CHAPTER 3
THEORY

3.1. ASSUMPTIONS

The following assumptions are used in the development of the model:

1. The flow is steady and uniform, and the limiting case of a maximum (or forming) flood is considered.
2. The channel is straight, of constant cross-section, and symmetrical with respect to it center. It has a central flat-bed region of width $B_f$, two curving bank regions of total width $B_s$, a center depth $D_c$, and a longitudinal slope $S$. Figure 2.4 is a definition diagram of the assumed channel cross-section.
3. The channel is composed of the same sediment all over its boundary. The sediment is specified by a coefficient of friction $\mu$, lift-to-drag ratio $\beta$, and grain sizes $d_{90}$ and $d_{50}$; the grain sizes such that 90 and 50% of the sediment is finer respectively.
4. The location of any point on the boundary of the channel is defined by its vertical depth below the water surface $D$, and its distance from the channel center $y$. The sign convention followed by the model is shown in Figure 2.4.
5. Shear stress $\tau$ acting on the channel boundary is expressed in the form of stress-depth $\delta = \tau/\rho g S$.
6) Terms are non-dimensionalized to make the solutions more generally applicable. They are indicated by the superscript $^*$. It is assumed that the channel will support bedload transport only, and that this bedload transport will occur over the entire flat-bed region. The curving banks are assumed to be stable, with all bank particles in a state of incipient motion. For this to happen, the stress distribution must be above critical over the flat-bed region, and at critical over the entire bank region.

The scheme that determines the geometry and stress-depth distribution of an optimal stable gravel channel is divided into three parts. Firstly, the profile of the bank is generated by solving the coupled equations of momentum-diffusion and force-balance, for a given value of critical stress-depth in the flat-bed region. Once the bank profile has been generated, the actual and critical stress-depth distributions along the bank can be determined by means of the momentum-diffusion equation.

Secondly, the width of the bed region which corresponds to this bank profile is determined. This is done by applying the momentum-diffusion equation from the center of the flat-bed region to the junction point. In addition, it is required that the values of stress-depth and lateral momentum flux yielded at the junction point, match the corresponding junction point values resulting from the generated bank profile. Once matching conditions at the junction point are achieved, the correct flat-bed width and stress-depth distribution will have been determined.

Thirdly, the bed and bank regions are put together, and the momentum-diffusion equation is applied to determine the actual stress-depth distribution over the entire channel half-width. The actual stress-depth distribution has to satisfy two boundary conditions. At the center of the channel, the slope of the stress-depth distribution must be equal to zero. At the water margin, the stress depth must be equal to zero.
3.2. DETERMINATION OF BANK PROFILE

The momentum-diffusion equation gives the actual shear stress exerted by the flowing water on a sediment particle lying on the boundary of the channel. It can be expressed in dimensionless form as follows (Diplas and Vigilar, 1992):

\[
\delta^* = A_0^{0.5} D^* \left(1 + \frac{1}{2} A_1 \right) + D^* \psi \frac{d^2 \delta^*}{dy^*^2} + \left( A_2 + 2 \right) \psi D^* \frac{dD^*}{dy^*} + D^* \frac{d\psi}{dy^*} \frac{d\delta^*}{dy^*}
\]  

(2)

where

\[
\delta^* = \frac{\delta}{D_c}; \text{ dimensionless stress depth}
\]

\[
D^* = \frac{D}{D_c}; \text{ dimensionless vertical depth}
\]

\[
y^* = \frac{y}{D_c}; \text{ dimensionless lateral distance from center of channel}
\]

\[
\psi = \left( \frac{4 + j}{24(2 - j)} \right) \ln \left( \frac{30 D^*}{k} \right) - \frac{5}{36(2 - j)} \left( 1 + \frac{1}{2 \ln \left( 30 D^*_n/k \right) - \frac{17}{3}} \right)
\]

\[
j = -D_n \frac{d^2 D^*}{dy^*} \left( 1 + \left( \frac{dD^*}{dy^*} \right)^2 \right)^{1.5}
\]

k = equivalent sand grain roughness

D_n = normal channel depth

\[
\frac{d\psi}{dy^*} = \left( \frac{10 - 3A_4(4 - A_2)}{36(A_2 + 2)(2A_4 - \frac{17}{3})^2} \right) \left( \frac{A_5}{A_3} \right) \left( \frac{30}{k^*} \right) + \left( 1 + \frac{1}{2A_4 - \frac{17}{3}} \right) C
\]

k^* = \frac{k}{D_c}; \text{ dimensionless equivalent sand grain roughness}

\[
C = \left( \frac{4 - A_2}{24(A_2 + 2)} \right) \left( \frac{A_5}{A_3} \right) \left( \frac{30}{k^*} \right) + \left( \frac{5}{36} - \frac{A_4}{4} \right) \left( \frac{A_5}{(A_2 + 2)^2} \right)
\]

A_1 = \left( \frac{dD^*}{dy^*} \right)^2 + 1

A_2 = \frac{D^* \frac{d^2 D^*}{dy^*^2}}{A_1 \ \frac{dy^*^2}{dy^*}}

A_3 = \frac{30D^*A_1^{0.5}}{k^*}

A_4 = \ln(A_3)

A_5 = (A_2 + 1)A_1^{0.5} \frac{dD^*}{dy^*}
The force-balance equation gives the critical stress at a given point on the channel boundary. The form of the force-balance equation used here is derived from the expression developed by Ikeda (1982):

\[
\delta_{crb}^* = \left( -\frac{r\mu}{A_i^{0.5} + A_8^{0.5}} \right) \delta_{cr}^*
\]

where

\[
\delta_{crb}^* = \frac{\delta_{crb}}{D_c}; \text{ dimensionless critical stress depth at a point on the bank}
\]

\[
\delta_{cr}^* = \frac{\delta_{cr}}{D_c}; \text{ dimensionless critical bed stress depth}
\]

\[
\delta_{cr} = \text{critical bed stress depth}
\]

\[
r = \mu \beta
\]

\[
\mu = \text{submerged coefficient of static friction of channel material}
\]

\[
\beta = \text{lift-to-drag ratio}
\]

\[
A_7 = \mu^2 - r^2 + 1
\]

\[
A_8 = \frac{A_7}{A_1} + r^2 - 1
\]

Since every particle on the bank of an optimal stable channel is in a state of incipient motion, they each experience critical stress. Thus, by combining Equations 2 and 3, critical stress is ensured over the entire bank region. The resulting equation is thus the governing equation for determining the profile of the bank region.

\[
\frac{d^3D^*}{dy^{*3}} = \frac{D_{3a} + D_{3b}}{D_{3c}}
\]

where

\[
D_{3a} = \frac{\delta^*}{\psi D^2} - \left( \frac{(\psi_1 + A_{12}) + dD^* (A_2 + 2)}{\psi D^*} \right) \frac{d\delta^*}{dy^*} - \frac{A_1^{0.5}}{\psi D^*} \left( 1 + \frac{1}{2} A_2 \right)
\]

\[
D_{3b} = \left\{ 2 \left( \frac{dD^*}{dy^*} \frac{d^2D^*}{dy^{*2}} \right) \left[ \frac{1.5r\mu}{A_i^{2.5}} + A_7 \left( \frac{A_7}{2A_i^{1.5} A_{1}^{4}} - \frac{2}{A_8^{0.5} A_i^{3}} \right) \right] - A_9 \left( \frac{d^2D^*}{dy^{*2}} \right)^2 \right\} \delta_{cr}^* \left( 1 - r\mu \right)
\]

\[
D_{3c} = \frac{A_9 \delta_{cr}^*}{(1 - r\mu)} \frac{dD^*}{dy^*} + \left( \frac{A_{10} D^*}{4} \right) \frac{A_i \psi}{\psi_D^*}
\]

\[
\psi_1 = \frac{10 - 3A_4 (4 - A_2)}{36 (A_2 + 2)(2A_i - \frac{12}{7})} \left( \frac{A_5}{A_3} \right) \left( \frac{30}{\bar{k}} \right)
\]
This is a non-linear, non-homogeneous, third order ordinary differential equation. Normally, such an equation would require three boundary conditions to be solved. However, the width of the bank is initially unknown. This means that the location of the water margin, where boundary conditions are to be specified, is not yet known. Thus an additional boundary condition is necessary to solve the governing equation. The four boundary conditions are as follows:

1. dimensionless depth, \( D^* = 1 \). 
2. lateral slope, \( dD^*/dy^* = 0 \). 

The first boundary condition at the junction point is due to the fact that the water depth at that point is equal to the center depth \( D_c \). Thus by definition, \( D^* = D_c/D_c = 1 \). The second boundary condition stems from the fact that the lateral slope at the junction point is zero, since it is the point where the flat-bed region begins.

At the water margin, \( y^* = (B^*_t + B^*_s)/2 \):

3. \( D^* = 0 \). 
4. \( dD^*/dy^* = -\mu \). 

The water margin is the point at which the water surface and channel boundary meet. Thus the water depth is zero at this point; yielding the first boundary condition there. The second boundary condition is based on the assumption that at the water margin, the sediment composing the boundary makes an angle with the horizontal equal to its angle of repose \( \phi \). This means that the lateral slope at that point is equal to the tangent of \( \phi \), which happens to be the sediment’s coefficient of friction \( \mu \). For consistency with the sign convention that has been adopted (positive downward and to the right), a negative sign must be affixed to \( \mu \).

In order to determine the bank profile of an optimal stable channel, the following parameters must be specified: coefficient of friction \( \mu \), lift-to-drag ratio \( \beta \), grain sizes \( d_{90} \) and \( d_{50} \), critical bed stress-depth \( \delta_c^* \), dimensionless critical bed stress \( \tau_c^* \) (\( = \delta_c S/R_s d_{50} \) where \( R_s = (\rho_s - \rho)/\rho \), \( \rho_s \) being the sediment’s mass density), water discharge \( Q \), and longitudinal channel slope \( S \) or sediment discharge \( Q_s \). Once these are given, a Runge-Kutta-Merson scheme can be implemented to solve Equation 4. This numerical method will be discussed more thoroughly in the next chapter. At this point, it is sufficient to say that by using the aforementioned method, values of dimensionless depth \( D^* \), and its derivatives \( dD^*/dy^* \), \( d^2D^*/dy^*2 \), and \( d^3D^*/dy^*3 \), can be obtained at grid points along the bank. These are used to generate the bank profile. They are also back-substituted in Equations 2 and 3 to solve for dimensionless stress-depth \( \delta^* \), and its derivatives \( d\delta^*/dy^* \) and \( d^2\delta^*/dy^*2 \); as well as dimensionless critical stress-depth \( \delta_{crb}^* \), and its derivatives...
d\(\delta_{cr}^*/dy^*\), and \(d^2\delta_{cr}^*/dy^{*2}\). With these, the actual and critical stress-depth distributions for the bank region can be plotted.

It should be noted that the generated bank profile satisfies both the momentum-diffusion and force-balance equations. It therefore follows that the actual stress-depth distribution for this bank profile should coincide with its critical stress-depth distribution.

### 3.3. DETERMINATION OF BED WIDTH

The governing equation for the flat-bed region is Equation 2. This is a non-homogeneous, second order, ordinary differential equation, which normally would require two boundary equations. However, the width of the flat-bed region still needs to be determined. Thus, three boundary conditions are required. They are as follows:

At the channel's center, \(y^* = 0\):

1. stress-depth gradient, \(d\delta^*/dy^* = 0\). (9)

The preceding boundary condition results from the symmetry of the channel cross-section about its center. This dictates that the channel’s stress-depth distribution be symmetrical about the channel’s center as well. Therefore, the slope of the stress-depth distribution has to be zero at that point.

At the junction of the bed and bank regions, \(y^* = B^*/2\):

2. dimensionless stress depth, \(\delta^* = \delta_{cr}^*\), and (10)
3. lateral momentum flux, \(\psi D^* = \{(dD^*/dy^*)^2+1\}(d\delta^*/dy^*)\) are continuous. (11)

These last two conditions provide matching between the bed and bank solutions. It must be mentioned that the last condition is different from that used by Parker (1978). He specified the stress-depth gradient to be continuous at the junction point. Analysis of the boundary condition to be used for this model will show that stress-depth gradient is, in fact, discontinuous at the junction point. This matching condition is analogous to that used for the case of steady heat conduction at the interface of a composite medium (Stakgold, 1968). At the interface, it is not temperature gradient which is continuous but rather, heat flux (which is a function of temperature gradient). The discontinuity in temperature gradient is due to the material discontinuity that occurs at the interface. In a similar manner, at the junction point of a channel, it is not stress-depth gradient which is continuous but rather, lateral momentum flux (which is a function of stress-depth gradient). The discontinuity in the stress-depth gradient at the junction point is due to the geometric discontinuity that occurs there; the curvature at that point is different depending on whether the point is approached from the side of the bed region or that of the bank region. Since \(d^2D^*/y^{*2}\) is an indication of curvature, it follows that it will be discontinuous at the junction point. This in turn causes \(\psi\) in Equation 11 to have two values at the junction point. Analyzing Equation 11, it can be seen that aside from \(\psi\) and stress-depth gradient \(d\delta^*/dy^*\), all other terms are continuous at the junction point. Thus, for Equation 11 to hold, \(d\delta^*/dy^*\) must be discontinuous at the junction point.

Over the flat-bed region, \(D^* = 1\), and \(dD^*/dy^* = d^2D^*/y^{*2} = d^3D^*/y^{*3} = 0\). However, because the bank starts to curve upward at the junction point, there is a sudden jump in the value of \(d^2D^*/y^{*2}\) there; from zero to some negative value. For this to happen, the third derivative \(d^3D^*/dy^{*3}\) at the junction point must experience a "surge" in magnitude. In order to take this behavior into
account, a modified Dirac-delta function is used to represent \( \frac{d^3D}{dy^3} \) over a very tiny interval just before the junction point (Figure 3.1).

\[
\frac{d^3D^*}{dy^3} = \frac{1 + \cos\left(\pi x^*/\Delta^*\right)}{\Delta^*} \left( \frac{d^2D^*}{dy^2} \right)_{y^* = y^*_j} + \frac{(x^* + \Delta^*)}{\Delta^*} \left( \frac{d^3D^*}{dy^3} \right)_{y^* = y^*_j} \quad \text{for } |x^*| < \Delta^*/2
\]

(12)

where

\( \Delta^* = \) dimensionless delta function interval

\( y^*_j = \) dimensionless lateral distance of junction point from center of channel

\( x^* = y^* - y^*_j + \Delta^*/2 \); dimensionless distance from center of delta function interval

Figure 3.1 Modified Dirac delta function
This will cause the value of \(d^2D/dy^2\) to change from zero at the beginning of the interval, to a negative value at the end of the interval (the junction point). The negative value for \(d^2D/dy^2\) at the end of the interval must coincide with the junction point value of \(d^2D/dy^2\) determined in the bank solution. Thus, the Dirac-delta function will depend on this value of \(d^2D/dy^2\).

The Dirac-delta function is modified in the sense that it is not symmetrical like the conventional Dirac-delta function. The reason for this is that at the beginning of the interval, the value of \(d^3D/dy^3\) is zero, however, at the end of the interval, the bank solution yields a \(d^3D/dy^3\) value which is, in general, not equal to zero. This non-zero value at the junction point cannot be ignored since it is significantly large compared to other \(d^3D/dy^3\) values along the bank region. However, this junction point value is negligible compared to the peak \(d^3D/dy^3\) value given by the delta function (which is of the order \(10^6\)). Thus, even though the delta function is modified in order to take this non-zero value of \(d^3D/dy^3\) at the junction point into account, the resulting function still practically satisfies the following requirements of a true Dirac-delta function (Bender and Orzag, 1978):

\[
\int_{-\infty}^{\infty} d^*(y^* - a^*) dy^* = 1 \quad (13a)
\]
\[
d^*(y^* - a^*) = 0 \quad \text{for} \quad y^* \neq a^* \quad (13b)
\]
\[
\int_{-\infty}^{\infty} f^*(y^*) d^*(y^* - a^*) dy^* = f^*(a^*) \quad \text{if} \quad y^* \text{is continuous at} \quad y^* = a^* \quad (13c)
\]

The flat-bed width of the optimal stable channel can be determined using an analytical solution for Equation 2 (the momentum-diffusion equation). Recall that anywhere on the flat-bed region, \(D^* = 1\), and \(dD^*/dy^* = d^2D^*/dy^2 = d^3D^*/dy^3 = 0\). Substituting these in Equation 2, and then solving it using the first two boundary conditions, yields the solution

\[
\delta^* = 1 + \frac{(\delta^*_c - 1)}{\cosh \left( \frac{B_f^*}{2\sqrt{\psi_0}} \right)} \cosh \left( \frac{y^*}{\sqrt{\psi_0}} \right) \quad (14)
\]

where

\[
\psi_0 = \left( \frac{1}{12} \ln \left( \frac{30}{k^*} \right) - \frac{5}{72} \right) \left( 1 + \frac{1}{2\ln(30/k^*)^\frac{17}{5}} \right)
\]

Equation 14 gives the dimensionless stress-depth at any point on the flat-bed, and thus can be used to generate the stress-depth distribution over that region. The only thing that needs to be done is to determine the flat-bed width \(B_f^*\).

Differentiating Equation 14 once with respect to \(y^*\) yields an expression for \(d\delta^*/dy^*\) in terms of \(\delta^*_c\) and \(B_f^*\).

\[
\frac{d\delta^*}{dy^*} = \frac{1}{\sqrt{\psi_0}} \frac{(\delta^*_c - 1)}{\cosh \left( \frac{B_f^*}{2\sqrt{\psi_0}} \right)} \sinh \left( \frac{y^*}{\sqrt{\psi_0}} \right) \quad (15)
\]

The lateral momentum flux on the bank side of the junction point is computed using the junction point value of \(d\delta^*/dy^*\) from the bank solution. Using the condition of continuous lateral
momentum flux at the junction point, the value of $d\delta^*/dy^*$ on the bed side of the junction point is obtained. Plugging this value of $d\delta^*/dy^*$, as well as that of $\delta_{cr}^*$, into Equation 15, and noting that at the junction point $y^* = B_j^*/2$,

$$
\frac{d\delta^*}{dy^*}_{jbd} = \frac{1}{\sqrt{\Psi_0}} (\delta_{cr} - 1) \tanh \left( \frac{B_j^*}{2\sqrt{\Psi_0}} \right)
$$

(16)

where

$$
\frac{d\delta^*}{dy^*}_{jbd} = \text{dimensionless stress-depth gradient at junction point; bed side}
$$

This equation can be rearranged and solved to give the value of $B_j^*$ directly:

$$
B_j^* = \sqrt{\Psi_0} \ln \left( \frac{1 + m}{1 - m} \right)
$$

(17)

where

$$
m = \frac{d\delta^*}{dy^*}_{jbd} \sqrt{\Psi_0} / (\delta_{cr}^* - 1)
$$

Once the flat-bed width is determined, the stress-depth distribution over the flat-bed region can be derived using Equation 14.

3.4. DETERMINATION OF STRESS-DEPTH DISTRIBUTION

The governing equation for this last phase of the solution is Equation 2. The stress-depth distribution over the entire channel boundary is generated by solving Equation 2 from the center of the channel to the water margin. Since the bank and bed widths of the optimal stable channel have already been determined, the total width of the channel is now known. Thus, only two boundary conditions are necessary.

At the center of the channel, $y^* = 0$:

1. $d\delta^*/dy^* = 0$. (18)

   This first boundary condition is again due to the symmetry of the stress-depth distribution about the center of the channel.

   At the water margin, $y^* = (B_j^* + B_i^*)/2$:

2. $\delta^* = 0$. (19)

   The second boundary condition results from the water depth being zero at the water margin. Substituting this value of depth in Equation 2 leads to a stress-depth equal to zero at the water margin.

   The stress-depth distribution for the entire channel width can actually be obtained simply by connecting the stress-depth distributions obtained from the bank and bed solutions. However, the aforementioned procedure is performed in order to verify that this composite stress-depth distribution is correct.