Reduced-Order Models for the Prediction of Unsteady Heat Release in Acoustically Forced Combustion

PhD Defense

Christopher R. Martin

Department of Mechanical Engineering
Virginia Polytechnic Institute & Statue University

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Motivation: Thermo-Acoustic Instabilities

- Blowoff
- Flashback
- Reduced efficiency
- Premature degradation of hardware
- Elevated emissions
Anatomy of an instability

- **Flame (Φ)**
- **Flame (velocity)**
- **Acoustics (flame)**
- **Acoustics (upstream)**

Inputs:
- $\Phi'$
- $u'_{inj}$
- $u'_{dump}$

Outputs:
- $\dot{q}$
Anatomy of a Flame

Inclined flames

- Weak resonance with delay
- Scales with $Str = \frac{L_f}{U}$

Flat flames

- $2^n d$-order resonance
- Damping and stability vary

Motivation

Background

Approach
Approach

Part I: Highly Turbulent Combustion

► Derive various formulations for a dynamic WSR
► Look for scaling that agrees with reality

Part II: Laminar Combustion

► Experimentally investigate a laminar flat flame
► Use an FE model to study the flame numerically
► Derive an analytical model to search for scaling quantities
► Search for insights into the turbulent case

Part III: Moderately Turbulent Combustion

► Use Reynolds-averaging to simplify a 1-D model
► Look for scaling that agrees with reality
Part I: Highly Turbulent Combustion

Formulation

Limiting Assumptions
Governing Equations

Three WSR Models

Simple WSR
Non-Simple Enthalpy WSR
Multi-Step Chemical Kinetic WSR

Summary
Limiting Assumptions

Highly turbulent implies very fast mixing.

- Spatial gradients are negligible in the reactor
- Incoming reactants are mixed instantly
- Leaving products are identical to the reactor contents
- The volume of the reactor is constant

Thus,

\[ Da_t = \frac{t_t}{t_r} < 1 \]
Governing Equations

\[ \dot{Y}_i + t_m^{-1} \left( Y_i - Y_{i,in} \right) = \zeta_i \]

\[ C_p \dot{T} + t_m^{-1} \sum_i Y_{i,in} \left( h_i - h_{i,in} \right) = - \sum_i h_i \zeta_i \]
Three WSR Models: Simple, Non-Simple Enthalpy, Multi-Step Kinetic
Simple WSR

Single-step reaction, Constant & equal $C_p$,

$$T = \Delta Tc + T_0, \quad Y_i = \Delta Y_i c + Y_{i,0}$$

$$c' + (1 + \epsilon u)c = Da R(c)$$

$$R(c) = A \exp \left( -\frac{T_a}{T} \right) (1 - c)$$

$$c_0 = Da R(c_0) \quad \quad c'_1 + (1 - Da R')c_1 = -u c_0$$
Simple WSR

Single-step reaction, Constant & equal $C_p$,

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Simple WSR

Single-step reaction, Constant & equal $C_p$,

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T = \Delta Tc + T_0, \quad Y_i = \Delta Y_i c + Y_{i,0}
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\]

\[
c_0 = Da R(c_0) \quad \quad c'_1 + (1 - Da R')c_1 = -u c_0
\]
Simple WSR Solution

Formulation

Assumptions
Equations

WSR Models
Simple WSR
NSH WSR
MS WSR

Summary
Non-Simple Enthalpy WSR

Single-step reaction

\[ T = \Delta T \tau + T_0, \quad Y_i = \Delta Y_i c + Y_{i,0} \]

\[ h(c, \tau) = (1 - c) h_r(\tau) + c h_p(\tau) \]

\[ \dot{m} = \dot{m}_0 (1 + \epsilon u(t)), \quad c_{in} = 0 + \epsilon v(t) \]
Non-Simple Enthalpy WSR

Single-step reaction

\[ T = \Delta T \tau + T_0, \quad Y_i = \Delta Y_i c + Y_{i,0} \]

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Non-Simple Enthalpy WSR

Single-step reaction

\[ T = \Delta T \tau + T_0, \quad Y_i = \Delta Y_i c + Y_{i,0} \]

\[ h(c, \tau) = (1 - c) h_r(\tau) + c h_p(\tau) \]

\[ \dot{m} = \dot{m}_0(1 + \epsilon u(t)), \quad c_{in} = 0 + \epsilon v(t) \]

\[ (C_p \Delta T) \tau' + (h_r(\tau) - h_r(0)) = \Delta h(\tau) Da \cdot \hat{R}(\tau, c) \]

\[ c' + c = Da \cdot \hat{R}(\tau, c). \]
Non-Simple Enthalpy WSR

Single-step reaction

\[ T = \Delta T \tau + T_0, \quad Y_i = \Delta Y_i c + Y_{i,0} \]

\[ h(c, \tau) = (1 - c) h_r(\tau) + c h_p(\tau) \]

\[ \dot{m} = \dot{m}_0 (1 + \epsilon u(t)), \quad c_{in} = 0 + \epsilon v(t) \]

\[ \tau_1' + \left[ 1 - \beta Da \cdot \hat{R}_\tau \right] \tau_1 + \left[ -\beta Da \cdot \hat{R}_c \right] c_1 = -u \beta c_0 \]

\[ + (\beta - \beta_0) v \]

\[ c_1' + \left[ -Da \cdot \hat{R}_\tau \right] \tau_1 + \left[ 1 - Da \cdot \hat{R}_c \right] c_1 = -uc_0 + v \]
Non-Simple Enthalpy WSR

Single-step reaction

\[ T = \Delta T \tau + T_0, \quad Y_i = \Delta Y_i c + Y_{i,0} \]

\[ h(c, \tau) = (1 - c) h_r(\tau) + c h_p(\tau) \]

\[ \dot{m} = \dot{m}_0 (1 + \epsilon u(t)), \quad c_{in} = 0 + \epsilon v(t) \]

\[ \hat{c} = \frac{C_{p,r}(\tau) - C_{p,p}(\tau)}{C_p(\tau, c)} \]

\[ \beta(\tau_0, c_0) = \frac{h_r(\tau_0) - h_p(\tau_0)}{C_p(\tau_0, c_0) \Delta T} \]
Non-Simple Enthalpy Solution

\[ \hat{c} = \frac{C_{p,r}(\tau) - C_{p,p}(\tau)}{C_p(\tau, c)} \]

\[ \beta(\tau_0, c_0) = \frac{h_r(\tau_0) - h_p(\tau_0)}{C_p(\tau_0, c_0) \Delta T} \]
Multi-Step Chemical Kinetic WSR

Constant & equal $C_p$

\[
X_0 = \begin{bmatrix} T_0 \\ Y_0 \end{bmatrix} \quad X_1 = \begin{bmatrix} T_1 \\ Y_1 \end{bmatrix} \quad q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}
\]

\[
X_0 - X_{0,\text{inlet}} = t_m A_1 \cdot q
\]

\[
\dot{X}_1 + (t_m^{-1} I - A_1 \cdot J) \cdot X_1 = B \cdot U.
\]

\[
q_1 = J \cdot X_1
\]

\[
\dot{q}_1 = (-t_m^{-1} I + J \cdot A_1) \cdot q_1 + J \cdot B \cdot U
\]
Multi-Step Chemical Kinetic WSR

Constant & equal $C_p$

\[
\begin{align*}
X_0 &= \left\{ \begin{array}{c}
T_0 \\
Y_0 
\end{array} \right\} & X_1 &= \left\{ \begin{array}{c}
T_1 \\
Y_1 
\end{array} \right\} & q &= \left\{ \begin{array}{c}
q_1 \\
\vdots \\
q_n 
\end{array} \right\} \\

X_0 - X_{0,\text{inlet}} &= t_m A_1 \cdot q \\
\dot{X}_1 + (t_m^{-1} I - A_1 \cdot J) \cdot X_1 &= B \cdot U. \\

q_1 &= J \cdot X_1 \\
\dot{q}_1 &= (-t_m^{-1} I + J \cdot A_1) \cdot q_1 + J \cdot B \cdot U
\end{align*}
\]
Multi-Step Chemical Kinetic WSR

Constant & equal $C_p$

$x_0 = \begin{bmatrix} T_0 \\ Y_0 \end{bmatrix}$ \hspace{1cm} $x_1 = \begin{bmatrix} T_1 \\ Y_1 \end{bmatrix}$ \hspace{1cm} $q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$

$x_0 - x_{0,\text{inlet}} = t_m A_1 \cdot q$

$\dot{x}_1 + (t_m^{-1} I - A_1 \cdot J) \cdot x_1 = B \cdot U.$

$q_1 = J \cdot x_1$

$\dot{q}_1 = (-t_m^{-1} I + J \cdot A_1) \cdot q_1 + J \cdot B \cdot U$
Multi-Step Chemical Kinetic WSR

Constant & equal $C_p$

\[ \begin{align*}
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\[ \mathbf{x}_0 - \mathbf{x}_{0,inlet} = t_m A_1 \cdot \mathbf{q} \]

\[ \dot{\mathbf{x}}_1 + (t_m^{-1} \mathbf{I} - A_1 \cdot \mathbf{J}) \cdot \mathbf{x}_1 = \mathbf{B} \cdot \mathbf{U}. \]

\[ q_1 = \mathbf{J} \cdot \mathbf{x}_1 \]

\[ \dot{q}_1 = (-t_m^{-1} \mathbf{I} + \mathbf{J} \cdot A_1) \cdot q_1 + \mathbf{J} \cdot \mathbf{B} \cdot \mathbf{U} \]
Multi-Step Results

Velocity Response

Equivalence Ratio Response
Summary of WSR Models

- **Simple WSR**
  \[
  \omega_c = t^{-1}_m \left[ 1 - Da \cdot \dot{R}'(c_0) \right] \\
  \approx -t^{-1}_r R'(c_0)
  \]

- **Non-Simple Enthalpy WSR**
  \[
  \omega_c = t^{-1}_m \left[ 1 - Da \cdot \dot{R}'(c_0) \right] \\
  \approx -t^{-1}_r R'(c_0)
  \]

- **Multi-Step Chemical Kinetic WSR**
  Multiple frequencies scaling with the various reaction times.
Part II: Laminar Combustion

Formulation

Finite Element Model

Analytical Model

Experimental Investigation

Comparison
Geometry & Limiting Assumptions

- 1-D symmetrical flow
- Single-step chemical kinetics
- *Non-unity Lewis number*

\[ T = \Delta T \tau + T_0 \quad Y_i = \Delta Y_i c + Y_{i,0} \]

- Neglect density changes in space

\[ \nabla \cdot u \approx 0 \]

- Typical non-dimensionalization

\[ \hat{t} = \frac{t}{t_r} \quad \hat{x} = \frac{x}{\delta} \quad \hat{u} = \frac{u}{S_L} \]

\[ \delta \equiv \sqrt{\alpha t_r} \quad S_L \equiv \sqrt{\frac{\alpha}{t_r}} \]
Geometry & Limiting Assumptions

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RO-Models for Acoustically Forced Flames

C. Martin

Formulation
Assumptions
Equations

Finite Element
Numerics
Static
Dynamic

Analytical
Simplifications
Solution
Analysis of Results

Experimental
Setup
Position
TF Measurements

Comparison
Static
Transfer Function
Frequency Scale

Geometry & Limiting Assumptions

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\[ \delta \equiv \sqrt{\alpha t_r} \quad S_L \equiv \sqrt{\frac{\alpha}{t_r}} \]
**Governing Equations**

\[
\dot{c} + \hat{u} c' - \frac{1}{Le} c'' = R(c, \tau)
\]
\[
\dot{\tau} + \hat{u} \tau' - \tau'' = R(c, \tau)
\]

\[
\hat{u} = \hat{U} + \epsilon \exp(j\Omega \hat{t})
\]

**Static**
\[
\hat{U} c_0' - \frac{1}{Le} c_0'' = R(c_0, \tau_0)
\]
\[
\hat{U} \tau_0' - \tau_0'' = R(c_0, \tau_0)
\]

**Dynamic**
\[
(j\Omega - R_c) c_1 + \hat{U} c_1' - \frac{1}{Le} c_1'' = R_\tau \tau_1 - c_0'
\]
\[
(j\Omega - R_\tau) \tau_1 + \hat{U} \tau_1' - \tau_1'' = R_c c_1 - \tau_0'
\]
Governing Equations

\[
\dot{c} + \hat{u} c' - \frac{1}{Le} c'' = R(c, \tau)
\]

\[
\dot{\tau} + \hat{u} \tau' - \tau'' = R(c, \tau)
\]

\[
\hat{u} = \hat{U} + \epsilon \exp(j\Omega \hat{t})
\]

Static

\[
\hat{U} c' - \frac{1}{Le} c'' = R(c_0, \tau_0)
\]

\[
\hat{U} \tau' - \tau'' = R(c_0, \tau_0)
\]

Dynamic

\[
(j\Omega - R_c) c_1 + \hat{U} c' - \frac{1}{Le} c'' = R_\tau \tau_1 - c_0
\]

\[
(j\Omega - R_\tau) \tau_1 + \hat{U} \tau' - \tau'' = R_c c_1 - \tau_0
\]
**Governing Equations**

\[
\dot{c} + \hat{u} c' - \frac{1}{Le} c'' = R(c, \tau)
\]
\[
\dot{\tau} + \hat{u} \tau' - \tau'' = R(c, \tau)
\]

\[
\hat{u} = \hat{U} + \epsilon \exp(j\Omega t)
\]

**Static**

\[
\hat{U} c'_0 - \frac{1}{Le} c''_0 = R(c_0, \tau_0)
\]
\[
\hat{U} \tau'_0 - \tau''_0 = R(c_0, \tau_0)
\]

**Dynamic**

\[
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\[
(j\Omega - R_{\tau}) \tau_1 + \hat{U} \tau'_1 - \tau''_1 = R_c c_1 - \tau'_0
\]
Governing Equations

\[
\dot{c} + \hat{u} c' - \frac{1}{Le} c'' = R(c, \tau)
\]

\[
\dot{\tau} + \hat{u} \tau' - \tau'' = R(c, \tau)
\]

\[\hat{u} = \hat{U} + \epsilon \exp(j\Omega \hat{t})\]

Static

\[
\hat{U} c'_0 - \frac{1}{Le} c''_0 = R(c_0, \tau_0)
\]

\[
\hat{U} \tau'_0 - \tau''_0 = R(c_0, \tau_0)
\]

Dynamic

\[
(j\Omega - R_c) c_1 + \hat{U} c'_1 - \frac{1}{Le} c''_1 = R_\tau \tau_1 - c'_0
\]

\[
(j\Omega - R_\tau) \tau_1 + \hat{U} \tau'_1 - \tau''_1 = R_c c_1 - \tau'_0
\]
Finite Element Model
Defining the Element

Ahrrenius reaction rate
- Constant choice of $t_r$ so that $R \approx O(1)$
- Linear interpolation functions
- Galerkin method of weighted residuals
- One-dimensional
A Brief Look at the Solver

Boundary Conditions
Initial Guess
System Parameters
Grid

Configuration Function

Element Function

Convergence Test Function

Solution

Solution: Node Values, Natural Boundary Values

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Solution

Solution: Node Values, Natural Boundary Values
Static Solution

$U = 0.7, \ Le = 2.0, \ 150$-point grid
Static Results

- Both estimates break down near $Le = 1$. 

Graphs showing various data points and curves for different $Le$ values.
Dynamic Solution

Dynamic Response

$|c_1|$, $|\tau_1|$

- Reaction rate
- Species
- Temperature

$U = 0.7$, $Le = 2.0$, $\Omega = 0.01$, 150-point grid
Dynamic Solution

\[ |c_1| \text{ and } |\tau_1| \]

\[ \text{phase (deg)} \]

\[ x / \delta \]

\[ U = 0.7, \; Le = 2.0, \; \Omega = 1.0, \; 150\text{-point grid} \]
**Internal Structure**

Heat release contribution from $c$ and $\tau$ oscillations

$$U = 0.8, \ Le = 2.0, \ (left) \ \Omega = 0.02, \ (right) \ \Omega = 10.0$$
Internal Structure

Front-half, back-half, and total heat release dynamics

\[ U = 0.8, \ Le = 2.0 \]
Dynamic Response

Heat release contribution from $c$ and $\tau$ oscillations

\[ Le = 2 \]
Dynamic Response

Complex root motion w.r.t. $\hat{U}$ for various $Le$

(left) $Le = 2$ only, (right) all $Le$
Natural frequency as a function of standoff distance

\[ \Omega \mu L^{-3} \]
Analytical Model
Ignition Reaction Model

Ahrrenius: \[ R = A \exp \left( -\frac{\tau_a}{\tau_0 + \tau} \right) (1 - c) \]

Ignition: \[ R = Ah (\tau - \tau_{ig}) (1 - c) \]

Ignition Advantages

- A jump condition is simpler than an exponential
- The steady solution is still $C^1$ continuous
- Already proven to give good results for $Le = 1$
Ignition Reaction Model

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Ignition Advantages

- A jump condition is simpler than an exponential
- The steady solution is still \( C^1 \) continuous
- Already proven to give good results for \( Le = 1 \)
Static Solution

Preheat

\[ 0 = Uc_0' - \frac{1}{Le} c_0'' \]
\[ 0 = U\tau_0' - \tau_0'' \]

Flame

\[ Uc_0' - \frac{1}{Le} c_0'' = A - Ac_0 \]
\[ U\tau_0' - \tau_0'' = A - Ac_0 \]
\[ \frac{c_0}{c_L} = \frac{\exp(U Le x) - 1}{\exp(U Le L) - 1} \]
\[ \frac{\tau_0}{\tau_{ig}} = \frac{\exp(U x) - 1}{\exp(U L) - 1} \]
\[ c_0 = 1 - \Delta c \exp(\Lambda(x - L)) \]
\[ \tau_0 = \tau_{ig} + \Delta \tau [1 - \exp(\Lambda(x - L))] \]
Static Solution

Preheat

\[ 0 = Uc_0' - \frac{1}{Le} c_0'' \]
\[ 0 = U\tau_0' - \tau_0'' \]

Flame

\[ Uc_0' - \frac{1}{Le} c_0'' = A - Ac_0 \]
\[ U\tau_0' - \tau_0'' = A - Ac_0 \]
\[ \frac{c_0}{c_L} = \frac{\exp(U Le x) - 1}{\exp(U Le L) - 1} \]
\[ \frac{\tau_0}{\tau_{ig}} = \frac{\exp(U x) - 1}{\exp(U L) - 1} \]
\[ c_0 = 1 - \Delta c \exp(\Lambda(x - L)) \]
\[ \tau_0 = \tau_{ig} + \Delta \tau [1 - \exp(\Lambda(x - L))] \]
**Static Solution**

**Preheat**

\[
0 = Uc_0' - \frac{1}{Le} c_0''
\]
\[
0 = U\tau_0' - \tau_0''
\]

**Flame**

\[
Uc_0' - \frac{1}{Le} c_0'' = A - Ac_0
\]
\[
U\tau_0' - \tau_0'' = A - Ac_0
\]
\[
c_0 = \exp (U Le x) - 1
\]
\[
c_L = \exp (U Le L) - 1
\]
\[
\tau_0 = \exp (U x) - 1
\]
\[
\tau_{ig} = \exp (U L) - 1
\]
Static Solution

Preheat

\[ 0 = Uc'_0 - \frac{1}{Le} c''_0 \]
\[ 0 = U\tau'_0 - \tau''_0 \]

Flame

\[ Uc'_0 - \frac{1}{Le} c''_0 = A - Ac_0 \]
\[ U\tau'_0 - \tau''_0 = A - Ac_0 \]

\[ c_0 = 1 - \Delta c \exp (\Lambda(x - L)) \]
\[ \tau_0 = \tau_{ig} + \Delta \tau [1 - \exp (\Lambda(x - L))] \]

\[ \Lambda = \frac{UL}{2} \left(1 - \sqrt{1 + \frac{4A}{U^2Le}}\right) \]
Static Solution

Preheat

\[ 0 = Uc'_0 - \frac{1}{Le} c''_0 \]
\[ 0 = U\tau'_0 - \tau''_0 \]

\[ \frac{c_0}{c_L} = \frac{\exp (U Le x) - 1}{\exp (U Le L) - 1} \]
\[ \frac{\tau_0}{\tau_{ig}} = \frac{\exp (U x) - 1}{\exp (U L) - 1} \]

Flame

\[ Uc'_0 - \frac{1}{Le} c''_0 = A - Ac_0 \]
\[ U\tau'_0 - \tau''_0 = A - Ac_0 \]

\[ c_0 = 1 - \Delta c \exp (\Lambda (x - L)) \]
\[ \tau_0 = \tau_{ig} + \Delta \tau [1 - \exp (\Lambda (x - L))] \]
Interfacial Matching

The offset distance, \( L \), and \( \Delta c \) are still unknown.

\[
(1 - \Delta c) U Le n(U Le L) = -\Delta c \Lambda \\
\tau_{ig} U n(U L) = -\beta \Delta c \Lambda
\]
Interfacial Matching

Alas! No exact solution.

- **High $Le$**

\[
\Delta c = \frac{-U \Lambda}{A}.
\]

\[
L = \frac{1}{U} \ln \left( \frac{1}{1 + \frac{U \tau_{ig}}{\Lambda \beta \Delta c}} \right)
\]

- **Low $Le$**

\[
\Delta c = 1 - \frac{\tau_{ig}}{\beta}
\]

\[
L = -\frac{1}{\Lambda} \left( \frac{\tau_{ig}/\beta}{1 - \tau_{ig}/\beta} \right)
\]
Interfacial Matching

When $U \rightarrow 1$, $L \rightarrow \infty$. The system of equations reduces precisely

$$A = \frac{Le}{4} \left[ \left( 1 + \frac{2a}{Le} \right)^2 - 1 \right]$$

$$a = \frac{\tau_{ig}}{1 - \tau_{ig}}$$
Static Results

![Graph showing static results comparison between Exact, High Le Estimate, Low Le Estimate, and Comparison Static and Dynamic results.](image)

- **Exact**
- **High Le Estimate**
- **Low Le Estimate**

**Axes:**
- **Le** (on the left graph)
- **U** (on the right graph)

**Legend:**
- Static
- Dynamic

**Graph Details:**
- Le ranges from 1 to 5.
- U ranges from 0 to 1.
- Δc values indicated on the graphs.

**Comparison Notes:**
- High Le Estimate
- Low Le Estimate
- Exact
- Static
- Dynamic
Static Results

- Exact
- High Le Estimate
- Low Le Estimate

Le = 1.0

$U$ vs. $T_F$

Low Le Estimate
High Le Estimate
Exact
**Dynamic Solution**

\[
(j\Omega - R_c) c_1 + \hat{U} c'_1 - \frac{1}{Le} c''_1 = R_\tau \tau_1 - c'_0
\]

\[
(j\Omega - R_\tau) \tau_1 + \hat{U} \tau'_1 - \tau''_1 = R_c c_1 - \tau'_0
\]

**Preheat**

\[
c_1 = p^+ \exp\left(k^+ (x - L)\right) + p^- \exp\left(k^- (x - L)\right) - \frac{c'_0}{j\Omega}
\]

\[
\tau_1 = b^+ \exp\left(d^+ (x - L)\right) + b^- \exp\left(d^- (x - L)\right) - \frac{\tau'_0}{j\Omega}
\]

**Flame**

\[
c_1 = P \exp\left(K (x - L)\right) - \frac{c'_0}{j\Omega}
\]

\[
\tau_1 = B \exp\left(D (x - L)\right) + PG \exp\left(K (x - L)\right) - \frac{\tau'_0}{j\Omega}
\]
Dynamic Solution

\[
\mathbf{M}(\Omega) \cdot \begin{Bmatrix} p^+ \\ p^- \\ P \\ b^+ \\ b^- \\ B \end{Bmatrix} = \begin{Bmatrix} c_0'(0) / j\Omega \\ \tau_0'(0) / j\Omega \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
\]
Dynamic Results

\[ Le = 2, \ \Omega = 0.1, \text{ (left) species (right) temperature} \]
Dynamic Results

\[ Le = 2 \]
Scaling
Eigen Values

Consider the homogeneous case:

\[
\begin{pmatrix}
\rho^+ \\
\rho^- \\
P \\
b^+ \\
b^- \\
B
\end{pmatrix}
\begin{pmatrix}
M(\lambda)
\end{pmatrix}
\Rightarrow
\det(M(\lambda)) = 0
\]
Eigen Values

\[
\log_{10} \left( \det (M) \det (M)^* \right) \text{ as a function of } \lambda \text{ in complex space}
\]
Experimental Investigation
Experimental Setup
Velocity Profile Measurements

- Approximately 5mm above the honeycomb
- The honeycomb is 7cm in diameter
- Mean velocities are 4cm/s, 6cm/s, and 8cm/s
Heat Release Rate

Parabolic mirror

Hg Vap. Lamp

Mirror

Parabolic mirror

Mirror

Screen

~10x magnification
Heat Release Rate

\[ L \approx C \left( \frac{U_c}{U_L} \right)^{-1.15} \left( 1 - \frac{U_c}{U_L} \right)^{-0.34} \]
Heat Release Rate

- The optics are focused to roughly a 4 cm diameter via the He-Ne laser.
- The fiber is connected to a spectrometer at 308 nm (OH*)
Transfer Function Measurements

\[ |Q_1| \]

\[ \angle Q_1 \]

\[ \gamma \]
Scaling

- RO-Models for Acoustically Forced Flames
- C. Martin

Formulation
- Assumptions
- Equations

Finite Element
- Numerics
- Static
- Dynamic

Analytical
- Simplifications
- Solution
- Analysis of Results

Experimental
- Setup
- Position
- TF Measurements

Comparison
- Static
- Transfer Function
- Frequency Scale
Comparison of Laminar Models and Data
Static Comparison

![Graph showing static comparison between experimental data, analytical model, and finite element model.](image)
Transfer Function Comparison
Frequency Comparison

- FE velocity is scaled by the blowoff velocity
- Experimental is scaled by methane flame diffusivity and reaction time
- Velocity disagreement is due to preheating from the ceramic
Preheating from the ceramic also explains the model’s instability.
Part III: Moderately Turbulent Combustion

Formulation
  Governing Equations
  Turbulent Ignition Model

Analytical Solution
  Static Solution
  Dynamic Solution

Results and Analysis
  Frequency Response
  Frequency Scaling
Governing Equations

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \nabla \cdot (\alpha \nabla T) = -\frac{\Delta T}{Y_{f,0}} \zeta_f
\]

\[
\frac{\partial Y_i}{\partial t} + \mathbf{u} \cdot \nabla Y_i - \nabla \cdot (D_i \nabla Y_i) = \zeta_i
\]

- Reynolds averaging eliminates uncorrelated oscillations
- Assume turbulence dominates molecular and thermal diffusion
Governing Equations

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \nabla \cdot (\alpha \nabla T) = -\frac{\Delta T}{Y_{f,0}} \zeta_f
\]

\[
\frac{\partial Y_i}{\partial t} + \mathbf{u} \cdot \nabla Y_i - \nabla \cdot (D_i \nabla Y_i) = \zeta_i
\]

- Reynolds averaging eliminates uncorrelated oscillations
- Assume turbulence dominates molecular and thermal diffusion

\[
\frac{\partial \langle T \rangle}{\partial t} + (U + v) \frac{\partial \langle T \rangle}{\partial x} - D_T \frac{\partial^2 \langle T \rangle}{\partial x^2} = -\frac{\Delta T}{Y_{f,0}} \langle \zeta_f \rangle
\]

\[
\frac{\partial \langle Y_i \rangle}{\partial t} + (U + v) \frac{\partial \langle Y_i \rangle}{\partial x} - D_T \frac{\partial^2 \langle Y_i \rangle}{\partial x^2} = \langle \zeta_i \rangle
\]
Governing Equations

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \nabla \cdot (\alpha \nabla T) = -\frac{\Delta T}{Y_{f,0}} \zeta_f \]

\[ \frac{\partial Y_i}{\partial t} + \mathbf{u} \cdot \nabla Y_i - \nabla \cdot (D_i \nabla Y_i) = \zeta_i \]

- Reynolds averaging eliminates uncorrelated oscillations
- Assume turbulence dominates molecular and thermal diffusion

\[ \dot{c} + (\hat{U} + \hat{v})c' - c'' = R(c) \]
Turbulent Ignition Model

\[ R(c) = A h(c - c_{ig}) h(1 - c) \]

- “Switches on” at \( c_{ig} \) and “switches off” when the fuel is exhausted
- Obeys Ranalli’s observation that heat release is proportional to volume
- Analytically simple
Analytical Solution
Static Solution

Preheat

\[ \hat{U}c_0' - c_0'' = 0 \]

Flame

\[ \hat{U}c_0' - c_0'' = A \]

Equilibrium

\[ \hat{U}c_0' - c_0'' = 0 \]
RO-Models for Acoustically Forced Flames

C. Martin

Formulation

Equations

Ignition Model

Analytical Solution

Static

Dynamic

Results

FRF

Scaling

Static Solution

Preheat

\[ c_0^{(p)} = c_{ig} \frac{\exp (\hat{U} x) - 1}{\exp (\hat{U} L_f) - 1} \]

Flame

\[ c_0^{(f)} = 1 - \frac{A}{U^2} [\exp (U(\hat{x} - L_e)) - 1] + \frac{A}{U}(\hat{x} - L_e) \]

Equilibrium

\[ c_0^{(e)} = 1 \]
Static Solution

- Interface matching applied just like with the PFR
- Yields leading and trailing edge positions
- The reaction coefficient is solved similarly (iteratively)
Static Solution

- Interface matching applied just like with the PFR
- Yields leading and trailing edge positions
- The reaction coefficient is solved similarly (iteratively)
Static Results

(Left) Flame leading and trailing edges  (right) flame thickness
Dynamic Solution

\[ j\omega c_1 + \hat{U}c'_1 - c''_1 = -c'_0 \]

- All three regions have the same governing equation
- The jump conditions and the particular solution shape the response

\[ c^{(p)}_1 = B^{(p)+} \exp (k^+ \hat{x}) + B^{(p)-} \exp (k^- \hat{x}) - \frac{c'_0}{j\omega} \]

\[ c^{(f)}_1 = B^{(f)+} \exp (k^+ (\hat{x} - L_e)) + B^{(f)-} \exp (k^- (\hat{x} - L_e)) - \frac{c'_0}{j\omega} \]

\[ c^{(e)}_1 = 0 \]
Interfacial Matching

\[ P(\Omega) \cdot M(\Omega)^{-1} = \left\{ \begin{array}{c} B^{(p)+} \\ B^{(p)-} \\ B^{(f)+} \\ B^{(f)-} \end{array} \right\} \]

- Jump conditions are accounted for in \( M \)
- Particular solution is accounted for in \( P \)
Dyamic Solution

- Discontinuities at $L_f$ and $L_e$
- Phase is almost constant
- Unsteady heat release is determined entirely at $L_f$ and $L_e$
Results and Analysis
Frequency Response

\[ |Q_1| \quad c_{ig} = 0.2 \]

\[ \angle Q_1 \quad 0.7 \]

\[ \text{delay} \quad 0.2 \]

\[ \Omega \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]
Eigen Values

\[ \text{det}(M(\lambda)) = 0 \]

\[ J \sinh(\beta L) \sinh(\beta L_f) + \beta \sinh(\beta L_e) = 0 \]

\[ \beta = \sqrt{\frac{\hat{U}^2}{4} + \lambda} \quad J = \frac{-\hat{U}}{1 - \exp(-\hat{U}L)} \]
Eigen Values

- Eigen values scale crudely as a power of $L_c$
  \[ \lambda_1 \propto L_c^{-3.2} \]

- Delay is determined entirely by pole locations
  \[ t_\infty = \sum_{n=1}^\infty \frac{1}{\lambda_n} \]
Unifying the Model Results
The Bhorgi Diagram

\[ Da = \frac{t_L}{t_r} = \left( \frac{L}{\delta} \right) \left( \frac{u_L}{S_L} \right)^{-1} \]

\[ Ka = \frac{t_r}{t_\eta} = \frac{\delta}{S_L} \frac{L}{u_L} Re^{1/2} \]

\[ = \left( \frac{u_L}{S_L} \right)^{3/2} \left( \frac{L}{\delta} \right)^{-1/2} \]
Highly Turbulent Combustion

Scales with the reaction time

\[ \Omega_c = \omega_c \, t_r \approx 1 \]
Moderately Turbulent Combustion

- Flat flames scale as a power law with distance
  \[ \Omega_c \propto \left( \frac{L_c}{\delta_t} \right)^{-m} = Da^{-m/2} \]

- Inclined flames scale with the Strouhal number
  \[ \Omega_c \propto \left( \frac{U}{S_L} \right) \left( \frac{L}{\delta} \right)^{-1} = Da^{-1} \]
Laminar Combustion

- Flat laminar flames scale as a power law with displacement

\[ \Omega_c \propto L^{-m} \]

- Inclined laminar flames scale with the Strouhal number

\[ \Omega_c \propto \frac{U}{L} \]

- Turbulent structures die off and do not play a major role in the laminar regime
The Extended Bhorgi Diagram

\[ \Omega = \omega S_L / \delta \]

- \( \text{WELL STIRRED} \)
- \( \text{DISTRIBUTED} \)
- \( \text{FLAMELET} \)

\[ \begin{align*}
\text{Re} &= 1 \\
\text{Da} &= 1 \\
\text{Ka} &= 1 \\
U &= S_L
\end{align*} \]
The Extended Bhorgi Diagram

\[ \Omega = \omega \delta / S_L \]

- WELL STIRRED
- DISTRIBUTED & FLAMELET

\[ Da = \frac{1}{m} \]
\[ m = -3 \]

Unification
Bhorgi
Overview
Extended Bhorgi
Future Work
Future Work
Additional Laminar Work

1. Experimental laminar transition study:
   Study the transition from flat, burner-stabilized flames to inclined flames.

2. Unify flamelet and laminar models:
   Write a 2-D model valid during the transition from flat to inclined flame.
3. Experimental turbulent transition study

Study the transition from flat, burner-stabilized flames to inclined flames.

4. Unify flamelet and Reynold-averaged models

Write a 2-D model valid during the transition from flat to inclined flame. What is the effect of turbulent flame thickness on the flamelet analysis?
Questions