CHAPTER I
INTRODUCTION

1.1 Bivariate Longevity (Failure) Modeling

People frequently discuss equipment behavior in terms of age and usage. Common examples are automobiles and automobile tires in which model year and accumulated mileage are usually both included in discussions of longevity. Less well recognized examples for which two measurement scales are quite important include factory equipment, power generation machines and aircraft. In fact, the use of many of the devices reliability specialists study is meaningfully described in terms of two measures.

Years of usage and mileage are not the only two quantities that might describe device longevity. We use the terms age and use here but these terms are generic and may represent quite different measures than duration of ownership and distance traveled. In the example of an automobile tire, age might correspond to accumulated mileage and usage might be measured as tread loss. Even more complicated measures such as current flow and thermal history may be appropriate for some integrated circuits. The point is that device life is a resource that may be best represented and for which the consumption may best be measured using a two (or higher) dimensional vector and the quantities that comprise the vector are specific to the equipment.

Some reliability models that respond to the need to include both age and use have been developed. However, nearly all of the models that have been developed are defined in a manner that permits them to be reduced to single dimensional models in which the independent variable is time. Some of the existing models are discussed further in Chapter 2. A structured examination of bivariate measures of equipment utilization has
not been performed so bivariate models have not been fully developed and some potentially useful model forms have not been defined at all.

Our intent in this research is to provide a framework for bivariate device modeling. We begin with the construction of bivariate failure models. Issues related to the creation of bivariate failure models that have and those that have not been developed are examined. We provide a brief taxonomy of bivariate device longevity model forms and define some example models to illustrate and explain some of the key issues. These bivariate failure models play a key role in our bivariate modeling. The key motivating factor is the fact that existing bivariate failure models are defined so that they can be collapsed into single variable life time distribution models despite the fact that bivariate models may be more representative or descriptive of device longevity.

1.2 Bivariate Maintenance Modeling

Maintenance can significantly affect the quality of products after they have been produced. A maintenance policy can be viewed as a combination of different maintenance actions such as inspection, repairs, and/or replacements; and may be classified as corrective (unscheduled) maintenance or preventive (scheduled) maintenance. A corrective maintenance policy includes all unscheduled maintenance actions performed as a result of system failure to restore the system to a specified condition. Preventive maintenance includes all scheduled maintenance actions performed to retain a system in a specified condition. Usually, the cost of maintenance and support is 60 to 75% of the total life-cycle cost of a system. Therefore, a preventive maintenance policy is of great importance in reducing this cost and lowering the risk of a catastrophic breakdown.

Most studies of maintenance policies (corrective or preventive) concentrate on systems with univariate lifetime distributions, i.e., based on univariate failure models.
However, in many practical situations, system lifetimes and maintenance policies depend on several factors. For example, it is common for automobiles to carry a maintenance policy for an oil change every three months or 3,000 miles, or a replacement of the camshaft timing belt every 60 months or 60,000 miles.

The cumulative usage of a system over different periods of time will have different effects on the system failure behavior. Furthermore, most systems deteriorate so that the use and time between repairs become shorter and shorter and the repair times increase. Some systems, like cars, airplanes, light bulbs, computers, or any object that can accumulate use due to environmental effects, may age or wear over time. A system may be in an inadequate operational condition after some amount of time or usage. The system may require a maintenance action to restore it back to a normal condition. The purpose is to prevent the occurrence of a catastrophic failure during its operation and to reduce the overall maintenance cost. However, very limited literature has addressed multidimensional failure models and their related maintenance policies.

For modeling bivariate maintenance policies, we first adapt and extend Hunter’s bivariate renewal theory [1974a] to facilitate the construction and analysis of bivariate maintenance models. Some basic results of the bivariate renewal theory are presented and applied to the bivariate maintenance models. Two types of bivariate maintenance policy are considered. These are corrective maintenance (CM) and preventive maintenance (PM) policies. The corresponding bivariate maintenance models are developed based on the bivariate device longevity models. Laplace transform techniques are used to analyze the models. For most of the developed models, we are not able to invert the transform explicitly. New numerical methods and algorithms that may be used to approximate the bivariate Laplace inverse transforms are identified.
1.3 Bivariate Availability Modeling

Availability has appeared to be an appropriate measure of the effectiveness of a maintained system. It considers the failure behaviors and the effects of maintenance actions. In the univariate case, availability is defined as the probability that the system is functioning satisfactorily at any point of time (Lie, Hwang, and Tillman [1977]). Bivariate availability can be defined and extended directly from the univariate availability.

Following bivariate device longevity and maintenance modeling, we consider the construction of bivariate device availability models. Based on the two types of maintenance policies, we develop the corresponding bivariate availability models. The Laplace transforms for the bivariate availability models are derived. Direct inversions of these transforms are not obtained here.

1.4 Approach to Bivariate Modeling

The approach to bivariate modeling may be organized into five key efforts:

(i) **Bivariate Failure Modeling.**
Model a single-unit system with bivariate longevity.

(ii) **Bivariate Renewal Modeling.**
Model a single-unit system with independent and identically distributed (i.i.d.) lifetimes and immediate/instantaneous maintenance services.
(iii) *Bivariate Corrective Maintenance Modeling.*

Model a single-unit system with i.i.d. lifetimes and i.i.d. corrective maintenance times.

(iv) *Bivariate Preventive Maintenance Modeling.*

Model a single-unit system with i.i.d. lifetimes and preventive maintenance as well as repair but both having distinct i.i.d. distributions.

(v) *Bivariate Availability Modeling.*

Model the bivariate availability measure for a single-unit system with maintenance services.

We describe each modeling stage in detailed as follows.

(i) *Bivariate Failure Modeling.*

Note that in this stage we focus on constructing bivariate failure models rather than maintenance policies. Our purpose is to improve univariate failure models by developing and analyzing bivariate failure models. We define the general structure of bivariate probability models of system failure. Several bivariate failure models are constructed to represent the possible correlation structures of the two system aging variables: time and usage. The behavior of these models is examined under the various correlation structures. The developed models will be used to analyze example maintenance problems.

(ii) *Bivariate Renewal Modeling.*

Consider a system with a bivariate random longevity, i.e., the longevity depends on the operating time and the usage. A failed system is immediately and instantaneously replaced by an i.i.d. new one. Thus, we assume that the maintenance is perfect and the
system operation can be modeled as a bivariate renewal process. Using bivariate renewal theory, we model and analyze the system failure behavior.

(iii) **Bivariate Corrective Maintenance Modeling.**

The bivariate renewal models developed in stage (ii) are extended to include repair times. The maintenance times (may include inspection, repair, and/or replace times) are assumed to be i.i.d. and the system after maintenance is assumed to be "as good as new." The combined bivariate lifetime-and-maintenance-time process may be modeled as an alternating bivariate renewal process or simply an ordinary bivariate renewal process. To construct this model, bivariate renewal theory may be used in the analysis of system failure behavior. Here we consider repair times but not preventive maintenance.

(iv) **Bivariate Preventive Maintenance Modeling.**

We modify and generalize the bivariate corrective maintenance models developed in stage (iii) to reflect the effects of preventive maintenance by developing maintenance models with i.i.d. preventive maintenance times and i.i.d. repair times. That is we construct bivariate preventive maintenance models. These models are examined under an age replacement preventive maintenance policy. Other policies such as group-replacement, opportunistic replacement, and/or the combination of policies may be modeled in a similar way.

(v) **Bivariate Availability Modeling.**

For the developed corrective and preventive maintenance models, we construct the Laplace transforms for their corresponding bivariate availability models. Some example bivariate availability functions are presented with the derived bivariate failure models. The idea of the quality of availability measures is defined in terms of bivariate availability models.
Stages (i) to (v) demonstrate a framework in bivariate reliability and maintenance modeling. The results from stages (i) to (v) provide a foundation for further study of bivariate and multivariate models.

1.5 Outline of Research

In Chapter 2, we provide a literature review for bivariate reliability modeling, bivariate renewal theory, and maintenance modeling for one-unit systems and multi-unit systems. There is very limited study in bivariate failure modeling; and none in bivariate maintenance modeling and bivariate availability measures.

In Chapter 3, we present a taxonomy of bivariate failure model classes and identify two classes as our focus. The stochastic function models and correlated models are developed and examined. Some issues related to model formulation, analysis, and further study are discussed. Some example numerical calculations are provided.

In Chapter 4, we study a bivariate renewal process and present its basic results based on Hunter’s bivariate renewal theorem. Some generalizations of this theorem are presented. We obtain the renewal function and renewal density for ordinary bivariate renewal, bivariate quasi-renewal, delayed bivariate renewal, and alternating bivariate renewal processes. Bivariate Laplace transforms associated with these renewal processes are obtained.

In Chapter 5, we present some examples for the bivariate corrective maintenance models. We apply bivariate renewal theory to corrective maintenance modeling. Renewal models and corrective maintenance models are obtained with consideration of the bivariate failure models developed in Chapter 3.

In Chapter 6, we further extend the corrective maintenance models to reflect the effects of preventive maintenance policies. An age replacement preventive maintenance is considered. General results are obtained. Examples of the bivariate preventive
maintenance models are presented in Chapter 7 with the consideration of the bivariate failure models developed in Chapter 3.

In Chapter 8, we develop bivariate availability for the bivariate corrective and preventive maintenance models. The Laplace transforms for the bivariate availability models are obtained. Issues related to the quality of availability measure are discussed. Examples for the bivariate availability models are presented in Chapter 9.

Conclusions and future research are summarized in Chapter 10.
CHAPTER II
LITERATURE REVIEW

Chapter II provides a historical overview of some of the existing literature on maintenance policies, bivariate renewal processes, and bivariate failure models. The existing literature about maintenance policies does not treat bivariate (or multivariate) maintenance in the sense studied here. Literature about availability measures in the two-dimensional case does not exist.

2.1 Introduction

In the last few decades, maintenance policies have been an active area of research. Pierskalla and Voelker [1976], Sherif and Smith [1981], Cho and Parlar [1991], and Murdock [1995] have presented a detailed survey of much of the existing work. Nachlas [1998], Barlow and Proschan [1975], and Gertsbakh [1977] presented maintenance and replacement models in a reliability context.

Maintenance problems can be classified by the complexities of the systems and their associated maintenance actions, namely, single-unit vs. multi-unit system maintenance problems, perfect vs. imperfect maintenance problems, and instantaneous vs. non-instantaneous repair maintenance problems. For example, the single-unit system maintenance model represents a single-unit system with independent and identically distributed (i.i.d.) lifetimes under perfect maintenance policies with immediate replacements. Typically, in this model, i.i.d. maintenance service times are also assumed and the system availability can be obtained. Furthermore, the i.i.d. assumptions (for both system lifetimes and maintenance times) can be generalized to model maintenance
problems with distinct maintenance service times. Multi-unit maintenance problems follow similar procedures and approaches. In both categories we can also consider different system configurations and/or structures to model more complicated equipment.

Before the further review of literature, we first provide some definitions of maintenance in brief.

2.1.1 Definitions of Maintenance

**Definition 2.1.1. Bathtub-shaped failure rate (BFR) distribution.**
A life distribution $F(t)$ is said to be a bathtub-shaped failure rate (BFR) distribution if:

$$F(t) \text{ is said to be a } \begin{cases} \text{decreasing failure rate (DFR)} & 0 \leq t < t_1 \\ \text{constant failure rate (CFR)} & t_1 \leq t < t_2 \\ \text{increasing failure rate (IFR)} & t_2 \leq t < \infty \end{cases}$$

The time interval $[0, t_1)$ is called the early failure or infant mortality period, the interval $[t_1, t_2)$ is called the useful or functional life, and the interval $[t_2, \infty)$ is called the wear-out period.

**Definition 2.1.2. Hazard function.**
The hazard function (or the failure rate function) is defined as:

$$z(t) = \frac{f(t)}{F(t)}. \quad (2.2)$$

where $f(t)$, the density function, exists and is defined as $\frac{d}{dt}F(t)$; and $F(t)$, the reliability (or survivor) function, is defined as $1 - F(t)$.
**Definition 2.1.3.** Decreasing, constant, and increasing failure rate (DFR, CFR, and IFR) distributions.

A life distribution $F(t)$ is said to be a DFR distribution if:

$$\frac{d}{dt} z(t) \leq 0, \text{ for } 0 \leq t < \infty.$$  \hspace{1cm} (2.3)

A life distribution $F(t)$ is said to be a CFR distribution if:

$$\frac{d}{dt} z(t) = 0, \text{ for } 0 \leq t < \infty.$$  \hspace{1cm} (2.4)

A life distribution $F(t)$ is said to be a IFR distribution if:

$$\frac{d}{dt} z(t) \geq 0, \text{ for } 0 \leq t < \infty.$$  \hspace{1cm} (2.5)

The function $z(t)$ is the hazard function (or the failure rate function).

**Definition 2.1.4.** Corrective Maintenance: perfect repair, minimal repair, and imperfect repair.

Let $z_0 = z(t = 0)$ be the hazard function for a new system. Then, defining $z(t) = \delta z_0$ as the hazard function for a system after service at time $t$. If $\delta = 1$, then the system undergoes a perfect repair. If $\delta = z(t)/z_0$, then the system undergoes a minimal repair. If $\delta = c$, where $c$ is a constant and $c > 1$, then the system is imperfectly repaired.

**Definition 2.1.5.** Preventive maintenance: age replacement and block replacement.

Under an age replacement policy, a unit $i$ is replaced, with an i.i.d. new one, upon failure or at age $Ta_i$, whichever comes first. Under a block replacement policy, a unit $i$ is replaced, with an i.i.d. new one, upon failure and at age $Tb_i, 2Tb_i, 3Tb_i, \ldots$
**Definition 2.1.6.** *Opportunistic maintenance.*

Opportunistic maintenance occurs when unit $i$ has reached an age $T_o$, and there are other units in the system being replaced due to either corrective maintenance (CM) or preventive maintenance (PM).

**Definition 2.1.7.** *Group maintenance.*

By replacing groups of failed units instead of replacing individual failed units, group maintenance is performed either when a fixed time interval $T_g$ is expired or when a fixed number of units ($N_g$) fail, whichever comes first.

### 2.2 Bivariate Reliability Models

There are certainly many ways to define a bivariate reliability model. In addition, there are probably several alternate ways to classify the model types. We feel that an informative classification scheme is based on the relationship between the two variables. Specifically, we distinguish between those models for which age and use are functionally related and those in which they are correlated rather than functionally dependent.

We further separate the models in which the two variables are functionally related on the basis of whether the functions are deterministic or stochastic. The models based on correlation of the two variables may be further classified by whether $\rho = 0$ or $\rho \neq 0$. Of course, from a reliability perspective, the case in which age and use are independent is unlikely to be practically meaningful.

Most of the previously developed bivariate reliability models treat the two variables as functionally related. Many of the models of wear process (e.g., Mercer [1961] and Lemoine and Wenocur [1985]) and several of those for cumulative damage (e.g., Barlow and Proschan [1975] and Birnbaum and Saunders [1969]) portray equipment reliability in
terms of deterministically defined deterioration occurring at random points in time. Important defining features of all of these models are:

1. they focus on a single failure mechanism or phenomenon,
2. the use variable determines failure by a use threshold value, and
3. they are ultimately reduced to univariate time models on the basis of the functional dependence. The focus of the models is to determine how much time will elapse before the use threshold is surpassed.

An interesting and not completely obvious observation is that while the wear process and cumulative damage models treat use as a function of time, the proportional hazards models (Leemis [1995]) treat age as a deterministic function of use covariates. Because the models based upon deterministic functions are reduced to univariate forms, they have been studied extensively and are not examined further here.

The analytical emphasis with models based on stochastic functions has also been their reduction to a single dimension - time. The stochastic wear models and cumulative damage models that treat damage magnitude as a random variable are all defined in a manner that permits focus on reliability in time. The same is true of the shot noise models, Lemoine and Wenocur [1986]. Even the diffusion process models (Cox [1962]) that really are expressed comprehensively in terms of both variables are analyzed in terms of first passage time to a failure state. Finally, the time dependent stress-strength interference models (Kapur and Lamberson [1977]) have the same characteristic that they are used to obtain a distribution in time.

It is appropriate to note that there are several papers that address bivariate and multivariate reliability models in a very different context than the one treated here. Specifically, Marshall and Olkin [1967a, 1967b] developed multivariate models for the reliability of series systems comprised of non-independent components. In these models, each variable corresponds to the age of one of the components. In the construction of the models, Marshall and Olkin provide some useful definitions but do not treat bivariate (or multivariate) reliability in the sense studied here.
It is Singpurwalla and Wilson [1993] who were the first to suggest the study of two-dimensional renewal processes with the development of bivariate failure models indexed by time and usage. They followed the results of Singpurwalla [1992] to introduce a generic bivariate model for failure and to present a general expression for a class of densities indexed by time and usage. They used this formula as a basis to simulate the density for a wide range of specifications for the usage process; and applied the bivariate failure models to the study of optimal warranty problems by using Monte Carlo simulations of the expected number of renewals (number of perfect repairs) via a bivariate warranty policy. They then formulated a game theoretic set-up for determining optimal warranties and used dynamic linear models to forecast warranty claims of systems indexed by time and usage.


It appears that the only effort to study a true bivariate reliability model was that of Eliashberg, Singpurwalla, and Wilson. [1997]. They developed models using both deterministic and stochastic functions and obtained useful results. Their focus was the determination of necessary warranty reserves for products such as automobiles. Under i.i.d. assumptions for system lifetime and usage, they used bivariate renewal theory to specify the probability generating function of the distribution of the number of warranty claims for an item. By utilizing Monte Carlo simulations, they then obtained approximate results of the number of warranty claims and their density functions for an item under imperfect repair. In our development of bivariate failure models, we include their models and show how their approach can be extended to define other models.

Singpurwalla and Wilson [1998] proposed a strategy for constructing bivariate reliability models that describe failure in terms of time and usage in a similar way to the models of Eliashberg, Singpurwalla, and Wilson [1997]. By using an additive hazard
function, they model the relationship between the variables and treat use as a time-dependent variable. They describe the evolution of usage by stochastic processes like the Poisson, the gamma, and the Markov additive; and developed the bivariate failure models for usage described by the Poisson, the doubly stochastic Poisson, and a compound Poisson process. They then use the model to solve the warranty problem by Bayesian statistical decision theory to obtain the optimal warranty. We include and extend their approach to the construction of bivariate failure models.

2.3 Bivariate Renewal Theorem

Chung [1952] was the first to study and extend the classical univariate renewal theorem to higher dimensions. Hunter [1974a, b] studied a renewal process in two dimensions and developed further results. He discussed the bivariate generating functions and bivariate Laplace transforms as the basic tools to generalize the classical theory of univariate renewal processes. He presented an example of a bivariate exponential distribution to illustrate the general theory and developed explicit expressions of the two-dimensional renewal function, as well as the correlation between the marginal univariate renewal counting processes. The renewal theorem for the multi-dimensional case can be found in Mode [1967], Spitzer [1986], and Steinebach and V.R. Eastwood [1996].

The mathematical development and applications of the classical univariate renewal theorem can be found in Çinlar [1975], Cox [1962], Ross [1996], and Smith [1958].
2.4 Maintenance models for one-unit systems

A single-unit model (or one-unit) represents a complicated system as a single entity. The term "single-unit" (or "one-unit") system used here means that the system (or product) is viewed collectively and that only the performance of the system as a whole is considered, not the performance of the individual components. Nevertheless, the system is a collection of the components so components affect the system. The system itself can be composed of one device or many dissimilar devices. A complex system can also be viewed as a one-unit system when the failure of any component is interpreted as failure of the entire system. In practice, sometimes it is difficult to obtain the reliability data for individual components; whereas data for the stochastic behavior of the entire system is available or easier to obtain. Valdez-Flores and Feldman [1989] provide a detailed survey of preventive maintenance models for stochastically deteriorating single-unit systems.

It is more important to note that a one-unit system is viewed and treated as a building block for a multi-unit system. The methods for analyzing single-unit systems form the basis for analyzing multiunit systems.

For one-unit (or single-unit) systems, Barlow and Proschan [1964] show that, assuming an IFR (DFR) (increasing (decreasing) failure rate) unit failure distribution, the number of failures in \([0, t]\) is stochastically larger (smaller) with an age replacement policy than with a block replacement policy. They also showed that the number of planned replacements and the total number of removals is always stochastically smaller under an age policy than under a block policy. Cox [1962] suggested allowing the system to remain inactive if a failure occurs in \([k-1]T + T_0, kT\) for any \(k\) and for some time \(T_0\). Sheu [1996] proposed a modified block replacement policy with two variables and general random minimal repair cost. He extended Cox's model such that if the unit fails in \([k-1]T, (k-1]T + T_0)\) it is either replaced by a new one or minimally repaired, and if it fails in \([k-1]T + T_0, kT\) it is either minimally repaired or remains inactive until the
next planned replacement. The model with two variables is transformed into a model with
one variable to obtain the optimal policy.

2.4.1 Imperfect maintenance

applications in imperfect maintenance. They defined a counting process \{N(t), t > 0\} and
let \(X_n\) denote the interarrival time between the \((n - 1)st\) and \(n\th\) events of the process, for
\(n \geq 1\). If the sequence of non-negative random variables \(\{X_n, n = 1, 2, 3, \ldots\}\) is
independent and \(X_n = \alpha^{n-1}Z_n\), for \(n = 1, 2, 3, \ldots\), where \(Z_n\) are independent and identically
distributed and \(\alpha > 0\) is a constant, then the counting process \(\{N(t), t > 0\}\) is said to be a
quasi-renewal process with parameter \(\alpha\) and first interarrival time \(X_1\). They obtained the
optimal imperfect block replacement policies for \(0 < \alpha \leq 1\) based on the expected
maintenance cost per renewal cycle and/or the limiting average availability.

Brown and Proschan [1983] considered an imperfect repair model in which an item
is repaired upon failure. With probability \(p\), the repair is a perfect repair, and with
probability \(1 - p\), the repair is minimal repair, i.e., after repair the system is "as bad as old". Block, Broges, and Savits [1985] extended the model of Brown and Proschan to the age-dependent imperfect repair model in which an item is repaired upon failure. With probability \(p(t)\), the repair is a perfect repair, and with probability \(q(t) = 1 - p(t)\), the repair
is a minimal repair, where \(t\) is the age at the failure time since the last perfect repair of the
item in use.

Lam [1988a, 1988b] considered a restricted replacement model with partial
repairs that also accounted for the repair times. He introduced the geometric process
which is a sequence of independent non-negative random variables, \(\{X_n, n = 1, 2, 3, \ldots\}\),
such that the distribution function of \(X_n\) is \(F(a^{n-1}x)\), where \(a\) is a positive constant. If \(a > 1\), then it is a decreasing geometric process, if \(a < 1\), it is an increasing geometric
process. Then, he studied the restricted repair replacement model for a deteriorating system, in which the successive survival times of the system form a geometric process and are stochastically non-increasing, whereas the consecutive repair times after failure also constitute a geometric process but are stochastically non-decreasing. He considered, simultaneously, two types of replacement strategies: (a) replace the system if and only if its total operating time attains a certain predetermined level, (b) replace the system at the Nth failure. He obtained explicit expressions for the long-run average costs of these strategies.

Stadje and Zuckerman [1990] considered additional monotone process replacement models. They presented two replacement policies, the policy \( T \) and policy \( N \). Under the replacement policy \( T \), a system is replaced at a time \( T \), which depends on the time from the installation or the last replacement. Under policy \( N \), a system is replaced at the time of the Nth failure. They showed that the optimal policy \( T^* \) belongs to the policy class \( N \) for the long-run average reward criterion.

Lam [1990] gave an explicit formula for determining the optimal policy \( N^* \) and in [1991] studied a repair replacement model for a stochastically deteriorating system. For the expected discounted reward case, he showed that the optimal replacement policy is of the form "replace at the time of the Nth failure." Lam [1992] generalized his earlier work [1988a, 1988b] to study a general repair replacement model in which two types of replacement policies are examined under the expected discounted reward criterion. He showed that the optimal policy \( N^* \) is at least as good as the optimal policy \( T^* \). Zhang [1994] generalized Lam's work [1988a] with the bivariate policy \((T, N)\) under which the system is replaced at wearing age \( T \) or at the time of the Nth failure, whichever occurs first. He considered the problem of choosing an optimal replacement policy \((T, N)^*\); such that, the long-run average cost per unit time is minimized. Under some mild conditions, he proved that the optimal policy \((T, N)^*\) is better than the optimal policy \( N^* \) or the optimal policy \( T^* \).
Mi [1998] compared system availability measures of a single-unit system based on stochastic orderings and classifications of lifetime distributions. The comparison results are useful in determining maintenance policies for improving or optimizing the system operation interval. Murdock [1995] applied renewal theory to develop an availability model over a finite time horizon for a continuously demanded component, which is maintained by an age replacement preventive maintenance policy. He showed that the optimal age replacement period in an infinite time horizon does not maximize average availability for all finite values of component economic life. The result is critical in life-cycle maintenance planning.

2.5 Maintenance models for multi-unit systems

In many applications, the optimal maintenance actions for one component often depend on the state of the other components and the system reliability requirements. The assumption of perfect maintenance to multi-unit systems is no longer valid unless we assume the renewal of all units (or components). Thus, the imperfectness of maintenance (for imperfect maintenance) and economic dependency (for opportunistic or group maintenance) are the major issues for modeling multi-unit system maintenance problems.

2.5.1 Imperfect maintenance

Shaked and Shanthikumar [1986] and Sheu and Griffith [1992] extended the univariate imperfect repair model to a multivariate imperfect repair model. They modeled systems with dependent components, having specific multivariate distributions, each of which undergoes imperfect repair. Shaked and Shanthikumar [1986] considered models of systems comprised of components having dependent lifetime distributions. Upon failure, the component is imperfectly repaired until it is perfectly repaired. They
then generalized their models to cover applications where more than one component can fail at the same time. Sheu and Griffith [1992] considered a bivariate non-age-dependent imperfect repair model in which two items start to function at the same time. Upon failure, an item undergoes a repair. They considered different external sources that cause the failure of these two items.

2.5.2 Opportunistic maintenance

Degbotse [1996] studied opportunistic age replacement policies for a two-unit system, and presented availability analysis using a nested renewal theory approach. The system availability function for the failure replacement policy, the opportunistic failure replacement policy, partial opportunistic age replacement policy, and the opportunistic age replacement model. He compared various replacement policies and showed that the failure replacement policy provides a higher availability than the other replacement policies.

Wang [1997] studied the \((\tau, T)\) opportunistic maintenance of \(k\)-out-of-\(n\) systems with the consideration of system reliability requirements. In his models only minimal repairs are performed on failed components before time \(\tau\) and corrective maintenance of all failed components are combined with preventive maintenance of all functioning but deteriorated components after \(\tau\). If the system survives to time \(T\) without perfect maintenance it will be subject to preventive maintenance at time \(T\). He derived the system asymptotic cost rate and availability with applications to aircraft engine maintenance.

2.5.3 Group maintenance

By replacing groups of failed units instead of replacing individual failed units, maintenance cost may be reduced. This cost saving, known as the economy of scale, results mostly from the quantity discount or reduction of maintenance set-up cost per unit. Sheu and Jhang [1996] studied a two-phase maintenance policy for a group of
identical repairable units, which incorporates minimal repair, overhaul, replacements, and downtime costs. They developed a model to calculate the long-run average cost per unit time for a generalized group maintenance policy.

Nachlas and Rao [1992] consider group maintenance for a series system. They develop a method for determining a maintenance schedule and grouping components. The method is approximate and heuristic and is based on a cost model which is defined in terms of component reliability measures. The modeling framework provides a analytical method for analyzing preventive maintenance scheduling problems and selecting optimal or near optimal system level maintenance plans.

2.6 Availability

Availability is usually used as the effectiveness or readiness measure for repairable and/or maintained systems because of its consideration of both reliability and maintainability. Barlow and Proschan [1975] define availability of a repairable system as “the probability that the system is operating at a specified time t.” Blanchard [1998] gives a qualitative definition of availability as “a measure of the degree of a system which is in the operable and committable state at the start of mission when the mission is called for at an unknown random point in time.” Lie, Hwang, and Tillman [1977] give a complete survey and systematic classification of availability. The classifications of availability are, first, depending on the time interval considered: (i) point (or instantaneous) availability, (ii) average availability, (iii) limiting (or steady-state) availability, (iv) limiting average availability, (Barlow and Proschan [1975], Lie, Hwang, and Tillman [1977], Nachlas [1998]); second, considering the types of downtime: (v) inherent availability, (vi) achieved availability, (vii) operational availability (Blanchard [1998], Lie, Hwang, and Tillman [1977]). Mi [1998] give some comparison results of
availability in the first category. The first classification of availability will be our focus in this dissertation and will be extended to two dimensions.


2.7 Remark on Literature Review

The literature review described above shows that many aspects in reliability modeling and maintenance analysis have been considered. Some of them have considered aspects of the problems we study but no one has addressed the full and coherent construction of bivariate failure repair and preventive maintenance models.
CHAPTER III
BIVARIATE FAILURE MODELING

In this chapter, issues related to the construction of bivariate reliability models and their application to maintenance planning are discussed. The distinction between bivariate failure models and models of first passage time to a failure threshold clarifies the motivation for the development of bivariate failure models. We identify two classes as our focus. The model classes examined here are those in which the two variables are related by a stochastic function and those in which the variables are simply correlated. Examples of the models of each of the two classes are defined. The general approach to model formulation is explained so that the reader may construct alternate forms.

In our view, the types of bivariate failure models described here provide a new way to study the reliability of equipment for which univariate measures are incomplete. Thus, a new area of reliability research is identified. The definitions we offer may be modified and the approach to model formulation we present may be used to define other models. We raise several open questions concerning model construction and analysis. Both conceptual definitions and analytical methods warrant further exploration.

3.1 Introduction

In Chapter 2, the discussion of bivariate reliability models provides our view of how bivariate failure models should be classified. It also indicates the types of models that already exist. We feel that the models that have been defined have been very useful in the study of equipment reliability but that there are problems for which they are not well suited. Specifically, the definition of preventive maintenance plans on the basis of
both use and age requires a bivariate failure model. Therefore, in this chapter, we explore
the definition of bivariate failure models. We consider both the cases of stochastic
functional relationships and simple correlation between the age and use variables.

Our intent in this chapter is to construct bivariate failure models and to examine the
issues related to the creation of the model types that have not yet been developed. We
define some example models and identify several questions that must be resolved in order
to create and analyze informative and useful bivariate models. We use our example
models to illustrate and explain some of the key issues and we suggest directions for
continuing study. The key motivating factor in all of our discussions is the fact that
existing bivariate models are defined so that they can be collapsed into single variable life
time distribution models despite the fact that bivariate models may be more representative
or descriptive of device longevity.

3.2 Notation

\( T \)  
  time to failure

\( U \)  
  use to failure

\( g(\cdot) \)  
  function relating use and time

\( \alpha, \beta, \gamma \)  
  parameters of the function \( g(\cdot) \)

\( \pi_a(\cdot) \)  
  density on the parameter \( a \)

\( \lambda(t), \eta(u) \)  
  age and use functions that determine the failure hazard

\( \rho \)  
  correlation coefficient

\( f_{T,U}(t,u), F_{T,U}(t,u) \)  
  bivariate failure density and distribution functions

\( F_{T,U}(t,u) \)  
  bivariate reliability functions

\( M_{T,U}(\theta_1, \theta_2) \)  
  moment generating function for \( F_{T,U}(t,u) \)

\( f_u(u) \)  
  marginal density on use at failure

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\[ f_{TV}(t) \quad \text{conditional density on age given usage} \]
\[ z_{TU}(t,u) \quad \text{bivariate failure hazard function} \]
\[ Z_{TU}(t,u) \quad \text{bivariate cumulative failure hazard function} \]
\[ z_{TV}(t) \quad \text{conditional hazard function on age given usage} \]
\[ \mu_t, \mu_u \quad \text{mean of the age and of the usage distributions} \]
\[ \sigma_t, \sigma_u \quad \text{standard deviations of the age and usage distributions} \]

### 3.3 Example Failure Models

The taxonomy of model types we discuss in Chapter 2 indicates that there are two model classes that are not yet well developed, are interesting and may be practically applicable. These are the models based on a stochastic functional relationship between the two variables and the models that represent the variables as correlated. We define example models in both of these classes below. Note that there are very many conceivable approaches to the construction of these models and that in order to initiate the study of bivariate models, we employ the simplest possible approaches here.

#### 3.3.1 Stochastic Functions

The definition of failure models on the basis of stochastic functions relating age and use starts with the specification of how the stochastic feature of the life variables is portrayed. We assume that the time and use to failure are related by the function \( u = g(t) \) and that the stochastic nature of this relationship can be represented by treating one or more of the parameters of \( g(t) \) as random variables. To illustrate this construction, we consider four example forms here:

(i) \( g(t) = \alpha t + \beta \)
(ii) \( g(t) = \alpha t^2 + \beta t + \gamma \)

(iii) \( g(t) = \alpha t^n \)

(iv) \( g(t) = \frac{e^{\omega t} - 1}{e^{\omega t} + \beta} \)

where the fourth form is the logistic model analyzed by Eliashberg, Singpurwalla, and Wilson [1997]. In each case, we introduce randomness into the function by treating the parameter \( \alpha \) as a random variable having distribution \( \pi_\alpha(\cdot) \). This imposes random variation on the extent of use experienced by any age. Consequently, both age and usage at failure are random variables. The use of the distribution \( \pi_\alpha(\cdot) \) to construct the marginal probability distribution on usage is accomplished using a transformation of variables. In general:

\[
    f_U (u) = f_{g(t)} (u) = \left| \frac{d\alpha(u)}{du} \right| \pi_\alpha (\alpha(u))
\]

(3.1)

For example, with \( g(t) = \alpha t + \beta \), solving for \( \alpha \) yields:

\[
    \alpha(u) = \frac{u - \beta}{t} \quad \text{and} \quad \frac{d\alpha(u)}{du} = \frac{1}{t},
\]

so:

\[
    f_U (u) = \frac{1}{t} \pi_\alpha \left( \frac{u - \beta}{t} \right).
\]

(3.2)

Once the marginal distribution on usage is obtained, we then construct the joint failure density using the conditioning relation:
and the conditional density $f_{T|U}(t)$ is obtained by using the well-known relationship between a univariate density and its hazard function:

$$f_{T|U}(t) = z_{T|U}(t)\exp\left[-\int_0^t z_{T|U}(x)dx\right] = z_{T|\xi(t)}(t)\exp\left[-\int_0^t z_{T|\xi(t)}(x)dx\right]$$  \hspace{1cm} (3.4)

We use this form specifically so that we can focus upon the hazard function in the definition of the failure model.

We assume that the conditional bivariate hazard function on age given usage may be stated as:

$$z_{T|U}(t|u) = \lambda(t) + \eta(u)$$  \hspace{1cm} (3.5)

so that the definitions of the functions $\lambda(t)$, $\eta(u)$, and $g(t)$ determine the conditional hazard and ultimately the bivariate life distribution. Here, in order to focus on the functions $g(t)$, we assume that $\lambda(t)$ and $\eta(u)$ are simple linear functions. Thus we use $\lambda(t) = \lambda t$ and $\eta(u) = \eta u$.

Under this modeling format, the bivariate life distribution corresponding to form (i) above is obtained by constructing:

$$z_{T|U}(x) = \lambda x + \eta \left( \frac{u - \beta}{t} x + \beta \right)$$  \hspace{1cm} (3.6)

and applying expressions (3.3) and (3.4) to obtain:
The same analytical approach yields:

$$f_{T,U}(t,u) = \frac{\lambda t + \eta u}{t} \exp \left\{ - \frac{\eta (u + \beta)}{2} t - \frac{\lambda}{2} t^2 \right\} \pi_a \left( \frac{u - \beta}{t} \right)$$  \hspace{1cm} (3.7)$$

and

$$f_{T,U}(t,u) = \frac{\lambda t + \eta u}{t^2} \exp \left\{ - \frac{\eta (u + 2\gamma)}{3} t - \frac{3\lambda + \eta \beta}{6} t^2 \right\} \pi_a \left( \frac{u - \beta t - \gamma}{t^2} \right)$$  \hspace{1cm} (3.8)$$

$$f_{T,U}(t,u) = \frac{\lambda t + \eta u}{t^n} \exp \left\{ - \frac{\eta u}{n+1} t - \frac{\lambda}{2} t^2 \right\} \pi_a \left( \frac{u}{t^n} \right)$$  \hspace{1cm} (3.9)$$

and

$$f_{T,U}(t,u) = \frac{(1+\beta)(\lambda t + \eta u)}{t(1-u)(1+\beta u)} \exp \left\{ - \frac{\lambda}{2} t^2 + \frac{\eta}{\beta} t - \left( \frac{\beta + 1}{\beta} \right) \left( \frac{1}{\ln \left( \frac{1+\beta u}{1-u} \right)} \right) \right\} \pi_a \left( \frac{1}{t^2} \ln \frac{1+\beta u}{1-u} \right)$$  \hspace{1cm} (3.10)$$

for cases (ii), (iii), and (iv) respectively. The details of how we arrive at these forms for the joint density are presented in Appendix A. Note that in case (iv), the definition of the use function limits the variable $U$ to $[0, 1]$ so the functions may require rescaling for some applications. Also, in cases (i) and (ii) the forms of the functions $g(t)$ imply a non-zero minimum value for usage.

Finally, observe that all four models are well defined and require only the specification of the density $\pi_a \left( \cdot \right)$ to be complete bivariate life distributions. On the other hand, for each of them, it is unlikely that a closed form expression can be obtained for the marginal distribution on age at failure.
3.3.2 Correlation

In many applications, the two life variables appear to be correlated rather than functionally dependent. The definition of models that can represent correlations in the life variables appears initially to be somewhat simpler than the construction above. We simply choose a bivariate distribution. However, it is important that the distribution be capable of accurately representing equipment behavior and in particular that it have marginal distributions that are consistent with experience. We have selected three example models that appear to hold promise for representing bivariate failure processes in which the two variables are correlated.

The first of the candidate models is the generalization of the bivariate exponential model defined by Baggs and Nagagaja [1996]. In this model, the reliability function is:

\[
F_{T,U}(t,u) = e^{-(\lambda t + \eta u)} \left(1 + \rho \left(1 - e^{-\lambda t} \right) \left(1 - e^{-\eta u}\right)\right)
\]

so the corresponding density function is:

\[
f_{T,U}(t,u) = \lambda \eta e^{-(\lambda t + \eta u)} \left(1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-(\lambda t + \eta u)}\right)\right),
\]

the joint hazard is:

\[
z_{T,U}(t,u) = \frac{\lambda \eta \left(1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-(\lambda t + \eta u)}\right)\right)}{\left(1 + \rho \left(1 - e^{-\lambda t} - e^{-\eta u} + e^{-(\lambda t + \eta u)}\right)\right)},
\]

and the marginal densities are:

\[
f_T(t) = \lambda e^{-\lambda t} \quad \text{and} \quad f_U(u) = \eta e^{-\eta u}
\]
which are both constituent exponentials regardless of the value of $\rho$.

A second model that is an obvious choice is the bivariate Normal. The density function for this model is well known to be:

$$f_{T,U}(t,u) = \frac{1}{2\pi\sigma_T\sigma_U\sqrt{1-\rho^2}} \exp \left\{ - \frac{1}{2(1-\rho^2)} \left[ \frac{(t-\mu_T)^2}{\sigma_T^2} - 2\rho \frac{(t-\mu_T)(u-\mu_U)}{\sigma_T\sigma_U} + \frac{(u-\mu_U)^2}{\sigma_U^2} \right] \right\}. \quad (3.14)$$

As is also well known, the marginal densities are Normal.

One final model that we wish to consider here is the one stated by Hunter [1974a] in a queueing context but also consistent with reliability interpretations:

$$f_{T,U}(t,u) = \frac{\lambda \eta}{1-\rho} I_0 \left( \sqrt{\frac{2\sqrt{\rho}}{1-\rho}} \sqrt{\lambda \eta \mu} \right) \exp \left\{ - \frac{\lambda t + \eta u}{1-\rho} \right\}. \quad (3.15)$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind of order $n$; and $\rho$ is positive. The marginal densities are

$$f_T(t) = \lambda e^{-\lambda t} \quad \text{and} \quad f_U(u) = \eta e^{-\eta u}.$$

### 3.4 Model Analysis

#### 3.4.1 Bivariate Probability Distributions

The starting point for the analysis of the bivariate failure models is the careful definition and interpretation of bivariate probabilities. There are some subtle and sometimes difficult questions and concepts that arise in the application of bivariate distributions to reliability. First, for the age and use variables $T$ and $U$ respectively, we
interpret the cumulative failure probability $F_{T,U}(t,u)$ as the probability that failure occurs by time $t$ and usage $u$, that is:

$$F_{T,U}(t,u) = \Pr[T \leq t, U \leq u].$$

(3.16)

One may interpret this probability as corresponding to the proportion of the population of devices that have longevity vector values at failure that do not exceed $(t, u)$ in either vector component. We emphasize this definition because of the fact that for a bivariate distribution, probability is generally computed over rectangles such as $[t_1 \leq T \leq t_2, u_1 \leq U \leq u_2]$. Consequently, for any specific longevity vector, $(t, u)$, the range of age and use values implies that there are four rectangles in the $(T, U)$ plane for which probabilities may be meaningfully calculated. In addition to the one in Eq. (3.16), the rectangle are $\Pr[T \leq t, U \geq u]$, $\Pr[T \geq t, U \leq u]$, and $\Pr[T \geq t, U \geq u]$. Figure 3.4.1 illustrates these rectangles. It is not obvious but relative to the cumulative probability $F_{T,U}(t,u)$, the probabilities:

$$\Pr[T \leq t, U > u] = \int_0^t \int_u^\infty f_{T,U}(s,v)dvds$$

(3.17)

and

$$\Pr[T > t, U \leq u] = \int_t^\infty \int_0^u f_{T,U}(s,v)dvds$$

(3.18)

are survival probabilities. They correspond to the proportions of the population that do not have longevity vectors inferior to $(t, u)$ either because their failure ages exceed $t$ or their failure usages exceed $u$. We do not have informative names for the probabilities represented by Eqs. (3.17) and (3.18) but have considered names such as marginal survival probabilities.
Figure 3.4.1  Bivariate Probability Distributions
A further point that is rather subtle is the fact that the reliability at \((t, u)\) does not include the probabilities represented by Eqs. (3.17) and (3.18). The reliability at longevity vector value \((t, u)\) corresponds to the proportion of the population for which failure age exceeds \(t\) and failure usage exceeds \(u\). Therefore, the reliability function corresponding to \(F_{T,U}(t,u)\) is:

\[
\bar{F}_{T,U}(t,u) = \Pr[T \geq t, U \geq u] = \int_t^\infty \int_u^\infty f_{T,U}(s,v)dvds \quad (3.19)
\]

Because it does not include the probabilities represented by expressions (3.17) and (3.18), we call this the reliability rather than the survivor function.

The apparent paradox in the definitions of \(F_{T,U}(t,u)\) and \(\bar{F}_{T,U}(t,u)\) arises from distinctions in point of observation. When considering the distribution, all positive valued longevity vectors can potentially occur and, across a population of devices, all do occur. Relative to the distribution, the cumulative probability at \((t, u)\) does not include devices for which either \(T\) exceeds \(t\) or \(U\) exceeds \(u\). On the other hand, all copies of a device population that have achieved a longevity of \((t, u)\) will have longevity vectors at failure that lie within the rectangle \(\left[t \leq T < \infty, u \leq U < \infty\right]\) so at \((t, u)\) the rectangles corresponding to the marginal survival probabilities are not accessible.

The computation of bivariate probabilities is reasonably clear. For any rectangle say \([t_1 \leq T \leq t_2, u_1 \leq U \leq u_2]\) in the plane, the probability of observing a failure at a point included in the rectangle is:

\[
\Pr[t_1 \leq T \leq t_2, u_1 \leq U \leq u_2] = F_{T,U}(t_2,u_2) - F_{T,U}(t_2,u_1) - F_{T,U}(t_1,u_2) + F_{T,U}(t_1,u_1) \quad (3.20)
\]

A useful special case of this expression applies to the reliability function that may be represented by:
\[
\tilde{F}_{T,U}(t,u) = \Pr[t \leq T < \infty, u \leq U < \infty] = 1 - F_U(u) - F_T(t) + F_{T,U}(t,u). \tag{3.21}
\]

Observe that this expression may also be used to compute cumulative probabilities in cases in which the reliability function is easier than the distribution function to analyze.

### 3.4.2 Hazard Functions

Very often, the first question that follows the definition of a probability model for device failure is that of the identity and behavior of the associated hazard function. This is because the construction and characterization of the hazard function associated with a life or longevity distribution is a key aspect of equipment reliability modeling and analysis. For a bivariate failure distribution, a return to first principles yields:

\[
z_{T,U}(t,u) = \lim_{\Delta t \to 0} \frac{\Delta u \Delta t}{\Pr[t \leq T \leq t + \Delta t, u \leq U \leq u + \Delta u | T > t, U > u]} = \lim_{\Delta t \to 0} \frac{\Delta u \Delta t}{\Pr[t \leq T \leq t + \Delta t, u \leq U \leq u + \Delta u | T > t, U > u]} = \frac{1}{\tilde{F}_{T,U}(t,u)} \lim_{\Delta u \to 0} \frac{F_{T,U}(t + \Delta t, u + \Delta u) - F_{T,U}(t + \Delta t, u) - F_{T,U}(t, u + \Delta u) + F_{T,U}(t, u)}{\Delta u \Delta t}
\]

\[
= \frac{1}{\tilde{F}_{T,U}(t,u)} \frac{\partial^2}{\partial t \partial u} \tilde{F}_{T,U}(t,u)
\]

\[
= \frac{f_{T,U}(t,u)}{\tilde{F}_{T,U}(t,u)}
\]

which is a very appealing result.
By analogy with the univariate case, one may want to construct bivariate failure models given the bivariate hazard. For the univariate case, this approach may be accomplished given the following relationships between $F$ and $z$:

$$z(t) = \ln F(t) \quad \text{or} \quad F(t) = e^{-z(t)}.$$  \hspace{1cm} (3.23)

Unfortunately, for the bivariate case, the above relationships do not exist except for the case where $t$ and $u$ are assumed to be independent, which is unlikely to be practically meaningful.

Naturally, the next question is whether or not the hazard function is increasing. Applying the Barlow and Proschan [1975] definition of MIFR (multivariate increasing failure rate) to the bivariate life distributions implies that a distribution is MIFR if and only if:

$$\frac{F_{T,U}(t+s,u+v)}{F_{T,U}(t,u)}$$

is non-increasing in $(t, u)$. The same statement applies to the marginal distributions, that is, it must also be the case that:

$$\frac{F_T(t+s)}{F_T(t)} \quad \text{and} \quad \frac{F_U(u+v)}{F_U(u)}$$

are non-increasing in $t$ and $u$, respectively. The application of these conditions is illustrated later in this paper.
3.4.3 Moments

A further question is how one computes the mean and other descriptive measures for a bivariate longevity distribution. The answer is that as with univariate distributions, one begins by constructing the moment generating function (or Laplace Transform) and then obtains moments as successive derivatives of the moment generating function. The moment generating function for the bivariate failure distribution is:

\[
M_{T,U}(\theta_1, \theta_2) = E[e^{\theta_1 T + \theta_2 U}] = \int_0^\infty \int_0^\infty e^{\theta_1 t + \theta_2 u} f_{T,U}(t,u) \, du \, dt
\]  

(3.26)

and the moments of the distribution are obtained as:

\[
E[t^k] = \frac{\partial}{\partial \theta_1^k} M_{T,U}(\theta_1, \theta_2) \bigg|_{\theta_1=0} \quad \text{and} \quad E[u^k] = \frac{\partial}{\partial \theta_2^k} M_{T,U}(\theta_1, \theta_2) \bigg|_{\theta_2=0}.
\]  

(3.27)

Examples of the use of these expressions are presented in the next section.

3.4.4 Renewal Functions

The final issue of critical importance to bivariate failure modeling is how convolutions are constructed and how bivariate renewal functions are defined and interpreted. Fortunately, the convolution theorem has been shown to extend directly to the bivariate case (Hunter [1974a]) so that:

\[
f_{T,U}^{(k)}(t,u) = \int_0^t \int_0^u f_{T,U}^{(k-1)}(t-s, u-v) f_{T,U}(s, v) \, dv \, ds.
\]  

(3.28)

On the other hand, the definition and interpretation of the associated counting process and the bivariate renewal function is less obvious and may depend upon the application.
3.5 Example Calculations of Bivariate Failure Models

The general concepts defined above can be illustrated by application to the example models. Of course, except for the bivariate exponential distribution of expression (3.12), none of the above bivariate models have closed form expressions for their cumulative probabilities. Thus, we must use numerical methods to compute probabilities, reliability values, and moments and to examine the behaviors of the hazard functions.

As examples of the analysis of the methods, consider for the stochastic function models the densities of Eqs. (3.7) and (3.10) and for the correlation models the bivariate exponential of Eq. (3.12) and the bivariate Normal in Eq. (3.14). In the case of the stochastic function models, assume \( \pi_a(\cdot) \) is a negative exponential density of the form:

\[
\pi_a(\cdot) = ce^{-ca}.
\]

Then, for the model of Eliashberg, Singpurwalla, and Wilson [1997], Eq. (3.10), arbitrarily setting the parameters to be \( \lambda = 10^{-6}, \beta = 10, \eta = 1.5 \times 10^{-6}, c = 1000 \) yields the following values for the cumulative distribution function as shown in Table 3.5.1. All the numerical results presented in this section are obtained by using the numerical integration function in Mathematica 4.0 (Wolfram Research 1999).
Table 3.5.1  CDF Values for Eq. (3.10).

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.5</th>
<th>0.8</th>
</tr>
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<td>0.116</td>
<td>0.117</td>
<td>0.118</td>
</tr>
<tr>
<td>1000</td>
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<td>0.364</td>
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<tr>
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</tr>
<tr>
<td>3000</td>
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<td>0.683</td>
<td>0.763</td>
<td>0.852</td>
<td>0.928</td>
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<tr>
<td>4000</td>
<td></td>
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<td>0.686</td>
<td>0.768</td>
<td>0.858</td>
<td>0.935</td>
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</tbody>
</table>

Corresponding reliability values are shown in Table 3.5.2.

Table 3.5.2  Reliability Values for Eq. (3.10).

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<th>0.3</th>
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</tr>
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<td>0.064</td>
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<tr>
<td>1000</td>
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</tr>
<tr>
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<td>0.065</td>
<td>0.048</td>
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<tr>
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<td>0.006</td>
<td>0.005</td>
<td>0.003</td>
</tr>
<tr>
<td>4000</td>
<td></td>
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<td>0.0002</td>
<td>0.0002</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

The hazard values are shown in Table 3.5.3.
Table 3.5.3   Hazard Values for Eq. (3.10).

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<th>0.3</th>
<th>0.5</th>
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</thead>
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<td>0.0037</td>
<td>0.00614</td>
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</table>

An interesting aspect of this distribution is that the test on the hazard function behavior is inconclusive because the hazard is increasing in age and decreasing and then increasing in usage.

As a basis of comparison, note that when $\lambda = 10^{-6}$, $\beta = 0$, $\eta = 1.5 \times 10^{-6}$, $c = 0.75$, the linear stochastic function of Eq. (3.7) yields the following cumulative failure probabilities as shown in Table 3.5.4.

Table 3.5.4   CDF Values for Eq. (3.7).

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</thead>
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<td>0.642</td>
<td>0.683</td>
<td>0.693</td>
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<td>0.978</td>
<td>0.994</td>
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</tr>
<tr>
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<td>0.979</td>
<td>0.995</td>
<td></td>
</tr>
</tbody>
</table>

The corresponding reliability values are shown in Table 3.5.5.
Table 3.5.5  Reliability Values for Eq. (3.7). (* = ×10\(^{-6}\))

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>0.4404</td>
<td>0.2533</td>
<td>0.0740</td>
<td>0.0198</td>
<td>0.0050</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.1786</td>
<td>0.0998</td>
<td>0.0285</td>
<td>0.0076</td>
<td>0.0019</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>0.0120</td>
<td>0.0052</td>
<td>0.0009</td>
<td>0.0002</td>
<td>0.00003</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>0.0003</td>
<td>0.0001</td>
<td>9.8891*</td>
<td>0.8781*</td>
<td>0.0764*</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>4.3270*</td>
<td>0.9354*</td>
<td>0.0425*</td>
<td>0.0019*</td>
<td>0.0008*</td>
<td></td>
</tr>
</tbody>
</table>

The values of hazard function are shown in Table 3.5.6. For this model, the hazard function is increasing in both variables.

Table 3.5.6 Hazard Values for Eq. (3.7). (* = ×10\(^{-6}\))

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>1.4714*</td>
<td>1.6028*</td>
<td>1.4733*</td>
<td>1.2079*</td>
<td>0.9456*</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>2.1047*</td>
<td>2.5418*</td>
<td>3.1797*</td>
<td>3.6612*</td>
<td>4.0621*</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>4.5417*</td>
<td>5.2497*</td>
<td>6.5370*</td>
<td>7.7366*</td>
<td>8.8919*</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>8.5270*</td>
<td>9.4970*</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00002</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>0.00001</td>
<td>0.00002</td>
<td>0.00002</td>
<td>0.00002</td>
<td>0.00002</td>
<td></td>
</tr>
</tbody>
</table>

The bivariate exponential, Eq. (3.12), is much easier to analyze because it can be integrated in closed form. The reliability is stated in equation (3.11) and the cumulative probabilities are given by:
Using parameter values of $\lambda = 0.0008$, $\eta = 0.0005$, and $\rho = 0.6$, we obtain the following values of cumulative failure probabilities as shown in Table 3.5.7.

Table 3.5.7 CDF Values for Eq. (3.12).

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td></td>
<td>0.096</td>
<td>0.161</td>
<td>0.239</td>
<td>0.279</td>
<td>0.301</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>0.147</td>
<td>0.252</td>
<td>0.383</td>
<td>0.454</td>
<td>0.494</td>
</tr>
<tr>
<td>2000</td>
<td></td>
<td>0.193</td>
<td>0.337</td>
<td>0.527</td>
<td>0.637</td>
<td>0.701</td>
</tr>
<tr>
<td>3000</td>
<td></td>
<td>0.210</td>
<td>0.370</td>
<td>0.586</td>
<td>0.715</td>
<td>0.792</td>
</tr>
<tr>
<td>4000</td>
<td></td>
<td>0.216</td>
<td>0.383</td>
<td>0.612</td>
<td>0.745</td>
<td>0.832</td>
</tr>
</tbody>
</table>

The reliability values are shown in Table 3.5.8.

Table 3.5.8 Reliability Values for Eq. (3.12).

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td></td>
<td>0.545</td>
<td>0.438</td>
<td>0.277</td>
<td>0.173</td>
<td>0.106</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>0.376</td>
<td>0.308</td>
<td>0.200</td>
<td>0.126</td>
<td>0.078</td>
</tr>
<tr>
<td>2000</td>
<td></td>
<td>0.174</td>
<td>0.146</td>
<td>0.097</td>
<td>0.062</td>
<td>0.039</td>
</tr>
<tr>
<td>3000</td>
<td></td>
<td>0.079</td>
<td>0.067</td>
<td>0.045</td>
<td>0.029</td>
<td>0.018</td>
</tr>
<tr>
<td>4000</td>
<td></td>
<td>0.036</td>
<td>0.030</td>
<td>0.020</td>
<td>0.013</td>
<td>0.008</td>
</tr>
</tbody>
</table>
In addition, for the stated parameter values, the hazard function has both increasing and decreasing behaviors. The values of hazard function are shown in Table 3.5.9.

Table 3.5.9 Hazard Values for Eq. (3.12). ($\times 10^{-7}$)

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u$</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td></td>
<td>4.2694</td>
<td>3.8731</td>
<td>3.3639</td>
<td>3.0754</td>
<td>2.9076</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>3.6015</td>
<td>3.0754</td>
<td>3.3625</td>
<td>3.2909</td>
<td>3.2503</td>
</tr>
<tr>
<td>2000</td>
<td></td>
<td>2.8957</td>
<td>3.1096</td>
<td>3.3613</td>
<td>3.4937</td>
<td>3.5680</td>
</tr>
<tr>
<td>3000</td>
<td></td>
<td>2.5920</td>
<td>2.9488</td>
<td>3.3608</td>
<td>3.5741</td>
<td>3.6927</td>
</tr>
<tr>
<td>4000</td>
<td></td>
<td>2.4581</td>
<td>2.8788</td>
<td>3.3605</td>
<td>3.6084</td>
<td>3.7457</td>
</tr>
</tbody>
</table>

Using the bivariate exponential of Hunter [1974], Eq. (3.15) and setting the parameters to be $\lambda = 0.0008$, $\eta = 0.0005$, and $\rho = 0.6$ as a basis of comparison, we obtain the values of cumulative failure probabilities as shown in Table 3.5.10.

Table 3.5.10 CDF Values for Eq. (3.15).

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u$</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td></td>
<td>0.126</td>
<td>0.205</td>
<td>0.283</td>
<td>0.313</td>
<td>0.323</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>0.181</td>
<td>0.304</td>
<td>0.444</td>
<td>0.505</td>
<td>0.532</td>
</tr>
<tr>
<td>2000</td>
<td></td>
<td>0.214</td>
<td>0.374</td>
<td>0.580</td>
<td>0.689</td>
<td>0.745</td>
</tr>
<tr>
<td>3000</td>
<td></td>
<td>0.220</td>
<td>0.389</td>
<td>0.619</td>
<td>0.750</td>
<td>0.824</td>
</tr>
<tr>
<td>4000</td>
<td></td>
<td>0.221</td>
<td>0.393</td>
<td>0.629</td>
<td>0.769</td>
<td>0.852</td>
</tr>
</tbody>
</table>
The reliability values are shown in Table 3.5.11.

Table 3.5.11  Reliability Values for Eq. (3.15).

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>0.564</td>
<td>0.470</td>
<td>0.310</td>
<td>0.195</td>
<td>0.118</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.398</td>
<td>0.349</td>
<td>0.250</td>
<td>0.167</td>
<td>0.105</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>0.184</td>
<td>0.172</td>
<td>0.140</td>
<td>0.104</td>
<td>0.072</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>0.081</td>
<td>0.078</td>
<td>0.069</td>
<td>0.056</td>
<td>0.042</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>0.034</td>
<td>0.033</td>
<td>0.031</td>
<td>0.027</td>
<td>0.021</td>
<td></td>
</tr>
</tbody>
</table>

The values of hazard function are shown in Table 3.5.12.

Table 3.5.12  Hazard Values for Eq. (3.15). ($\times 10^{-7}$)

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>4.930</td>
<td>3.557</td>
<td>1.695</td>
<td>0.7489</td>
<td>0.3147</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>3.953</td>
<td>3.520</td>
<td>2.281</td>
<td>1.2734</td>
<td>0.6509</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>2.223</td>
<td>2.691</td>
<td>2.681</td>
<td>2.075</td>
<td>1.395</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>1.164</td>
<td>1.782</td>
<td>2.460</td>
<td>2.441</td>
<td>2.023</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>0.602</td>
<td>1.120</td>
<td>2.034</td>
<td>2.488</td>
<td>2.460</td>
<td></td>
</tr>
</tbody>
</table>

For the stated parameter values, the hazard function behavior is inconclusive because the hazard is decreasing, increasing and then decreasing, and increasing in age; and decreasing, increasing and then decreasing in usage.
For the bivariate normal, Eq. (3.14) with parameters values $\mu_x = 3000$, $\mu_u = 3250$, $\sigma_x = 750$, $\sigma_u = 625$, and $\rho = 0.6$, we obtain the following values of cumulative failure probabilities as shown in Table 3.5.13.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u$</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>0.0093*</td>
<td>0.1343*</td>
<td>0.0002</td>
<td>0.0004</td>
<td>0.0004</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.0256*</td>
<td>0.4924*</td>
<td>0.0015</td>
<td>0.0036</td>
<td>0.0038</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>0.0478*</td>
<td>0.0001</td>
<td>0.0123</td>
<td>0.0721</td>
<td>0.0909</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>0.0498*</td>
<td>0.0002</td>
<td>0.0218</td>
<td>0.2656</td>
<td>0.4870</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>0.0499*</td>
<td>0.0002</td>
<td>0.0227</td>
<td>0.3411</td>
<td>0.8338</td>
<td></td>
</tr>
</tbody>
</table>

The reliability values are shown in Table 3.5.14.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u$</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>0.9995</td>
<td>0.9994</td>
<td>0.9970</td>
<td>0.6554</td>
<td>0.1150</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.9961</td>
<td>0.9960</td>
<td>0.9749</td>
<td>0.6552</td>
<td>0.1150</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>0.9088</td>
<td>0.9087</td>
<td>0.8983</td>
<td>0.6364</td>
<td>0.1148</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>0.4999</td>
<td>0.4999</td>
<td>0.4990</td>
<td>0.4210</td>
<td>0.1020</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>0.0912</td>
<td>0.0912</td>
<td>0.0912</td>
<td>0.0878</td>
<td>0.0401</td>
<td></td>
</tr>
</tbody>
</table>

The values of hazard function are shown in Table 3.5.15.
Table 3.5.15 Hazard Values for Eq. (3.14). \((\times10^{-7})\)

<table>
<thead>
<tr>
<th>(t)</th>
<th>(u)</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>0.0000</td>
<td>0.0002</td>
<td>0.0017</td>
<td>0.0003</td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.0000</td>
<td>0.0005</td>
<td>0.0110</td>
<td>0.0060</td>
<td>0.0002</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>0.0000</td>
<td>0.0004</td>
<td>0.0631</td>
<td>0.2420</td>
<td>0.0668</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0374</td>
<td>0.8897</td>
<td>1.3504</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0042</td>
<td>0.6455</td>
<td>3.8388</td>
<td></td>
</tr>
</tbody>
</table>

For the stated parameter values, the hazard function behavior is inconclusive because the hazard is increasing and then decreasing in age; and increasing and then decreasing in usage.

The purposes of these examples is to illustrate the fact that cumulative probabilities, reliability values, and hazard values can be conveniently computed. The models can be made to cover any appropriate scale by adjusting the parameters or reinterpreting the variables. The values enumerated also illustrate the fact that

\[
\left( F_{T,U}(t,u) + F_{T,U}(t,u) \right) < 1. \tag{3.31}
\]

The next aspect of the analysis of the bivariate failure models is the determination of the distribution moments. For the bivariate Normal distribution of Eq. (3.14), the analysis is straightforward as the moment generating function is known to be (Mood, Graybill, and Boes [1974]):

\[
M_{T,U}(\theta_1, \theta_2) = \exp \left\{ \theta_1 u_1 + \theta_2 u_2 + \frac{\theta_1^2 \sigma^2}{2} + 2\rho \theta_1 \theta_2 \sigma \sigma_u + \frac{\theta_2^2 \sigma^2}{2} \right\} \tag{3.32}
\]
and as anticipated, application of Eq. (3.27) yields the mean vector \( E[(t,u)] = (\mu_t, \mu_u) \).

Corresponding results occur for higher moments.

As a second example, note that the moment generating function for the bivariate exponential can be derived as:

\[
M_{T,U}(\theta_1, \theta_2) = \frac{\lambda \eta (1 + \rho)}{(\eta - \theta_2)(\lambda - \theta_1)} - \frac{2 \lambda \eta \rho}{(2 \eta - \theta_2)(\lambda - \theta_1)} + \frac{2 \lambda \eta \rho}{(2 \eta - \theta_2)(2 \lambda - \theta_1)}
\]

and as a result, the mean vector is \( E[(t,u)] = (1/\lambda, 1/\eta) \).

For the other example distributions, the construction of the moment generating function is considerably more difficult. In fact, for the models in Eqs. (3.7), (3.8), (3.9), and (3.10), closed form expressions for the moment generating functions do not exist. In each of those cases, analysis of moments is best performed by numerically computing the moments for the marginal distributions. For example, for the model of Eliashberg, Singpurwalla, and Wilson, [1997] with the same parameters as those used above, numerical computation of the expectation vector yields \( E[(t,u)] = (1252.45, 0.206) \).

Similarly, the linear stochastic function model has expectation vector of \((821.5, 833.5)\).

Unfortunately, a direct numerical approach will only work for the correlated function models in the construction of convolutions and renewal functions. It appears that convolutions for the stochastic function models will require the use of series expansions and numerical approximations such as those defined for the univariate Weibull distribution by Lomnicki [1966].

Even with an approach to computing convolutions, the definition of the renewal function is not straightforward. Hunter [1974] starts with marginal renewal counting processes and takes the minimum of those to obtain the bivariate counting process. That approach appears appropriate for the queueing applications that Hunter studies but does
not appear to conform to the sense of equipment reliability and maintenance analysis. We suggest that the counting process be defined over the age-usage plane so that $F_{T,U}^{(k)}(t,u)$ is interpreted as the cumulative probability that at the $k$th renewal event, the longevity vector does not exceed an age of $t$ or a usage of $u$. With this interpretation, the counting process, $N_{T,U}$ is well defined and we can use the conventional "time-frequency" duality to state probabilities such as:

$$\Pr[N_{T,U}(t,u) \geq k] = F_{T,U}^{(k)}(t,u).$$

(3.34)

Then, we should be able to use this probability statement to obtain renewal results that are presented in Chapter 4. In Chapter 5, we develop corrective maintenance models by using the results of bivariate renewal theory. In Chapter 6, we consider an age replacement preventive maintenance policy and develop bivariate preventive maintenance models. Examples for the preventive maintenance models are presented in Chapter 7. Then, we define an availability measure for equipment that has a bivariate longevity measure. This is presented in Chapter 8, with its examples presented in Chapter 9.
CHAPTER IV
BIVARIATE RENEWAL MODELING

In this chapter, a bivariate renewal theory developed by Hunter [1974a, b] is presented. Based on Hunter's work, we propose an ordinary bivariate renewal theory. We also extend Wang and Pham’s [1996] univariate quasi-renewal theory to the bivariate case. Basic results for the two types of processes and some generalizations of ordinary bivariate renewal theory are also obtained.

4.1 Notation

\( T_n \) the \( n \)th time to failure
\( U_n \) the \( n \)th use to failure
\( X_n \) the bivariate random vector, \((T_n, U_n)\)
\( S_n \) the time of the \( n \)th renewal
\( V_n \) the usage of the \( n \)th renewal
\( Y_n \) the bivariate random vector of both time and use of the \( n \)th renewal, \((S_n, V_n)\)
\( f_{TU}(t,u), F_{TU}(t,u) \) bivariate failure density and distribution functions
\( F^{(n)}(t,u) \) the \( n \)-fold convolution of \( F(t,u) \)
\( G^{(n)}(t,u) \) the convolution of \( F_1(t,u), F_2(t,u), \ldots, \) and \( F_n(t,u) \)
\( N_{TU}(t,u) \) the number of renewals by time \( t \) and usage \( u \)
\( M_{F}(t,u) \) the bivariate renewal function for \( F_{TU}(t,u) \)
4.2 Bivariate Renewal Process

Let \( \{ X_n = \{(T_n, U_n)\}, n = 1, 2, \ldots \} \), be a sequence of independent and identically distributed non-negative bivariate random vectors, with the common joint distribution function (j.d.f.) \( F(t, u) = P\{T_n \leq t, U_n \leq u\} \), and to avoid trivialities, assume that \( F(0,0) = P\{T_n = 0, U_n = 0\} < 1 \). Letting

\[
Y_0 = (0, 0), \quad Y_n = \sum_{i=1}^{n} X_i = \left( \sum_{i=1}^{n} T_i, \sum_{i=1}^{n} U_i \right) = (S_n, V_n), \quad n \geq 1, \quad (4.1)
\]

it follows that \( Y_n = (S_n, V_n) \) is the bivariate measure of both time and use of the \( n \)th renewal as illustrated in Figure 4.2.1. As the number of events by time \( t \) and usage \( u \) will equal the largest value of \( n \) for which the \( n \)th renewal occurs before or at time \( t \) and usage \( u \), it follows that \( N_{T,U}(t,u) \), the number of renewals by time \( t \) and usage \( u \), is given by

\[
N_{T,U}(t,u) = \sup \{ n : n \geq 0, S_n \leq t, V_n \leq u \} = \sup \{ n : n \geq 0, Y_n \leq (t,u) \}. \quad (4.2)
\]

Note that \( \{ N_{T,U}(t,u), t > 0, u > 0 \} \) is a counting process. Let \( T_n \) and \( U_n \) denote the interarrival time and usage, respectively, between the \((n-1)\)st and \( n \)th events of this process, \( n \geq 1 \).
Definition 4.2.1

Let the sequence of non-negative bivariate random vectors \( \{ X_i, i = 1, 2, \ldots \} \) be independent and identically distributed. The counting process \( \{ N_{T,U}(t,u), t > 0, u > 0 \} \) is called a **bivariate renewal counting process**.

For an example of a bivariate renewal process, suppose that there is an infinite supply of new tires whose age-mileage vectors, \( \{ X_i, i = 1, 2, \ldots \} \), are independent and identically distributed. Note that age (lifetime) and mileage (usage) may be correlated for a given tire. Suppose also that the tires are used one at a time, and when one fails, we immediately replace it with a new one. Under these conditions, the counting process \( \{ N_{T,U}(t,u), t > 0, u > 0 \} \) is a bivariate renewal counting process where \( N_{T,U}(t,u) \) represents the number of tires that have failed by time \( t \) and by total mileage \( u \).
4.3 Distribution of \(N_{T,U}(t,u)\) for a Bivariate Renewal Process

The distribution of \(N_{T,U}(t,u)\) can be obtained by first noting the important relationship that the number of renewals by time \(t\) and usage \(u\) is greater than or equal to \(n\) if, and only if, the \(n\)th renewal occurs before or at time \(t\) and usage \(u\). That is,

\[
\{N_{T,U}(t,u) \geq n\} \iff \{S_n \leq t, V_n \leq u\} \iff \{Y_n \leq (t,u)\}.
\]  (4.3)

From Eq. (4.3),

\[
P[N_{T,U}(t,u) = n] = P[N_{T,U}(t,u) \geq n] - P[N_{T,U}(t,u) \geq n + 1]
\]  (4.4)

\[
= P[Y_n \leq (t,u)] - P[Y_{n+1} \leq (t,u)].
\]

Since the random vectors, \(X_i, i = 1, 2, \ldots\), are independent and identically distributed with a common joint distribution \(F(t,u) = P(T_n \leq t, U_n \leq u)\), it follows that \(Y_n\) is distributed as \(F^{(n)}(t,u)\), the \(n\)-fold convolution of \(F(t,u)\) with itself. Thus, from Eq. (4.4), we obtain the following theorem.

**Theorem 4.3.1**

For \(t, u \geq 0\), and \(n \geq 0\),

\[
P[N_{T,U}(t,u) = n] = F^{(n)}(t,u) - F^{(n+1)}(t,u).
\]  (4.5)

**Proof.** See Hunter [1974], p. 386.
An important specific realization of Eq. (4.5) is:

\[ P\{N_{T,U}(t,u) = 0\} = F^{(0)}(t,u) - F^{(1)}(t,u) = 1 - F(t,u). \quad (4.6) \]

Note that in Eq.(4.6), the probability that the number of bivariate renewals before \((t, u)\) is zero is not equal to the reliability \(F(t,u)\). In fact, \(F(t,u) < 1 - F(t,u)\), as we have already shown in Chapter 3.

Let

\[ M_F(t,u) = E[N_{T,U}(t,u)] \quad (4.7) \]

represent the bivariate renewal function (or bivariate mean-value function). The following theorem illustrates the relationship between \(M_F(t,u)\) and \(F(t,u)\).

**Theorem 4.3.2**  For \(t, u \geq 0\), and \(n \geq 0\),

\[ M_F(t,u) = \sum_{n=1}^{\infty} F^{(n)}(t,u). \quad (4.8) \]

**Proof.** Let \(N_{T,U}(t,u) = \sum_{n=1}^{\infty} I_n(t,u)\), where

\[ I_n(t,u) = \begin{cases} 
1 & \text{if the } n\text{th renewal occurred in the rectangle } [0,t] \times [0,u], \\
0 & \text{otherwise}.
\end{cases} \]

Hence,

\[ E[N_{T,U}(t,u)] = E[\sum_{n=1}^{\infty} I_n(t,u)] \]
\[= \sum_{n=1}^{\infty} E[I_n(t,u)]\]
\[= \sum_{n=1}^{\infty} P[I_n(t,u) = 1]\]
\[= \sum_{n=1}^{\infty} P[S_n \leq t \cap V_n \leq u]\]
\[= \sum_{n=1}^{\infty} P[Y_n \leq (t,u)]\]
\[= \sum_{n=1}^{\infty} F^{(n)}(t,u)\]

where the interchange of expectation and summation is justified by the non-negativity of the \(I_n(t,u)\).

Transforming \(M_F(t,u)\) into the recursive form (cf. Nachlas [1998], p.99):

\[M_F(t,u) = \sum_{n=1}^{\infty} F^{(n)}(t,u)\]
\[= F^{(1)}(t,u) + \sum_{n=2}^{\infty} F^{(n)}(t,u)\]
\[= F(t,u) + \sum_{j=1}^{\infty} F^{(j+1)}(t,u)\]
\[= F(t,u) + \sum_{j=1}^{\infty} \int_{0}^{t} \int_{0}^{u} F^{(j)}(t-x, u-y)dF(x,y)\]
\[= F(t,u) + \int_{0}^{t} \int_{0}^{u} \sum_{j=1}^{\infty} F^{(j)}(t-x,u-y)dF(x,y)\]
\[= F(t,u) + \int_{0}^{t} \int_{0}^{u} M(t-x,u-y)dF(x,y),\]

Hence,

\[M_F(t,u) = F(t,u) + \int_{0}^{t} \int_{0}^{u} M(t-x,u-y)dF(x,y).\]  
(4.9)
Eq. (4.9) is called the integral equation of bivariate renewal theory.

Assuming that $F(t, u)$ is absolutely continuous, then the derivative of $M_F(t,u)$, denoted by $m_f(t,u)$, is called the bivariate renewal density function,

$$m_f(t,u) = \frac{\partial^2}{\partial t \partial u} M_F(t,u) = \sum_{n=1}^{\infty} f^{(n)}(t,u)$$

$$= f(t,u) + \int_0^t \int_0^u m(t-x,u-y)f(x,y)dxdy,$$  \hspace{1cm} (4.10)

and $m_f(t,u)$ represents the probability of a renewal at any point in the time-usage plane.

Note that if we let the means and variances of $T$ and $U$ be $(\mu_T, \mu_U)$ and $(\sigma_T^2, \sigma_U^2)$, respectively, then

$$m_f(t,u) = \sum_{n=1}^{\infty} f^{(n)}(t,u) \sim \sum_{n=1}^{\infty} \phi(t,u)$$  \hspace{1cm} (4.11)

where $\phi(t,u)$ is the bivariate normal density with mean $(n\mu_T, n\mu_U)$, variance $(n\sigma_T^2, n\sigma_U^2)$, and correlation coefficient $\rho$.

### 4.4 Bivariate Quasi-Renewal Process

Renewal theory provides a fundamental tool for constructing simple maintenance models. The simplicity comes from the i.i.d. assumption of the time between successive events. Thus, after maintenance, the equipment is assumed to be "as good as new." However, in practice, after maintenance the lifetime of the equipment will become shorter and shorter while its maintenance time may become longer and longer. In order to use
renewal theory to model a gradual deterioration, Wang and Pham [1996, 1997] developed a univariate quasi-renewal process in which the i.i.d. assumption is reduced to assume only that the successive interarrival times are independent. The quasi-renewal process is also called a geometric process by Lam Yeh [1988b]. In this section, we extend their results to a bivariate quasi-renewal process and present some basic results.

**Definition 4.4.1**

Let \( X_n = (T_n, U_n), n = 1, 2, ... \) be a stochastic process such that \( T_n \) and \( U_n \) denote the interarrival time and usage, respectively, between the \((n-1)st\) and \(n\)th events of a counting process \( \{ N_{T,U}(t,u), t > 0, u > 0 \} \). Then the counting process is said to be a bivariate quasi-renewal process if and only if:

\[
X_n = (\zeta_T(n)T_n, \zeta_U(n)U_n), \ n = 1, 2, ...
\]

where \((T_n, U_n)\) are i.i.d. random vectors; and \(\zeta_T(n)\) and \(\zeta_U(n)\) are non-negative functions.

In words, a bivariate quasi-renewal process is a stochastic process where the successive interarrival events are independent and are decreasing, upon each renewal, by a fraction \((1-\zeta_T(n))\) for \(T_n\) and \((1-\zeta_U(n))\) for \(U_n\), for both \(\zeta_T(n)\) and \(\zeta_U(n) < 1\); or increasing by a fraction \((\zeta_T(n)-1)\) for \(T_n\) and \((\zeta_U(n)-1)\) for \(U_n\), for both \(\zeta_T(n)\) and \(\zeta_U(n) > 1\). Thus, for both \(\zeta_T(n)\) and \(\zeta_U(n)\) equal to 1 the bivariate quasi-renewal process is the ordinary bivariate renewal process. In solving maintenance problems, the bivariate quasi-renewal process can be used to model a bivariate stochastic maintenance process, with both \(\zeta_T(n)\) and \(\zeta_U(n) < 1\). For both \(\zeta_T(n)\) and \(\zeta_U(n) > 1\), bivariate
quasi-renewal process can be applied to model a reliability growth process in product development and/or a burn-in program.

Note that the extension to the bivariate case is not the only extension to the univariate quasi-renewal process that we can see. We also suggest that more general forms for both \( \zeta_T(n) \) and \( \zeta_U(n) \) can be defined for the univariate and bivariate cases but that we will not pursue these here.

If we let \( (\zeta_T(n), \zeta_U(n)) = (\zeta_T^{n-1}, \zeta_U^{n-1}) \) where \( \zeta_T^{n-1} \) and \( \zeta_U^{n-1} \) are non-negative constants, and assume that the joint probability density function (pdf), cumulative density function (cdf), survival function (sf), and hazard function (hf) of the random vector \( X_1 = (T_1, U_1) \), are \( f_1(t,u) \), \( F_1(t,u) \), \( \bar{F}_1(t,u) \), and \( z_1(t,u) \), respectively. Then, it follows that the pdf, cdf, sf, hf, mean, and variance of random vector \( X_n \), for \( n = 2, 3, 4, \ldots \) are given by:

\[
f_n(t,u) = \frac{1}{\zeta_T^{n-1} \zeta_U^{n-1}} f_1(\zeta_T^{1-n} t, \zeta_U^{1-n} u) \quad (4.13)
\]

\[
F_n(t,u) = F_1(\zeta_T^{1-n} t, \zeta_U^{1-n} u) \quad (4.14)
\]

\[
\bar{F}_n(t,u) = \bar{F}_1(\zeta_T^{1-n} t, \zeta_U^{1-n} u) \quad (4.15)
\]

\[
z_n(t,u) = \frac{1}{\zeta_T^{n-1} \zeta_U^{n-1}} z_1(\zeta_T^{1-n} t, \zeta_U^{1-n} u) \quad (4.16)
\]

\[
E[X_n] = \frac{\zeta_T^{n-1} \zeta_U^{n-1}}{\zeta_T \zeta_U} E[X_1] \quad (4.17)
\]

\[
Var(X_n) = \frac{\zeta_T^{2n-2} \zeta_U^{2n-2}}{\zeta_T \zeta_U} Var(X_1). \quad (4.18)
\]

Because the non-negativity of \( X_1 \) and the fact \( X_1 \) is not identically 0, we conclude that \( E[X_n] = \mu_1 \neq 0 \).
**Theorem 4.4.1** If \( f_1(t,u) \) belongs to MIFR (MDFR) then \( f_n(t,u) \) is also MIFR (MDFR) for \( n = 2, 3, 4, \ldots \).

**Proof.** Suppose that the hazard function of \( X_1 \) is differentiable with respect to \( t \) and \( u \). From Eq. (4.16) the derivative of the hazard function of \( X_n \) is given by:

\[
z'_n(t,u) = \frac{1}{\xi_T^{n-2} - \xi_U^{n-2}} z_1'(\xi_T^{1-n} t, \xi_U^{1-n} u).
\]  
(4.19)

From Eq. (4.19), we see that if \( z_1(t,u) \) is increasing (decreasing) then \( z_n(t,u) \) is also increasing (decreasing). This completes the proof.

---

**4.5 Distribution of \( N_{T,U}(t,u) \) for a Bivariate Quasi-Renewal Process**

Consider a bivariate quasi-renewal process. The distribution of \( N_{T,U}(t,u) \) can be obtained by first noting the important relationship that the number of renewals by time \( t \) is greater than or equal to \( n \) if, and only if, the \( n \)th quasi-renewal occurs before or at time \( t \) and usage \( u \). That is,

\[
\{ N_{T,U}(t,u) \geq n \} \leftrightarrow \{ S_n \leq t, V_n \leq u \} \leftrightarrow \{ Y_n \leq (t,u) \}.
\]  
(4.20)

where \( Y_0 = (0, 0) \), and

\[
Y_n = \sum_{i=1}^{n} X_i = \left( \sum_{i=1}^{n} \xi_T (i) T_i, \sum_{i=1}^{n} \xi_U (i) U_i \right) = (S_n, V_n), n \geq 1.
\]  
(4.21)
It follows that $Y_n$ is the bivariate measure of both time and use of the $n$th quasi-renewal as in an ordinary bivariate renewal process. Since the number of events by time $t$ and usage $u$ will equal the largest value of $n$ for which the $n$th quasi-renewal occurs before or at time $t$ and usage $u$, we have that $N_{T,U}(t,u)$, the number of quasi-renewals by time $t$, is given by

$$N_{T,U}(t,u) = \sup \{ n : n \geq 0, S_n \leq t, V_n \leq u \} = \sup \{ n : n \geq 0, Y_n \leq (t,u) \}.$$  \hspace{1cm} (4.22)

From Eq. (4.20), we obtain

$$P\{N_{T,U}(t,u) = n\} = P\{N_{T,U}(t,u) \geq n\} - P\{N_{T,U}(t,u) \geq n + 1\}$$  \hspace{1cm} (4.23)

$$= P\{Y_n \leq (t,u)\} - P\{Y_{n+1} \leq (t,u)\}$$

$$= G^{(n)}(t,u) - G^{(n+1)}(t,u) ,$$

where $G^{(n)}(t)$, the convolution of the interarrival distributions $F_1(t,u), F_2(t,u), \ldots, F_n(t,u)$, i.e., $G^{(n)}(t) = F_1(t,u) * F_2(t,u) * \cdots * F_n(t,u)$.

Since the random variables $X_n$, $n = 1, 2, \ldots$, are independently distributed with distributions $F_n(t,u) = P\{X_n \leq (t,u)\} = P\{\zeta_{S_T}(n)T_n \leq t, \zeta_{T,V}(n)U_n \leq u\}$, it follows that $Y_n$ is distributed as $G^{(n)}(t)$, the convolution of the interarrival distributions $F_1(t,u), F_2(t,u), \ldots, F_n(t,u)$. Therefore, the following theorem is obtained:

**Theorem 4.5.1** For $t, u \geq 0$, and $n \geq 0$,

$$P\{N_{T,U}(t,u) = n\} = G^{(n)}(t,u) - G^{(n+1)}(t,u) .$$  \hspace{1cm} (4.24)
**Proof.** Given above.

An important specific realization of Eq. (4.24) is:

\[ P\{N_{T,U}(t,u) = 0\} = G^{(0)}(t,u) - G^{(1)}(t,u) \]
\[ = 1 - F_1(t,u). \]  

(4.25)

Let

\[ M_{G}(t,u) = E[N_{T,U}(t,u)], \]  

(4.26)

where \( M_{G}(t,u) \) represents the quasi-renewal function (or mean-value function). The following theorem illustrates the relationship between \( M_{G}(t,u) \) and \( G^{(n)}(t,u) \).

**Theorem 4.5.2** For \( t \geq 0, \ u \geq 0, \) and \( n \geq 0, \)

\[ M_{G}(t,u) = \sum_{n=0}^{\infty} G^{(n)}(t,u). \]  

(4.27)

**Proof.** The proof of this theorem is similar as of Theorem 4.3.2. Here, we present a shorter proof as follows.

\[ M_{G}(t,u) = E[N_{T,U}(t,u)] \]
\[ = \sum_{n=0}^{\infty} nP\{N(t,u) = n\} \]
\[ = \sum_{n=0}^{\infty} n\left[G^{(n)}(t,u) - G^{(n+1)}(t,u)\right] \] (by Theorem 4.5.1)
\[ = \sum_{n=0}^{\infty} G^{(n)}(t,u). \]  

(4.27)
Transforming \( M_G(t, u) \) into the recursive form:

\[
M_G(t, u) = \sum_{n=1}^{\infty} G^{(n)}(t, u)
\]

\[
= G^{(1)}(t, u) + \sum_{n=2}^{\infty} G^{(n)}(t, u)
\]

\[
= F_1(t, u) + \sum_{j=1}^{\infty} G^{(j+1)}(t, u)
\]

\[
= F_1(t, u) + \sum_{j=1}^{\infty} \int_0^t \int_0^u G^{(j)}(t-x, u-y) dF_1(x, y)
\]

\[
= F_1(t, u) + \int_0^t \sum_{j=1}^{\infty} G^{(j)}(t-x, u-y) dF_1(x, y)
\]

\[
= F_1(t, u) + \int_0^t \int_0^u G_G(t-x, u-y) dF_1(x, y).
\]

Hence,

\[
M_G(t, u) = F_1(t, u) + \int_0^t \int_0^u G_G(t-x, u-y) dF_1(x, y). \tag{4.28}
\]

Eq. (4.28) is called the integral equation of quasi-renewal theory.

Assuming that \( F_1(t, u) \) is absolutely continuous, then the derivative of \( M_G(t, u) \), denoted by \( m_G(t, u) \), is called the quasi-renewal density function,

\[
m_G(t, u) = \frac{d}{dt} M_G(t, u) = \sum_{n=1}^{\infty} G^{(n)}(t, u)
\]

\[
= f_1(t, u) + \int_0^t \int_0^u m_G(t-x, u-y) f_1(x, y) dx dy,
\]

and \( m_G(t, u) \) represents the probability of a quasi renewal at any point in the time-usage plane.
4.6 Bivariate Laplace Transforms

In bivariate renewal theory, the longevity random variables represent the interarrival intervals and only assume non-negative values. The bivariate Laplace transform of an arbitrary function $\phi(t, u)$ is defined as:

$$\phi^*(s, v) = L_{s,v}\{\phi(t, u)\} = \int_0^\infty \int_0^\infty e^{-st-vu} \phi(x, y) dx dy. \quad (4.30)$$

For a joint distribution function $F(t, u)$, the Laplace-Stieltjes transform is defined as:

$$F^*(s, v) = L_{s,v}\{F(t, u)\} = \int_0^\infty \int_0^\infty e^{-st-vu} dF(t, u) = \int_0^\infty \int_0^\infty e^{-st-vu} f(t, u)dt du = f^*(s, v) = L_{s,v}\{f(t, u)\}. \quad (4.31)$$

Assume that the joint distribution function $F(t, u)$ is adequately continuous and differentiable, i.e.,

$$F(t, u) = \int_0^t \int_0^u f(x, y)dy dx, \; 0 < t, u < \infty.$$ 

The following theorem indicates the relationship between the Laplace transform of $F(t, u)$ and its density function $f(t, u)$.

**Theorem 4.6.1** Let $F^*_{T,U}(s, v)$ and $f^*_{T,U}(s, v)$ be the Laplace transforms of $F_{T,U}(t, u)$ and its density function $f_{T,U}(t, u)$. Then

$$F^*_{T,U}(s, v) = \frac{1}{sv} f^*_{T,U}(s, v). \quad (4.32)$$
Proof.

\[ F^*(s,v) = L_{s,v} \{ F(t,u) \} = L_{s,v} \left\{ \int_0^t \int_0^u f(x,y) \, dy \, dx \right\} = \int_{0}^{\infty} \int_{0}^{\infty} e^{-st-uv} \int_{0}^{t} \int_{0}^{u} f(x,y) \, dy \, dx \, dt \, du \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} \int_{x}^{\infty} e^{-st-uv} f(x,y) \, dy \, dx \, dt \, du \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{0}^{\infty} e^{-st-uv} \, du \right) \, dt - \int_{0}^{\infty} \int_{0}^{\infty} e^{-st-uv} \, du \, dt - \int_{x}^{\infty} \int_{0}^{\infty} e^{-st-uv} \, du \, dt - \int_{x}^{\infty} \int_{0}^{\infty} e^{-st-uv} \, du \, dt \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} \left( \frac{1}{sv} \left( 1 - e^{-s} \right) \left( 1 - e^{-v} \right) \right) \, f(x,y) \, dy \, dx \]

\[ = \frac{1}{sv} \int_{0}^{\infty} \int_{0}^{\infty} \left( e^{-s-x-y} \right) \, f(x,y) \, dy \, dx \]

\[ = \frac{1}{sv} F^*(s,v) \]

From Eqs. (4.7), (4.8), (4.30), and (4.32), if we write,

\[ M^*_F(s,v) = L_{s,v} \left\{ M_F(t,u) \right\} = \int_{0}^{\infty} \int_{0}^{\infty} e^{-st-uv} M_F(t,u) \, dt \, du , \]

then it follows that,

\[ M^*_F(s,v) = L_{s,v} \left\{ M_F(t,u) \right\} = F^*(s,v) + L_{s,v} \left\{ \int_{0}^{t} \int_{0}^{u} M_F(t-x,u-y) \, dy \, dx \right\} \]

\[ = F^*(s,v) + \int_{0}^{\infty} \int_{0}^{\infty} e^{-st-uv} \int_{0}^{t} \int_{0}^{u} M_F(t-x,u-y) \, dy \, dx \, dt \]

\[ = F^*(s,v) + \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{t} \int_{0}^{u} e^{-st-uv} M_F(t-x,u-y) \, dy \, dx \, dt \]

\[ = F^*(s,v) + \int_{0}^{\infty} \int_{0}^{\infty} \left[ \int_{x}^{\infty} \int_{y}^{\infty} e^{-s(t-x)-v(u-y)} M_F(t-x,u-y) \, du \, dv \right] e^{-s-x-y} f(x,y) \, dy \].
Let \( w = t - x \) and \( z = u - y \) so \( dw = dt \) and \( dz = du \) for \( w \in [0, \infty) \) and \( z \in [0, \infty) \), then

\[
M_F^*(s,v) = \mathcal{F}(s,v) + \int_0^\infty \int_0^\infty e^{-svz} M_F(w,z) dzdw e^{-sxv} f(x,y) dydx
\]

\[
= \mathcal{F}(s,v) + \int_0^\infty \int_0^\infty M_F^*(s,v) e^{-svz} f(x,y) dydx
\]

\[
= \mathcal{F}(s,v) + M_F^*(s,v) f^*(s,v) . \quad (4.33)
\]

Eq. (4.33) can be solved to yield the following theorem:

**Theorem 4.6.2** Let \( M_F^*(s,v) \) be the Laplace transform of \( M_F(t,u) \). Then

\[
M_F^*(s,v) = \frac{\mathcal{F}(s,v)}{1 - f^*(s,v)} = \frac{f^*(s,v)}{sv[1 - f^*(s,v)]} = \frac{\mathcal{F}(s,v)}{1 - svF^*(s,v)} \quad (4.34)
\]

and

\[
F^*(s,v) = \frac{M_F^*(s,v)}{1 + svM_F^*(s,v)} \quad \text{or} \quad f^*(s,v) = \frac{svM_F^*(s,v)}{1 + svM_F^*(s,v)} . \quad (4.35)
\]

The corresponding results for \( m_F^*(s,v) \) and \( f^*(s,v) \) are shown in the following theorem.

**Theorem 4.6.3** Let \( m_F^*(s,v) \) be the Laplace transform of \( m_F(t,u) \). Then

\[
m_F^*(s,v) = \frac{f^*(s,v)}{1 - f^*(s,v)} \quad (4.36)
\]

and
\[ f^*(s,v) = \frac{m_F^*(s,v)}{1 + m_F^*(s,v)}. \]  

(4.37)

It is important that knowledge of the bivariate renewal function \( M_F(t,u) \) implies complete knowledge of all aspects of the bivariate renewal process (see Hunter [1974], pp. 387). The results of theorems 4.6.2 and 4.6.3 correspond to the univariate renewal theory (Cox [1962], pp.46), which are very appealing.

For the bivariate quasi-renewal process with \((\zeta_T(n), \zeta_U(n)) = (\zeta_T^{n-1}, \zeta_U^{n-1})\), we obtain the following results for the Laplace-Stieltjes transforms.

**Theorem 4.6.4** Let \( F_n^*(s,v) \) be the Laplace-Stieltjes transform for \( F_n(t,u) \), the distribution for \( X_n \). Let \( M_G^*(s,v) \) be the Laplace-Stieltjes transforms for \( M_G(t,u) \). Let \( m_G^*(s,v) \) be the Laplace transform for \( m_G(t,u) \). Then

\[
F_n^*(s,v) = LS_{s,v}\{F_n(t,u)\} = \int_0^{\infty} \int_0^{\infty} e^{-s n_1 u - s n_2 v} dF_n(t,u) = F_1^* \left( \zeta_T^{n-1} s, \zeta_U^{n-1} v \right),
\]

(4.38)

\[
M_G^*(s,v) = LS_{s,v}\{M_F(t,u)\} = \sum_{n=1}^{\infty} G_{s,v}^*(n) (s,v)
\]

\[
= \sum_{n=1}^{\infty} F_1^* (s,v) \cdot F_1^* (\zeta_T s, \zeta_U v) \cdots F_1^* (\zeta_T^{n-1} s, \zeta_U^{n-1} v),
\]

(4.39)

and
\[ m_G^* (s, v) = \sum_{n=1}^\infty g^{s(n)} (s, v) = \sum_{n=1}^\infty f_1^* (s, v) \cdot f_1^* (\zeta_T, s, \zeta_U, v) \cdots f_1^* (\zeta_T, s, \zeta_T, s, \zeta_T, v) . \quad (4.40) \]

4.7 Delayed (or Modified) Bivariate Renewal Processes

Consider a bivariate renewal process where the first renewal has a different distribution than subsequent renewals. Suppose that we start observing a bivariate renewal process at some point \((t, u) > 0\). If a renewal does not occur at \((t, u)\), then the joint distribution of the time and usage we must wait until the first observed renewal will not be the same as the remaining interarrival distributions.

Let \( \{ X_n = (T_n, U_n), n = 1, 2, ... \} \), be a sequence of independent non-negative bivariate random vectors with \( X_1 \) having joint distribution \( F_1 \), and \( X_i \) having common joint distribution \( F \), for \( i > 1 \). Let \( Y_0 = (0,0), Y_n = \sum_{i=1}^n X_i = (\sum_{i=1}^n T_i, \sum_{i=1}^n U_i) = (S_n, V_n) \), for \( n = 1, 2, ... \). The number of renewals by time \( t \) and usage \( u \), is defined by

\[ ND_{T,U} (t, u) = \sup \{ n : n \geq 0, S_n \leq t, V_n \leq u \} = \sup \{ n : n \geq 0, Y_n \leq (t, u) \} . \quad (4.41) \]

**Definition 4.7.1**

Let the sequence of non-negative bivariate random vectors \( \{ X_i, i = 1, 2, ... \} \) be independently distributed with \( X_1 \) having joint distribution \( F_1 \), and \( X_i \) having common joint distribution \( F, i = 1, 2, ... \). The counting process \( \{ ND_{T,U} (t, u), t > 0, u > 0 \} \) is called a delayed (or modified) bivariate renewal counting process.
When $F_1 = F$, we have an ordinary bivariate renewal process. As in the ordinary case, we have

$$P\{ND_{T,U}(t,u) = n\} = P\{ND_{T,U}(t,u) \geq n\} - P\{ND_{T,U}(t,u) \geq n + 1\}$$

$$= P\{Y_n \leq (t,u)\} - P\{Y_{n+1} \leq (t,u)\}$$

$$= F_1 * F^{(n-1)}(t,u) - F_1 * F^{(n)}(t,u)$$

where $F_1 * F^{(n)}(t,u)$ is the convolution of $F_1$ and $F^{(n)}$.

Let

$$MD(t,u) = E[ND_{T,U}(t,u)] .$$

Then it is easy to derive the relationship between $MD(t,u)$ and $F(t,u)$ that

$$MD(t,u) = \sum_{n=1}^{\infty} F_1 * F^{(n-1)}(t,u)$$

and

$$MD(t,u) = F_1(t,u) + \int_0^t \int_0^u MD(t-x,u-y)dF(x,y)$$

or

$$md(t,u) = f_1(t,u) + \int_0^t \int_0^u md(t-x,u-y)f(x,y)dydx.$$
Taking Laplace transforms of Eq. (4.45), by Eq. (4.32) and assuming that the joint distribution function $F(t,u)$ is adequately continuous and differentiable with density $f(t,u)$, we obtain

$$MD^*(s,v) = F_1^*(s,v) + MD^*(s,v) f^*(s,v). \quad (4.47)$$

Eq. (4.47) can be solved to yield the following theorem.

**Theorem 4.7.1** Let $MD^*(s,v)$ be the Laplace transform of $MD(t,u)$. Then

$$MD^*(s,v) = \frac{F_1^*(s,v)}{1 - f^*(s,v)} = \frac{f_1^*(s,v)}{sv[1 - f^*(s,v)]} = \frac{F_1^*(s,v)}{1 - svF^*(s,v)}. \quad (4.48)$$

Similarly, from Eq. (4.44), we obtain the following.

**Theorem 4.7.2** Let $md^*(s,v)$ be the Laplace transform of $md(t,u)$, then

$$md^*(s,v) = \frac{f_1^*(s,v)}{1 - f^*(s,v)}. \quad (4.49)$$

### 4.8 Alternating Bivariate Renewal Processes

Consider a system that can be in one of two states: $E_1$ or $E_2$ (e.g., operating or repair; on or off). Initially it is in $E_1$ and it remains in $E_1$ for a time $T_1$ and a usage $U_1$; it then fails and remains in $E_2$ for a time $R_1$ and a usage $Q_1$; it then goes on to $E_1$ for a
time $T_2$ and for a usage $U_2$; then off to $E_2$ for a time $R_2$ and a usage $Q_2$; and so on. Suppose that the random vectors $(T_n, U_n, R_n, Q_n), n \geq 0$, are independent and identically distributed. That is, all the sequences of random variables $\{T_n\}, \{U_n\}, \{R_n\},$ and $\{Q_n\}$ are i.i.d., but $\{T_n\}, \{U_n\}, \{R_n\},$ and $\{Q_n\}$ are allowed to be dependent. In other words, each time the system goes on everything starts over again, but when it goes off we allow the length of off-time to depend on the previous operating-time and/or operating-usage. The resulting process is called an alternating bivariate renewal process.

Let $X_n = \{(T_n, U_n)\}$ and $Y_n = \{(R_n, Q_n)\}, n = 1, 2, \ldots$, be vectors of i.i.d. non-negative bivariate random variables with the common joint distribution functions $F(t,u) = P[T_n \leq t, U_n \leq u]$ and $G(t,u) = P[R_n \leq t, Q_n \leq u]$, respectively; and to avoid trivialities, assume that $F(0,0) = P[T_n = 0, U_n = 0] < 1$ and $G(0,0) = P[R_n = 0, Q_n = 0] < 1$.

Consider the sequence of renewals formed transitions from state $E_2$ to state $E_1$. This is an ordinary bivariate renewal process in which each renewal happens at the sum of $((T_n + R_n), (U_n + Q_n)), n \geq 1$. Thus, the first renewal occurs at $((T_1 + R_1), (U_1 + Q_1))$, the second at $((T_1 + R_1), (U_1 + Q_1)) + ((T_2 + R_2), (U_2 + Q_2))$, and so on. By Eq. (4.32), taking as the distribution of the renewal-time the convolution of $F(t,u)$ and $G(t,u)$, with Laplace transform $\mathcal{L}\{F(t,u)G(t,u)\}$, the mean number of $E_2$ renewals in rectangle $(0, 0) \times (t, u)$, denoted by $M_{II}(t,u)$, satisfies

\[
M_{II}^*(s,v) = \frac{F^*(s,v)G^*(s,v)}{1 - f^*(s,v)g^*(s,v)} = \frac{f^*(s,v)g^*(s,v)}{sv[1 - f^*(s,v)g^*(s,v)]} = \frac{F^*(s,v)G^*(s,v)}{1 - svF^*(s,v)G^*(s,v)}.
\]

Similarly, if we consider solely renewals of $E_1$ state, we have a delayed (or modified) bivariate renewal process, in which the probability density function of first renewal is $f(t,u)$, and of all subsequent renewals is the convolution of $f(t,u)$ and $g(t,u)$. 

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Thus, by Eq. (4.48), the expected number of $E_1$ renewals in rectangle $(0, 0) \times (t, u)$, denoted by $M_1(t, u)$, satisfies

$$M_1^*(s, v) = \frac{F^*(s, v)}{1 - f^*(s, v)g^*(s, v)} = \frac{f^*(s, v)}{sv[1 - f^*(s, v)g^*(s, v)]} = \frac{F^*(s, v)}{1 - svF^*(s, v)G^*(s, v)}.$$  \hfill (4.51)

The corresponding renewal densities have Laplace transforms

$$m_k^*(s, v) = svM_k^*(s, v), \text{ for } k = I, II.$$  \hfill (4.52)

We summarize the results for an alternating bivariate renewal theorem in the following theorem.

**Theorem 4.8.1** For an alternating bivariate renewal process, the expected number of $E_1$ and $E_2$ renewals satisfies

$$M_1^*(s, v) = \frac{F^*(s, v)}{1 - f^*(s, v)g^*(s, v)} = \frac{f^*(s, v)}{sv[1 - f^*(s, v)g^*(s, v)]} = \frac{F^*(s, v)}{1 - svF^*(s, v)G^*(s, v)},$$

and

$$M_2^*(s, v) = \frac{F^*(s, v)G^*(s, v)}{1 - f^*(s, v)g^*(s, v)} = \frac{f^*(s, v)g^*(s, v)}{sv[1 - f^*(s, v)g^*(s, v)]} = \frac{F^*(s, v)G^*(s, v)}{1 - svF^*(s, v)G^*(s, v)}.$$
The corresponding renewal densities have Laplace transforms

\[ m_k^*(s,v) = svM_k^*(s,v), \text{ for } k = I, II. \]
CHAPTER V
BIVARIATE CORRECTIVE MAINTENANCE MODELING

In this chapter we apply the results of the bivariate renewal theory to construct bivariate corrective maintenance (CM) models for a single-unit system. We limit the construction to the bivariate failure models with correlated dependence relations between the two variables. With only the consideration of corrective maintenance actions, the results of the bivariate renewal theory can be applied directly. We obtain the Laplace transforms of the renewal functions for the bivariate CM models.

5.1 Notation

\( R_n \) the repair/replace duration after the \( n \)th renewal

\( g_{R}(t, u), g_{R}(t) \) the density functions for \( R_n \)

\( H_{T,U}(t, u) \) the common distribution of \( X_n + R_n \)

\( h_{T,U}(t, u) \) the probability density function of \( H_{R}(t, u) \)

\( H^{*}_{T,U}(t, u), h^{*}_{T,U}(t, u) \) the Laplace transforms for \( H_{R}(t, u) \) and \( h_{T,U}(t, u) \)

\( M_{H}(t, u), M^{*}_{H}(t, u) \) the bivariate renewal function for \( H_{R}(t, u) \) and its Laplace transform

\( m_{H}(t, u), m^{*}_{H}(t, u) \) the bivariate renewal density function for \( h_{R}(t, u) \) and its Laplace transform
Assumptions and Descriptions of Bivariate Corrective Maintenance Models

Consider a single-unit system with an independent and identically distributed (i.i.d.) non-negative bivariate random lifetime denoted by \( X_n = \{(T_n, U_n)\}, n = 1, 2, \ldots \), with a common bivariate joint distribution function given by \( F(t,u) = P(T_n \leq t, U_n \leq u) \). Letting

\[
Y_0 = (0, 0), \quad Y_n = \sum_{i=1}^n X_i = \left(\sum_{i=1}^n T_i, \sum_{i=1}^n U_i\right) = (S_n, V_n), \quad n \geq 1, \quad (5.1)
\]

it follows that \( Y_n = (S_n, V_n) \) is the bivariate measure of both time and use of the \( n \)th renewal as illustrated in Figure 5.2.1. Assume systems fail permanently and independently. A failed system is either repaired or replaced by an i.i.d. new one. Under a replacement policy, a system is immediately replaced upon failure. Under a repair policy, a system is immediately repaired upon failure. Both replacement and repair maintenance actions are assumed to be perfect so that after maintenance the failed system is restored to an “as good as new” state. Thus, the system is said to be renewed after each maintenance action.

Let \( T_n \) and \( U_n \) be the operating time and usage after the \((n-1)\)st maintenance, respectively. Let \( R_n \) be the repair/replace time after the \( n \)th renewal. Assume \( \{T_n, n = 1, 2, \ldots\} \), \( \{U_n, n = 1, 2, \ldots\} \), and \( \{R_n, n = 1, 2, \ldots\} \) are stochastic processes with sequences of i.i.d. non-negative random variables where \( \{T_n\} \), \( \{U_n\} \), and \( \{R_n\} \) may be dependent.

Note that \( R_n \) can be bivariate or univariate. Our goal is to develop and construct bivariate corrective maintenance models indexed by time and usage based on the bivariate failure models and the bivariate renewal theory. Figure 5.2.1 shows a realization of a bivariate repair/replacement stochastic process.

We consider the correlated bivariate failure models developed in Chapter 3 as our joint distribution functions in construction of bivariate CM models.
Figure 5.2.1 A realization of a bivariate repair/replacement stochastic process.
5.3 Corrective Maintenance Models with Instantaneous Repair

In this section we consider the bivariate corrective maintenance (CM) models with instantaneous repair, i.e., $R_e = 0$, $n = 1, 2, \ldots$. That is, we assume the repair/replacement is negligible. Therefore, the bivariate CM models can be constructed directly from the results of bivariate renewal theory. We call these CM models bivariate renewal models to distinguish between the CM models with instantaneous repair and with non-negligible repair.

5.3.1 Bivariate Exponential renewal Model I

Assume that the distribution function of $X_n$, $F(t,u)$, is Baggs and Nagagaja's bivariate exponential with joint density function:

$$f_{T,U}(t,u) = \lambda \eta e^{-(\lambda t + \eta u)} \left( 1 + \rho \left( 1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-(\lambda t + \eta u)} \right) \right).$$  \hspace{1cm} (5.2)

The Laplace transform of Eq. (5.2) is:

$$f_{T,U}^*(s,v) = \frac{\lambda \eta [(v + 2\eta)(s + 2\lambda) + s\nu \rho]}{(v + \eta)(s + \lambda)(v + \nu)(s + \nu \lambda)}.$$  \hspace{1cm} (5.3)

From Eq. (4.34), we obtain the Laplace transform of the bivariate renewal function:

$$M_{T,U}^*(s,v) = \frac{\lambda \eta [(v + 2\eta)(s + 2\lambda) + s\nu \rho]}{sv[(v + 2\eta)(s + 2\lambda)(s + \nu \lambda) + s\nu \lambda \rho]}.$$  \hspace{1cm} (5.4)

and the Laplace transform of renewal density function:
Eqs. (5.4) or (5.5) are the models for immediate and instantaneous repair/replacement maintenance.

5.3.2 Bivariate Exponential Renewal Model II

Assume that the distribution function of \( X_n, F(t,u) \), is Hunter's bivariate exponential with joint density function:

\[
    f_{r,u}(t,u) = \frac{\lambda \eta \left( \sqrt{\frac{1-\rho}{\rho}} \right) \exp \left\{ -\frac{\lambda t + \eta u}{1-\rho} \right\}}{1-\rho} \tag{5.6}
\]

where \( I_n(\cdot) \) is the modified Bessel function of the first kind of order \( n \); and \( \rho \) is positive.

The Laplace transform of Eq. (5.6) is:

\[
    f^*(s,v) = \left[ \left( \frac{S}{\lambda} + 1 \right) \left( \frac{\nu}{\eta} + 1 \right) - \frac{sv\rho}{\lambda\eta} \right]^{-1} \tag{5.7}
\]

From Eq. (4.34), we obtain the Laplace transform of the bivariate renewal function:

\[
    M_r^*(s,v) = \left[ sv \left( \frac{S}{\lambda} + \frac{\nu}{\eta} + \frac{sv}{\lambda\eta} \right) (1-\rho) \right]^{-1} \tag{5.8}
\]

and the Laplace transform of renewal density function:
\[ m_f^*(s, v) = \left[ \frac{s + v + sv}{\lambda \eta + \lambda \eta (1 - \rho)} \right]^{-1}. \] (5.9)

Eq. (5.9) may be inverted (Hunter [1974]) to give an explicit expression for \( m_f(t, u) \),

\[ m_f(t, u) = \frac{\lambda \eta}{1 - \rho} I_0 \left( \frac{2\sqrt{\lambda \eta tu}}{1 - \rho} \right) \exp \left\{ -\frac{\lambda t + \eta u}{1 - \rho} \right\}. \] (5.10)

where \( I_n(\cdot) \) is the modified Bessel function of the first kind of order \( n \); and \( \rho \) is positive.

Equations (5.8), (5.9), or (5.10) are the models for immediate and instantaneous repair/replacement maintenance. Note that in this model we have the inverse of the bivariate Laplace transform, but this is rare.

5.3.3 Bivariate Normal Renewal Model

Assume that the distribution function of \( X_n, F(t, u) \), is bivariate normal with joint density function:

\[ f_{t,u}(t, u) = \frac{1}{2\pi \sigma_i \sigma_u \sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ \frac{(t - \mu_u)^2}{\sigma_i^2} - 2\rho \left( \frac{t - \mu_u (u - \mu_u)}{\sigma_i \sigma_u} + \frac{(u - \mu_u)^2}{\sigma_u^2} \right) \right] \right\}. \] (5.11)

The Laplace transform of Eq. (5.11) is:

\[ f^*(s, v) = \exp \left[ -sv \mu_u + \frac{s^2 \sigma_i^2}{2} + 2s \rho v \sigma_i \sigma_u + v^2 \sigma_u^2 \right]. \] (5.12)

From Eq. (4.34), we obtain the Laplace transform of the bivariate renewal function:
\[ M^*_F(s, v) = \frac{\exp\left[-s\mu_r - v\mu_u + \frac{1}{2} \left(s^2\sigma_r^2 + 2\rho sv\sigma_r\sigma_u + v^2\sigma_u^2\right)\right]}{sv\left[1 - \exp\left[-s\mu_r - v\mu_u + \frac{1}{2} \left(s^2\sigma_r^2 + 2\rho sv\sigma_r\sigma_u + v^2\sigma_u^2\right)\right]\right]}, \quad (5.13) \]

and the Laplace transform of renewal density function:

\[ m^*_f(s, v) = \frac{\exp\left[-s\mu_r - v\mu_u + \frac{1}{2} \left(s^2\sigma_r^2 + 2\rho sv\sigma_r\sigma_u + v^2\sigma_u^2\right)\right]}{1 - \exp\left[-s\mu_r - v\mu_u + \frac{1}{2} \left(s^2\sigma_r^2 + 2\rho sv\sigma_r\sigma_u + v^2\sigma_u^2\right)\right]}. \quad (5.14) \]

Equations (5.13) and (5.14) are the models for immediate and simultaneous repair/replacement maintenance.

5.4 Corrective Maintenance Models

In this section we consider the bivariate corrective maintenance (CM) models with repair that is not negligible. Two approaches are presented. First, by the results of alternating bivariate renewal theory, the bivariate CM models can be constructed to obtain the Laplace transforms for the bivariate renewal function and renewal density. Second, we construct the CM models by developing the common distribution of \( X_n + R_n \) and applying the results of ordinary bivariate renewal theory to derive the Laplace transforms for renewal functions and densities.

5.4.1 Bivariate Exponential CM Model I

We consider that the repair/replacement is not negligible and assume that \( R_n \) is distributed either \( g_R(t, u) \) or \( g_R(t) \), where
\[ g_R(t,u) = \lambda_i \eta_i e^{-(\lambda_i + \eta_i)u} \left( 1 + \rho_i \left( 1 - 2e^{-\lambda_i t} - 2e^{-\eta_i u} + 4e^{-(\lambda_i + \eta_i)u} \right) \right) \]  
(5.15)

and

\[ g_R(t) = \lambda_i e^{-\lambda_i t}. \]  
(5.16)

The corresponding Laplace transforms for Eqs. (5.15) and (5.16) are:

\[ g^*_R(s,v) = \frac{\lambda_i \eta_i \left( (v + 2\eta_i)(s + 2\lambda_i) + sv \rho_i \right)}{((v + \eta_i)(s + \lambda_i)(v + 2\eta_i)(s + 2\lambda_i))} \]  
(5.17)

and

\[ g^*_R(s) = \frac{\lambda_i}{\lambda_i + s}. \]  
(5.18)

State \( E_1 \) indicates the system is operating (on); and state \( E_2 \) indicates it is in the repair/replacement (off) state. Then by the results of the alternating bivariate renewal process, i.e., Eqs. (4.50) and (4.51), we obtain the following models for the number of \( E_1 \) and \( E_2 \) renewals.

\[ M^*_1(s,v) = \frac{f^*(s,v)}{sv[1 - f^*(s,v)g^*_R(s,v)]} \]  
(5.19)

and

\[ M^*_2(s,v) = \frac{f^*(s,v)g^*_R(s,v)}{sv[1 - f^*(s,v)g^*_R(s,v)]}. \]  
(5.20)

For \( g_R(t) = \lambda_i e^{-\lambda_i t} \), we obtain that
\[ M_k^*(s,v) = \frac{(\lambda + s\eta\lambda_{1}[(v + 2\eta)(s + 2\lambda) + sv\rho])}{sv(v + 2\eta)(s + 2\lambda)[(v + \eta)(s + \lambda)(s + \lambda_{1}) + \lambda_{1}\lambda\eta]} - s^2v^2\eta\lambda\lambda_{1}\rho, \quad (5.21) \]

and

\[ M_{II}^*(s,v) = \frac{\eta\lambda\lambda_{1}[(v + 2\eta)(s + 2\lambda) + sv\rho]}{sv(v + 2\eta)(s + 2\lambda)[(v + \eta)(s + \lambda)(s + \lambda_{1}) + \lambda_{1}\lambda\eta]} - s^2v^2\eta\lambda\lambda_{1}\rho. \quad (5.22) \]

For \( g_{k}(t,u) = \lambda_{e}\eta_{e}e^{-(\lambda_{e}t+\eta_{e}u)}\left(1 + \rho_{e}\left(1 - 2e^{-\lambda_{e}t} - 2e^{-\eta_{e}u} + 4e^{-(\lambda_{e}t+\eta_{e}u)}\right)\right) \), we obtain

\[ M_{I}^*(s,v) = \frac{\lambda\eta[B][C]}{sv[A][B] - \lambda\eta\lambda\eta sv[C][D]}, \quad (5.23) \]

and

\[ M_{II}^*(s,v) = \frac{\lambda\eta_{1}\lambda\eta[C][D]}{sv[A][B] - \lambda\eta_{1}\lambda\eta sv[C][D]}, \quad (5.24) \]

where

\[ [A] = (s + \lambda)(s + 2\lambda)(v + \eta)(v + 2\eta), \]

\[ [B] = (s + \lambda_{1})(s + 2\lambda_{1})(v + \eta_{1})(v + 2\eta_{1}), \]

\[ [C] = (v + 2\eta)(s + 2\lambda) + sv\rho, \text{ and} \]

\[ [D] = (v + 2\eta_{1})(s + 2\lambda_{1}) + sv\rho_{1}. \]

The corresponding renewal densities have Laplace transforms

\[ m_k^*(s,v) = svM_k^*(s,v), \text{ for } k = I, \ II. \quad (5.25) \]
If we consider the common distribution of \( X_n + R_n \), and denote it as \( H_{T,U}(t,u) \) with j.d.f. \( h_{T,U}(t,u) \), then \( H_{T,U}(t,u) \) is the convolution of \( F_{T,U}(t,u) \). Let \( M_H(t,u) \) be the bivariate renewal function corresponding to the underlying distribution \( H_{T,U}(t,u) \), i.e., \( M_H(t,u) = \sum_{n=1}^{\infty} H_{T,U}^{(n)}(t,u) \). Let \( m_H(t,u) \) be the corresponding renewal density function. Let \( H^*_R(s,v) \), \( M^*_H(s,v) \), and \( m^*_H(s,v) \) be the Laplace transforms for \( H_{T,U}(t,u) \), \( M_H(t,u) \), and \( m_H(t,u) \). Then from Eq. (4.34), we obtain the Laplace transform of the bivariate renewal function:

\[
M^*_H(s,v) = \frac{H^*_R(s,v)}{1 - h^*_R(s,v)} = \frac{h^*_R(s,v)}{s v (1 - h^*_R(s,v))},
\]

where

\[
h^*_R(s,v) = \frac{\lambda \lambda \eta \eta_i [(v + 2 \eta_i)(s + 2 \lambda_i) + sv p_i] [(v + 2 \eta)(s + 2 \lambda) + sv p]}{[(v + \eta_i)(s + \lambda_i)(v + 2 \eta_i)(s + 2 \lambda_i)] [(v + \eta)(s + \lambda)(v + 2 \eta)(s + 2 \lambda)]} \quad \text{for} \quad g_R(t,u)
\]\n
(5.27)

or

\[
h^*_R(s,v) = \frac{\lambda \lambda \eta \ [(v + 2 \eta)(s + 2 \lambda) + sv p]}{[(v + \eta)(s + \lambda)(v + 2 \eta)(s + 2 \lambda)] \ (\lambda_i + s)} \quad \text{for} \quad g_R(t).
\]\n
(5.28)
5.4.2 Bivariate Exponential CM Model II

We consider that the repair/replacement is not negligible and assume that \( R_n \) is distributed either \( g_R(t,u) \) or \( g_R(t) \), where

\[
g_R(t,u) = \frac{\lambda_i \eta}{1 - \rho_1} I_0 \left( \frac{2\sqrt{\rho_1}}{1 - \rho_1} \sqrt{\lambda_i \eta_i t u} \right) \exp \left\{ - \frac{\lambda_i t + \eta_i u}{1 - \rho_1} \right\} \tag{5.29}
\]

and

\[
g_R(t) = \lambda_i e^{-\lambda_i t}. \tag{5.30}
\]

State \( E_1 \) indicates the system is operating (on); and state \( E_2 \) indicates it is in the repair/replacement (off) state. Then by the results of the alternating bivariate renewal process, i.e., Eqs. (5.19) and (5.20), we obtain the following models for the numbers of \( E_1 \) and \( E_2 \) renewals.

For \( g_R(t) = \lambda_i e^{-\lambda_i t} \), we obtain that

\[
M^*_I(s,v) = \frac{\lambda \eta (s + \lambda)}{sv \{ (s + \lambda)(v + \eta) - sv \rho (s + \lambda) - \lambda \eta \lambda_i \}} \tag{5.31}
\]

\[
M^*_II(s,v) = \frac{\lambda \eta \lambda_i}{sv \{ (s + \lambda)(v + \eta) - sv \rho (s + \lambda) - \lambda \eta \lambda_i \}}. \tag{5.32}
\]

For \( g_R(t,u) = \frac{\lambda_i \eta_i}{1 - \rho_1} I_0 \left( \frac{2\sqrt{\rho_1}}{1 - \rho_1} \sqrt{\lambda_i \eta_i t u} \right) \exp \left\{ - \frac{\lambda_i t + \eta_i u}{1 - \rho_1} \right\}, \) we obtain...
The corresponding renewal densities have Laplace transforms

\[ m_k^*(s, v) = svM_k^*(s, v), \text{ for } k = I, II. \] (5.35)

If we consider the common distribution of \( X_n + R_n \) and denote it as \( H_{T,U}(t,u) \) with j.d.f. \( h_{T,U}(t,u) \), then \( H_{T,U}(t,u) \) is the convolution of \( F_{T,U}(t,u) \) and \( G_k(t,u) \) (or \( G_k(t) \)). Let \( M_{H}(t,u) \) be the bivariate renewal function corresponding to the underlying distribution \( H_{T,U}(t,u) \), i.e., \( M_{H}(t,u) = \sum_{n=1}^{\infty} H_{T,U}^{(n)}(t,u) \). Let \( m_{H}(t,u) \) be the corresponding renewal density function. Let \( H_{T,U}^*(s,v), \ M_{H}^*(s,v), \) and \( m_{H}^*(s,v) \) be the Laplace transforms for \( H_{T,U}(t,u) \), \( M_{H}(t,u) \), and \( m_{H}(t,u) \). Then from Eq. (4.34), we obtain the Laplace transform of the bivariate renewal function:

\[ M_{H}^*(s,v) = \frac{H_{T,U}^*(s,v)}{1-h_{T,U}^*(s,v)} = \frac{h_{T,U}^*(s,v)}{sv(1-h_{T,U}^*(s,v))}, \] (5.36)

where

\[ h_{T,U}^*(s,v) = \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{v}{\eta} + 1 \right) - svp \left[ \left( \frac{s}{\lambda_1} + 1 \right) \left( \frac{v}{\eta_1} + 1 \right) - svp \right] \right]^{-1} \text{ for } g_k(t,u) \] (5.37)
or

\[ h_{T,v}(s,v) = \left( \frac{\lambda_1}{\lambda_1 + s} \right) \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{v}{\eta} + 1 \right) - \frac{s v \rho}{\lambda \eta} \right]^{-1} \text{ for } g_R(t). \quad (5.38) \]

5.4.3 Bivariate Normal CM Model

Now, we consider that the repair/replacement is not negligible and assume that \( R_n \) is distributed either \( g_R(t,u) \) or \( g_R(t) \), where

\[
g_R(t,u) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho_R^2}} \exp \left\{ -\frac{1}{2(l - \rho_R^2)} \left[ \frac{(t - \mu_1)^2}{\sigma_1^2} \right] - \frac{2 \rho_R}{\sigma_1 \sigma_2} \left[ (t - \mu_1)(u - \mu_2) + \frac{(u - \mu_2)^2}{\sigma_2^2} \right] \right\} \tag{5.39}
\]

and

\[
g_R(t) = \frac{1}{\sqrt{2\pi \sigma_1}} \exp \left\{ -\frac{(t - \mu_1)^2}{2\sigma_1^2} \right\} \tag{5.40}
\]

State \( E_1 \) indicates the system is operating (on); and state \( E_2 \) indicates it is in the repair/replacement (off) state. Then by the results of the alternating bivariate renewal process, i.e., Eqs. (5.19) and (5.20), we obtain the following models for the numbers of \( E_1 \) and \( E_2 \) renewals.

For \( g_R(t) = \frac{1}{\sqrt{2\pi \sigma_1}} \exp \left\{ -\frac{(t - \mu_1)^2}{2\sigma_1^2} \right\} \), we obtain that
\[ M^*_l(s,v) = \frac{\exp[-s\mu_1 - v\mu_u + \frac{1}{2}(s^2\sigma_i^2 + 2\rho sv\sigma_i\sigma_u + v^2\sigma_u^2)]}{sv[1 - \exp[-s(\mu_1 + \mu_i) - v\mu_u + \frac{1}{2}(s^2(\sigma_i^2 + \sigma_u^2) + 2\rho sv\sigma_i\sigma_u + v^2\sigma_u^2)]]} \]

(5.41)

\[ M^*_u(s,v) = \frac{\exp[-s(\mu_1 + \mu_i) - v\mu_u + \frac{1}{2}(s^2(\sigma_i^2 + \sigma_u^2) + 2\rho sv\sigma_i\sigma_u + v^2\sigma_u^2)]}{sv[1 - \exp[-s(\mu_1 + \mu_i) - v\mu_u + \frac{1}{2}(s^2(\sigma_i^2 + \sigma_u^2) + 2\rho sv\sigma_i\sigma_u + v^2\sigma_u^2)]]}. \]

(5.42)

For \( g_k(t,u) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho_k^2}} \exp\left\{ -\frac{1}{2(1 - \rho_k^2)} \left[ \frac{(t - \mu_1)^2}{\sigma_1^2} - 2\rho_k \frac{(t - \mu_1)(u - \mu_2)}{\sigma_1\sigma_2} + (u - \mu_2)^2 \right] \right\}, \)

we obtain

\[ M^*_l(s,v) = \frac{\exp[-s\mu_1 - v\mu_u + \frac{1}{2}(s^2\sigma_i^2 + 2\rho sv\sigma_i\sigma_u + v^2\sigma_u^2)]}{sv[1 - \exp[-s(\mu_1 + \mu_i) - v\mu_u + \frac{1}{2}(s^2(\sigma_i^2 + \sigma_u^2) + 2\rho sv\sigma_i\sigma_u + v^2\sigma_u^2)]]} \]

(5.43)

and

\[ M^*_u(s,v) = \frac{\exp[-s(\mu_1 + \mu_i) - v\mu_u + \frac{1}{2}(s^2(\sigma_i^2 + \sigma_u^2) + 2sv(\rho\sigma_1\sigma_u + \rho_x\sigma_1\sigma_z) + v^2(\sigma_u^2 + \sigma_z^2))]}{sv[1 - \exp[-s(\mu_1 + \mu_i) - v\mu_u + \frac{1}{2}(s^2(\sigma_i^2 + \sigma_u^2) + 2sv(\rho\sigma_1\sigma_u + \rho_x\sigma_1\sigma_z) + v^2(\sigma_u^2 + \sigma_z^2))]]}. \]

(5.44)

The corresponding renewal densities have Laplace transforms

\[ m^*_k(s,v) = svM^*_k(s,v), \text{ for } k = I, II. \]
If we consider the common distribution of $X_n + R_n$ and denote it as $H_{T,U}(t,u)$ with j.d.f. $h_{T,U}(t,u)$, then $H_{T,U}(t,u)$ is the convolution of $F_{T,U}(t,u)$ and $G_R(t,u)$ (or $G_R(t)$). Let $M_H(t,u)$ be the bivariate renewal function corresponding to the underlying distribution $H_{T,U}(t,u)$, i.e., $M_H(t,u) = \sum_{n=1}^{\infty} H_{n,T,U}^{(n)}(t,u)$. Let $m_H(t,u)$ be the corresponding renewal density function. Let $H_{T,U}^*(s,v)$, $M_H^*(s,v)$, and $m_H^*(s,v)$ be the Laplace transforms for $H_{T,U}(t,u)$, $M_H(t,u)$, and $m_H(t,u)$. Then from Eq. (4.34), we obtain the Laplace transform of the bivariate renewal function:

$$M_H^*(s,v) = \frac{H_{T,U}^*(s,v)}{1-h_{T,U}^*(s,v)} = \frac{h_{T,U}^*(s,v)}{sv(1-h_{T,U}^*(s,v))}, \quad (5.46)$$

where

$$h_{T,U}^*(s,v) = \exp\left[-s(\mu_r + \mu_1) - v(\mu_u + \mu_2) + \frac{1}{2}\left(s^2(\sigma_r^2 + \sigma_1^2) + 2sv\sigma_r\sigma_1 + v^2(\sigma_u^2 + \sigma_2^2)\right]\right]$$

for $g_R(t,u) \quad (5.47)$

or

$$h_{T,U}^*(s,v) = \exp\left[-s(\mu_r + \mu_1) - v\mu_u + \frac{1}{2}\left(s^2(\sigma_r^2 + \sigma_1^2) + 2sv\sigma_r\sigma_1 + v^2(\sigma_u^2 + \sigma_2^2)\right]\right] \quad \text{for} \quad g_R(t) \quad (5.48)$$

5.5 Discussion of the Results of Bivariate Corrective Maintenance Models

Sections 5.3. and 5.4 show that the bivariate corrective maintenance models can be developed using bivariate renewal theory. For the cases in which the repair duration is
negligible, we model bivariate corrective maintenance as an ordinary bivariate renewal process. When the repair duration is not negligible, we model bivariate corrective maintenance as an alternating bivariate renewal process. Another way to model the corrective maintenance with non-negligible repair is to construct the common distribution of operating state and repair state and to model it as an ordinary bivariate renewal process.

The Laplace transforms for the bivariate renewal function and renewal density function are obtained for the three different correlated bivariate failure models for both bivariate renewal models and bivariate corrective maintenance models. In each case, except for the renewal density of Hunter’s bivariate renewal model (Eq. (5.10)), the direct inverse of the Laplace transform is very difficult, and there is no available algorithm for inverting a bivariate Laplace transform numerically. As indicating in the end of Chapter 3, one possible way to obtain the inverse transforms is the use of series expansions and numerical approximations such as those defined for the univariate Weibull distribution by Lomnicki [1966].

We do not continue to explore the inversion of the renewal models. We will move on and develop bivariate preventive maintenance models for an age replacement policy. Our goal is to construct and develop bivariate availability models, which are presented in Chapters 8 and 9, based on the results of corrective and preventive maintenance models. The inversion of Laplace transforms will be expected to be more difficult and closed forms for the inversions may not exist. Numerical and other inversion methods form part of the body of future work that may be performed by researchers interested in our models.
CHAPTER VI
BIVARIATE PREVENTIVE MAINTENANCE MODELING

In this chapter, we consider the effects of preventive maintenance (PM) upon our renewal and corrective maintenance (CM) models. We generalize the renewal and CM models with i.i.d. lifetimes and preventive maintenance as well as repair assuming both have i.i.d. distributions. These models are examined under an age-replacement policy. Our goal is to construct bivariate preventive maintenance models that can serve as a basis for developing bivariate availability models. We obtain the Laplace transforms for the renewal functions of the bivariate PM models.

6.1 Notation

\((T,U)\) the policy period for the bivariate age replacement preventive maintenance

\(gr_{T,U}(t,u), Gr_{T,U}(t,u)\) bivariate repair-time density and distribution functions

\(gp_{T,U}(t,u), Gp_{T,U}(t,u)\) bivariate PM-time density and distribution functions

\(f_{cm_{T,U}}(t,u)\) the bivariate probability density function for the longevity between renewals in the CM cycle

\(f_{pm_{T,U}}(t,u)\) the bivariate probability density function for the longevity between renewals in the PM cycle

\(f_{ARPM}(t,u)\) the bivariate probability density function for the longevity between renewals in the ARPM cycle
Consider a single-unit system with an independent and identically distributed (i.i.d.) non-negative bivariate random lifetime denoted by \(X_n = \{(T_n, U_n)\}, n = 1, 2, \ldots\), with a common bivariate joint distribution function given by \(F(t, u) = P(T_n \leq t, U_n \leq u)\). Letting

\[Y_0 = (0, 0), \quad Y_n = \sum_{i=1}^{n} X_i = \left(\sum_{i=1}^{n} T_i, \sum_{i=1}^{n} U_i\right) = (S_n, V_n), \quad n \geq 1, \quad (6.1)\]

it follows that \(Y_n = (S_n, V_n)\) is the bivariate measure of both time and use of the \(n\)th renewal as illustrated in Figure 6.2.1.

Assume systems fail permanently and independently. Under an age replacement policy, if a system fails before time \(T\) and usage \(U\) since the last maintenance action then it will be immediately repaired. Otherwise, if the system survives until time \(T\) or usage \(U\) (whichever comes first) since the last maintenance action then it will be immediately to
begin being replaced by an i.i.d. new one. The repair actions (corrective maintenance) are
i.i.d. distributed with bivariate repair-time density $gr_{T,U}(t,u)$. The replacement actions
(preventive maintenance) are i.i.d. distributed with bivariate PM-time density $gp_{T,U}(t,u)$.

We assume that there is no logistics delay time (LDT) or administrative delay time
(ADT). Both replacement and repair maintenance actions are assumed to be perfect.
That is, after maintenance the system is restored to an "as good as new" state. Thus, the
system is said to be renewed after each maintenance action. Figure 6.2.1 shows these two
different maintenance cycles, namely, the corrective maintenance (CM) cycle and the
preventive maintenance (PM) cycle.

Let $T_n$ and $U_n$ be the operating time and usage after the $(n-1)$th maintenance,
respectively. Let $R_n = (R_{T_n}, R_{U_n})$ be the repair time and usage after the $n$th renewal. Let
$P_n = (P_{T_n}, P_{U_n})$ be the replacement time after the $n$th renewal. Assume that \{ $T_n$, $n = 1, 2, \ldots$ \}, \{ $U_n$, $n = 1, 2, \ldots$ \}, \{ $R_{T_n}$, $n = 1, 2, \ldots$ \}, \{ $R_{U_n}$, $n = 1, 2, \ldots$ \}, \{ $P_{T_n}$, $n = 1, 2, \ldots$ \},
and \{ $P_{U_n}$, $n = 1, 2, \ldots$ \} are stochastic processes with sequences of i.i.d. non-negative
random variables. Our goal is to develop and construct bivariate preventive maintenance
models indexed by time and usage based on the bivariate renewal models. Figure 6.2.1
shows a specific realization of a bivariate age replacement preventive maintenance
(ARPM) stochastic process.
Failure before \((T, U)\),

i.e., \((T_i, V_i) < (T, U)\)

Corrective maintenance Cycle

System survives

Preventive Maintenance Cycle

Unit survive until \((T_j, U_j)\)

i.e., \((T_j, U_j) = (t, u)\) for \(t = T\) or \(u = U\)

System survives

Figure 6.2.1 A realization of a bivariate ARPM stochastic process.
6.3 Construction of Preventive Maintenance Models

6.3.1 The CM Cycle

We first further consider the two maintenance cycles. In a CM cycle, the system fails before \((T, U)\) since the last maintenance action finished. Upon failure, the failed system is immediately repaired. Thus, the longevity of the system is always less than \((T, U)\) in a CM cycle (see Figure 6.3.1). The lifetime probability density function is truncated at \((T, U)\). This requires the system failure density function, \(f_{T,U}(t,u)\), to be normalized by the factor, \(F_{T,U}(T,U)\), which is the probability that the system fails before \((T, U)\). The truncated failure density function is \(\frac{f_{T,U}(t,u)}{F_{T,U}(T,U)}\) for \((t,u) \leq (T,U)\) and zero otherwise.

Let \(f_{cm_{T,U}}(t,u)\) be the probability density function for the longevity between renewals in the CM cycle. Then \(f_{cm_{T,U}}(t,u)\) is the convolution of the truncated failure density function, \(\frac{f_{T,U}(t,u)}{F_{T,U}(T,U)}\), and the CM repair-time probability density function, \(g_{r_{T,U}}(t,u)\).

That is,

\[
f_{cm_{T,U}}(t,u) = \left( \frac{f_{T,U}(t,u)}{F_{T,U}(T,U)} \right) * g_{r_{T,U}}(t,u). \tag{6.2}
\]

Taking the Laplace transforms of both sides of Eq. (6.2) and applying the convolution theorem for truncated Laplace transforms (Murdock [1995], pp.38), we obtain

\[
f_{cm_{T,U}}^*(s,v) = L_{s,v} \left( \left( \frac{f_{T,U}(t,u)}{F_{T,U}(T,U)} \right) * g_{r_{T,U}}^*(s,v) \right) = \frac{1}{F_{T,U}(T,U)} \left( \int_0^T \int_0^U e^{-sv} f_{T,U}(t,u) du dt \right) g_{r_{T,U}}^*(s,v) \tag{6.3}
\]
Corrective maintenance Cycle

Corrective Maintenance Point
(Failure Point)
(Ti,Ui) ≤ (T,U)

Age Replacement Policy
Renewal Points

Figure 6.3.1 CM cycle of a bivariate ARPM stochastic process.
Let the truncated (or partial) Laplace transform of \( f_{T,U}(t,u) \) be denoted by \( f^*_{T,U}(s,v;T,U) \), then

\[
\begin{align*}
 f^*_{T,U}(s,v;T,U) &= \int_0^T \int_0^U e^{-st-vu} f_{T,U}(t,u) du dt \\
 &= \int_0^\infty \int_0^U e^{-st-vu} f_{T,U}(t,u) du dt - \int_0^\infty \int_0^T e^{-st-vu} f_{T,U}(t,u) du dt \\
 &\quad - \int_T^\infty \int_0^U e^{-st-vu} f_{T,U}(t,u) du dt - \int_0^T \int_U^\infty e^{-st-vu} f_{T,U}(t,u) du dt \\
 &= f^*_{T,U}(s,v) - \left( \int_0^\infty \int_0^T e^{-s(\tau-T)-v(u-U)} f_{T,U}(\tau-T,u-U) d\tau du \right) - 0 - 0 \\
 &= f^*_{T,U}(s,v) - e^{st+vu} \int_0^\infty \int_0^T e^{-s(\tau-T)-v(u-U)} f_{T,U}(\tau-T,u-U) d\tau du . \tag{6.4}
\end{align*}
\]

6.3.2 The PM Cycle

In a PM cycle, the system survives until time \( T \) or usage \( U \), whichever comes first, since the last maintenance action finished, i.e., \((T_n, U_n) \leq (T, U)\). Upon reaching \( T \) or \( U \), the system is immediately replaced. Thus, in this cycle the longevity of the system is a set of boundary points in \([0,T] \times [0,U]\) (see Figure 6.3.2), i.e.,

\[
\left\{ (T_n, U_n) \right\} \text{ for } T_n = T, U_n \in [0,U] \text{ or } T_n \in [0,T], U_n = U .
\]

The lifetime probability density function of this longevity may be treated as a bivariate Dirac Delta function or bivariate Unit Impulse function (see Appendix B for detailed description; and see Spiegel [1965], pp.8; Murdock [1995], pp. 60 for the univariate case). We designate this function as \( \delta(t-\alpha, u-\beta) \) which has a singularity of infinite value at

\[
\begin{cases}
 t = \alpha \\
 0 \leq u \leq \beta
\end{cases}
\]

and is equal to zero at all other values of \((t, u)\).
Figure 6.3.2 PM cycle of a bivariate ARPM stochastic process.
The bivariate Dirac Delta function for this PM cycle, \( \delta(t - T, u - U) \), has the following properties:

(i) \( \int_0^\infty \int_0^\infty \delta(t - T, u - U) dt du = 1 \)

(ii) \( \int_0^\infty \int_0^\infty \delta(t - T, u - U) \phi(t, u) dt du = \phi(T, U) \) for any continuous function \( \phi(t, u) \).

Property (i) gives the bivariate Dirac Delta function the characteristic of a bivariate probability density function. Property (ii) helps in taking the Laplace transform of the bivariate Dirac Delta function. The Laplace transform of \( \delta(t - T, u - U) \) is

\[
L_{s,v}\{\delta(t - T, u - U)\} = \int_0^\infty \int_0^\infty \delta(t - T, u - U) e^{-st - uv} dt du = e^{-sT - uv} .
\]  

Let \( f_{pm_{T,U}}(t, u) \) be the probability density function for the longevity between renewals in the PM cycle. Then \( f_{pm_{T,U}}(t, u) \) is the convolution of the bivariate Dirac Delta function, \( \delta(t - T, u - U) \), and the PM replacement-time probability density function is \( g_{p_{T,U}}(t, u) \), that is,

\[
f_{pm_{T,U}}(t, u) = (\delta(t - T, u - U)) * g_{p_{T,U}}(t, u) .
\]  

(6.6)

Taking the Laplace transforms of both sides of Eq. (6.6) and applying the convolution theorem for Laplace transforms, we obtain

\[
f_{pm_{T,U}}^*(s, v) = L_{s,v}\{\delta(t - T, u - U)\} g_{p_{T,U}}^*(s, v) = e^{-sT - uv} \left( g_{p_{T,U}}^*(s, v) \right) .
\]  

(6.7)
6.3.3 The Renewal Function for the ARPM stochastic process

Now we consider the ARPM stochastic process with randomly mixed CM and PM cycles. Let \( f_{\text{ARPM}} (t,u) \) be the probability density function for the longevity between renewals for the ARPM stochastic process and \( M_{\text{ARPM}} (t,u) \) be the renewal function of \( f_{\text{ARPM}} (t,u) \). Let \( f^*_{\text{ARPM}} (s,v) \) and \( M^*_{\text{ARPM}} (t,u) \) be the Laplace transforms for \( f_{\text{ARPM}} (t,u) \) and \( M_{\text{ARPM}} (t,u) \), respectively. From Eq. (4.34), we have:

\[
M^*_{\text{ARPM}} (s,v) = \frac{f^*_{\text{ARPM}} (s,v)}{sv(1 - f^*_{\text{ARPM}} (s,v))}, \quad (6.8)
\]

and the Laplace transform of renewal density function, \( m_{\text{ARPM}} (t,u) \):

\[
m^*_{\text{ARPM}} (s,v) = \frac{f^*_{\text{ARPM}} (s,v)}{(1 - f^*_{\text{ARPM}} (s,v))}. \quad (6.9)
\]

Equations (6.8) and (6.9) are the PM models for the age replacement preventive maintenance policy. The derivation of \( f^*_{\text{ARPM}} (s,v) \) is presented in the next section.

6.3.4 Derivation of \( f^*_{\text{ARPM}} (s,v) \)

A maintenance action is either a CM repair or a PM replacement. The Laplace transform, \( f^*_{\text{ARPM}} (s,v) \), may be derived by conditioning on the longevity of the system at the point of system failure. Referring to Figure 6.2.1, the following two conditions are used:
System failed before it has been in operation for time $T$ and usage $U$ since the last maintenance action, i.e., $(T_n, U_n) < (T, U)$.

(i) System survived until it has been in operation for time $T$ or usage $U$ (whichever comes first) since the last maintenance action, i.e., $(T_n, U_n) \geq (T, U)$.

Using conditions (i) and (ii), we may derive the Laplace transform, $f_{\text{ARPM}}^*(s, v)$,

\[
f_{\text{ARPM}}^*(s, v) = f_{\text{ARPM}}^*(s, v)\{T_n < T, U_n < U\}\Pr\{T_n < T, U_n < U\} \\
+ f_{\text{ARPM}}^*(s, v)\{T_n \geq T, U_n \geq U\}\Pr\{T_n \geq T, U_n \geq U\} \\
+ f_{\text{ARPM}}^*(s, v)\{T_n < T, U_n \geq U\}\Pr\{T_n < T, U_n \geq U\} \\
+ f_{\text{ARPM}}^*(s, v)\{T_n \geq T, U_n < U\}\Pr\{T_n \geq T, U_n < U\}.
\]

(6.10)

Note that $\Pr\{T_n < T, U_n \geq U\}$ and $\Pr\{T_n \geq T, U_n < U\}$ are both equal to zero. Thus, Eq. (6.10) can be simplified as:

\[
f_{\text{ARPM}}^*(s, v) = f_{\text{ARPM}}^*(s, v)\{T_n < T, U_n < U\}\Pr\{T_n < T, U_n < U\} \\
+ f_{\text{ARPM}}^*(s, v)\{T_n \geq T, U_n \geq U\}\Pr\{T_n \geq T, U_n \geq U\}
\]

(6.11)

where

\[
f_{\text{ARPM}}^*(s, v)\{T_n < T, U_n < U\} = f_{\text{cm}}^*_{T,U}(s, v),
\]

\[
f_{\text{ARPM}}^*(s, v)\{T_n \geq T, U_n \geq U\} = f_{\text{pm}}^*_{T,U}(s, v),
\]

$\Pr\{T_n < T, U_n < U\} = F_{T,U}^*(T, U)$, and

$\Pr\{T_n \geq T, U_n \geq U\} = F_{T,U}^*(T, U)$.

Thus, Eq. (6.11) can be expressed as:
\[ f_{\text{ARPM}}^*(s,v) = f_{cm_{T,U}}^*(s,v)F_{T,U}(T,U) + f_{pm_{T,U}}^*(s,v)\bar{F}_{T,U}(T,U). \] (6.12)

From Eqs. (6.3) and (6.6), we obtain the following:

\[ f_{\text{ARPM}}^*(s,v) = \frac{1}{F_{T,U}(T,U)} \left( \int_0^T \int_0^U e^{-sT-uv} f_{T,U}(t,u) du dt \right) \left( g_{T,U}^*(s,v) \right) F_{T,U}(T,U) + e^{-sT-uv} \left( g_{P_{T,U}}^*(s,v) \right) \bar{F}_{T,U}(T,U). \] (6.13)

Thus,

\[ f_{\text{ARPM}}^*(s,v) = \left( f_{T,U}^*(s,v;T,U) \right) \left( g_{T,U}^*(s,v) \right) + e^{-sT-uv} \left( g_{P_{T,U}}^*(s,v) \right) \bar{F}_{T,U}(T,U), \] (6.14)

where

\[ f^*(s,v;T,U) = \left( \int_0^T \int_0^U e^{-sT-uv} f_{T,U}(t,u) du dt \right) \] (6.15)

is the bivariate truncated Laplace transforms for \( f_{T,U}(t,u) \).

Note that Eq. (6.14) is the expression for the Laplace transform of the longevity between renewals for the ARPM stochastic process. That is, the longevity between renewals for the ARPM process is the summation of two convolutions. The first one is related to the CM cycle. That is the convolution of the truncated lifetime distribution and the CM repair-time (i.e., failure repair) distribution. The second is related to the PM cycle. That is, the convolution of the bivariate Dirac Delta function and the PM replacement-time (i.e., preventive replacement) distribution multiplied by the reliability function that the system survives at least \( (T,U) \) longevity units. Equation (6.14) stands as an important result to our research in bivariate maintenance modeling. It is also an
attractive general result that definitely corresponds to Murdock’s univariate ARPM results (see Murdock [1995], pp. 62). Based on this result, we can derive the Laplace transforms for the renewal functions and renewal density functions for ARPM models. The successful derivation of Eq. (6.14) makes further research on bivariate availability possible.

The difficulty involved in this bivariate renewal approach to preventive maintenance modeling is the inversion of the bivariate Laplace transforms. It is unlikely to lead to an explicit answers for the inverse transforms except for some problems with special functions. Some examples of the derivation of the renewal function and renewal density function are presented in Chapter 7.
CHAPTER VII
EXAMPLES OF AGE REPLACEMENT PREVENTIVE MAINTENANCE MODELS

In this chapter, we use the correlated bivariate failure models (Chapter 3) as the joint lifetime distribution functions and apply the general results of age replacement preventive maintenance (ARPM) (Chapter 6) to construct the ARPM models. We consider two cases. The first case is that we assume the longevity random variables $X_n$, the repair random variables $R_n$, and the preventive maintenance random variables $P_n$ are i.i.d. with the same distribution function. The second case is to assume that $X_n$, $R_n$, and $P_n$ are independent and with distributions from the same family. We derive the Laplace transforms for the renewal function and renewal density function. These results will serve as a basis for developing bivariate availability models.

7.1 Assumptions and Descriptions of ARPM Models

Consider a single-unit system with an independent and identically distributed (i.i.d.) non-negative bivariate random lifetime denoted by $X_n = \{(T_n, U_n)\}, n = 1, 2, ..., $ with a common bivariate joint distribution function given by $F(t,u) = P\{T_n \leq t, U_n \leq u\}$. Assume systems fail permanently and independently. Under an age replacement policy, if a system fails before time $T$ and usage $U$ since the last maintenance action then it will be immediately repaired. Otherwise, if the system survives until time $T$ or usage $U$ (whichever comes first) since the last maintenance action then it will be immediately to
begin being replaced by an i.i.d. new one. The repair actions (corrective maintenance) are i.i.d. distributed with bivariate repair-time density $g_{rT,U}(t,u)$. The replacement actions (preventive maintenance) are i.i.d. distributed with bivariate PM-time density $g_{pT,U}(t,u)$. We assume that there is no logistics delay time (LDT) or administrative delay time (ADT). Both replacement and repair maintenance actions are assumed to be \textit{perfect}, that is, after maintenance the system is restored to an "\textit{as good as new}" state. Thus, the system is said to be renewed after each maintenance action.

Let $X_n = \{(T_n, U_n)\}$ be the operating time and usage after the $(n - 1)$th maintenance, respectively. Let $R_n = \{(Rt_n, Ru_n)\}$ be the repair time and usage after the $n$th renewal. Let $P_n = \{(Pt_n, Pu_n)\}$ be the replacement time after the $n$th renewal. Assume that $\{X_n\}$, $\{R_n\}$, and $\{P_n\}$, for $n = 1, 2, \ldots$, are stochastic processes with sequences of i.i.d. non-negative random variables. Our goal is to develop and construct bivariate preventive maintenance models indexed by time and usage based on the bivariate renewal models.

### 7.2 ARPM Models with i.i.d. $X_n$, $R_n$, and $P_n$

In this section, we use the correlated bivariate failure models developed in Chapter 3 as our joint lifetime distribution functions as well as the CM repair-time and PM replacement-time distributions to obtain the ARPM models. We assume that $\{X_n\}$, $\{R_n\}$, and $\{P_n\}$, for $n = 1, 2, \ldots$, are i.i.d. with the same distribution function.
7.2.1 Bivariate Exponential ARPM Model I

Assume that $X_n$, $R_n$, and $P_n$ are i.i.d. with the same distribution function, $F(t,u)$ which is Baggs and Nagagaja’s bivariate exponential with joint density function:

$$f_{T,U}(t,u) = \lambda \eta e^{-(\lambda t + \eta u)} \left(1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-(\lambda t + \eta u)}\right)\right), \quad (7.1)$$

and its Laplace transform:

$$f_{T,U}^*(s,v) = \frac{\lambda \eta \left[(v + 2\eta)(s + 2\lambda) + sv\rho\right]}{(v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)}. \quad (7.2)$$

From Eq. (6.14), we obtain the Laplace transform of $f_{\text{ARPM}}^*(s,v)$:

$$f_{\text{ARPM}}^*(s,v) = f_{T,U}^*(s,v)\left[f_{T,U}^*(s,v;T,U) + e^{-sT-vU} F_{T,U}(T,U)\right]. \quad (7.3)$$

From Eq. (6.4), the partial Laplace transform $f_{T,U}^*(s,v;T,U)$ for the bivariate exponential failure function can be obtained as follows:

$$f_{T,U}^*(s,v;T,U) = f_{T,U}^*(s,v) - e^{sT+vU} \int_0^\infty \int_0^\infty e^{-sT-vU} f_{T,U}^*(\tau,\tau,\tau,U)d\tau d\nu$$

$$= f_{T,U}^*(s,v) - e^{sT+vU} \int_0^\infty \int_0^\infty e^{-sT-vU} \left(e^{(\lambda T+\eta U)\tau} f_{T,U}^*(\tau,\tau,\tau,\tau,U)\right)d\tau d\nu$$

$$= f_{T,U}^*(s,v) - e^{(s+\lambda,\lambda,\tau)+(v+\eta)\tau} \int_0^\infty \int_0^\infty e^{-sT-vU} f_{T,U}^*(\tau,\tau,\tau) d\tau d\nu$$

$$= f_{T,U}^*(s,v) - e^{(s+\lambda,\lambda,\tau)+(v+\eta)\tau} f_{T,U}^*(s,v)$$

$$= f_{T,U}^*(s,v) \left[1 - e^{(s+\lambda,\lambda,\tau)+(v+\eta)\tau}\right]. \quad (7.4)$$
Substituting Eq. (7.4) into Eq. (7.3), we obtain

$$f_{ARPM}^*(s, v) = f_{T,U}^*(s, v) \left[ f_{T,U}^*(s, v) \left[ 1 - e^{(s + \lambda)T + (v + \eta)U} \right] + e^{-\xi T - \eta U} \bar{F}_{T,U}(T, U) \right].$$  \hspace{1cm} (7.5)

From Eq. (3.11), we have

$$\bar{F}_{T,U}(t, u) = e^{-(\lambda + \eta)u} \left[ 1 + \rho \left( 1 - e^{-\lambda u} \right) \left( 1 - e^{-\eta u} \right) \right].$$  \hspace{1cm} (7.6)

Substituting Eq. (7.6) into Eq. (7.5), we obtain

$$f_{ARPM}^*(s, v) = f_{T,U}^*(s, v) \left[ f_{T,U}^*(s, v) \left[ 1 - e^{(s + \lambda)T + (v + \eta)U} \right] + e^{-(s + \lambda)T - (v + \eta)U} \left[ 1 + \rho \left( 1 - e^{-\lambda T} \right) \left( 1 - e^{-\eta U} \right) \right] \right]$$

$$= \left( f_{T,U}^*(s, v) \right)^2 + \left( f_{T,U}^*(s, v) \right)^2 e^{(s + \lambda)T + (v + \eta)U}$$

$$+ f_{T,U}^*(s, v) e^{-(s + \lambda)T - (v + \eta)U} \left[ 1 + \rho \left( 1 - e^{-\lambda T} \right) \left( 1 - e^{-\eta U} \right) \right].$$  \hspace{1cm} (7.7)

Thus,

$$f_{ARPM}^*(s, v) = \left( f_{T,U}^*(s, v) \right)^2 + \left( f_{T,U}^*(s, v) \right)^2 e^{sT + \xi U} C_1 + f_{T,U}^*(s, v) e^{-\xi T - \eta U} C_2,$$  \hspace{1cm} (7.8)

where

$$f_{T,U}^*(s, v) = \frac{s \eta \left[ s \eta + 2 \eta (s + 2 \lambda) + sv \rho \right]}{s \eta + 2 \eta (s + \lambda) + sv \rho}.$$

$$C_1 = e^{\lambda T + \eta U}, \text{ and } C_2 = e^{-\lambda T - \eta U} \left[ 1 + \rho \left( 1 - e^{-\lambda T} \right) \left( 1 - e^{-\eta U} \right) \right].$$

Substituting Eq. (7.8) into Eqs. (6.8) and (6.9), $M_{ARPM}^*(t, u)$ and $m_{ARPM}^*(t, u)$ can be obtained. The inverse of Eq. (7.8) can be obtained by using Mathematica (Wolfram
Research, Inc.). The inversion of \( \left(f_{T,U}^*(s,v)\right)^2 \) in Eq. (7.8) is the 2-fold convolution of \( f_{T,U}(t,u) \), i.e., \( f_{T,U}^{(2)}(t,u) \). The inversion of other terms of Eq. (7.8) may be obtained by using the \textit{Linear Shifting property} of Laplace transform (see Nelson [1995]), that is,

\[
e^{-as-bv} f^*(s,v) = L_{s,v} \{ f(t-a,u-b) \}. \tag{7.9}
\]

Then Eq. (7.8) is inverted as

\[
f_{ARPM}(t,u) = [A] + C_1[B] + C_2[C], \tag{7.10}
\]

where

\[
[A] = \lambda \eta e^{-2(\nu+\eta u)} \left\{ 8 \rho \left[ 1 + 2 \rho (1 + \lambda t)(1 + \eta u) \right] \\
+ 4 \rho e^{\lambda t} \left[ \lambda t (1 + \rho + \rho \eta u) - 2(1 + 2 \rho (1 + \eta u)) \right] \\
+ 4 \rho e^{\eta u} \left[ \eta u (1 + \rho + \rho \lambda t) - 2(1 + 2 \rho (1 + \lambda t)) \right] \\
+ e^{\lambda t + \eta u} \left[ -4 \rho\left[-2 - 4 \rho + \eta u (1 + \rho)\right] + \lambda t (1 + \rho) \left[-4 \rho + \eta u (1 + \rho)\right]\right] \right\}, \tag{7.11}
\]

\[
[B] = \lambda \eta e^{-2(\lambda (t+T)+\eta (u+U))} \left\{ 8 \rho \left[ 1 + 2 \rho (1 + \lambda (t+T))(1 + \eta (u+U)) \right] \\
+ 4 \rho e^{\lambda (t+T)} \left[ \lambda t (1 + \rho + \rho \eta (u+U)) - 2(1 + 2 \rho (1 + \eta (u+U))) \right] \\
+ 4 \rho e^{\eta (u+U)} \left[ \eta (u+U) (1 + \rho + \rho \lambda (t+T)) - 2(1 + 2 \rho (1 + \lambda (t+T))) \right] \\
+ e^{\lambda (t+T) + \eta (u+U)} \left[ -4 \rho \left[-2 - 4 \rho + \eta (u+U) (1 + \rho)\right] + \lambda (t+T) (1 + \rho) \left[-4 \rho + \eta (u+U) (1 + \rho)\right]\right] \right\}, \tag{7.12}
\]

\[
[C] = \lambda \eta e^{-[\lambda (t-T)+\eta (u-U)]} \left[ 1 + \rho \left( 1 - 2 e^{-\lambda (t-T)} - 2 e^{-\eta (u-U)} + 4 e^{-[\lambda (t-T)+\eta (u-U)]}\right) \right], \tag{7.13}
\]

\[t=104\]
However, it is not possible to directly invert the Laplace transforms for the renewal function and renewal density function. The infinite series expansions used by Lomnicki [1966] may be a possible way to obtain the approximate inverse Laplace transforms. Existing numerical methods for inverting the univariate Laplace transforms may be an alternative way to obtain the inversions. Unfortunately, we cannot find available software that can numerically solve bivariate inversions.

### 7.2.2 Bivariate Normal ARPM Model

Assume that \( X_n, R_n, \) and \( P_n \) are i.i.d. with the same distribution function, \( F(t,u) \) which is bivariate normal with joint density function:

\[
f_{T,U}(t,u) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(t-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(t-\mu_1)(u-\mu_2)}{\sigma_1\sigma_2} + \frac{(u-\mu_2)^2}{\sigma_2^2}\right]\right\}
\]

(7.14)

and its Laplace transform:

\[
f^*(s,v) = \exp\left[-s\mu_1 - v\mu_2 + \frac{1}{2}(s^2\sigma_1^2 + 2\rho sv\sigma_1\sigma_2 + v^2\sigma_2^2)\right].
\]

(7.15)

From Eq. (7.3), we have

\[
f_{ARPM}^*(s,v) = f_{T,U}^*(s,v)\left[f_{T,U}^*(s,v;T,U) + e^{-sT-vU}F_{T,U}(T,U)\right].
\]

(7.16)

The truncated Laplace transform \( f_{T,U}^*(s,v;T,U) \) for the bivariate normal failure function can be obtained as follows:
\[ f^*_{T,U}(s,v;T,U) = L_{s,v}\left\{ f_{T,U}(t,u;T,U) \right\} \]

\[ = L_{s,v}\left\{ \frac{f_{T,U}(t,u)}{\Phi(T,U;\mu_U,\mu_u,\sigma_u,\rho)} \right\} \]

\[ = \frac{f^*_{T,U}(s,v)}{\Phi(T,U;\mu_U,\mu_u,\sigma_u,\rho)} \]

\[ = C_f f^*_{T,U}(s,v) . \quad (7.17) \]

Substituting Eq. (7.17) into Eq. (7.16), we obtain

\[ f^*_{ARPM}(s,v) = f^*_{T,U}(s,v)\left[ C_1 f^*_{T,U}(s,v) + C_2 e^{-sT-vU} \right] \quad (7.18) \]

where

\[ C_1 = \left[ \Phi(T,U;\sigma_u,\mu_u,\rho) \right]^{-1} \]

and

\[ C_2 = \left[ 1 - \Phi(T,\sigma_u,\mu_u) - \Phi(U;\sigma_u,\mu_u) + \Phi(T,\sigma_u,\mu_u,\rho) \right] . \]

Thus,

\[ f^*_{ARPM}(s,v) = C_1 \left( f^*_{T,U}(s,v) \right)^2 + f^*_{T,U}(s,v)e^{-sT-vU} C_2 . \quad (7.19) \]

Substituting Eq. (7.19) into Eqs. (6.8) and (6.9), \( M^*_{ARPM}(t,u) \) and \( m^*_{ARPM}(t,u) \) can be obtained.

Note that the inversion of \( \left( f^*_{T,U}(s,v) \right)^2 \) in Eq. (7.19) is the 2-fold convolution of \( f_{T,U}(t,u) \), i.e., \( f^{(2)}_{T,U}(t,u) \), which is a normal distribution with means \( 2\mu_U,2\mu_u \), variances
\(2\sigma_i^2, 2\sigma_u^2\), and correlation coefficient \(\rho\). The inverse of the other term in Eq. (7.19) is a normal distribution with means \(\mu_i + T, \mu_u + U\), variances \(\sigma_i^2, \sigma_u^2\), and correlation coefficient \(\rho\).

### 7.2.3 Bivariate Exponential ARPM Model II

Assume that \(X_n, R_n, P_n\) are i.i.d. with the same distribution function, \(F(t,u)\) which is Hunter’s bivariate exponential with joint density function:

\[
f_{T,U}(t,u) = \frac{\lambda \eta}{1 - \rho} I_0 \left( \frac{2\sqrt{\rho}}{1 - \rho} \sqrt{\lambda \eta tu} \right) \exp \left\{ -\frac{\lambda t + \eta u}{1 - \rho} \right\}, \tag{7.20}\]

where \(I_n()\) is the modified Bessel function of the first kind of order \(n\), \(\rho\) is positive, and its Laplace transform:

\[
f^*(s,v) = \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{\nu}{\eta} + 1 \right) - \frac{sv \rho}{\lambda \eta} \right]^{-1}. \tag{7.21}\]

From Eq. (7.3), we have

\[
f_{ARPM}^*(s,v) = f_{T,U}^*(s,v) \left[ f_{T,U}^*(s,v;T,U) + e^{-sT-vU} F_{T,U}(T,U) \right]. \tag{7.22}\]

The partial Laplace transform \(f_{T,U}^*(s,v;T,U)\) for the Hunter’s bivariate exponential failure function may be obtained as follows:

\[
f_{T,U}^*(s,v;T,U) = L_{s,v} \left\{ f_{T,U}(t,u;T,U) \right\} = L_{s,v} \left\{ \frac{f_{T,U}(t,u)}{F(T,U)} \right\} = \frac{f_{T,U}^*(s,v)}{F(T,U)}
\]
\[ C_1 f_{T,U}^* (s,v) \]  \hspace{1cm} (7.23)  

where \( C_1 = \left[ F(T,U) \right]^{-1} \).

Substituting Eq. (7.23) into Eq. (7.22), we obtain

\[ f_{ARPM}^* (s,v) = f_{T,U}^* (s,v) \left[ C_1 f_{T,U}^* (s,v) + C_2 e^{-sT-vU} \right] \]  \hspace{1cm} (7.24)  

where \( C_1 = \left[ F(T,U) \right]^{-1} \) and \( C_2 = \bar{F}(T,U) \).

Thus,

\[ f_{ARPM}^* (s,v) = C_1 \left( f_{T,U}^* (s,v) \right)^2 + f_{T,U}^* (s,v) e^{-sT-vU} C_2 . \]  \hspace{1cm} (7.25)  

Substituting Eq. (7.25) into Eqs. (6.8) and (6.9), \( M_{ARPM}^* (t,u) \) and \( m_{ARPM}^* (t,u) \) can be obtained.

The inversion of \( \left( f_{T,U}^* (s,v) \right)^2 \) in Eq. (7.25) is the 2-fold convolution of \( f_{T,U}^* (t,u) \), i.e., \( f_{T,U}^{(2)} (t,u) \) which is given by

\[ f_{T,U}^{(n)} (t,u) = \left[ \frac{(\lambda \eta)^n}{(1 - \rho) \Gamma(n)} \right] \left[ \frac{tu}{\rho \lambda u} \right]^{(n-1)/2} I_{n-1} \left( \frac{2 \sqrt{\rho \lambda u}}{1 - \rho} \right) \exp \left\{ -\frac{\lambda t + \eta u}{1 - \rho} \right\} \text{ for } n = 2, \]  \hspace{1cm} (7.26)  

where \( I_n (\cdot) \) is the modified Bessel function of the first kind of order \( n \), \( \rho \) is positive.
Equation (7.26) is known as Kibble’s bivariate gamma distribution (see Hunter [1974], pp. 389). The inversion of other terms of Eq. (7.26) may be obtained by using the Linear Shifting property of Laplace transform (see Nelson [1995]), that is,

\[ e^{-as-bs} f^*(s, v) = L_{s,v} \left\{ f(t-a, u-b) \right\}. \]

The Eq.(7.25) can be inverted as:

\[ f_{ARPM}(t,u) = C_1[A] + C_2[B], \quad (7.27) \]

where

\[ [A] = f^{(2)}_{T,U}(t,u) = \left[ \frac{(\lambda)\gamma}{(1-\rho)} \right] \left[ \frac{tu}{\rho \mu} \right]^{1/2} I_1 \left( \frac{2\sqrt{\rho}}{1-\rho} \sqrt{\lambda \eta tu} \right) \exp \left\{ -\frac{\lambda t + \eta u}{1-\rho} \right\}, \quad (7.28) \]

where \( I_n (\cdot) \) is the modified Bessel function of the first kind of order \( n \), \( \rho \) is positive,

\[ [B] = \frac{\lambda \eta}{1-\rho} I_0 \left( \frac{2\sqrt{\rho}}{1-\rho} \sqrt{\lambda \eta (t-T)(u-U)} \right) \exp \left\{ -\frac{\lambda (t-T) + \eta (u-U)}{1-\rho} \right\}, \quad (7.29) \]

\[ C_1 = \left[ F(T,U) \right]^{-1} \text{ and } C_2 = \tilde{F}(T,U). \]

The results in the above examples indicate that we can only obtain the inverses for the \( f_{ARPM}^*(s,v) \). The inverses of \( M_{ARPM}^*(t,u) \) and \( m_{ARPM}^*(t,u) \) are still not possible and require the use of numerical methods.
7.3 ARPM Models with independent but distinct $X_n$, $R_n$, and $P_n$

In this section, we use the correlated bivariate failure models developed in Chapter 3 as our joint lifetime distribution functions as well as the CM repair-time and PM replacement-time distributions to obtaining the ARPM models. We assume that $X_n$, $R_n$, and $P_n$ are independent but with distinct distribution functions.

7.3.1 Bivariate Exponential ARPM Model I

Assume that $X_n$, $R_n$, and $P_n$ are independent but with distinct distribution functions from the same family. $X_n$, $R_n$, and $P_n$ are distributed as $F_{T,U}(t,u)$, $Gr_{T,U}(t,u)$, and $Gp_{T,U}(t,u)$, respectively. $F_{T,U}(t,u)$, $Gr_{T,U}(t,u)$, and $Gp_{T,U}(t,u)$ are Baggs and Nagagaja's bivariate exponential with joint density function, respectively:

$$f_{T,U}(t,u) = \lambda \eta e^{-(\lambda t + \eta u)} \left(1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-(\lambda t + \eta u)}\right)\right). \quad (7.30)$$

$$gr_{T,U}(t,u) = \lambda \eta e^{-(\lambda t + \eta u)} \left(1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-(\lambda t + \eta u)}\right)\right). \quad (7.31)$$

$$gp_{T,U}(t,u) = \lambda_p \eta_p e^{-(\lambda_p t + \eta_p u)} \left(1 + \rho \left(1 - 2e^{-\lambda_p t} - 2e^{-\eta_p u} + 4e^{-(\lambda_p t + \eta_p u)}\right)\right). \quad (7.32)$$

and their Laplace transforms:

$$f^{*}_{T,U}(s,v) = \frac{\lambda \eta [(v+2\eta)(s+2\lambda) + s\nu \rho]}{(v+\eta)(s+\lambda)(v+2\eta)(s+2\lambda)}. \quad (7.33)$$
\begin{equation}
g_{T,U}^*(s,v) = \frac{\lambda_r \eta_r \left[(v + 2\eta_r)(s + 2\lambda_r) + sv\rho_r\right]}{(v + \eta_r)(s + \lambda_r)(v + 2\eta_r)(s + 2\lambda_r)}. \quad (7.34)
\end{equation}

\begin{equation}
g_{p_{T,U}}^*(s,v) = \frac{\lambda_p \eta_p \left[(v + 2\eta_p)(s + 2\lambda_p) + sv\rho_p\right]}{(v + \eta_p)(s + \lambda_p)(v + 2\eta_p)(s + 2\lambda_p)}. \quad (7.35)
\end{equation}

From Eq. (6.14), we obtain the Laplace transform of \( f_{ARPM}^*(s,v) \):

\begin{equation}
f_{ARPM}^*(s,v) = (f_{T,U}^*(s,v; T,U))\left(g_{T,U}^*(s,v) + e^{-\lambda T-vU}\left(g_{p_{T,U}}^*(s,v)\right)\bar{F}_{T,U}(T,U)\right). \quad (7.36)
\end{equation}

From Eq. (7.4), the partial Laplace transform \( f_{T,U}^*(s,v; T,U) \) for the bivariate exponential failure function can be obtained:

\begin{equation}
f_{T,U}^*(s,v) = f_{T,U}^*(s,v)\left(1 - e^{-(\lambda \gamma)T+(\nu \eta)U}\right) \quad (7.37)
\end{equation}

Substituting Eq. (7.37) into Eq. (7.36), we obtain

\begin{equation}
f_{ARPM}^*(s,v) = f_{T,U}^*(s,v)g_{T,U}^*(s,v)\left(1 - e^{-(\lambda \gamma)T+(\nu \eta)U}\right) + e^{-\lambda T-vU}g_{p_{T,U}}^*(s,v)\bar{F}_{T,U}(T,U) \quad (7.38)
\end{equation}

From Eq. (3.11), we have

\begin{equation}
\bar{F}_{T,U}(T,U) = e^{-(\lambda T+\nu U)}\left(1 + \rho\left(1 - e^{-\lambda T}\right)\left(1 - e^{-\nu U}\right)\right). \quad (7.39)
\end{equation}

Substituting Eq. (7.39) into Eq. (7.38), we obtain
\[ f_{\text{ARP}}^*(s,v) = f_{T,U}^*(s,v)g_{T,U}^*(s,v)\left\{1-e^{(s+\lambda)T+(v-\eta)U}\right\} \\
+ gp_{T,U}^*(s,v)e^{-(s+\lambda)T-(v-\eta)U}\left[1+\rho\left(1-e^{-\lambda T}\right)(1-e^{-\eta U})\right] \] 

(7.40)

Thus,

\[ f_{\text{ARPM}}^*(s,v) = f_{T,U}^*(s,v)g_{T,U}^*(s,v) - f_{T,U}^*(s,v)g_{T,U}^*(s,v)e^{sT+vU}C_1 \\
+ gp_{T,U}^*(s,v)e^{-sT-vU}C_1^{-1} + gp_{T,U}^*(s,v)e^{-sT-vU}C_2 \]

(7.41)

where

\[ f_{T,U}^*(s,v) = \frac{\lambda\eta\left[(v+2\eta)(s+2\lambda)+sv\rho\right]}{(v+\eta)(s+\lambda)(v+2\eta)(s+2\lambda)}. \]

\[ g_{T,U}^*(s,v) = \frac{\lambda\eta\left[(v+2\eta)(s+2\lambda)+sv\rho\right]}{(v+\eta)(s+\lambda)(v+2\eta)(s+2\lambda)}. \]

\[ gp_{T,U}^*(s,v) = \frac{\lambda\eta\left[(v+2\eta)(s+2\lambda)+sv\rho\right]}{(v+\eta)(s+\lambda)(v+2\eta)(s+2\lambda)}. \]

\[ C_1 = e^{\lambda T+vU}, \text{ and } C_2 = e^{-\lambda T-vU}\left[1+\rho\left(1-e^{-\lambda T}\right)(1-e^{-\eta U})\right]. \]

Substituting Eq. (7.41) into Eqs. (6.8) and (6.9), \( M_{\text{ARPM}}^*(t,u) \) and \( m_{\text{ARP}}^*(t,u) \) can be obtained.

In Eq. (7.41), the inverse of \( f_{T,U}(s,v)g_{T,U}^*(s,v) \) is the convolution of \( f_{T,U}(t,u) \) and \( g_{T,U}(t,u) \). Using the Linear Shifting property of Laplace transform, the inverse of \( f_{T,U}^*(s,v)g_{T,U}^*(s,v)e^{sT+vU} \) is the convolution of \( f_{T,U}(t,u) \) and \( g_{T,U}(t+T,u+U) \). The inversion of other terms of Eq. (7.41) can also be obtained. The Eq.(7.41) is inverted as
\[ f_{\text{ARPM}}(t,u) = f^* gr_{T,U}(t,u) - C_1 f^* gr_{T,U}(t + T, u + U) + (C_1^{-1} + C_2)g_{P_{T,U}}(t - T, u - U), \]

(7.42)

where \( f^* gr_{T,U}(t,u) \) is the convolution of \( f_{T,U}(t,u) \) and \( gr_{T,U}(t,u) \) and \( f^* gr_{T,U}(t + T, u + U) \) is the convolution of \( f_{T,U}(t,u) \) and \( gr_{T,U}(t + T, u + U) \),

\[ C_1 = e^{\lambda T + \eta U}, \quad \text{and} \quad C_2 = e^{-\lambda T - \eta U} \left( 1 + \rho \left( 1 - e^{-\lambda T} \right) \left( 1 - e^{-\eta U} \right) \right). \]

However, it is not possible to directly invert the Laplace transforms for the renewal function and renewal density function. The infinite series expansions used by Lomnicki [1966] may be a possible way to obtain the approximate inverse Laplace transforms. Existing numerical methods for inverting the univariate Laplace transforms may be an alternative way to obtain the inversions. Unfortunately, we cannot find available software that can solve bivariate inversions numerically.

### 7.3.2 Bivariate Normal ARPM Model

Assume that \( X_n, R_n, \) and \( P_n \) are independent but with distinct distribution function from the same family. \( X_n, R_n, \) and \( P_n \) are distributed as \( F_{T,U}(t,u) \), \( Gr_{T,U}(t,u) \), and \( Gp_{T,U}(t,u) \), respectively. \( F_{T,U}(t,u) \), \( Gr_{T,U}(t,u) \), and \( Gp_{T,U}(t,u) \) are bivariate normal with joint density function, respectively:

\[ f_{T,U}(t,u) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(t-\mu_x)^2}{\sigma_x^2} - 2\rho \frac{(t-\mu_x)(u-\mu_u)}{\sigma_x \sigma_u} + \frac{(u-\mu_u)^2}{\sigma_u^2} \right] \right\}. \]

(7.43)
\[ g_{T,U}(t,u) = \frac{1}{2\pi \sigma \sigma_2 \sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ \frac{(t - \mu_1)^2}{\sigma_1^2} - 2\rho \frac{(t - \mu_1)(u - \mu_2) + (u - \mu_2)^2}{\sigma_1 \sigma_2} \right] \right\}, \]  
(7.44)

\[ g_{T,U}(t,u) = \frac{1}{2\pi \sigma \sigma_4 \sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ \frac{(t - \mu_3)^2}{\sigma_3^2} - 2\rho \frac{(t - \mu_3)(u - \mu_4) + (u - \mu_4)^2}{\sigma_3 \sigma_4} \right] \right\}, \]  
(7.45)

and their Laplace transforms:

\[ f_{T,U}^*(s, v) = \exp \left\{ -s\mu_1 - \nu\mu_u + \frac{1}{2}\left(s^2 \sigma_1^2 + 2\rho sv \sigma_u \sigma_v + v^2 \sigma_u^2 \right) \right\}, \]  
(7.46)

\[ g_{T,U}^*(s, v) = \exp \left\{ -s\mu_1 - \nu\mu_2 + \frac{1}{2}\left(s^2 \sigma_1^2 + 2\rho sv \sigma_2 \sigma_v + v^2 \sigma_2^2 \right) \right\}, \]  
(7.47)

\[ g_{pT,U}^*(s, v) = \exp \left\{ -s\mu_3 - \nu\mu_4 + \frac{1}{2}\left(s^2 \sigma_3^2 + 2\rho sv \sigma_4 \sigma_v + v^2 \sigma_4^2 \right) \right\}. \]  
(7.48)

From Eq. (6.14), we obtain the Laplace transform of \( f_{ARPM}^*(s, v) \):

\[ f_{ARPM}^*(s, v) = \left( f_{T,U}^*(s, v; T, U) \right) \left( g_{T,U}^*(s, v) \right) + e^{-sT} \left( g_{pT,U}^*(s, v) \right) \Phi(T, U), \]  
(7.49)

The truncated Laplace transform \( f_{T,U}^*(s, v; T, U) \) for the bivariate normal failure function is:

\[ f_{T,U}^*(s, v; T, U) = \frac{f_{T,U}^*(s, v)}{\Phi(T, U; \mu_1, \mu_u, \sigma_1, \sigma_u, \rho)}. \]  
(7.50)
Substituting Eq. (7.50) into Eq. (7.49), we obtain

\[ f_{\text{ARPM}}^* (s,v) = f_{T,U}^* (s,v) gr_{T,U}^* (s,v) C_1 + e^{-sT-vU} gp_{T,U}^* (s,v) C_2 \]  \hspace{1cm} (7.51)

where

\[ f_{T,U}^* (s,v) = \exp\left[-s\mu_1 - v\mu_2 + \frac{1}{2} \left( s^2 \sigma_1^2 + 2 \rho_{sv} \sigma_1 \sigma_2 + v^2 \sigma_2^2 \right) \right], \]

\[ gr_{T,U}^* (s,v) = \exp\left[-s\mu_1 - v\mu_2 + \frac{1}{2} \left( s^2 \sigma_1^2 + 2 \rho_{sv} \sigma_1 \sigma_2 + v^2 \sigma_2^2 \right) \right], \]

\[ gp_{T,U}^* (s,v) = \exp\left[-s\mu_3 - v\mu_4 + \frac{1}{2} \left( s^2 \sigma_3^2 + 2 \rho_{sv} \sigma_3 \sigma_4 + v^2 \sigma_4^2 \right) \right], \]

\[ C_1 = \left[ \Phi(T,U;\sigma_1,\sigma_2,\mu_1,\mu_2,\rho) \right]^{-1}. \]

and

\[ C_2 = \left[ I - \Phi(T;\sigma_1,\mu_1) - \Phi(U;\sigma_2,\mu_2) + \Phi(T,U;\sigma_1,\sigma_2,\mu_1,\mu_2,\rho) \right]. \]

Substituting Eq. (7.51) into Eqs. (6.8) and (6.9), \( M_{\text{ARPM}}^* (t,u) \) and \( m_{\text{ARPM}}^* (t,u) \) can be obtained.

In Eq. (7.51), the inverse of \( f_{T,U}^* (s,v) gr_{T,U}^* (s,v) \) is the convolution of \( f_{T,U}^* (t,u) \) and \( gr_{T,U}^* (t,u) \), i.e., \( f_{T,U}^* \ast gr_{T,U}^* (t,u) \) which is a normal distribution with means \( \mu_1 + \mu_1 \) and \( \mu_2 + \mu_2 \), variances \( \sigma_1^2 + \sigma_1^2 \) and \( \sigma_2^2 + \sigma_2^2 \), and correlation coefficient \( \rho \). Using the Linear Shifting property of Laplace transform, the inverse of \( gp_{T,U}^* (s,v) e^{sT+vU} \) is \( gr_{T,U}^* (t-T,u-U) \) which is a normal distribution with mean \( \mu_3 + T \) and \( \mu_4 + U \), variances \( \sigma_3^2 \) and \( \sigma_4^2 \), and correlation coefficient \( \rho \).
7.3.3 Bivariate Exponential ARPM Model II

Assume that $X_n$, $R_n$, and $P_n$ are independent but with distinct distribution functions from the same family. $X_n$, $R_n$, and $P_n$ are distributed as $F_{T,U}(t,u)$, $Gr_{T,U}(t,u)$, and $Gp_{T,U}(t,u)$, respectively. $F_{T,U}(t,u)$, $Gr_{T,U}(t,u)$, and $Gp_{T,U}(t,u)$ are bivariate normal with joint density function, respectively:

$$f_{T,U}(t,u) = \frac{\lambda_r \eta_r}{1 - \rho} I_0 \left( \frac{2 \sqrt{\rho_r} \sqrt{\lambda_r \eta_r t u}}{1 - \rho_r} \right) \exp \left\{ - \frac{\lambda_r t + \eta_r u}{1 - \rho_r} \right\}, \quad (7.52)$$

$$gr_{T,U}(t,u) = \frac{\lambda_r \eta_r}{1 - \rho_r} I_0 \left( \frac{2 \sqrt{\rho_r} \sqrt{\lambda_r \eta_r t u}}{1 - \rho_r} \right) \exp \left\{ - \frac{\lambda_r t + \eta_r u}{1 - \rho_r} \right\}, \quad (7.53)$$

$$gp_{T,U}(t,u) = \frac{\lambda_p \eta_p}{1 - \rho_p} I_0 \left( \frac{2 \sqrt{\rho_p} \sqrt{\lambda_p \eta_p t u}}{1 - \rho_p} \right) \exp \left\{ - \frac{\lambda_p t + \eta_p u}{1 - \rho_p} \right\}, \quad (7.54)$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind of order $n$, and $\rho, \rho_r, \rho_p$ are positive; and their Laplace transforms:

$$f^*(s,v) = \left[ \frac{s}{\lambda} + 1 \left( \frac{v}{\eta} + 1 \right) - \frac{s \sqrt{\rho}}{\lambda \eta} \right]^{-1}. \quad (7.55)$$

$$gr^*(s,v) = \left[ \frac{s}{\lambda_r} + 1 \left( \frac{v}{\eta_r} + 1 \right) - \frac{s \sqrt{\rho_r}}{\lambda_r \eta_r} \right]^{-1}. \quad (7.56)$$
\[
gp^*(s, v) = \left[ \left( \frac{s}{\lambda_p} + 1 \right) \left( \frac{v}{\eta_p} + 1 \right) \right]^{-1} \frac{s v p}{\lambda_p \eta_p}. \tag{7.57}
\]

From Eq. (6.14), we obtain the Laplace transform of \( f_{ARPM}^*(s, v) \):

\[
f_{ARPM}^*(s, v) = \left( f_{T,U}^*(s, v; T, U) \right) (g_{T,U}^* (s, v)). \tag{7.58}
\]

The truncated Laplace transform \( f_{T,U}^* (s, v; T, U) \) for the Hunter’s bivariate exponential failure function is:

\[
f_{T,U}^*(s, v; T, U) = \frac{f_{T,U}^*(s, v)}{F(T, U)}. \tag{7.59}
\]

Substituting Eq. (7.59) into Eq. (7.58), we obtain

\[
f_{ARPM}^*(s, v) = f_{T,U}^*(s, v) g_{T,U}^* (s, v) C_1 + e^{-sT-vU} g_{T,U}^* (s, v) C_2 \tag{7.60}
\]

where

\[
f^*(s, v) = \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{v}{\eta} + 1 \right) \right]^{-1} \frac{s v p}{\lambda \eta},
\]

\[
g_{T,U}^* (s, v) = \left[ \left( \frac{s}{\lambda_r} + 1 \right) \left( \frac{v}{\eta_r} + 1 \right) \right]^{-1} \frac{s v p}{\lambda_r \eta_r},
\]

\[
g_{T,U}^* (s, v) = \left[ \left( \frac{s}{\lambda_p} + 1 \right) \left( \frac{v}{\eta_p} + 1 \right) \right]^{-1} \frac{s v p}{\lambda_p \eta_p},
\]

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\( C_1 = [F(T,U)]^{-1}, \)

and

\( C_2 = \bar{F}(T,U). \)

Substituting Eq. (7.60) into Eqs. (6.8) and (6.9), \( M_{ARPM}^*(t,u) \) and \( m_{ARPM}^*(t,u) \) can be obtained.
CHAPTER VIII
BIVARIATE AVAILABILITY MODELING

In this chapter, some general bivariate availability models are presented. We develop bivariate availability models for the bivariate maintenance policies based on the results of previous chapters. The bivariate availability models for CM and PM are treated separately. For CM models, the bivariate availability measure is straightforward and can be derived from the bivariate reliability function and the renewal function using the integral equation of bivariate renewal theory. For PM models with ARPM policy, the bivariate availability measure can be derived by conditioning on the time-usage durations and is based on the results of CM models. Some general results and the Laplace transforms for the bivariate availability models are obtained. Issues concerning the quality of availability measures are also discussed.

8.1 Notation

\( A(t,u) \) the bivariate point availability
\( A \) the bivariate limiting availability
\( A_{\text{avg}} (T,U) \) the bivariate average availability
\( A_{\infty} \) the bivariate limiting average availability
\( A_{\text{avg}} (T_2 - T_1, U_2 - U_1) \) the bivariate interval average availability
\( A(t,u;T,U) \) the bivariate point availability for ARPM with \((T,U)\)-policy
\( Q[A(t)], Q[A(u)] \) the quality of availability measures
8.2 Definitions of Bivariate Availability

Literature that treats bivariate availability does not exist. For evaluating the bivariate maintenance models, we need to extend the univariate availability models to two dimensions. To develop bivariate availability models, we begin with the definitions of bivariate availability measures.

**Definition 8.2.1**

The *bivariate point availability*, $A(t, u)$, of a system is the probability that it is functioning at any point in time $t$ and usage $u$ and is given by

$$A(t, u) = P[I(t, u) = 1] = E[I(t, u)],$$

where $I(t, u)$ is the system status and is defined as

$$I(t, u) = \begin{cases} 
1, & \text{if the device is operating at time } t \text{ and usage } u, \\
0, & \text{otherwise.}
\end{cases}$$

Note that if repair is not allowed, then $A(t, u)$ reduces to reliability, $F(t, u)$, the probability that the device operates without failure during $[0, t] \times [0, u]$.

**Definition 8.2.2**

The *bivariate limiting availability*, $A$, of a system is the limit of the point availability and is given by

$$A = \lim_{t \to \infty} \lim_{u \to \infty} A(t, u).$$
The bivariate limiting availability, $A$, may be thought of as the proportion of time that the device is operational in the long term.

**Definition 8.2.3**

The *bivariate average availability*, $A_{\text{avg}}(T,U)$, of a system over a rectangle $[0, T] \times [0, U]$ is given by

$$A_{\text{avg}}(T,U) = \frac{1}{TU} \int_0^T \int_0^U A(t,u) \, du \, dt. \quad (8.3)$$

Note that the bivariate average availability in $[0, T] \times [0, U]$ is the expected proportion of time and usage the system is operating during $[0, T] \times [0, U]$.

Bivariate interval average availability may be obtained for intervals $[T_1, T_2]$ and $[U_1, U_2]$ as

$$A_{\text{avg}}(T_2 - T_1, U_2 - U_1) = \frac{1}{(T_2 - T_1)(U_2 - U_1)} \int_{T_1}^{T_2} \int_{U_1}^{U_2} A(t,u) \, du \, dt. \quad (8.4)$$

**Definition 8.2.4**

The *bivariate limiting average availability*, $A_{\infty}$, of a device is the limit of the bivariate average availability and is given by

$$A_{\infty} = \lim_{T \to \infty} \lim_{U \to \infty} A_{\text{avg}}(T,U) = \lim_{T \to \infty} \lim_{U \to \infty} \frac{1}{TU} \int_0^T \int_0^U A(t,u) \, du \, dt. \quad (8.5)$$
8.3 Bivariate Availability Model for Bivariate Corrective Maintenance Models

8.3.1 Development of Bivariate Availability Models

We develop a bivariate availability model for the corrective maintenance (CM) model studied in Chapter 5 (see Figure 5.2.1). The CM model is described as follows. Consider a single-unit system with an independent and identically distributed (i.i.d.) non-negative bivariate random lifetime denoted by \( X_n = \{ (T_n, U_n) \}, n = 1, 2, \ldots \), with a common bivariate joint distribution function given by \( F(t, u) = P(T_n \leq t, U_n \leq u) \).

Letting

\[
Y_0 = (0, 0), \quad Y_n = \sum_{i=1}^{n} X_i = \left( \sum_{i=1}^{n} T_i, \sum_{i=1}^{n} U_i \right) = (S_n, V_n), \quad n \geq 1, \quad (8.6)
\]

it follows that \( Y_n = (S_n, V_n) \) is the bivariate measure of both time and use of the \( n \)th renewal (see Figure 5.2.1). Assume systems fail permanently and independently. A failed system is either repaired or replaced by an i.i.d. new one. Under a corrective maintenance policy, a system is immediately replaced/repaired upon failure. Both replacement and repair maintenance actions are assumed to be perfect. That is, after maintenance the failed system is restored to an "as good as new" state. Thus, the system is said to be renewed after each maintenance action. Let \( T_n \) and \( U_n \) be the operating time and usage after the \((n - 1)\)st maintenance, respectively. Let \( R_n \) be the repair/replace time-usage duration after the \( n \)th renewal. Assume \( \{ T_n, n = 1, 2, \ldots \} \), \( \{ U_n, n = 1, 2, \ldots \} \), and \( \{ R_n, n = 1, 2, \ldots \} \) are stochastic processes with sequences of i.i.d. non-negative random variables; \( \{ T_n \} \), \( \{ U_n \} \), and \( \{ R_n \} \) may be dependent. Our goal is to develop and construct bivariate availability models for the bivariate corrective maintenance models.
indexed by time and usage based on the bivariate failure models and the bivariate renewal theory.

Let \( R_n \) be distributed either \( g_R(t,u) \) or \( g_R(t) \) and \( H_{T,U}(t,u) \) the common distribution of \( X_n + R_n \) with j.d.f. \( h_{T,U}(t,u) \). Then \( H_{T,U}(t,u) \) is the convolution of \( F_{T,U}(t,u) \) and \( G_R(t,u) \) (or \( G_R(t) \)). Let \( M_H(t,u) \) be the bivariate renewal function corresponding to the underlying distribution \( H_{T,U}(t,u) \), i.e., \( M_H(t,u) = \sum_{n=1}^{\infty} H_{T,U}^{(n)}(t,u) \).

Let \( m_H(t,u) \) be the corresponding renewal density function. Let \( H^*_T(s,v) \), \( M^*_H(s,v) \), and \( m^*_H(s,v) \) be the Laplace transforms for \( H_{T,U}(t,u) \), \( M_H(t,u) \), and \( m_H(t,u) \). Then from Eq. (4.32), we obtain the Laplace transform of the bivariate renewal function:

\[
M^*_H(s,v) = \frac{H^*_T(s,v)}{1 - h^*(s,v)}. \tag{8.7}
\]

Next, we assume that the system that is in operation at \((t, u) = (0, 0)\) is new and is functioning. The system will be available at any future point \((t, u)\) if:

(i) it survives until \((t, u)\) or

(ii) it fails and is repaired/replaced for some duration of time and usage by a duration before \((t, u)\) and the copy of system installed at that point survives until \((t, u)\).

Then by Theorem 4.3.2 and Eq. (4.8), it is easy to show that the point availability may be expressed as

\[
A(t,u) = \overline{F}(t,u) + \int_0^t \int_0^u \overline{F}(t,x,u-y) dM_H(x,y), \tag{8.8}
\]
where $\tilde{F}(t,u)$ is the system reliability function and $M_H(t,u)$ is the renewal function.

Equation (8.8) may be solved by taking bivariate Laplace transforms of both sides. Before proceeding, we rewrite Eq. (8.8) by assuming that $M_H(t,u)$ is adequately continuous and differentiable with density $m_H(t,u)$ then

$$A(t,u) = \tilde{F}(t,u) + \int_0^t \int_0^u \tilde{F}(t-x,u-y)m_H(x,y)dydx.$$  \hspace{1cm} (8.9)

The integral part of Eq. (8.9) is a convolution of the system reliability function and the renewal density function of the process. Taking Laplace transforms, we obtain

$$A^*(s,v) = \tilde{F}^*(s,v) + \tilde{F}^*(s,v)m_H^*(s,v),$$  \hspace{1cm} (8.10)

or

$$A^*(s,v) = \tilde{F}^*(s,v)(1 + m_H^*(s,v)).$$  \hspace{1cm} (8.11)

By Eq. (4.36), $m_H^*(s,v) = \frac{h^*(s,v)}{1-h^*(s,v)}$, substitute into Eq. (8.11), we derive

$$A^*(s,v) = \frac{\tilde{F}^*(s,v)}{1-h^*(s,v)}.$$  \hspace{1cm} (8.12)

where $\tilde{F}^*(s,v)$ is the Laplace transform for the bivariate reliability function and $h^*(s,v)$ is the Laplace transform for the probability density function for the time between renewals, which is the convolution of system longevity and corrective maintenance time. Equation (8.11) can be used to avoid the cases when the renewal density functions are
complicated. From the previous chapters, we will also expect difficulties in inverting the Laplace transforms. A direct inversion of the Laplace transform for the bivariate availability function will sometimes be impossible.

### 8.4 Bivariate Availability Model for Bivariate Age Replacement Preventive Maintenance Models

In this section, we study the preventive maintenance effects in modeling bivariate availability. We consider the age replacement preventive maintenance (ARPM) model in Chapter 6 and develop the Laplace transform for its bivariate availability model. The assumptions and description of the ARPM model are as follows.

Consider a single-unit system with an independent and identically distributed (i.i.d.) non-negative bivariate random lifetime denoted by \( X_n = \{(T_n, U_n)\}, n = 1, 2, \ldots \), with a common bivariate joint distribution function given by \( F(t,u) = P\{T_n \leq t, U_n \leq u\} \). Letting

\[
Y_0 = (0, 0), \quad Y_n = \sum_{i=1}^n X_i = \left( \sum_{i=1}^n T_i, \sum_{i=1}^n U_i \right) = (S_n, V_n), \quad n \geq 1, \quad (8.13)
\]

it follows that \( Y_n = (S_n, V_n) \) is the bivariate measure of both time and use of the \( n \)th renewal as illustrated in Figure 6.2.1. Assume systems fail permanently and independently. Under an age replacement policy, if a system failed before time \( T \) or usage \( U \) (whichever comes first) since the last maintenance action then repair begins immediately, otherwise, if the system survives until time \( T \) or usage \( U \) (whichever comes first) since the last maintenance action then replacement begins immediately and is replaced by an i.i.d. new one. The repair actions (corrective maintenance) are i.i.d. distributed with bivariate repair-time density \( g_{r,T,U}(t,u) \). The replacement actions
(preventive maintenance) are i.i.d. distributed with bivariate PM-time density \( g_{pT,U}(t,u) \). We assume that there is no logistics delay time (LDT) or administrative delay time (ADT). Both replacement and repair maintenance actions are assumed to be \textit{perfect}, that is, after maintenance the system is restored to an "as good as new" state. Thus, the system is said to be renewed after each maintenance action.

Let \( T_n \) and \( U_n \) be the operating time and usage after the \((n - 1)\)th maintenance, respectively. Let \( R_n = (Rt_n, Ru_n) \) be the repair time and usage after the \(n\)th renewal. Let \( P_n = (Pt_n, Pu_n) \) be the replacement time after the \(n\)th renewal. Assume that \( \{T_n, n = 1, 2, \ldots\}, \{U_n, n = 1, 2, \ldots\}, \{Rt_n, n = 1, 2, \ldots\}, \{Ru_n, n = 1, 2, \ldots\}, \{Pt_n, n = 1, 2, \ldots\}, \) and \( \{Pu_n, n = 1, 2, \ldots\} \) are stochastic processes with sequences of i.i.d. non-negative random variables. Our goal is to develop and construct bivariate availability models indexed by time and usage.

\[8.4.1\] Development of Bivariate Availability Models for ARPM Policy

We consider two types of time-usage regions, the first \( t < T \) and \( u < U \); and the second \( t \geq T \) and \( u \geq U \), where \((T, U)\) is the period of age replacement preventive maintenance policy, i.e., if a system failed before time \( T \) or usage \( U \) (whichever comes first) since the last maintenance action then it will immediately begin to be repaired; otherwise, if the system survives until time \( T \) or usage \( U \) (whichever comes first) since the last maintenance action then it will immediately begin to be replaced by an i.i.d. new one. Note that as shown in Section 6.3.4, we do not need to consider other regions because \( \Pr\{T_n < T, U_n \geq U\} \) and \( \Pr\{T_n \geq T, U_n < U\} \) are both equal to zero (also see Figure 6.2.1).

In the first region, \( t < T \) and \( u < U \), all maintenance actions are due to system failure. A system replacement due to preventive maintenance will not have occurred. A system will only be subject to corrective maintenance, i.e., failure repair, in the first
region. Therefore, from the results of Section 6.3.1, the bivariate availability function for \( t < T \) and \( u < U \) can be expressed as

\[
A(t,u) = \bar{F}(t,u) + \int_0^t \int_0^u \bar{F}(t-x,u-y)m_{ARPM}(x,y)dydx
\]

(8.14)

where \( \bar{F}(t,u) \) is the bivariate reliability function, and \( m_{ARPM}(t,u) \) is the renewal density function for the common distribution of ARPM process with distribution \( F_{ARPM}(t,u) \).

In the second region, \( t \geq T \) and \( u \geq U \), replacements that occur may be due to system failure or to preventive maintenance. At the point of a maintenance action, the farthest point back since the last renewal is \((t - T, u - U)\) because of the \((T, U)\) ARPM policy. Thus, our region is restricted to \([t - T,t] \times [u - U,u]\). The expression for the bivariate availability function for \([t - T,t] \times [u - U,u]\) is:

\[
A(t,u) = \int_{t-T}^t \int_{u-U}^u \bar{F}(t-x,u-y)m_{ARPM}(x,y)dydx.
\]

(8.15)

Thus, the bivariate availability for \([t - T,t] \times [u - U,u]\) is the probability the system functions at \((t, u)\) and this is equal to the probability that a renewal occurs at \((x, y)\) (represented by \( m_{ARPM}(x,y) \)) and the system survives the remaining period to \((t, u)\) (represented by \( \bar{F}(t-x,u-y) \)). In this interval, the renewals (due to system failure or scheduled preventive maintenance) may occur continuously during \([t - T,t] \times [u - U,u]\).

Therefore, from Eqs. (8.14) and (8.15), the bivariate availability for an ARPM policy with policy period \((T,U)\) is:

\[
A(t,u;T,U) = \begin{cases} 
\bar{F}(t,u) + \int_0^t \int_0^u \bar{F}(t-x,u-y)m_{ARPM}(x,y)dydx, & \text{for } t < T, u < U \\
\int_{t-T}^t \int_{u-U}^u \bar{F}(t-x,u-y)m_{ARPM}(x,y)dydx, & \text{for } t \geq T, u \geq U 
\end{cases}
\]

(8.16)
Equation (8.18) represents the probability that a single-unit system functions at \((t,u)\) with an age replacement preventive maintenance period of \((T,U)\). We treat the policy \((T,U)\) as parameters so that the availability at a given point, \((t,u)\), depends on the ARPM policy \((T,U)\). In the next section, we develop the Laplace transform for \(A(t,u;T,U)\).

### 8.4.2 Bivariate Laplace Transform of the Bivariate Availability for the ARPM Policy

In this section, we develop the Laplace transform of the bivariate availability function as shown in Eq. (8.16). Let the Laplace transform of \(A(t,u;T,U)\) be denoted by \(A^*(s,v;T,U)\),

\[
A^*(s,v;T,U) = \int_0^\infty \int_0^\infty e^{-st-vu} A(t,u;T,U) dtdu. \tag{8.17}
\]

Substituting Eq. (8.16) into Eq. (8.17), we obtain

\[
A^*(s,v;T,U) = \left[ \int_0^T \int_0^U e^{-st-vu} \left( F(t,u) + \int_0^t \int_0^u F(t-x,u-y)m_{ARPM}(x,y)dydx \right) dtdx \right] \\
+ \left[ \int_0^\infty \int_0^\infty e^{-st-vu} \left( \int_{t-T}^t \int_{u-U}^u F(t-x,u-y)m_{ARPM}(x,y)dydx \right) dtdx \right] \\
= \left[ \int_0^T \int_0^U e^{-st-vu} F(t,u) dtdx \right] \\
+ \left[ \int_0^T \int_0^U e^{-st-vu} \left( \int_{t-T}^t \int_{u-U}^u F(t-x,u-y)m_{ARPM}(x,y)dydx \right) dtdx \right] \\
+ \left[ \int_0^\infty \int_0^\infty e^{-st-vu} \left( \int_{t-T}^t \int_{u-U}^u F(t-x,u-y)m_{ARPM}(x,y)dydx \right) dtdx \right] \tag{8.18}
\]
Let $w = t - x$ and $z = u - y$, then $dw = dt$ and $dz = du$, and

\[
A^*(s, v; T, U) = \left[ \int_0^T \int_0^U e^{-st-vu} F(t,u) dudt \right] + \left[ \int_0^T \int_0^U m_{\text{ARPM}}(x,y) \left( \int_T^t \int_U^u e^{-st-vu} F(t-x, u-y) dudt \right) dydx \right]
\]

\[
+ \left[ \int_0^T \int_0^U m_{\text{ARPM}}(x,y) \left( \int_T^t \int_U^u e^{-st-vu} F(t-x, u-y) dudt \right) dydx \right]
\]

\[
+ \left[ \int_T^\infty \int_0^U m_{\text{ARPM}}(x,y) \left( \int_T^t \int_U^u e^{-st-vu} F(t-x, u-y) dudt \right) dydx \right].
\]

(8.21)

Define the bivariate truncated (or partial) Laplace transform of $\bar{F}(t,u)$ as

\[
= \left[ \int_0^T \int_0^U e^{-st-vu} \bar{F}(t,u) dudt \right] + \left[ \int_0^T \int_0^U e^{-st-vu} m_{\text{ARPM}}(x,y) dydx \right] \left[ \int_0^T \int_0^U e^{-st-vu} \bar{F}(w,z) dzdw \right].
\]

(8.23)
\[
\overline{F}^*(s,v;T,U) = \left[ \int_0^T \int_0^U e^{-sv} \overline{F(t,u)} \, du \, dt \right],
\]  
(8.24)

and the bivariate Laplace transform of \( m_{ARPM}^* (x, y) \) as

\[
m_{ARPM}^*(s,v) = \int_0^\infty \int_0^\infty e^{-sx-xy} m_{h} (x,y) \, dy \, dx.
\]  
(8.25)

We obtain

\[
A^*(s,v;T,U) = \overline{F}^*(s,v;T,U) + m_{ARPM}^*(s,v) \overline{F}^*(s,v;T,U) \\
= \overline{F}^*(s,v;T,U) \left[ 1 + m_{ARPM}^*(s,v) \right]
\]  
(8.26)

By Eq. (6.9), \( m_{ARPM}^* (s,v) = \frac{f_{ARPM}^* (s,v)}{1 - f_{ARPM}^* (s,v)} \), substitute into Eq. (6.9), we derive

\[
A^*(s,v;T,U) = \frac{\overline{F}^*(s,v;T,U)}{1 - f_{ARPM}^* (s,v)},
\]  
(8.27)

where

\[
f_{ARPM}^* (s,v) = \left( f_{T,U}^* (s,v;T,U) \right) \left( g_{T,U}^* (s,v) \right) + e^{-sv} \left( g_{P,T,U}^* (s,v) \right) \overline{F}_{T,U} (T,U).
\]  
(8.28)

For some distributions without closed form distribution functions, we may express \( A^*(s,v;T,U) \) as
\[ \mathbf{A}^{*} (s,v;T,U) = \frac{F(T,U) - f_T^*(s;T) - f_U^*(v;U) + f^*_{T,U}(s,v;T,U)}{sv(1 - f^*_{ARPM}(s,v))} \tag{8.29} \]

where \( f_T^*(s;T) \) and \( f_U^*(v;U) \) are the truncated Laplace transforms for the marginals.

Eq. (8.29) can be shown by the following lemma.

**Lemma 6.4.1**  The truncated Laplace transform for the reliability function satisfies:

\[ \bar{F}^*(s,v;T,U) = \frac{1}{sv} \left[ F(T,U) - f_T^*(s;T) - f_U^*(v;U) + f^*_{T,U}(s,v;T,U) \right]. \tag{8.30} \]

**Proof.**

\[
\begin{align*}
\bar{F}^*(s,v;T,U) &= \int_{0}^{T} \int_{0}^{U} e^{-st-vu} \bar{F}(t,u) du dt = \int_{0}^{T} \int_{0}^{U} \left( \int_{0}^{\infty} \int_{0}^{\infty} f(x,y) dy dx \right) e^{-st-vu} du dt \\
&= \int_{0}^{T} \int_{0}^{U} \left( \int_{0}^{\infty} \int_{0}^{\infty} f(x,y) du dy \right) e^{-st-vu} dx dy \\
&= \int_{0}^{T} \int_{0}^{U} \left( \int_{0}^{\infty} \int_{0}^{\infty} e^{-st-vu} du dy \right) f(x,y) dx dy \\
&= \int_{0}^{T} \int_{0}^{U} \left( \frac{1-e^{-sx}}{s} \right) \left( \frac{1-e^{-vy}}{v} \right) f(x,y) dx dy \\
&= \frac{1}{sv} \left[ \int_{0}^{T} \int_{0}^{U} f(x,y) dx dy - \int_{0}^{T} \int_{0}^{U} e^{-sx} f(x,y) dx dy \\
&\quad - \int_{0}^{T} \int_{0}^{U} e^{-vy} f(x,y) dy dx + \int_{0}^{T} \int_{0}^{U} e^{-sx-ty} f(x,y) dy dx \right] \\
&= \frac{1}{sv} \left[ F(T,U) - f_T^*(s;T) - f_U^*(v;U) + f^*_{T,U}(s,v;T,U) \right].
\end{align*}
\]
Note that the Laplace inverse transforms for the bivariate availability model (Eqs. (8.27) or (8.29)) for the ARPM policy are expected to be very difficult to obtain. In the next chapter, we explore some examples of the bivariate availability to derive the results based on the Laplace transforms.

8.5 Quality of Availability Measure

In this section, we discuss some advantages of using the bivariate availability function and define the quality of availability measure. We begin with the concepts of bivariate availability, \( A(t, u) \).

8.5.1 Definition of Quality of Availability Measure

Let \( A(t, u) \) be the probability that a system is functioning at any point \((t, u)\). Figure 8.5.1 shows the contours of an example bivariate normal availability function. By using the univariate (marginal) availability, we discard the information provided by bivariate availability models, which may be more representative and descriptive of system effectiveness. Actually, the univariate measures of availability are the marginals of the bivariate availability, which are also indicated in Figure 8.5.1. These univariate availability measures do not consider the effects of the other variable. For example, a usual time domain univariate availability function gives the measure of a system effectiveness in spite of the wear and tear accumulated on the system. As a result, two systems with the same age, but with different amounts of accumulated wear, will always have the same availability measure. If we consider the conditional availability \( A_{t|T}(u | t = T) \) (see Figure 8.5.1) then it is obvious that given a fixed value of \( t = T \), the
system availability is distributed as the conditional availability $A_{u|t}(u|t=T)$ which is a function of $u$, the usage. We define the quality of availability as follows.

**Definition 8.5.1**

The *quality of an availability measure*, $Q[A(t)]$ or $Q[A(u)]$ of a device is the conditional availability given by

$$Q[A(t)] = CA_{u|t}(u|t = T) = C \frac{A(t, U)}{A(U)},$$

(8.31)

or

$$Q[A(u)] = CA_{u|t}(u|t = T) = C \frac{A(T, u)}{A(T)},$$

(8.32)

where $A(t) = \int_0^\infty A(t, u)du$, $A(U) = \int_0^\infty A(t, U)dt$, and $C$ is a positive constant.

From Definition 8.5.1, once we obtain the bivariate availability then the quality of availability measure can be expressed as, for example, the bivariate availability (given time $T$) times a positive constant divided by its marginal (given time $T$). In this case, we have the quality of availability measure as a function of usage as shown in Eq. (8.32). Thus the quality of availability can be measured by its usage. The constant $C$ is used to amplify the scale.

We can apply Eqs. (8.31) and (8.32) to describe the equipment availability quality and obtain more information about the equipment effectiveness for which univariate measures are inadequate and incomplete.
Figure 8.5.1 Contours of a bivariate availability distribution showing the marginal distributions, $A_t(t)$ and $A_u(u)$, and the conditional availability distribution, $A_{t|u}(u|T)$. 
In this chapter, examples for the bivariate availability models are presented. We develop bivariate availability models for the bivariate maintenance policies based on the results of previous chapters. The examples for the bivariate availability models are treated separately for the CM and PM models. Laplace transforms for the example bivariate availability models are derived.

9.1 Bivariate Availability Model for Bivariate Corrective Maintenance Models

Consider a single-unit system with an independent and identically distributed (i.i.d.) non-negative bivariate random lifetime denoted by $X_n = \{(T_n, U_n)\}$, $n = 1, 2, \ldots$, with a common bivariate joint distribution function given by $F(t,u) = P\{T_n \leq t, U_n \leq u\}$. Letting

$$Y_0 = (0, 0), \quad Y_n = \sum_{i=1}^n X_i = \left(\sum_{i=1}^n T_i, \sum_{i=1}^n U_i\right) = (S_n, V_n), \quad n \geq 1,$$

it follows that $Y_n = (S_n, V_n)$ is the bivariate measure of both time and use of the $n$th renewal (see Figure 6.2.1). Assume systems fail permanently and independently. A failed system is either repaired or replaced by an i.i.d. new one. Under a corrective maintenance policy, a system is immediately replaced/repaired upon failure. Both replacement and
repair maintenance actions are assumed to be perfect so that after maintenance the failed system is restored to an "as good as new" state. Thus, the system is said to be renewed after each maintenance action. Let \( R_n \) be the repair/replace time after the \( n \)th renewal. Assume \( \{ T_n, n = 1, 2, \ldots \} \), \( \{ U_n, n = 1, 2, \ldots \} \), and \( \{ R_n, n = 1, 2, \ldots \} \) are stochastic processes with sequences of i.i.d. non-negative random variables. \( \{ T_n \} \), \( \{ U_n \} \), and \( \{ R_n \} \) may be dependent. Our goal is to develop and construct bivariate availability models for the bivariate corrective maintenance models indexed by time and usage based on the bivariate failure models and bivariate renewal theory. Let \( R_n \) be distributed either \( \delta_{uT} g \) or \( \delta_{T} g \) with

\[
\delta_{uT} g \quad \text{or} \quad \delta_{T} g \quad \text{and} \quad \delta_{uT} R \quad \text{or} \quad \delta_{T} R.
\]

9.1.1 Bivariate Exponential (Naggs and Nagagaja) Case

We consider the correlated bivariate failure models developed in Chapter 3 as our joint distribution functions to construct the bivariate availability models. Assume that the distribution function of \( X_n, F(t,u) \), is Baggs and Nagagaja's bivariate exponential with joint density function:

\[
f_{T,U}(t,u) = \lambda \eta e^{-(\lambda t+\eta u)} \left(1 + \rho (1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-(\lambda t+\eta u)})\right). \tag{9.2}
\]

Let \( R_n \) be distributed either \( g_R(t,u) \) or \( g_R(t) \) with

\[
g_R(t,u) = \lambda_1 \eta_1 e^{-(\lambda_1 t+\eta_1 u)} \left(1 + \rho_1 (1 - 2e^{-\lambda_1 t} - 2e^{-\eta_1 u} + 4e^{-(\lambda_1 t+\eta_1 u)})\right) \tag{9.3}
\]

and

\[
g_R(t) = \lambda_1 e^{-\lambda_1 t}. \tag{9.4}
\]

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Let $H_{T,U}(t,u)$ be the common distribution of $X_n + R_n$ with j.d.f. $h_{T,U}(t,u)$. Then $H_{T,U}(t,u)$ is the convolution of $F_{T,U}(t,u)$ and $G_R(t,u)$ (or $G_R(t)$). The Laplace transforms of $h_{T,U}(t,u)$ are

$$h^*(s,v) = \frac{\lambda \eta \left[ (v + 2\eta)(s + 2\lambda) + sv\rho \right]}{[(v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)](v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)}$$

for $g_R(t,u)$ (9.5)

or

$$h^*(s,v) = \frac{\lambda \eta \left[ (v + 2\eta)(s + 2\lambda) + sv\rho \right]}{[(v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)](\lambda + s)}$$

for $g_R(t)$ (9.6)

The reliability function is

$$\bar{F}_{T,U}(t,u) = e^{-(\lambda + \eta)u} \left(1 + \rho \left(1 - e^{-\lambda u}\right)\left(1 - e^{-\eta u}\right)\right)$$

with the corresponding Laplace transform

$$\bar{F}^*_{T,U}(t,u) = \frac{(s + 2\lambda)(v + 2\eta) + \lambda \eta \rho}{(s + \lambda)(s + 2\lambda)(v + \eta)(v + 2\eta)}$$

(9.8)

From Eq. (8.12), $A^*(s,v) = \frac{\bar{F}^*(s,v)}{1 - h^*(s,v)}$, we obtain the Laplace transforms for the bivariate availability:

$$A^*(s,v) = \frac{[C]}{[D] - [E]}$$

for $g_R(t,u)$ (9.9)

where
\[ [C] = [(s + 2\lambda)(v + 2\eta) + \lambda\eta \rho \llbracket (s + \lambda_1)(v + \eta_1)(s + 2\lambda_1)(v + 2\eta_1) \rrbracket, \]
\[ [D] = [(s + \lambda_1)(v + \eta_1)(s + 2\lambda_1)(v + 2\eta_1)(s + \lambda)(v + \eta)(s + 2\lambda)(v + 2\eta)], \] and
\[ [E] = \lambda\lambda_1\eta\eta_1[(s + 2\lambda_1)(v + 2\eta_1) + \eta\rho \llbracket (s + 2\lambda)(v + 2\eta) + \eta\rho \rrbracket]. \]

Similarly,

\[
A^*(s,v) = \frac{(s + 2\lambda)(v + 2\eta) + \lambda\eta\rho}{(s + \lambda)(s + 2\lambda)(v + \eta)(v + 2\eta) - \lambda\lambda_1\eta\eta_1[(s + 2\lambda)(v + 2\eta) + \eta\rho]} \text{ for } g_R(t). 
\]

(9.10)

The direct inversion of \( A^*(s,v) \) is not an easy task and it may be necessary to use numerical methods to invert the bivariate Laplace transforms.

9.1.2 Bivariate Exponential (Hunter) Case

Assume that the distribution function of \( X_n \), \( F(t,u) \), is Hunter’s bivariate exponential with joint density function:

\[
f_{T,U}(t,u) = \frac{\lambda\eta}{1 - \rho} I_0 \left( \frac{2\sqrt{\rho}}{1 - \rho} \sqrt{\lambda\eta tu} \right) \exp \left\{ -\frac{\lambda t + \eta u}{1 - \rho} \right\} \]

(9.11)

where \( I_n(\cdot) \) is the modified Bessel function of the first kind of order \( n \); and \( \rho \) is positive.

Let \( R_n \) be distributed either \( g_R(t,u) \) or \( g_R(t) \) with

\[
g_R(t,u) = \frac{\lambda_1\eta_1}{1 - \rho_1} I_0 \left( \frac{2\sqrt{\rho_1}}{1 - \rho_1} \sqrt{\lambda_1\eta_1 tu} \right) \exp \left\{ -\frac{\lambda_1 t + \eta_1 u}{1 - \rho_1} \right\} \]

(9.12)
and

\[ g_R(t) = \lambda_t e^{-\lambda t}. \]  

(9.13)

Let \( H_{T,U}(t,u) \) be the common distribution of \( X_n + R_n \) with j.d.f. \( h_{T,U}(t,u) \). Then \( H_{T,U}(t,u) \) is the convolution of \( F_{T,U}(t,u) \) and \( G_R(t,u) \) (or \( G_R(t) \)). The Laplace transforms of \( h_{T,U}(t,u) \) are

\[ h^*(s,v) = \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{v}{\eta} + 1 \right) - \frac{s \eta v}{\lambda \eta} \right]^{-1} \left[ \left( \frac{s}{\lambda_i} + 1 \right) \left( \frac{v}{\eta_i} + 1 \right) - \frac{s \eta_i v_i}{\lambda_i \eta_i} \right]^{-1} \]

\[ = \left[ \frac{\lambda \eta}{(s + \lambda)(v + \eta) - s \eta v} \right] \left[ \frac{\lambda_i \eta_i}{(s + \lambda_i)(v + \eta_i) - s \eta_i v_i} \right] \text{ for } g_R(t,u), \]  

(9.14)

or

\[ h^*(s,v) = \frac{\lambda_i}{(\lambda_i + s)} \left[ \frac{s}{\lambda} + 1 \right] \left( \frac{v}{\eta} + 1 \right) - \frac{s \eta v}{\lambda \eta} \right]^{-1} \]

\[ = \frac{\lambda_i}{(\lambda_i + s)} \left[ \frac{\lambda \eta}{(s + \lambda)(v + \eta) - s \eta v} \right] \text{ for } g_R(t). \]  

(9.15)

The reliability function can be written as

\[ F_{T,U}(t,u) = 1 - F_T(t) - F_U(u) + F_{T,U}(t,u) \]  

(9.16)

where \( F_T(t), F_U(u) \) are exponential marginals (see Section 3.3.2). Thus, the Laplace transform for the reliability function can be expressed as
\[
\bar{F}_{s,v}^* (s,v) = L_{s,v} \left\{ 1 - F_T (t) - F_U (u) + F_{T,U}^* (t,u) \right\}.
\] (9.17)

That is,

\[
\bar{F}_{s,v}^* (s,v) = \frac{1}{sv} - F_T^* (s) - F_U^* (v) + F_{T,U}^* (s,v)
\]
\[
= \frac{1}{sv} - \left[ \frac{1}{s} - \frac{1}{s + \lambda} \right] - \left[ \frac{1}{v} - \frac{1}{v + \eta} \right] + \left[ \frac{s}{\lambda} + \frac{1}{\eta} + 1 \right] - \frac{s v \rho}{\lambda \eta} \right]^{-1}
\]
\[
= \frac{1}{sv} - \frac{s + \lambda - 1}{s(s + \lambda)} - \frac{v + \eta - 1}{v(v + \eta)} + \frac{\lambda \eta}{(s + \lambda)(v + \eta) - s v \rho}.
\] (9.18)

From Eqs. (8.12), (9.14), (9.15), and (9.18), we obtain the Laplace transforms of the bivariate availability

\[
A^* (s,v) = \frac{1}{sv} - \frac{s + \lambda - 1}{s(s + \lambda)} - \frac{v + \eta - 1}{v(v + \eta)} + \frac{\lambda \eta}{(s + \lambda)(v + \eta) - s v \rho} + \frac{\lambda \eta}{(s + \lambda \eta)(v + \eta) - s v \rho} \quad \text{for } g_R (t,u)
\] (9.19)

and

\[
A^* (s,v) = \frac{1}{sv} - \frac{s + \lambda - 1}{s(s + \lambda)} - \frac{v + \eta - 1}{v(v + \eta)} + \frac{\lambda \eta}{(s + \lambda)(v + \eta) - s v \rho} + \frac{\lambda \eta}{(s + \lambda \eta)(v + \eta) - s v \rho} \quad \text{for } g_R (t).
\] (9.20)

The direct inversion of \( A^* (s,v) \) is not an easy task and it may be necessary to use numerical methods to invert the bivariate Laplace transform.
9.1.3 Bivariate Normal Case

Assume that the distribution function of \( X_n, F(t,u) \), is bivariate normal with joint density function:

\[
f_{T,U}(t,u) = \frac{1}{2\pi \sigma \sigma_u \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(t-\mu_1)^2}{\sigma_1^2} - 2\rho \frac{(t-\mu_1)(u-\mu_u)}{\sigma_1\sigma_u} + \frac{(u-\mu_u)^2}{\sigma_u^2} \right] \right\}. \quad (9.21)
\]

Let \( R_n \) be distributed either \( g_R(t,u) \) or \( g_R(t) \) with

\[
g_R(t,u) = \frac{1}{2\pi \sigma \sigma_u \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(t-\mu_1)^2}{\sigma_1^2} - 2\rho \frac{(t-\mu_1)(u-\mu_2)}{\sigma_1\sigma_2} + \frac{(u-\mu_2)^2}{\sigma_2^2} \right] \right\} \quad (9.22)
\]

and

\[
g_R(t) = \frac{1}{\sqrt{2\pi \sigma_1}} \exp \left\{ -\frac{(t-\mu_1)^2}{2\sigma_1^2} \right\}. \quad (9.23)
\]

Let \( H_{T,U}(t,u) \) be the common distribution of \( X_n + R_n \) with j.d.f. \( h_{T,U}(t,u) \). Then \( H_{T,U}(t,u) \) is the convolution of \( F_{T,U}(t,u) \) and \( G_R(t,u) \) (or \( G_R(t) \)). The Laplace transforms of \( h_{T,U}(t,u) \) are

\[
h^*(s,v) = \exp \left[ -s(\mu_1 + \mu_u) - v(\mu_u + \mu_2) + \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 \right) + 2sv\sigma_1\sigma_u + v^2(\sigma_u^2 + \sigma_2^2) \right]
\]

for \( g_R(t,u) \) \quad (9.24)

or
\( h^*(s,v) = \exp \left[ -s(\mu_i + \mu_i) - v\mu_u + \frac{1}{2} \left( s^2\sigma_i^2 + \sigma_u^2 \right) + 2\rho sv\sigma_i \sigma_u + v^2\sigma_u^2 \right] \) for \( g_R(t) \).

(9.25)

The bivariate normal reliability function can not be expressed in a closed form. However, from Eqs. (9.17) and (4.32), the Laplace transform for the reliability function can be expressed as

\[
\bar{F}_{T,U}^*(s,v) = \frac{1}{sv} - \frac{f_T^*(s) - f_U^*(v) + \bar{F}_{T,U}^*(s,v)}{sv} \\
= \frac{1 - v f_T^*(s) - s f_U^*(v) + f_{T,U}^*(s,v)}{sv} 
\]

(9.26)

where

\[
f_T^*(s) = \exp \left[ -\mu_i s + \frac{1}{2} \left( s^2\sigma_i^2 \right) \right] \quad \text{and} \quad f_U^*(v) = \exp \left[ -\mu_u v + \frac{1}{2} \left( v^2\sigma_u^2 \right) \right]
\]

(9.27)

are Laplace transforms for normal marginals and

\[
f_{T,U}^*(s,v) = \exp \left[ -s\mu_i - v\mu_u + \frac{1}{2} \left( s^2\sigma_i^2 + 2\rho sv\sigma_i \sigma_u + v^2\sigma_u^2 \right) \right] 
\]

(9.28)

is Laplace transform for bivariate normal.

From Eqs. (8.12), (9.24), (9.25), and (9.26), we obtain the Laplace transforms of the bivariate availability.
The direct inversion of $A^*(s,v)$ is not an easy task and it may be necessary to use numerical methods to invert the bivariate Laplace transforms.

### 9.2 Examples of Bivariate Availability Functions for Age Replacement Preventive Maintenance (ARPM) Models with i.i.d. $X_n, R_n,$ and $P_n$

In this section, we use the correlated bivariate failure models developed in Chapter 3 as the joint longevity, corrective maintenance (repair) time, and preventive maintenance (replacement) time distributions to develop the bivariate availability functions for ARPM models developed in Chapter 7.

Consider a single-unit system with an independent and identically distributed (i.i.d.) non-negative bivariate random lifetime denoted by $X_n = \{(T_n, U_n)\}, n = 1, 2, \ldots,$ with a common bivariate joint distribution function given by $F(t,u) = P(T_n \leq t, U_n \leq u).$ Assume systems fail permanently and independently. Under an age replacement policy, if a system failed before time $T$ or usage $U$ ( whichever comes first ) since the last maintenance action then it will be immediately repaired, otherwise, if the system survives until time $T$ or usage $U$ ( whichever comes first ) since the last maintenance action then it will be immediately replaced by an i.i.d. new one. The repair actions (corrective maintenance) are i.i.d. distributed with bivariate repair-time density $g_{r_T,U}(t,u).$ The replacement actions (preventive maintenance) are i.i.d. distributed with bivariate PM-time density $g_{P_T,U}(t,u).$ We assume that there is no logistics delay time (LDT) or administrative delay time (ADT). Both replacement and repair maintenance actions are
assumed to be perfect. After maintenance, the system is restored to an "as good as new" state. Thus, the system is said to be renewed after each maintenance action.

Let $T_n$ and $U_n$ be the operating time and usage after the $(n - 1)$st maintenance, respectively. Let $R_n = (R_{t_n}, R_{u_n})$ be the repair time and usage after the $n$th renewal. Let $P_n = (P_{t_n}, P_{u_n})$ be the replacement time after the $n$th renewal. Assume that $\{T_n, n = 1, 2, \ldots\}$, $\{U_n, n = 1, 2, \ldots\}$, $\{R_n, n = 1, 2, \ldots\}$, $\{R_{t_n}, n = 1, 2, \ldots\}$, $\{P_{t_n}, n = 1, 2, \ldots\}$, and $\{P_{u_n}, n = 1, 2, \ldots\}$ are stochastic processes with sequences of i.i.d. non-negative random variables. Our goal is to develop and construct bivariate availability models indexed by time and usage.

9.2.1 Bivariate Availability for Bivariate Exponential ARPM Model I

Assume that $X_n, R_n,$ and $P_n$ are i.i.d. with the same distribution function, $F(t,u)$ which is Baggs and Nagagaja's bivariate exponential with joint density function:

$$f_{T,U}(t,u) = \lambda \eta e^{-(\lambda t + \eta u)} \left(1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-(\lambda t + \eta u)} \right) \right), \quad (9.30)$$

and its Laplace transform

$$f^*_{T,U}(s,v) = \frac{\lambda \eta [(v + 2\eta)(s + 2\lambda) + sv \rho]}{(v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)}. \quad (9.31)$$

From the results of Chapter 8, we have

$$f^*_{\text{ARPM}}(s,v) = \left(f^*_{T,U}(s,v;T,U)\right)f^*_{T,U}(s,v) + e^{-sT-vU} \left(fp^*_{T,U}(s,v)\right)\overline{F}_{T,U}(T,U) \quad (9.32)$$
and obtain the Laplace transform of \( f^*_{ARPM}(s,v) \):

\[
f^*_{ARPM}(s,v) = f^*_{T,U}(s,v) \left[ f^*_{T,U}(s,v;T,U) + e^{-\lambda(T-U)} \tilde{F}_{T,U}(T-U) \right].
\] (9.33)

From Eq. (7.4), the truncated Laplace transform \( f^*_{T,U}(s,v;T,U) \) for the bivariate exponential failure function is equal to \( f^*_{T,U}(s,v)(1 - e^{\lambda(T-U) + (v+\eta)U}) \). After substituting into Eq. (9.33), we obtain

\[
f^*_{ARPM}(s,v) = f^*_{T,U}(s,v) \left[ f^*_{T,U}(s,v)(1 - e^{\lambda(T-U) + (v+\eta)U}) + e^{-\lambda(T-U)} \tilde{F}_{T,U}(T,U) \right] \] (9.34)

\[
= f^*_{T,U}(s,v) \left[ f^*_{T,U}(s,v)(1 - e^{\lambda(T-U) + (v+\eta)U}) + e^{-\lambda(T-U) - (v+\eta)U} \left( 1 + \rho \left( 1 - e^{-\lambda T} \right) \left( 1 - e^{-\eta U} \right) \right) \right]
\] (9.35)

\[
= \left( f^*_{T,U}(s,v) \right)^2 + f^*_{T,U}(s,v)e^{\lambda(T-U) + (v+\eta)U} \]
\[
+ f^*_{T,U}(s,v)e^{-\lambda(T-U) - (v+\eta)U} \left( 1 + \rho \left( 1 - e^{-\lambda T} \right) \left( 1 - e^{-\eta U} \right) \right)
\] (9.36)

\[
= \left( f^*_{T,U}(s,v) \right)^2 + f^*_{T,U}(s,v)e^{\lambda(T-U) + \eta U}C_1 + f^*_{T,U}(s,v)e^{-\lambda(T-U) - \eta U}C_2
\] (9.37)

where

\[
f^*_{T,U}(s,v) = \frac{\lambda \eta \left[ (v+2\eta) \left( s + 2\lambda \right) + s \nu \rho \right]}{(v+\eta)(s+\lambda)(v+2\eta)(s+2\lambda)},
\]

\[C_1 = e^{\lambda T + \eta U}, \text{ and}
\]

\[C_2 = e^{-\lambda T - \eta U} \left( 1 + \rho \left( 1 - e^{-\lambda T} \right) \left( 1 - e^{-\eta U} \right) \right).
\]
On the other hand, the truncated Laplace transform for the marginals are \( f_T^*(s)(1 - e^{(s+\lambda)T}) \) and \( f_U^*(v)(1 - e^{(v+\eta)U}) \). Thus, we have

\[
\overline{F}_{T,U}(s,v;T,U) = \frac{1}{sv} \left[ F(T,U) - f_T^*(s;T) - f_U^*(v;U) + f_{T,U}^*(s,v;T,U) \right]
\]

\[
= \frac{1}{sv} \left[ F(T,U) - f_T^*(s)(1 - e^{(s+\lambda)T}) - f_U^*(v)(1 - e^{(v+\eta)U}) + f_{T,U}^*(s,v)(1 - e^{(s+\lambda)T+(v+\eta)U}) \right].
\]  

(9.38)

Substituting Eqs. (9.37) and (9.38) into

\[
A^*(s,v) = \frac{F(T,U) - f_T^*(s;T) - f_U^*(v;U) + f_{T,U}^*(s,v;T,U)}{sv\left(1 - f_{ARPM}^*(s,v)\right)},
\]  

(9.39)

we obtain

\[
A^*(s,v) = \frac{F(T,U) - f_T^*(s)(1 - e^{(s+\lambda)T}) - f_U^*(v)(1 - e^{(v+\eta)U}) + f_{T,U}^*(s,v)(1 - e^{(s+\lambda)T+(v+\eta)U})}{sv\left[1 - f_{T,U}^*(s,v)(f_{T,U}^*(s,v) + e^{(s+\eta)U}C_1 + e^{-sT+vU}C_2)\right]}
\]

(9.40)

where

\[
f_T^*(s) = \frac{\lambda}{\lambda + s},
\]

\[
f_U^*(v) = \frac{\eta}{\eta + v},
\]

\[
f_{T,U}^*(s,v) = \frac{\lambda\eta[(v + 2\eta)(s + 2\lambda) + sv\eta]}{(v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)}. 
\]
\[ C_1 = e^{\lambda T \eta U}, \text{ and} \\
C_2 = e^{-\lambda T \eta U} \left( 1 + \rho \left( 1 - e^{-\lambda T} \right) \left( 1 - e^{-\eta U} \right) \right). \]

Note that in this example the bivariate availability for ARPM policy, Eq. (9.40), is so complicated that new numerical methods are needed for the inverse transform and that even with new methods inversion will be computationally difficult.

### 9.2.2 Bivariate Availability for Bivariate Normal ARPM Model

Assume that \( X_n, R_n, \) and \( P_n \) are i.i.d. with the same distribution function, \( F(t,u) \) which is bivariate normal with joint density function:

\[
f_{T,U}(t,u) = \frac{1}{2\pi \sigma_t \sigma_u \sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ \frac{(t - \mu_t)^2}{\sigma_t^2} - 2\rho \frac{(t - \mu_t)(u - \mu_u)}{\sigma_t \sigma_u} + \frac{(u - \mu_u)^2}{\sigma_u^2} \right] \right\} \tag{9.41}
\]

and its Laplace transform:

\[
f^*(s,v) = \exp \left[ -s \mu_t - v \mu_u + \frac{1}{2} \left( s^2 \sigma_t^2 + 2 \rho sv \sigma_t \sigma_u + v^2 \sigma_u^2 \right) \right]. \tag{9.42}
\]

From Eq. (8.29), the bivariate availability for bivariate normal ARPM model is given by

\[
A^*(s,v) = \frac{F(T,U) - f^*_T(s;T) - f^*_U(v;U) + f^*_{T,U}(s,v;T,U)}{sv \left( 1 - f^*_{ARPM}(s,v) \right)} \tag{9.43}
\]

where
\[ f_{\text{ARPM}}^*(s,v) = C_1 \left( f_{T,U}^*(s,v) \right)^2 + f_{T,U}^*(s,v) e^{-\lambda \tau - \eta v} C_2, \]

\[ f_{T,U}^*(s,v;T,U) = C_1 f_{T,U}^*(s,v), \]

\[ f_T^*(s;T) = C_1 f_T^*(s), \text{ and} \]

\[ f_U^*(v;U) = C_4 f_U^*(v) \]

with

\[ C_1 = \left[ \Phi(T; \sigma, \mu) \right]^{-1}, \]

\[ C_2 = \left[ 1 - \Phi(T; \sigma, \mu) - \Phi(U; \sigma, \mu) + \Phi(T,U; \sigma, \mu, \rho) \right], \]

\[ C_3 = \left[ \Phi(T; \sigma, \mu) \right]^{-1}, \text{ and} \]

\[ C_4 = \left[ \Phi(U; \sigma, \mu) \right]^{-1}. \]

Note that in this example the bivariate availability for ARPM policy, Eq. (9.43), is so complicated that new numerical methods are needed for the inverse transform and that even with new methods inversion will be computationally difficult.

### 9.2.3 Bivariate Availability for Bivariate Exponential ARPM Model II

Assume that \( X_n, R_n, \) and \( P_n \) are i.i.d. with the same distribution function, \( F(t,u) \) which is Hunter's bivariate exponential with joint density function:

\[ f_{T,U}(t,u) = \frac{\lambda \eta}{1-\rho} I_0 \left( \frac{2 \sqrt{\rho}}{1-\rho} \sqrt{\lambda \eta} \right) \exp \left\{ -\frac{\lambda t + \eta u}{1-\rho} \right\} \]  \hspace{1cm} (9.44)
where $I_n(\cdot)$ is the modified Bessel function of the first kind of order $n$; and $\rho$ is positive, and its Laplace transform:

$$f^*(s,v) = \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{v}{\eta} + 1 \right) - \frac{s\rho}{\lambda\eta} \right]^{-1}. \tag{9.45}$$

From Eq. (8.29), the bivariate availability for bivariate ARPM model is given by

$$A^*(s,v) = \frac{F(T,U) - f^*_T(s;T) - f^*_U(v;U) + f^*_{T,U}(s,v;T,U)}{sv(1 - f^*_{\text{ARPM}}(s,v))} \tag{9.46}$$

where

$$f^*_{\text{ARPM}}(s,v) = C_1 \left( f^*_{T,U}(s,v) \right)^2 + f^*_T(s,v)e^{-sT-vU} C_2,$$

$$f^*_{T,U}(s,v;T,U) = C_1 f^*_{T,U}(s,v),$$

and

$$f^*_T(s;T) = C_3 f^*_T(s), \quad f^*_U(v;U) = C_4 f^*_U(v),$$

with

$$C_1 = [F(T,U)]^{-1}, \quad C_2 = \overline{F(T,U)}, \quad C_3 = [F(T)]^{-1}, \text{ and } C_4 = [F(U)]^{-1}.$$
9.3 Examples of Bivariate Availability Functions for ARPM Models with independent but distinct $X_n$, $R_n$, and $P_n$

In this section, we use the correlated bivariate failure models developed in Chapter 3 as the joint longevity, corrective maintenance (repair) time, and preventive maintenance (replacement) time distributions to develop the bivariate availability functions for ARPM models developed in Chapter 7.

Consider a single-unit system with an independent and identically distributed (i.i.d.) non-negative bivariate random lifetime denoted by $X_n = \{(T_n, U_n)\}, n = 1, 2, \ldots$, with a common bivariate joint distribution function given by $F(t,u) = P[T_n \leq t, U_n \leq u]$. Assume systems fail permanently and independently. Under an age replacement policy, if a system failed before time $T$ or usage $U$ (whichever comes first) since the last maintenance action then it will be immediately repaired, otherwise, if the system survives until time $T$ or usage $U$ (whichever comes first) since the last maintenance action then it will be immediately replaced by an i.i.d. new one. The repair actions (corrective maintenance) are i.i.d. distributed with bivariate repair-time density $g_{r,T,U}(t,u)$. The replacement actions (preventive maintenance) are i.i.d. distributed with bivariate PM-time density $g_{p,T,U}(t,u)$. We assume that there is no logistics delay time (LDT) or administrative delay time (ADT). Both replacement and repair maintenance actions are assumed to be perfect. After maintenance, the system is restored to an "as good as new" state. Thus, the system is said to be renewed after each maintenance action.

Let $T_n$ and $U_n$ be the operating time and usage after the $(n - 1)$st maintenance, respectively. Let $R_n = (R_{t,n}, R_{u,n})$ be the repair time and usage after the $n$th renewal. Let $P_n = (P_{t,n}, P_{u,n})$ be the replacement time after the $n$th renewal. $X_n$, $R_n$, and $P_n$ are independent but with distinct distributions. Assume that $\{T_n, n = 1, 2, \ldots\}$, $\{U_n, n = 1, 2, \ldots\}$, $\{R_{t,n}, n = 1, 2, \ldots\}$, $\{R_{u,n}, n = 1, 2, \ldots\}$, $\{P_{t,n}, n = 1, 2, \ldots\}$, and $\{P_{u,n}, n = 1, 2, \ldots\}$
are stochastic processes with sequences of i.i.d. non-negative random variables. Our goal is to develop and construct bivariate availability models indexed by time and usage.

9.3.1 Bivariate Availability for Bivariate Exponential ARPM Model I

Assume that \( X_n, R_n, \) and \( P_n \) are independent but with distinct distribution functions from the same family. \( X_n, R_n, \) and \( P_n \) are distributed as \( F_{T,U}(t,u), Gr_{T,U}(t,u), \) and \( Gp_{T,U}(t,u), \) respectively. \( F_{T,U}(t,u), Gr_{T,U}(t,u), \) and \( Gp_{T,U}(t,u) \) are Baggs and Nagagaja’s bivariate exponential with joint density function, respectively:

\[
f_{T,U}(t,u) = \lambda \eta e^{-\{\lambda t + \eta u\}} \left[1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-\{\lambda t + \eta u\}}\right)\right]. \tag{9.47}
\]

\[
gr_{T,U}(t,u) = \lambda, \eta e^{-\{\lambda t + \eta u\}} \left[1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-\{\lambda t + \eta u\}}\right)\right]. \tag{9.48}
\]

\[
gp_{T,U}(t,u) = \lambda, \eta e^{-\{\lambda t + \eta u\}} \left[1 + \rho \left(1 - 2e^{-\lambda t} - 2e^{-\eta u} + 4e^{-\{\lambda t + \eta u\}}\right)\right]. \tag{9.49}
\]

and their Laplace transforms:

\[
f_{T,U}^{*}(s,v) = \frac{\lambda \eta \left[(v + 2\eta)(s + 2\lambda) + sv \rho\right]}{(v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)}. \tag{9.50}
\]

\[
gr_{T,U}^{*}(s,v) = \frac{\lambda, \eta \left[(v + 2\eta)(s + 2\lambda) + sv \rho\right]}{(v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)}. \tag{9.51}
\]

\[
gp_{T,U}^{*}(s,v) = \frac{\lambda, \eta \left[(v + 2\eta)(s + 2\lambda) + sv \rho\right]}{(v + \eta)(s + \lambda)(v + 2\eta)(s + 2\lambda)}. \tag{9.52}
\]
From Chapter 8, we have

\[ A^*(s,v;T,U) = \frac{F^*(s,v;T,U)}{1 - f^*_{ARPM}(s,v)}. \]  \hspace{1cm} (9.53)

From Lemma 8.4.1, we have

\[ \bar{F}^*_{T,U}(s,v;T,U) = \frac{1}{sv} \left[ F(T,U) - f^*_T(s;T) - f^*_U(v;U) + f^*_T(s,v;T,U) \right] \]  \hspace{1cm} (9.55)

In Eq. (9.55), for the Baggs and Nagahaja's bivariate exponential, we have \( f^*_T(s)(1 - e^{(s+\lambda)T}) \) and \( f^*_U(v)(1 - e^{(v+\eta)U}) \) for the truncated Laplace transform of the marginals; and \( f^*_T(s,v)(1 - e^{(s+\lambda)T+(v+\eta)U}) \) for the truncated Laplace transform. Thus,

\[ \bar{F}^*_{T,U}(s,v;T,U) = \frac{1}{sv} \left[ F(T,U) - f^*_T(s)(1 - e^{(s+\lambda)T}) - f^*_U(v)(1 - e^{(v+\eta)U}) + f^*_T(s,v)(1 - e^{(s+\lambda)T+(v+\eta)U}) \right] \]  \hspace{1cm} (9.56)

From Eq. (7.41), we obtain the Laplace transform of \( f^*_{ARPM}(s,v) \):

\[ f^*_{ARPM}(s,v) = f^*_{T,U}(s,v)g^*_{T,U}(s,v) - f^*_{T,U}(s,v)g^*_{T,U}(s,v)e^{sT+\nu U} C_1 + g^*_{T,U}(s,v)e^{-sT-\nu U} C_1^{-1} + g^*_{T,U}(s,v)e^{-sT-\nu U} C_2 \]  \hspace{1cm} (9.57)

where

\[ f^*_{T,U}(s,v) = \frac{\lambda \eta[(v+2\eta)(s+2\lambda)+sv\rho]}{(v+\eta)(s+\lambda)(v+\eta)(s+2\lambda)}. \]
Substituting Eqs. (56) and (57) into Eq. (53), we can obtain the Laplace transform for the bivariate availability for ARPM policy with bivariate exponentials. The inverse of this bivariate Laplace transform is so complicated that new numerical methods are needed for the inverse transform and that even with new methods inversion will be computationally difficult.

9.3.2 Bivariate Availability for Bivariate Normal ARPM Model

Assume that $X_n$, $R_n$, and $P_n$ are independent but with distinct distribution functions from the same family. $X_n$, $R_n$, and $P_n$ are distributed as $F_{T,U}(t,u)$, $Gr_{T,U}(t,u)$, and $Gp_{T,U}(t,u)$, respectively. $F_{T,U}(t,u)$, $Gr_{T,U}(t,u)$, and $Gp_{T,U}(t,u)$ are Baggs and Nagagaja's bivariate exponential with joint density function, respectively:

$$f_{T,U}(t,u) = \frac{1}{2\pi\sigma_x\sigma_u\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \left[ \frac{(t-\mu_x)^2}{\sigma_x^2} - 2\rho \frac{(t-\mu_x)(u-\mu_u)}{\sigma_x \sigma_u} + \frac{(u-\mu_u)^2}{\sigma_u^2} \right] \right\},$$

(9.58)
\[
g_{r,t,u}(t,u) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ \frac{(t - \mu_t)^2}{\sigma_1^2} - 2\rho \frac{(t - \mu_t)(u - \mu_u)}{\sigma_1 \sigma_2} + \frac{(u - \mu_u)^2}{\sigma_2^2} \right] \right\},
\]
\[
g_{r,t,u}(t,u) = \frac{1}{2\pi \sigma_3 \sigma_4 \sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ \frac{(t - \mu_t)^2}{\sigma_3^2} - 2\rho \frac{(t - \mu_t)(u - \mu_u)}{\sigma_3 \sigma_4} + \frac{(u - \mu_u)^2}{\sigma_4^2} \right] \right\}.
\]

From Chapter 8, we have

\[
A^*(s,v;T,U) = \frac{F^*(s,v;T,U)}{1 - f_{\text{ARPM}}^*(s,v)}.
\]

From Lemma 8.4.1, we have

\[
F_{T,U}^*(s,v;T,U) = \frac{1}{sv} \left[ F(T,U) - f_T^*(s;T) - f_U^*(v;U) + f_{T,U}^*(s,v;T,U) \right]
\]

where

\[
F(T,U) = \Phi(T,U;\sigma_1,\sigma_2,\mu_1,\mu_2,\rho)
\]

\[
f_{T,U}^*(s,v;T,U) = C_1 f_{T,U}^*(s,v),
\]

\[
f_T^*(s;T) = C_2 f_T^*(s), \text{ and}
\]

\[
f_U^*(v;U) = C_3 f_U^*(v)
\]

with
\[ C_1 = \left[ \Phi(T; \sigma, \mu, \mu, \rho) \right]^{-1}, \]
\[ C_2 = \left[ \Phi(T; \sigma, \mu) \right]^{-1}, \text{ and} \]
\[ C_3 = \left[ \Phi(U; \sigma, \mu) \right]^{-1}. \]

From Eq. (7.51), we have the Laplace transform of \( f_{ARPM}^*(s,v) : \)

\[ f_{ARPM}^*(s,v) = f_{T,U}^*(s,v) g_{T,U}^*(s,v) C_1 + e^{-sT-vU} g_{P_{T,U}}^*(s,v) C_4 \quad (9.63) \]

where

\[ f_{T,U}^*(s,v) = \exp\left[ -s\mu - v\mu_u + \frac{1}{2} (s^2\sigma_t^2 + 2s\sigma_m\sigma_1 + v^2\sigma_u^2) \right], \]
\[ g_{T,U}^*(s,v) = \exp\left[ -s\mu - v\mu_2 + \frac{1}{2} (s^2\sigma_1^2 + 2s\sigma_m\sigma_2 + v^2\sigma_2^2) \right], \]
\[ g_{P_{T,U}}^*(s,v) = \exp\left[ -s\mu - v\mu_4 + \frac{1}{2} (s^2\sigma_3^2 + 2s\sigma_m\sigma_4 + v^2\sigma_4^2) \right], \]

\[ C_1 = \left[ \Phi(T; \sigma, \mu, \mu, \rho) \right]^{-1}, \]

and

\[ C_4 = \left[ I - \Phi(T; \sigma, \mu) - \Phi(U; \sigma, \mu) + \Phi(T, U; \sigma, \mu, \mu, \rho) \right]. \]

Substituting Eqs. (62) and (63) into Eq. (61), we can obtain the Laplace transform for the bivariate availability for ARPM policy with bivariate normals. The inverse of this bivariate Laplace transform is so complicated that new numerical methods are needed for the inverse transform and that even with new methods inversion will be computationally difficult.
9.3.3 Bivariate Availability for Bivariate Exponential ARPM Model II

Assume that \( X_n, R_n, \) and \( P_n \) are independent but with distinct distribution functions from the same family. \( X_n, R_n, \) and \( P_n \) are distributed as \( F_{T,U}(t,u), Gr_{T,U}(t,u), \) and \( Gp_{T,U}(t,u), \) respectively. \( F_{T,U}(t,u), Gr_{T,U}(t,u), \) and \( Gp_{T,U}(t,u) \) are Hunter’s bivariate exponential with joint density function, respectively:

\[
f_{T,U}(t,u) = \frac{\lambda_I}{1-\rho} I_0 \left( \frac{2\sqrt{\rho}}{1-\rho} \sqrt{\lambda_I \eta_I t u} \right) \exp \left\{ -\frac{\lambda_I + \eta_I u}{1-\rho} \right\}, \tag{9.64}
\]

\[
g_{r_{T,U}}(t,u) = \frac{\lambda_I}{1-\rho_r} I_0 \left( \frac{2\sqrt{\rho_r}}{1-\rho_r} \sqrt{\lambda_I \eta_I t u} \right) \exp \left\{ -\frac{\lambda_I t + \eta_I u}{1-\rho_r} \right\}, \tag{9.65}
\]

\[
g_{p_{T,U}}(t,u) = \frac{\lambda_I}{1-\rho_p} I_0 \left( \frac{2\sqrt{\rho_p}}{1-\rho_p} \sqrt{\lambda_I \eta_I t u} \right) \exp \left\{ -\frac{\lambda_I t + \eta_I t u}{1-\rho_p} \right\}, \tag{9.66}
\]

where \( I_n(\cdot) \) is the modified Bessel function of the first kind of order \( n, \) and \( \rho, \rho_r, \rho_p \) are positive.

From Chapter 8 we obtain the bivariate availability as

\[
A^*(s,v; T,U) = \frac{\bar{F}^*(s,v; T,U)}{1 - f_{ARPM}(s,v)}. \tag{9.67}
\]

From Chapter 7, we have
\[ f_{ARP M}^*(s,v) = f_{T,U}^*(s,v) g_{T,U}^*(s,v) C_1 + e^{-sT-vU} g_{T,U}^*(s,v) C_2 \] (9.68)

where

\[ f^*(s,v) = \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{v}{\eta} + 1 \right) - \frac{s v p}{\lambda \eta} \right]^{-1} \]

\[ g_{r}^*(s,v) = \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{v}{\eta} + 1 \right) - \frac{s v p}{\lambda \eta} \right]^{-1} \]

\[ g_{p}^*(s,v) = \left[ \left( \frac{s}{\lambda} + 1 \right) \left( \frac{v}{\eta} + 1 \right) - \frac{s v p}{\lambda \eta} \right]^{-1} \]

\[ C_1 = [F(T,U)]^{-1} \]

and

\[ C_2 = \bar{F}(T,U) \]

From Lemma 8.4.1, we have

\[ \bar{F}_{T,U}^*(s,v,T,U) = \frac{1}{s v} \left[ F(T,U) - f^*_T(s;T) - f^*_U(v;U) + f_{T,U}^*(s,v,T,U) \right] \] (9.69)

where

\[ f_{T,U}^*(s,v;T,U) = C_1 f_{T,U}^*(s,v) \]

\[ f^*_T(s;T) = C_3 f^*_T(s), \text{ and} \]

\[ f^*_U(v;U) = C_4 f^*_U(v) \]

with
Substituting Eqs. (68) and (69) into Eq. (67), we can obtain the Laplace transform for the bivariate availability for ARPM policy with bivariate normals. The inverse of this bivariate Laplace transform is so complicated that new numerical methods are needed for the inverse transform and that even with new methods inversion will be computationally difficult.
CHAPTER X

CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

10.1 Conclusions

This research presents a framework for the construction and development of bivariate failure models, bivariate renewal (corrective maintenance) models, bivariate preventive maintenance models, and bivariate availability models. The underlying theories are developed and the basic results are presented. The most significant observation is that this framework provides a new way to study the reliability of equipment for which univariate measures are inadequate and incomplete. Therefore, a new area of reliability research is identified.

10.1.1 Bivariate Failure Modeling

We present a taxonomy of bivariate model classes and identify two classes as our focus. The model classes examined here are those where the two variables are related by a stochastic function and those where the variables are simply correlated. Issues related to the construction of bivariate reliability models and their application to maintenance planning are discussed.

Examples of the models of each of the two classes, i.e., stochastic functions (functional dependence) and correlated functions (correlated dependence), are defined. The general approach to model formulation is explained so that alternate forms may be introduced.
10.1.2 Bivariate Renewal Modeling

(i) Basic Results for Bivariate Renewal Theory.

Extend and apply Hunter’s bivariate renewal theory to maintenance modeling. Some results are obtained for bivariate quasi-renewal process, delayed bivariate renewal process, and alternating bivariate renewal process.

(ii) Asymptotic Results for Bivariate Renewal Theory.

Some asymptotic results for ordinary and quasi-renewal processes are obtained. The bivariate renewal function and density are also obtained.

(iii) Bivariate Corrective Maintenance Models.

Use the results of bivariate renewal theory and apply bivariate failure models to construct bivariate corrective maintenance models. Bivariate failure models with correlated dependence are used because of their mathematical simplicity, but closed forms are not possible in most cases. Maintenance models based on stochastic dependence will require the use of numerical approaches.

10.1.3 Bivariate Preventive Maintenance Modeling

Bivariate Age Replacement Preventive Maintenance (ARPM) Models.

We extend the univariate age replacement preventive maintenance model (Murdock [1995]) by using the developed bivariate failure models and bivariate renewal theorems. The general results of the bivariate ARPM model is analogous to the univariate case. The Laplace transforms for ARPM are derived for example bivariate failure, repair, and replacement distributions.

10.1.4 Bivariate Availability Modeling

We define different bivariate availability measures. The bivariate availability models for corrective maintenance policies are developed. The bivariate availability models...
models for a bivariate age-replacement preventive maintenance policy are also developed. We obtain the Laplace transforms of bivariate availability for example bivariate distributions.

10.1.5 Quality of Availability Measures

We define the quality of availability measures as the conditional availability given a known value of the variable. Thus, for a fixed point of time, univariate availability is a conditional probability density function of usage. Equipment at the same age, under the same operating conditions with different usage, will have a different availability measure. This may be considered as the effects of wear processes. The quality of availability measures may be considered as a system effectiveness and/or performance measure.

10.2 Future Research Directions

10.2.1 Extensions of the Bivariate Failure Models

By analogy with the univariate case, we may consider the construction of bivariate failure models from their hazard functions. This is done by solving the differential equation of hazard and reliability, i.e.,

\[
\frac{\partial^2}{\partial t \partial u} \tilde{F}(t, u) = \tilde{F}(t, u) z(t, u). \tag{10.1}
\]

This approach requires techniques for solving a general second order partial differential equations. For more than three decades solving Eq. (10.1) has been and is still an open problem (see Gindikin and Volevich [1996]).
10.2.2 Extensions of the Bivariate Renewal Models

Generalize the bivariate renewal models to deal with more general and practical maintenance problems. For example, consider multi-unit systems with different system structures (parallel or series). For the 2-unit system case, there are three renewal processes, i.e., a renewal process for the system and renewal processes for the components. More general results of bivariate renewal theory may be obtained. Extension to multi-dimensional renewal theory is also possible. The results of multivariate renewal theory may be applied to multi-unit systems.

Another direction is to generalized bivariate renewal theory to develop bivariate regenerative processes or planar point processes.

10.2.3 Extensions of the Bivariate Maintenance Models

Construct maintenance models based on the bivariate failure models with a stochastic dependence relation between time and usage variables. This approach to the construction of bivariate maintenance models will require the use of numerical methods.

Modify and extend the bivariate maintenance models to analyze the effects of other preventive maintenance policies, e.g., block replacement, opportunistic replacement, group replacement, and combinations of different replacement policies. Generalize the assumptions of i.i.d. lifetime and maintenance times. That is, we construct bivariate preventive maintenance models with distinct lifetimes, corrective maintenance times, and preventive maintenance times.

Construct age replacement preventive maintenance models by using different bivariate failure models other than the bivariate correlated models that we have used (i.e., the bivariate exponential of Baggs and Nagagaja [1996], the bivariate exponential of Hunter [1974], and the bivariate normal).
10.2.4 Bivariate Availability Models for Other Preventive Maintenance Models

Develop availability models for block, opportunistic, and group replacement preventive maintenance models. Availability models for combinations of different maintenance policies may also be considered.

10.2.5 Extension to Non-homogeneous (and/or non-renewal) Processes

In the construction of bivariate CM, PM, and availability models, one may consider using non-homogeneous and/or non-renewal processes to extend the bivariate renewal models.

10.2.6 Optimal Bivariate Preventive Maintenance Policy

Obtain the optimal bivariate preventive maintenance policy, e.g., optimal age replacement policy \((T^*, U^*)\), based on:

(i) the expected total maintenance cost \((TMC)\) per unit time,
(ii) the bivariate availability measure,
(iii) the expected \(TMC\) per unit time subject to the restriction of availability requirements.

10.2.7 Comparisons of Different Preventive Maintenance Models

Model different preventive maintenance policies and obtain the comparison results of different policies. Possible combinations of preventive maintenance policies are:

(i) age-block replacement policy,
(ii) age-opportunistic replacement policy,
(iii) age-group replacement policy, and
(iv) block-opportunistic replacement policy,
(v) block-group replacement policy, and
(vi) group-opportunistic replacement policy.

Optimal combination(s) of preventive maintenance policies may be accomplished through the utilization of the bivariate availability models.

10.2.8 Comparisons of Existing Univariate Models and Bivariate Models

Define a general mapping from the bivariate model space to a univariate time model so that the bivariate model can be compared to previous models.

10.2.9 Numerical Methods and Algorithms for Bivariate Laplace Inverse Transforms

The analysis and evaluation of bivariate models require the extension of the existing numerical methods and algorithms for bivariate functions. The most important one is to develop new numerical methods and algorithms that are effective and efficient in inverting bivariate Laplace inverse transforms. For example, Hwang and Lu [1999] presented numerical inversion of bivariate Laplace transforms by fast Hartley transform (FHT) computations. Computer packages for bivariate FHT computations may be developed for solving bivariate Laplace inverse transform problems. This will allow us to model and analyze bivariate maintenance policies and availability with more complicated bivariate failure models.

10.2.10 Simulation

Simulation models may be developed as a tool to facilitate the analysis and evaluation of the developed bivariate models. The advantage of doing this is to eliminate the difficulties involving the direct analysis of the bivariate models. Thus, bivariate simulation models may be used as a pedagogical device to reinforce and verify analytic solutions.