Appendix B Equations for Residual Forces

Although numerous papers pertaining to the FE equations governing pore fluid flow in a deforming porous solid are available, none outline the equations required to calculate the residual forces. In this section, these equations are developed for a nonlinear incremental finite element program that employs a modified Newton-Raphson solution scheme.

The time integration of Equations 2.5 and 2.6 is performed using the following approximation:

\[
\int_{t}^{t+\Delta t} \chi \, dt = \alpha \Delta t \frac{t^+\Delta \chi}{\Delta t} + (1 - \alpha) \Delta t \frac{t^\chi}{\Delta t}
\]

B.1

for \(0 \leq \alpha \leq 1\). From Equation B.1, the following are developed:

\[
t^+\alpha \Delta \chi = \frac{t^+\Delta \chi}{\Delta t} - \frac{t^\chi}{\Delta t}
\]

B.2

and

\[
t^+\alpha \Delta \chi = (1 - \alpha) t^\chi + \alpha t^+\Delta \chi
\]

B.3
### Table B.1. Time Integration Parameters

<table>
<thead>
<tr>
<th>Value $\alpha$</th>
<th>Difference Scheme</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Forward or Euler</td>
<td>Conditionally</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>Crank-Nicolson</td>
<td>Unconditionally</td>
</tr>
<tr>
<td>$%$</td>
<td>Galerkin</td>
<td>Unconditionally</td>
</tr>
<tr>
<td>1</td>
<td>Backward</td>
<td>Conditionally</td>
</tr>
</tbody>
</table>

Table B.1 gives the most common difference schemes adopted by the selection of a given value of $\alpha$. Equation 3.5 may be written for a given time $t + \alpha \Delta t$ as:

$$
t^{\alpha \Delta t} [K] t^{\alpha \Delta t} \ddot{u} - t^{\alpha \Delta t} [L] t^{\alpha \Delta t} \dot{\pi} = t^{\alpha \Delta t} \dot{R} .
$$

Equations B.2 and B.3 are introduced into Equation B.4 to produce the following

$$
t^{\alpha \Delta t} [K] \left\{ t^{\alpha \Delta t} u - t \dot{u} \right\} - t^{\alpha \Delta t} [L] \left\{ t^{\alpha \Delta t} \pi - t \pi \right\} = t^{\alpha \Delta t} R - \dot{R} .
$$

Multiplying by $\Delta t$ and collecting terms, one obtains

$$
t^{\alpha \Delta t} [K] \left\{ t^{\alpha \Delta t} u - t \dot{u} \right\} - t^{\alpha \Delta t} [L] \left\{ t^{\alpha \Delta t} \pi - t \pi \right\} = t^{\alpha \Delta t} R - \dot{R} .
$$

In general, Equation B.6 represents nonlinear behavior. The relationship may be linearized with the following expressions:

$$
t^{\alpha \Delta t} u^{(i)} = t^{\alpha \Delta t} u^{(i-1)} + \delta u^{(i)} \quad \text{B.7}
$$

$$
t^{\alpha \Delta t} \pi^{(i)} = t^{\alpha \Delta t} \pi^{(i-1)} + \delta \pi^{(i)} \quad \text{B.8}
$$

where $i$ represents the current iteration, and the initial conditions are $t^{\alpha \Delta t} u^{(0)} = t \dot{u}$ and
$t+\Delta t \pi^{(0)} = t \pi$. This linearization can be used as the first step in a Newton-Raphson iteration (Bathe 1982). If Equations B.7 and B.8 are substituted into Equation B.6 and the terms for iterations $(i-1)$ and $(0)$ are brought to the right hand side of the equation, then one obtains:

\[
(t+\Delta t) [K] \delta u^{(i)} - (t+\Delta t) [L] \delta \pi^{(i)} = (t+\Delta t) R - t R \\
- (t+\Delta t) [K] \left\{ (t+\Delta t) u^{(i-1)} - (t+\Delta t) u^{(0)} \right\} \\
+ (t+\Delta t) [L] \left\{ (t+\Delta t) \pi^{(i-1)} - (t+\Delta t) \pi^{(0)} \right\} .
\]

Recognizing that

\[
(t+\Delta t) R = (t+\Delta t) [K] (t+\Delta t) u^{(0)} - (t+\Delta t) [L] (t+\Delta t) \pi^{(0)}
\]

Equation B.9 may be written in terms of the incremental or accumulative stresses and pore pressures as

\[
(t+\Delta t) [K] \delta u^{(i)} - (t+\Delta t) [L] \delta \pi^{(i)} = \\
(t+\Delta t) R - (t+\Delta t) F^{(i-1)} + (t+\Delta t) C^{(i-1)}
\]

where

\[
(t+\Delta t) F^{(i-1)} = \int_V B^T (t+\Delta t) \sigma^{(i-1)} dV
\]

and

\[
(t+\Delta t) C^{(i-1)} = (t+\Delta t) [L] (t+\Delta t) \pi^{(i-1)}.\]

Equation B.11 is one of the two equations required to solve for pore fluid flow in a deforming nonlinear porous solid. The second equation is developed from Equation 3.6 and is written at time $t+\alpha \Delta t$ as:

\[
[H] (t+\alpha \Delta t) \pi - [S] (t+\alpha \Delta t) \pi - [L]^T (t+\alpha \Delta t) \dot{u} = (t+\alpha \Delta t) Q.
\]

Equations B.2 and B.3 are introduced into Equation B.12 to produce
Appendix B  Equations for Residual Forces

\[
[H]{(1 - \alpha)^t \pi + \alpha^{t + \Delta t} \pi} - [S]\left\{\frac{t^{\Delta t} \pi - t \pi}{\Delta t}\right\}
\]

\[
- [L]^T \left\{\frac{t^{\Delta t} u - t u}{\Delta t}\right\}
\]

\[
= (1 - \alpha)^t Q + \alpha^{t + \Delta t} Q .
\]

Collecting terms and multiplying by \(\Delta t\) one obtains:

\[
\{ -[S] + \alpha \Delta t [H] \}^{t + \Delta t} \pi - [L]^T \left\{t^{\Delta t} u - t u\right\}
\]

\[
+ \{[S] + (1 - \alpha) \Delta t [H]\} t \pi = P
\]

where \(P = \Delta t\{ (1 - \alpha)^t Q + \alpha^{t + \Delta t} Q \}\). If the following equations

\[
\hat{S} = -[S] + \alpha \Delta t [H]
\]

and

\[
\hat{H} = [S] + (1 - \alpha) \Delta t [H] = \Delta t [H] - \hat{S}
\]

are substituted into Equation B.14 for simplification, then one obtains

\[
\hat{S}^{t + \Delta t} \pi - [L]^T \hat{S}^{t + \Delta t} u = P - \hat{S}^{t \pi} - [L]^T t \pi .
\]

Equation B.17 must now be formulated for general nonlinear behavior using the same linearization process applied to Equation B.6, i.e., Equations B.7 and B.8 must be substituted. This operation produces

\[
^{t + \Delta t} \hat{S}^{t + \Delta t} \pi^{(i-1)} + \delta \pi^{(i)} - ^{t + \Delta t}[L]^T \left\{^{t + \Delta t} u^{(i-1)} + \delta u^{(i)}\right\}
\]

\[
= ^{t + \Delta t} P - ^{t + \Delta t} \hat{H}^{t + \Delta t} \pi^{(0)} - ^{t + \Delta t}[L]^T^{t + \Delta t} u^{(0)} .
\]
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Collecting the $\delta u$ and $\delta \pi$ terms on the left hand side of the equation, one gets

$$
t^+ \Delta t \hat{S} \delta \pi^{(i)} - t^+ \Delta t [ L ]^T \delta u^{(i)} = t^+ \Delta t \bar{P} - t^+ \Delta t \hat{H} t^+ \Delta t \pi^{(0)}$$

Equations B.15 and B.16 can be used to eliminate the terms $\hat{H}$ and $\hat{S}$ from the right hand side of Equation B.19 to produce

$$
t^+ \Delta t \hat{S} \delta \pi^{(i)} - t^+ \Delta t [ L ]^T \delta u^{(i)} = t^+ \Delta \bar{P}$$

Equations B.11 and B.20 are written in matrix form for increment $t+\Delta t$ as:

$$
\begin{bmatrix}
K & -L \\
-L^T & -S + \alpha \Delta t H
\end{bmatrix}
\begin{bmatrix}
\delta u^{(i)} \\
\delta \pi^{(i)}
\end{bmatrix}
= 
\begin{bmatrix}
R - F^{(i-1)} + C^{(i-1)} \\
P - \Delta t [ H ] \pi^{(0)} - G^{(i-1)} + M^{(i-1)}
\end{bmatrix}
$$

This is the system of equations that must be solved to calculate displacements and pore fluid pressures in a deforming porous solid.