the direction of the infinitesimal mechanism.

As discussed above, the mobility and determinacy predicted by Kutzbach’s and Maxwell’s equations do not appear to apply to tensegrity structures. This infinitesimal mechanism motion replaces the finite mechanism motion when the mechanism is in a position on the boundary of nonassembly. This principle can be clearly seen in the four-bar planar example, but also exists for tensegrity structures when in a tensegrity position on the boundary of a nonassembly region. Thus, the discrepancy in the mobility and determinacy equations can be resolved by taking the infinitesimal mechanisms into account.

**Move to Tensegrity Position**

The Geometry chapter discussed how adjusting a single link length could move a non-stable tensegrity structure into a stable tensegrity position. This also means that all but one of the link lengths in a tensegrity structure can be chosen at random, and a tensegrity position can still be found by adjusting the length of the remaining link. However, there are some degenerative cases in which the tensegrity position cannot be found by adjusting only one link length.

The number of link lengths that must be changed to reach a tensegrity position is determined from the nonassembly region. If the tensegrity structure starts in a position that is sufficiently close to a nonassembly region, only one length adjustment will ever
have to be made. However, if the structure starts in a position that is not close to a nonassembly region, two or more links may have to be adjusted in length in order to reach a position on the boundary of nonassembly.

If one link is not sufficient and a single tension link is shortened, there are two possibilities. First, it will shorten until it gets to zero length and a nonassembly region corresponding to a tensegrity position will not be found. Second, shortening a tension link can collapse all or part of the structure to a one or two-dimensional structure that will have a degenerate stable tensegrity position. If one link is not sufficient and a compression link is lengthened, it will continue to lengthen until all or part of the structure collapses and nonassembly is created in only one or two dimensions.

When the T-6 icosahedron tensegrity model was built, the original link lengths were built such that shortening one of the tension ropes did not bring the structure to its tensegrity position. When one rope was shortened, part of the structure collapsed into a one-dimensional structure with two ropes running along one of the bars. When two ropes were shortened simultaneously, part of the structure collapsed into a two-dimensional structure. That exercise helped visualize and confirm the reasoning discussed above.

The greatest benefit from understanding that the tensegrity positions lie on the boundary of nonassembly is that it allows a new and easier method of determining the stable tensegrity positions. The ideas and methods used for this position finding are described in the following chapter.
Chapter 5, Position Finding

Finding the tensegrity positions that will yield a stable tensegrity structure is one of the most difficult tasks in the understanding and use of tensegrity structures. Many papers refer to the conditions under which these positions exist as prestressability conditions. Sultan, Corless, and Skelton define them as “The conditions under which a tensegrity structure yields an equilibrium configuration with all cables in tension and all bars in compression under no external loads” [2001]. The prestressable configurations can be found using the areas of nonassembly that were discussed above.

The ideal method for finding tensegrity positions would be to find and define a region of nonassembly for the truss. Tensegrity positions for that truss would be any position on the boundary of the nonassembly region. Unfortunately, completely defining the region of nonassembly is difficult for the complex three-dimensional tensegrity structures. The region would be based on the specific links that would be allowed to change length. Allowing links to change simultaneously makes this a very complex problem. Fortunately it is relatively easy to find specific positions on the boundary of nonassembly.

Method/Theory

When models of tensegrity structures were built, it was found that as the last tension link was shortened, the structure moved from a collapsed bundle of bars and cables to a tensegrity structure that allows prestress. As discussed above, this is because the structure moves into a position on the edge of nonassembly. However, as noted above, there are exceptions when certain configurations are not sufficiently close to the boundary of nonassembly. Those exceptions can cause problems in finding the tensegrity positions if they are not taken into consideration. Understanding the kinematic process by which the unstable structure moves into a stable tensegrity position on the boundary of nonassembly allows the stable positions to be found. Thus, tensegrity positions can be found by adjusting the length of a link in a kinematic mathematical model until the
structure moves into position on the edge of assembly. This method was used to write software to find stable tensegrity positions for different tensegrity structures.

**Finding Nonassembly Boundary**

MATLAB software for finding general tensegrity positions was developed based on the above theory. The program starts with a set of members connected in truss form that may or may not be in a tensegrity position. A numerical root finding method is used to find the position of the truss that satisfies the kinematic constraints for each link, an example of which is shown in Equation (3) above. A link is chosen to be adjusted in a numerical attempt to move the structure into the tensegrity position. If the structure starts in a truss position in which assembly is possible, then the chosen link will be changed in length (tension member shortened or compression member lengthened) to move the structure towards the boundary of non-closure. The chosen link is adjusted in length until the closure equations (kinematic constraints) begin to become unsolvable. This shows that the structure is on the boundary of a nonassembly volume and has reached a prestressable configuration. If the structure were to have started with link lengths such that assembly is not possible, the chosen link would be changed in length (tension member lengthened or compression member shortened) to move back towards an assembly region until all constraint equations just began to be met. This would also cause the structure to be on the boundary of nonassembly. Using this approach, the tensegrity position corresponding to the given set of link lengths can always be found. At this point, a new set of link lengths (all but one) can be chosen to make a new structure, and the process can be repeated to find the corresponding tensegrity position.

The code described above is based on a general method whose principles hold true for any tensegrity structure. Additionally, the code was written in a general form such that it could be applied to any potential tensegrity structure. An m file can be created for any tensegrity structure that describes the number and type of members as well as their connectivities. The code uses that file to find tensegrity positions for the specified tensegrity structure.
Statics Check

To verify that the truss is indeed in a tensegrity position, a check is performed using static force analysis. A system of equations is written for the structure by summing the forces at all of the nodes. This system of equations can be written in matrix form as:

$$ST = 0$$  \hspace{1cm} (4)

where $S$ is the statics matrix describing the geometry and $T$ is the vector of unknown member forces (tensions in tension members and compression forces in compression members). The equation is set equal to the resultant loads on each node, which for the equilibrium position of the tensegrity structure are all set to zero, to find the initial prestressable configuration. This equation is solved for a set of nonzero cable tensions that will yield a solution. The set of vectors that, when multiplied by the statics matrix, yield zero are called the null space of the $S$ matrix. The equation will only have a non-trivial solution when the determinant of the $S$ matrix (or $S^T S$ for non square matrices) is zero. For a determinant of zero, the possible results could yield one or more infinities of solutions. For each vector solution, any scalar multiple of that vector is also a solution, since a scalar times a vector of zeros is still zero. In addition, if more than one vector solution is found, any linear combination of the vectors is also a solution. This is shown in Equation 5:

$$ST = 0 \Rightarrow k_1 (ST_1) + k_2 (ST_2) + k_3 (ST_3) = k_1 (0) + k_2 (0) + k_3 (0) = 0$$  \hspace{1cm} (5)

where $k$ is any scalar. Singular value decomposition of the statics matrix allows the null space of the matrix to be found. An $m \times n$ matrix $A$ can be decomposed into a diagonal matrix $S$ containing the singular values of $A$ and orthogonal unitary matrices $U$ and $V$ such that

$$A = U \ S \ V^T$$  \hspace{1cm} (6)
The rank of the matrix \( A \) is equal to the number of nonzero singular values in \( S \). For each zero or extremely small singular value there exists a family of solutions in the null space. This null space of \( A \) is equal to the column vectors of \( V \) that correspond to the zero entries in the diagonal of the \( S \) matrix. Thus, for the statics matrix, the member forces that allow for prestress can be found by performing the SVD and extracting the null space vectors.

Software was developed to perform this statics check. The code first sets up the statics matrix, which is formed by summing the forces at all nodes. The determinant of the statics matrices is first taken to check for the existence of an answer. The program then solves for the null space of the \( S \) matrix to find the set of member forces that would yield a stable configuration. This is done by performing the SVD on the statics matrix. The number of vectors in the null space shows the number of infinities of solutions. This shows the number of tension vectors that can be picked. Linear combinations of these will yield all possible force solutions. The \( T \) vector (member forces) is then checked to verify that all forces are positive (i.e. all tension members are in tension and all compression members are in compression).
Application to Examples

T-3 Prism

These programs were first applied to the T-3 tensegrity structure and tested on a “perfect” tensegrity position that can be found for the end polygon family of structures as discussed in the Geometry chapter. The position finding program showed that no adjustment was necessary and the structure was in a tensegrity position. The program also outputs a drawing of the solution position; this is shown below in Figure 7. When the statics check was done, the determinant of the statics matrix proved to be zero. In addition, the statics analysis yielded a single vector with positive values for member forces, indicating that the tension members are in tension and the compression members are in compression. As expected from symmetry, the top and bottom tension members were all equal, as were the side tension members and three compression members. The test of the programs with the truss in the original prestressable position showed that the programs worked as intended.

![Figure 7. T-3 Tensegrity Structure in Original “Perfect” Tensegrity Position](image-url)

(a) Perspective View  (b) Top View

Figure 7. T-3 Tensegrity Structure in Original “Perfect” Tensegrity Position

(a) Perspective View (b) Top View
The program was next run with the T-3 truss in a non-tensegrity position. The program adjusted the length of a pre-specified link until a tensegrity position was achieved. The statics check verified this tensegrity position. The structure drawn by the program is shown below in Figure 8.

![Figure 8. T-3 Tensegrity Structure Changed to Tensegrity Position From Original Truss Position](a) Perspective View (b) Top View

Several different link lengths were selected to be gradually changed by the program. Different link lengths were also selected for adjustment to find the tensegrity positions. One example of this change is shown in Figure 9. The path (shown in magenta) taken by one point in the tensegrity structure as it moves from position to position is shown.

Figure 10 shows part of the display of the program after running. The code correctly found tensegrity positions for all adjustments.
The Mobility of this structure calculated from the Kutzbach mobility equation is as follows:

Mobility = 0

Link #4 has been changed in length from 5.25 to:
changel = 5.2500 5.1250 5.0000 4.8750 4.7500

Link #5 has been correspondingly adjusted in length to become a tensegrity structure from 5.5 to:
adjustl = 5.3100 5.4500 5.5900 5.7500 5.9100

The statics check showed the following results for the determinate (Det Approximately=0 Signifies Tensegrity Position)

Checkr = 1.0e-012 *
0.0505 -0.0037 0.1018 0.3379 -0.1648

Figure 9. T-3 Tensegrity Structure After Change in Length of Link #4 (Side Link)

Figure 10. Display of Position Finding Program for Change in Length of Link #4
**T-4 Prism**

A “perfect” tensegrity position was also found for the T-4 prism from the method discussed in the Geometry chapter. The position finding program was run for this structure and also showed that the structure was in a tensegrity position. The output drawing of the structure in its “perfect” tensegrity position is shown below in Figure 11.

![T-4 Prism Diagram](image)

Figure 11. T-4 Tensegrity Structure in Original “Perfect” Tensegrity Position
(a) Perspective View (b) Top View

A T-4 prism structure was found based on a set of link lengths measured from a model. The link lengths were changed by somewhat before being entered into the program so the structure did not start in a tensegrity prism. The position finding program was again able to successfully move the structure into a tensegrity position by adjusting one link length. The new structure is shown in Figure 12 below. The T-4 prism is slightly more complex (greater number of links and joints) than the T-3 prism. It also has the added complexity of two degrees-of-freedom when not in a tensegrity position. These introduced more numerical error that will be discussed in the Numerical Accuracy section. However, this method still worked with the T-4 prism structure and showed that the programs worked as intended.
An initial tensegrity position for the T-6 icosahedron was not found in any of the literature. In order to find an initial position to test the position finding program, a symmetric model was used. An equation was written for the distance between the bar ends that would be joined by tension members. This distance is the same for all tension members in the symmetric structure. The distance equation was solved for the location of the bars (distance from the center of the structure) that would minimize the distance (length of tension members). This yielded a structure that was initially in a tensegrity position.

The T-6 icosahedron in the initial tensegrity position was first tested using the static check software. This check showed that the structure was indeed in a tensegrity position. The structure was then put into the position finding program. The program made some slight adjustments in link lengths to move the structure closer to nonassembly in the range defined by the software code. The structure was still in approximately the same tensegrity position and the statics check verified this. However, there was more
significant numerical error. The T-6 icosahedron is much more complex than the T-3 or T-4 prisms and this introduces more numerical error. The structure drawn by the programs is shown below in Figure 13.

The position finding software was next run using a structure in a non-tensegrity position that was still close to the boundary of nonassembly. When the code was first run for this case numerical accuracy became a significant problem. The solution converged to a position judged to be on the boundary of nonassembly. However, the statics check did not show that the structure was in a tensegrity position. The actual numbers are shown and discussed in the next section.

In order to minimize the numerical error and overcome the resulting problems, changes were made in the code. The details of those changes will be discussed in the Numerical Accuracy section. The position finding program was rerun with the new code for the structure both in the original tensegrity position and the non-tensegrity position. With the

Figure 13. T-6 Icosahedron Tensegrity Structure in Original “Perfect” Tensegrity Position
new code the numerical error for the original tensegrity position was much smaller and the distance from the perfect position was much smaller. The link length difference from the exact tensegrity position was only one thousandth of an inch for the structure with ten-inch bars. Manufacturing and assembly tolerances for such a structure would be on this same order. For the structure starting in a non-tensegrity position an actual tensegrity position was found. The position finding method was again shown to be successful for the T-6 icosahedron.

**Numerical Accuracy**

As the position finding programs above were based on numerical methods, the numerical accuracy of the results had to be addressed. The boundary of nonassembly found by the program was defined to occur when the kinematic constraint equations no longer yielded a solution. The exact numerical values of the error in those equations that defined the boundary had to be determined and verified. The low boundary value had to be high enough so that a structure that could assemble was not judged to be in the nonassembly region from numerical error. The high value also had to be low enough so that a structure judged on the boundary was not actually far away from being able to assemble with resulting errors in the tensegrity position.

The main kinematic model was based on numerically finding the roots of the constraint equations to find the position of the truss for given link lengths. There is an inherent numerical error in that process, and this error increases dramatically with the number of members and resulting constraint equations. The T-3 prism has only 12 members, while the T-6 icosahedron has 30 members. This caused difficulties in both numerical accuracy and computing time (hours vs. minutes) for the more complex structures.

The error from the constraint equations is a combination of any actual error from the links not being able to assemble and the numerical error discussed above. The constraint equations error is used to judge the position of the structure on the boundary of nonassembly. Thus, if the numerical error is significant it will cause the program to predict a tensegrity position that is some distance from the actual position. If the
kinematic program is controlled properly it will converge to a solution with minimal numerical error.

As noted above, the T-6 structure had numerical problems in converging to a tensegrity position. This was the result of numerical error in the constraint equations error. After the kinematic program converged on a solution with a resulting error, the program was run again starting from the last converged solution. A different solution was found with a much smaller error. This shows that the numerical errors generated in solving the constraint equations error are limiting the accuracy of the solution. As discussed in the above section the code was changed to minimize this error. The changes made were: the step size for link length change was shortened, the tolerances were tightened on the numerical solver for the kinematic constraint equations, and the kinematic solver program was rerun when the constraint equations error was large. These changes greatly minimized the numerical error and allowed more exact tensegrity positions to be found. As discussed above, the errors were reduced to the same order as manufacturing tolerances.

The values of the determinant of the statics matrix to be considered as zero also had to be determined and adjusted. In addition, the zeros in the singular value decomposition used to find the null space tension vector had to be determined. Some example values for the different tensegrity structures that must be compared to numerical “zero” values are
shown in Table 1 below. The differences between the results from the old program and the new improved code are obvious. As seen in the table the more complex structures have more error and thus larger numbers that in an exact solution should be exactly zero.

Table 1 Numerical Values of Tensegrity Position “Zeros”

<table>
<thead>
<tr>
<th>Tensegrity Structure Type</th>
<th>Determinate Value at Tensegrity Position</th>
<th>Last Singular Value at Tensegrity Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-3 Perfect</td>
<td>2.5e-11</td>
<td>5.3e-9</td>
</tr>
<tr>
<td>T-3 Adjusted</td>
<td>1.6e-13</td>
<td>1.4e-9</td>
</tr>
<tr>
<td>T-4 Perfect</td>
<td>1.3e-13, (3.9e-7)</td>
<td>8.6e-9, (8.9e-6)</td>
</tr>
<tr>
<td>T-4 Adjusted</td>
<td>6.2e-13</td>
<td>1.1e-8</td>
</tr>
<tr>
<td>T-6 Perfect (Statics Check)</td>
<td>6.4e-13</td>
<td>2.0e-16</td>
</tr>
<tr>
<td>T-6 Perfect (Adjusted)</td>
<td>1.5e-11, (4.1e-6)</td>
<td>3.1e-8, (5.4e-4)</td>
</tr>
<tr>
<td>T-6 Adjusted</td>
<td>1.3e-9, (1.3e+3)</td>
<td>2.7e-7, (1.5e-1)</td>
</tr>
</tbody>
</table>

* Numbers in parentheses (#) are from the old code before numerical improvements were made

The numerical errors discussed above must be taken into consideration when using and developing the position finding program. These errors can be minimized and made insignificant by reducing tolerances and running the numerical optimization or root finding programs until complete convergence is found. Significant time and computing power may be required for complex structures. Numerical errors were reduced to acceptable (insignificant based on manufacturing tolerances) levels for all structures considered in this work.
Chapter 6, Conclusion

Tensegrity structures are fundamentally overconstrained kinematic devices. Tensegrity structures can be formed from a variety of statically determinant or underconstrained devices. In all cases, however, the tensegrity structures exist only in special limiting geometric configurations where freedoms are lost in the base device. Past approaches used to find these tensegrity structure positions were based on static force analysis. This approach is useful for checking a given device to see if it is a tensegrity position, but it does not reveal the mobility of the underlying device. In addition, attempting to form the statics matrix without knowing the link lengths and thus the node positions is a complicated and difficult process. This study, which approached the problem of finding tensegrity positions from a kinematic viewpoint, has yielded some important results. Tensegrity structures can exist and allow prestress because the tensegrity positions exist on the boundaries of nonassembly of the structure. This understanding also allows the tensegrity positions to be easily found.
References


Hanaor, A. Aspects of design of double-layer tensegrity domes Space Structures. v 7 n 2 1992. p 101-113


Snelson, K. Snelson on the tensegrity invention Space Structures. v 11 n 1-2 1996. p 43-48


Vita

William B. Whittier was born on October 9, 1974 in Logan, Utah. He is the son of W. Dee Whittier and Mary Lou Whittier. Will grew up as the oldest of eight children in Blacksburg, Virginia. After graduating from Blacksburg High School in 1992, he went on to begin a Bachelors degree at Brigham Young University in Provo, Utah. In 1993 Will took a two-year leave of absence from his studies to serve a mission in Argentina for The Church of Jesus Christ of Latter-Day Saints. After returning from South America, Will finished his Bachelors degree in Mechanical Engineering at Brigham Young University in June 1998. That fall Will was married to Nicole Fawcett. After graduation, he took a job with Bechtel Bettis Inc. at the Naval Reactors Facility. Will worked in the facility engineering group in Idaho Falls, Idaho for two and one half years before returning to school to pursue a Master’s degree. Will completed his Master’s degree in Mechanical Engineering at Virginia Polytechnic Institute and State University in December 2002.