Chapter 10

Bayesian Designs for the Two Regressor Model

§10.1 Motivation

While optimal and conditionally optimal designs can be found in terms of IECs, the main effect parameters must still be known in order to convert the IECs to natural units. Some information is often available about these parameters because they only involve knowledge about each regressor individually. However, unless a great deal of experimentation has been previously performed, the researcher will not be able to provide an accurate guess for each parameter. This inability to accurately determine the values of $\beta_1, \cdots, \beta_k$ brings about the need for a design which accounts for the variability around these guesses. This can be accomplished via the same Bayesian design techniques used in the single regressor case. Recall that either a uniform or a normal prior was placed on $\beta_1$ as developed from a range on an EC given by the experimenter. Designs were then cataloged by the ratio of the endpoints of the uniform distribution or the value of sigma in the normal case. In this chapter, the Bayesian D-optimal designs for the two regressor no interaction and interaction models will be considered. The uniform prior on each of the parameters is used.
For the $k$-regressor case independent uniform priors will be placed on $\beta_1, \ldots, \beta_k$. The Bayesian D-optimal design is the one which maximizes the expected value of the negative of the Bayes risk over a prior as shown in (10.1.1)

$$\max_{\delta \in \Delta} \int R(\delta, \beta) \pi(\beta) d\beta$$

(10.1.1)

In (10.1.1), $R(\delta, \beta) = ||\beta||$, $\pi(\beta) = \pi_1(\beta_1) \pi_2(\beta_2) \cdots \pi_k(\beta_k)$, and $\pi_i = U(c_i, d_i)$. Note that a prior is not placed on $\beta_0$. Recall that the design for the single regressor model was invariant to the choice of the prior on the parameter $\beta_0$. The same property holds in the multiple regressor case because $\beta_0$ is not directly tied to the $x_{li}$ values in the information matrix.

§ 10.2 The Bayesian D-Optimal Design for the Two Regressor No Interaction Model

The two regressor model will serve to illustrate the principles of Bayesian design in the context of the multiple regressor model. Consider the two regressor no interaction model given in

$$y_i = c_{\beta_0} + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$$

(10.2.1)

Let $\beta_1 \sim U(c_1, d_1)$ and $\beta_2 \sim U(c_2, d_2)$. Substituting $R(\delta, \beta) = ||\beta||$ gives the Bayesian design criterion as (10.2.2)

$$\max_{\delta \in \Delta} \int \lambda_1 \lambda_2 \lambda_3 (x_{i1} x_{i2} - x_{i1} x_{23} - x_{i2} x_{21} + x_{13} x_{22} - x_{13} x_{21} + x_{12} x_{23})^2 d\beta_1 d\beta_2$$

(10.2.2)

(Note that the factor $n_1 n_2 n_3$ was removed from this expression since it is obvious that $n_1 = n_2 = n_3 = \frac{N}{3}$ is the allocation scheme which maximizes the criterion.) To simplify this expression, assume that the Bayesian design takes on the same basic structure as the non-Bayesian design, i.e. one point at the control and two pure component points. Without loss of generality, this
can be accomplished by letting \( \lambda_1 = 1 \) and \( x_{11} = x_{21} = x_{12} = x_{23} = 0 \). Expression (10.2.2) then becomes

\[
\int_p \int_p e^{\beta_1 x_{11}} e^{\beta_2 x_{22}} \left( x_{13} x_{22} \right)^2 \frac{d\beta_1 d\beta_2}{(d_1 - c_1)(d_2 - c_2)} = F_1 F_2
\]

(10.2.3)

where

\[
F_1 = \int e^{\beta_1 x_{11}} x_{13}^2 \frac{d\beta_1}{p_1(d_1 - c_1)} = \frac{x_{13}}{(d_1 - c_1)} \left( e^{d_1 x_{13}} - e^{c_1 x_{13}} \right)
\]

(10.2.4)

and

\[
F_2 = \int e^{\beta_2 x_{22}} x_{22}^2 \frac{d\beta_2}{p_2(d_2 - c_2)} = \frac{x_{22}}{(d_2 - c_2)} \left( e^{d_2 x_{22}} - e^{c_2 x_{22}} \right).
\]

(10.2.5)

So the design which maximizes \( F_1 \) and \( F_2 \) in turn maximizes the Bayesian D-optimality criterion. Fortunately, these points are not difficult to determine and, in fact, they have already been determined. Each factor, \( F_1 \) and \( F_2 \), corresponds to the Bayesian criterion in the single regressor case where the control was substituted as the second level. This means that the IECs corresponding to \( x_{13} \) based on the ratio of \( \frac{c_1}{d_1} \) and \( x_{22} \) on the ratio of \( \frac{c_2}{d_2} \) are the same as the ECs which produced the single regressor Bayesian D-optimal designs in Chapter 3. These designs are tabled again below.

Table 10.2.1. Non-Control IEC Levels for Bayesian D-Optimal Designs on the Two Regressor No Interaction Model.

<table>
<thead>
<tr>
<th>Ratio</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>IEC</td>
<td>13.53</td>
<td>12.80</td>
<td>11.44</td>
<td>9.96</td>
<td>8.55</td>
<td>7.24</td>
<td>6.07</td>
</tr>
</tbody>
</table>

These IECs are cataloged by the ratio of the endpoints of the uniform prior on \( \beta_j \) or \( \frac{c_j}{d_j} \) and thus, the entire design is determined by the 2 ratios of the uniform priors on the parameters. As one
might guess, the ratio constancy proof given in Chapter 3 applies to these designs as well based on
the ratios of \( \frac{c_j}{d_j} \) for each individual parameter.

In order to better understand how these designs would be used an example is provided. Based on two separate ranges on their respective IECs, it is determined that \( \beta_1 \sim U(-4,-2) \) and \( \beta_2 \sim U(-3,-1) \), so the ratio for \( \beta_1 \) is 2 and the ratio for \( \beta_2 \) is 3. In Table 10.2.1, the IECs listed under the ratios indicate that \( x_{13} = \text{IEC}_{11.44} \) and \( x_{22} = \text{IEC}_{8.55} \). Thus, in tabular form, the Bayesian D-optimal design for \( \beta_1 \sim U(-4,-2) \) and \( \beta_2 \sim U(-3,-1) \) is

<table>
<thead>
<tr>
<th>( i )</th>
<th>( p_i )</th>
<th>( x_{1i} )</th>
<th>( x_{2i} )</th>
<th>Contour</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>IEC(_{100})</td>
<td>IEC(_{100})</td>
<td>MEC(_{100})</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>IEC(_{100})</td>
<td>IEC(_{8.55})</td>
<td>MEC(_{8.55})</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>IEC(_{11.44})</td>
<td>IEC(_{100})</td>
<td>MEC(_{11.44})</td>
</tr>
</tbody>
</table>

The optimality of this particular design was verified using Bayesian equivalence theory. The techniques used to derived the equivalence theory function are detailed in Appendix G.

§10.3 The Bayesian D-Optimal Design for the Two Regressor Interaction Model

The interaction case follows a similar pattern in its development. The priors are placed on the main effects parameters in the same way as the no interaction design shown in expression (10.1.1). A point prior of 0 is placed on the interaction parameter, \( \beta_{12} \). Assuming a factorial structure for this design by letting \( \lambda_1 = 1 \) and \( x_{11} = x_{21} = x_{12} = x_{23} = 0 \), the Bayesian D-optimality criterion is given by

\[
\int \int \frac{e^{\beta_1 x_{13}} e^{\beta_2 x_{22}} e^{\beta_{14} x_{14} + \beta_{24} x_{24}} (x_{13} x_{22} x_{14} x_{24})^2}{(d_1 - c_1)(d_2 - c_2)} d\beta_1 d\beta_2
\]

Thus decomposing the expression into two factors, \( F_1 \) and \( F_2 \), we have
Thus, maximizing $F_1$ and $F_2$ separately will in turn maximize the expected value of the negative of the Bayes risk over the prior. Since $F_1$ and $F_2$ are essentially the same, one table is provided below which gives the values of the optimal IECs corresponding to the ratios of the endpoints of the uniform prior. While no formal proof is offered, it is easy to see that a ratio constancy proof similar to that shown in Chapter 3 for the single regressor case also holds in this case. Numerous test values of $c_j$ and $d_j$ confirm this.

<table>
<thead>
<tr>
<th>Ratio</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>IEC</td>
<td>12.08</td>
<td>9.48</td>
<td>6.95</td>
<td>4.91</td>
<td>3.41</td>
<td>2.80</td>
<td></td>
</tr>
</tbody>
</table>

Thus, in tabular form, the Bayesian D-optimal design for $\beta_1 \sim U(-4,-2)$ and $\beta_2 \sim U(-3,-1)$ which has ratios of 2 and 3 respectively is given in Table (10.3.2)
§10.4 Extensions of Bayesian Design Techniques in the $k$-regressor Model

A cursory glance at the application of Bayesian design techniques has been shown in this chapter. This glance provides a springboard for the application of these methods under a variety of circumstances. One of these circumstances is the use of a different type of prior on the main effects parameters such as a normal or gamma distribution. Bayesian designs could also be formulated in conjunction with the interaction optimality and $D_c$-optimality criteria. In addition, Bayesian designs which account for LOF testing could be developed as well as Bayesian optimal fractional factorial designs. There is no doubt that Bayesian designs which account for variability around parameter guesses would be useful in all of these situations.