Chapter 7: Dual Modeling in the Presence of Constant Variance

7.A Introduction

An underlying premise of regression analysis is that a given response variable changes systematically and smoothly due to an underlying functional relationship with a set of regressor variables. The primary goal of the analysis is to provide an estimate of this functional relationship. Traditionally, a key assumption in developing an estimate of the underlying function has been that the conditional variance of the response variable remains constant over all combinations of the regressor variables. If the assumption of constant variance is not valid for a particular process, there are many types of analyses to choose from which are designed to account for this variance heterogeneity. Ignoring heteroscedasticity not only leads to an estimate of the regression function which is less precise than an analysis which accounts for the non-constant variance but it also leads to faulty inferences regarding the underlying function.

Dual model analysis is one approach to analyzing heteroscedastic data. The distinguishing characteristic of dual model analysis is the assumption that, like the response variable of interest, the variance of this response also changes systematically and smoothly due to an underlying functional relationship with a set of regressor variables. A question which has not been addressed in the literature concerns the repercussions of a dual model analysis when the assumption of non-constant variance has been violated. In other words, if the variance of the response variable is indeed constant, what impact is there when the data is analyzed via dual modeling techniques? The discussion in this chapter seeks to address this question for the four dual modeling techniques being considered in this research (Parametric, Nonparametric Residual-based, Nonparametric Difference-based, and Dual Model Robust Regression).

7.B Model Assumptions and Motivation

To study this issue, we will consider data which is generated from the underlying model

$$y_i = 2(x_i - 5.5)^2 + 5x_i + \gamma \sin \left( \frac{\pi(x_i - 1)}{2.25} \right) + \varepsilon_i$$

(7.B.1)

where $\gamma \in [0,1]$ and $\varepsilon_i \sim N(0,16)$. Notice that, apart from the error term, this is the same model for the response considered in Examples 1, 2, and 3 of Chapter 6. If $\gamma = 0.0$, the optimal approach to data analysis would be to specify the following quadratic model

$$y_i = \beta_0 + \beta_1x_i + \beta_2x_i^2 + \varepsilon_i$$

(7.B.2)
where $\varepsilon_i \sim \text{N}(0, \sigma^2)$, and perform ordinary least squares. However, we will consider the scenario in which the user assumes non-constant variance and approaches the analysis from a dual modeling perspective. For purposes of this discussion, we will assume that the following models are specified by the user in the parametric and DMRR procedures:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + g^{1/2}(z_i; \theta) \varepsilon_i$$  \hspace{1cm} (7.B.3)$$

$$\sigma_i^2 = g^{1/2}(z_i; \theta) = \exp\{\theta_0 + \theta_1 z_i + \theta_2 z_i^2\}.$$  \hspace{1cm} (7.B.4)$$

Recall that there are two basic philosophies regarding variance modeling; the residual-based philosophy and the difference-based philosophy. The residual-based philosophy takes the position that the correct data to be used in modeling the variance is the set of residuals from the fit to the process mean. The idea is that once the effect of the mean has been removed from the raw data, any remaining structure is due to process variability. Thus, if the variance of the response is constant, then the residuals from the means fit will contain no structure and the estimated variance will be that of a constant function. In the case of parametric dual modeling, the hope is that the EWLS fit to the mean will remove all of the structure in the raw data, leaving unstructured EWLS residuals. If the set of squared EWLS residuals have no structure, then the analysis of the model

$$e_i^{2(\text{ewls})} = \exp\{\theta_0 + \theta_1 z_i + \theta_2 z_i^2\} + \eta_i$$  \hspace{1cm} (7.B.5)$$

should result in estimates of $\theta_1$ and $\theta_2$ which are not significantly different than 0, implying constant process variance ($\sigma_i^2 = \exp\{\theta_0\}$). As a result, the EWLS fit will use constant weights and thus should be very close to the fit produced by OLS. However, as we have seen, something that is critical to the success of residual-based variance estimation is that the residuals from the means fit are free from bias. In Examples 1 and 3 of Chapter 6 it was observed that even slight misspecification of the means model introduced enough bias to dramatically influence the variance model analysis. Thus, as we will see for $\gamma > 0$, parametric dual modeling will have the same problems encountered in Chapter 6.

Nonparametric residual-based variance estimation utilizes the residuals from a an LLR fit to the mean and LLR, while somewhat variable in its fit to the mean, does not suffer from the bias problems experience with EWLS. The main drawback to nonparametric residual-based variance estimation is that the variance fit is often quite variable due to the absence of a specified function to stabilize it. Thus, although there may be no obvious structure in the squared LLR residuals, the nonparametric residual-based estimate may indicate some variance structure simply due to the variation inherent in nonparametric function estimation.

The other residual-based variance estimate is the DMRR variance estimate known as VMRR. Unlike the estimate of variance obtained in parametric dual modeling, VMRR is robust to the user’s specification of a process means model. This is due to the fact that VMRR relies on the squared MMRR residuals to model variance. Thus, regardless of the form of the underlying
means function, VMRR uses data which is essentially bias free. If the process variance is constant, the MMRR residuals should have little or no structure and the analysis of

$$e_i^{2(\text{mmrr})} = \exp\{\theta_0 + \theta_1 z_i + \theta_2 z_i^2\} + \eta_i$$  \hspace{1cm} (7.B.6)

will result in estimates of $\theta_1$ and $\theta_2$ which are not significantly different than 0. Thus, the parametric portion of VMRR should resemble $\exp\{\hat{\theta}_0^{(\text{mmrr})}\}$. Since the process variance is constant, there is no need to mix parametric and nonparametric techniques to provide the final VMRR estimate. Recall that this was only done when the parametric fit failed to capture part of the trend in the underlying variance function and the addition of a nonparametric portion was necessary to recapture this “lack-of-fit”. Thus the optimal value for $\lambda\sigma_o$ should be close to zero and the final VMRR estimate will be given as $\hat{\sigma}_i^2(\text{vmrr}) = \exp\{\hat{\theta}_0\}$.

The philosophy of difference-based variance estimation is that information about the process variance should not depend on first estimating the mean. Instead variance information is assumed to be contained in the pseudo-residuals which are constructed by taking differences of the raw data in successive neighborhoods of the x-space. If the process variance is constant, these difference measures (the pseudo-residuals) should not change much from neighborhood to neighborhood. The hope is that the local linear fit to these squared pseudo-residuals will indicate very little, if any structure in the underlying variance function.

Comparisons of the dual modeling techniques will follow the same format used for the examples in Chapter 6. First, the procedures will be compared graphically using a given data set. Then, more general comparisons will be made based on the IVMSE and IMMSE values for each of the procedures.

### 7.C Graphical Comparisons

To compare the dual modeling procedures graphically, consider a random data set generated from the underlying model

$$y_i = 2(x_i - 5.5)^2 + 5x_i + 7.5 \sin\left(\frac{\pi (x_i - 1)}{2.25}\right) + \epsilon_i$$  \hspace{1cm} (7.C.1)

at 61 evenly spaced x-values from 0 to 10, where $\epsilon_i \sim N(0,1)$. Figure 7.C.1 provides a scatter plot of the raw data along with the true underlying function. The EWLS, LLR, and MMRR estimates of the underlying function are pictured in Figures 7.C.2, 7.C.3, and 7.C.4, respectively. All three estimates are overlayed in Figure 7.C.5. Notice that as expected, the EWLS fit is smooth but it fails to capture the dips which were introduced into the true function by the sine
Figure 7.C.1  Scatter plot of a random data set generated from the model in equation 7.C.1.

Figure 7.C.2  Plot of true underlying function and generated data set along with the EWLS fit.
Figure 7.C.3  Plot of true underlying function and generated data set along with the LLR fit.

Figure 7.C.4  Plot of true underlying function and generated data set along with the MMRR fit.
The LLR and MMRR estimates are almost equivalent and do a good job capturing the structure of the underlying function.

Recall that for dual modeling procedures which rely on residual-based variance estimation, a well estimated means function is critical. In fact, for many practical situations, dual modeling only occurs after the researcher detects heteroscedasticity through plots of the means model residuals. If the process variance is constant and the means model has been fit with little or no bias, then no distinct patterns will be revealed in the residuals. In such cases, the researcher generally assumes constant variance and dual model analysis will not be considered.

One common plot used to detect heteroscedasticity is to plot the logarithm of the squared means model residuals versus those variables thought to influence process variance. Figures 7.C.6 and 7.C.7 show these plots for residuals corresponding to the EWLS and MMRR fits (respectively) to the data set currently under consideration. A plot of the logarithm of the squared LLR residuals is not given since LLR and MMRR give practically identical fits to the raw data. Its not surprising that the plot involving the MMRR residuals do not reveal any distinct patterns because the process variance is constant and there was very little bias in the MMRR means fit. However, notice the quadratic pattern in Figure 7.C.6. The observed trend in Figure 7.C.6 can be checked for significance by a simple OLS analysis of the model

\[
\ln\left(\epsilon_i^2\right) = \theta_0 + \theta_1 z_i + \theta_2 z_i^2 + \varepsilon_i. \quad (7.C.2)
\]
The OLS analysis of the model in (7.C.2), using the current set of squared EWLS residuals, reveals that the quadratic term is significant at the 0.0026 level and the linear term is significant at the 0.0035 level. Given this result, the user is led to believe that the data is heteroscedastic and the models given in equations (7.B.3) and (7.B.4) appear appropriate for describing the underlying process. However, this is clearly the wrong approach for the analysis for two reasons: first, the user has misspecified the form of the underlying means function and second, the user is assuming non-constant variance when in actuality, the underlying process variance is constant.

![Figure 7.C.6](image_url)

Figure 7.C.6  Plot of $\ln(e_i^{2(\text{ewls})})$ vs. $z_i$
For comparison purposes, a similar analysis was done for the model in (7.C.2) using the set of $n \ln(e_{i}^{2\text{ (mmrr)}})$ as data. In this case, a p-value of 0.2409 was associated with the quadratic term, while the p-value associated with the linear term was 0.1613. With such large p-values, the user very likely would assume constant variance and dual modeling would never have been considered. This example illustrates an important result. The potential mistake of conducting dual model analysis in the wake of constant variance can often be avoided simply by doing a good job of estimating the underlying regression function.

Assuming that the researcher actually prescribes a dual model analysis for the current data set, the parametric, nonparametric residual-based, nonparametric difference-based, and VMRR variance estimates are plotted in Figures 7.C.8, 7.C.9, 7.C.10, and 7.C.11, respectively. Figure 7.C.12 shows all four competing variance estimates plotted simultaneously. Notice the difference in the parametric and VMRR (based on $b_{\sigma_{o}} = 1.0$ and $\lambda_{\sigma_{o}} = 0.0$) estimates of the variance function. This difference reinforces the results found in the two OLS analyses of the model in (7.C.2) (one analysis used the set of $n \ln(e_{i}^{2\text{ (ewls)}})$ and the other used the set of $n \ln(e_{i}^{2\text{ (mmrr)}})$ as data). The parametric variance estimate appears to have a highly significant quadratic pattern while the VMRR variance estimate follows only a slight (insignificant) quadratic pattern. The nonparametric residual-based estimate (based on $b_{\sigma_{o}} = 0.07799$) also follows a slight quadratic pattern.
pattern but the fit is not as smooth (more variable) than the VMRR estimate. For this example, the nonparametric difference-based variance estimate (based on $b\sigma = 1.0$) offers the best estimate of the underlying variance function. However, keep in mind that the comparisons made thus far have only been based on one of infinitely many possible data sets that can be generated from the model in (7.B.1).

A more complete comparison of the four dual modeling procedures can be made by comparing the MMIMSE and VMIMSE values for various sample sizes and different degrees of model misspecification. In the next section, the MMIMSE and VMIMSE

![Figure 7.C.8](image)

**Figure 7.C.8** Plot of the parametric variance estimate using the squared EWLS residuals from the means fit shown in Figure 7.C.2.
Figure 7.C.9  Plot of the nonparametric residual-based variance estimate using the squared LLR residuals from the fit shown in Figure 7.C.3.

Figure 7.C.10  Plot of the nonparametric difference-based variance estimate using the squared pseudo-residuals formed from the raw data.
Figure 7.C.11  Plot of the VMRR variance estimate using the squared MMRR residuals from the fit shown in Figure 7.C.4.

Figure 7.C.12  Plot of the parametric, nonparametric residual-based, nonparametric difference-based, and VMRR estimates of variance.
values will be compared for three different sample sizes \((n = 21, 41\) and 61) and five different degrees of model misspecification \((\gamma = 0, 2.5, 5.0, 7.5\) and 10.0). The accuracy of the theoretical MMIMSE and VMIMSE values will be checked by comparing these values to their associated simulated integrated mean square error values.

### 7.D Numerical Comparisons

Tables 7.D.1 and 7.D.2 provide the numerical results of interest for this study. Notice from Table 7.D.2 that except for moderate to extreme model misspecification \((\gamma \geq 5.0)\) in the small sample case \((n = 21)\), the optimal mixing parameter used in the VMRR variance estimate \((\lambda, \sigma_o)\) is equal to zero. This implies that the parametric portion of the VMRR estimate is sufficient and little or no structure is found in the set of residuals from the parametric fit. Recall that this is the desired value of \(\lambda, \sigma_o\) in the case of constant process variance. Also note from Table 7.D.2 the values of \(b, \mu, \lambda, \sigma_o\) for the model robust fit to the mean (MMRR). As the quadratic model becomes less adequate in describing the underlying regression function \((\gamma > 0)\), the value of \(b, \mu\) becomes smaller and \(\lambda, \mu, \sigma_o\) gets closer to one. This is also a promising result since greater structure in the residuals from the parametric fit to the data (a result from \(\gamma > 0)\) needs a smaller and smaller bandwidth to capture the structure. Once the structure (the bias) in the parametric fit is recaptured, we want the mixing parameter, \(\lambda, \mu, \sigma_o\), to be close to one in order to add back this lack of fit.

The most interesting results are found in Table 7.D.1 where the MMIMSE and VMIMSE values are given for each of the procedures. First, notice how the MMIMSE and VMIMSE values (given in bold) get closer to their respective simulated integrated mean square error values as the sample size gets larger. This not only provides a sense of the asymptotic behavior of the theoretical values but it also provides a check for the accuracy of the formulas from which the MMIMSE and VMIMSE values were obtained. Another interesting point can be observed by comparing the parametric and VMRR MMIMSE values to the MMIMSE value for OLS when there is no model misspecification \((\gamma = 0)\). Recall that OLS should provide the optimal MMIMSE here since it assumes constant variance in developing its estimate of the mean. However, the parametric and MMRR MMIMSE values are extremely close to that for OLS. This indicates that although a variance function was specified in the parametric and VMRR procedures, the variance function was estimated to be nearly constant for each procedure. The MMIMSE values for LLR are more than twice than those for OLS, parametric dual modeling, and DMRR when \(\gamma = 0)\).

A result that was observed in all three examples of Chapter 6 is again found in this example. Notice that as \(\gamma\) increases in magnitude from zero (model misspecification), the quality of the OLS and parametric estimates of the underlying regression function greatly diminishes. In fact, with only slight model misspecification \((\gamma = 2.5)\) and a large sample
Table 7.C.1 Simulated mean square error values for optimal mean and variance fits from 300 Monte Carlo runs (Means model misspecified and variance is constant). Theoretical MMIMSE and VMIMSE values are in bold.

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Table 7.C.2  Optimal bandwidths and mixing parameters chosen by minimizing the AVEMMSE and AVEMMSE values for the nonparametric and model robust procedures in the constant variance scenario.

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size \( n = 61 \), the MMIMSE values for OLS and parametric dual modeling are almost double the MMIMSE value for MMRR. Also notice that as \( \gamma \) increases, the performance of the LLR estimate approaches the quality of the MMRR estimate. Overall though, MMRR outperforms the other dual modeling procedures in terms of MMIMSE. This overall best performance is also seen with VMRR, the variance component of the dual model robust procedure. The true VMIMSE values, as obtained from the simulation study, for VMRR are consistently superior to those VMIMSE values for each of the other dual modeling procedures.